Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(7): 12194–12210.
DOI: 10.3934/mbe.2023543
Received: 17 April 2023
Revised: 11 May 2023
Accepted: 12 May 2023
Published: 17 May 2023

Research article

Bifurcation analysis of a reaction-diffusion-advection predator-prey system with delay

Honghua Bin[1], **Daifeng Duan**[2,*] **and Junjie Wei**[1,2]

[1] Department of Mathematics, Jimei University, Jimei, Fujian 361021, China

[2] Department of Mathematics, Harbin Institute of Technology, Weihai, Shandong 264209, China

* **Correspondence:** Email: duandf2016@163.com.

Abstract: A diffusive predator-prey system with advection and time delay is considered. Choosing the conversion delay τ as a bifurcation parameter, we find that as τ varies, the system will generate Hopf bifurcation. Then, for the reaction diffusion model proposed in this paper, we use an improved center manifold reduction method and normal form theory to derive an algorithm for determining the direction and stability of Hopf bifurcation. Finally, we provide simulations to illustrate the effects of time delay τ and advection α on system behaviors.

Keywords: conversion delay; diffusion; advection; Hopf bifurcation

1. Introduction

To present a more realistic dynamic behavior in the predator-prey model, it is necessary to consider the addition of time delay and spatial distribution of the population in the ecosystem when modeling. Time delays play a crucial role in the stability or instability of prey and predators' densities. Therefore, predator-prey models with diffusion and time delay have received widespread attention, see [1–10]. There have been some articles introducing the bifurcation theory of delayed reaction-diffusion models describing biological or chemical reactions, see [11–17].

Due to natural phenomena such as crustal movement, volcanic eruption or human migration activities, the flow rate of aquatic animal habitats will change dramatically over time. This will change the dominant position of the species, which may lead to a population from occupying absolute advantage to coexisting with other species, or going extinct. Therefore, exploring how water flow velocity regulates the coexistence of two species is an important topic [18–21]. In summary, we establish the following predator-prey model with time delay, diffusion, and advection.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u(x,t)}{\partial x^2} - \alpha\frac{\partial u(x,t)}{\partial x} + u(x,t)[a - bu(x,t) - cv(x,t)], & x \in (0,L), t > 0, \\ \frac{\partial v(x,t)}{\partial t} = \varepsilon d\frac{\partial^2 v(x,t)}{\partial x^2} - \varepsilon\alpha\frac{\partial v(x,t)}{\partial x} + \gamma v(x,t)[u(x,t-\tau) - \beta], & x \in (0,L), t > 0, \\ d\frac{\partial u(0,t)}{\partial x} - \alpha u(0,t) = -\alpha\beta,\ u(L,t) = \beta, & t \geq 0, \\ d\frac{\partial v(0,t)}{\partial x} - \alpha v(0,t) = -\alpha\frac{a-b\beta}{c},\ v(L,t) = \frac{a-b\beta}{c}, & t \geq 0, \end{cases} \tag{1.1}$$

where $d, \varepsilon, \alpha, a, b, c, \tau, \gamma$ and β all represent nonnegative constants, α and $\varepsilon\alpha$ are the advection transport velocities of the prey and predator, respectively. The specific biological significance of other parameters can be found in reference [22]. We should point out that the diffusion coefficients (advection transport velocities) are proportionate is a need of mathematical technique.

We would like to mention that the model (1.1) with $d = \alpha = 0$ has been analysed by several researchers, see [23–25]. When $\alpha = 0$, taking the conversion rate γ from prey to predator as the bifurcation parameter, Wu [26] gave the existence of Hopf bifurcation under the Neumann boundary. In addition, under the Dirichlet boundary condition, Liu and Wei [22] studied the properties of Hopf bifurcation by selecting other parameters. In 2001, Faria [2] extended the normal form method in [27–29] to a class of predator-prey systems with delay, diffusion and Neumann boundary conditions. Due to incorporating the advection term into the predator-prey system, we must extend the method proposed by Faria. Based on this idea, we derived an algorithm to determine the properties of Hopf bifurcation, provided the direction of Hopf bifurcation and the stability and instability of the bifurcation periodic solution. When solving eigenvalue problems, the specific form of the eigenvalue cannot be directly calculated, so we use $\{\mu_n\}_{n\geq 1}$ to represent it, and the minimum eigenvalue is not 0. The corresponding characteristic functions become more complex in form compared to those without convection terms. The Laplace operator is self-adjoint, and the operator after considering advection is non self-adjoint. In the process of using the normal form method, we represented it with A_v and recalculated the adjoint operator A_v^* by combining linear equations and boundary conditions. We also investigated the impact of advection rate on the system, with advection rate α and time delay τ as the main parameters. Our main findings are that, due to factors such as diffusion, advection, and time delay, the system can generate Hopf bifurcation under mode-1, where the stable steady-state solution becomes unstable and generates spatially non-uniform oscillations.

The structure of the remaining chapters is arranged as follows. In Sections 2 and 3, we derived an algorithm that can determine the direction of Hopf bifurcation and whether the bifurcation periodic solutions are stable or not of model (1.1). In Section 4, we found that the system generates spatially non-uniform oscillations under the mode-1. In addition, under the influence of advection, the system may exhibit spatial non-homogeneous periodic oscillations or steady-state solutions.

2. Stability of positive steady state and existence of Hopf bifurcations

Obviously, the model (1.1) has a unique constant positive steady state $(\beta, (a - b\beta)/c)$ when $a > b\beta$. To ensure that the steady-state solution has a biological explanation, we assume

$$(H_1) \quad a > b\beta.$$

For the convenience of calculations, let $\tilde{u} = u - \beta$, $\tilde{v} = v - (a - b\beta)/c$. Furthermore, to simplify the symbols, remove the wavy lines on u and v. Then (1.1) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial x} - b\beta u - c\beta v - bu^2 - cuv, \quad x \in (0, L), t > 0, \\ \frac{\partial v}{\partial t} = \varepsilon d\frac{\partial^2}{\partial x^2} - \varepsilon\alpha\frac{\partial v}{\partial x} + \frac{\gamma(a-b\beta)}{c}u(x, t-\tau) + \gamma u(x, t-\tau)v, \quad x \in (0, L), t > 0, \\ d\frac{\partial u(0,t)}{\partial x} - \alpha u(0, t) = 0,\ u(L, t) = 0, \quad t \geq 0, \\ d\frac{\partial v(0,t)}{\partial x} - \alpha v(0, t) = 0,\ v(L, t) = 0, \quad t \geq 0. \end{cases} \tag{2.1}$$

The linearization of (2.1) around the origin is

$$\begin{cases} \frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial x} - b\beta u - c\beta v, \quad x \in (0, L), t > 0, \\ \frac{\partial v}{\partial t} = \varepsilon d\frac{\partial^2 v}{\partial x^2} - \varepsilon\alpha\frac{\partial v}{\partial x} + \frac{\gamma(a-b\beta)}{c}u(x, t-\tau), \quad x \in (0, L), t > 0, \end{cases} \tag{2.2}$$

Define the real-valued Sobolev space

$$X := \{(\phi, \psi) \in H^2(0, L) \times H^2(0, L) | d\frac{\partial\phi(0)}{\partial x} - \alpha\phi(0) = 0,\ \phi(L) = 0, d\frac{\partial\psi(0)}{\partial x} - \alpha\psi(0) = 0,\ \psi(L) = 0\},$$

and

$$X_{\mathbb{C}} = X \oplus \mathrm{i}X = \{y_1 + \mathrm{i}y_2 | y_1, y_2 \in X\}.$$

We have known that the eigenvalues of following problem

$$\begin{cases} d\frac{\partial^2\varphi(x)}{\partial x^2} - \alpha\frac{\partial\varphi(x)}{\partial x} = -\mu\varphi(x) \\ d\frac{\partial\varphi(0)}{\partial x} - \alpha\varphi(0) = 0,\ \ \varphi(L) = 0, \end{cases} \tag{2.3}$$

are given by $\{\mu_n\}_{n\geq 1}$ with

$$\frac{\alpha^2}{4d} < \mu_1 < \cdots < \mu_n < \mu_{n+1} < \cdots,$$

and $\lim_{n\to\infty} \mu_n = \infty$. For each n, the following equation

$$\tan\frac{L\sqrt{d\mu - \frac{\alpha^2}{4}}}{d} = -\frac{2\sqrt{d\mu - \frac{\alpha^2}{4}}}{\alpha}. \tag{2.4}$$

can determine μ_n. The associated eigenfunction of μ_k is

$$b_k = \frac{\varphi_k}{||\varphi_k||}, \tag{2.5}$$

where

$$\varphi_k = e^{\frac{\alpha}{2d}x}\Big(\cos\frac{\sqrt{d\mu_k - \frac{\alpha^2}{4}}}{d}x + \frac{\alpha}{2\sqrt{d\mu_k - \frac{\alpha^2}{4}}}\sin\frac{\sqrt{d\mu_k - \frac{\alpha^2}{4}}}{d}x\Big), k = 1, 2, \cdots,$$

and

$$||\varphi_k|| = \Big(\int_0^L \varphi_k^2 dx\Big)^{\frac{1}{2}}.$$

Then the characteristic equations of (2.2) are given by

$$\det(\lambda Id + \begin{pmatrix} \mu_k + b\beta & c\beta \\ -\frac{\gamma(a-b\beta)}{c}e^{-\lambda\tau} & \varepsilon\mu_k \end{pmatrix}) = 0, \quad k = 1, 2, \cdots .$$

That is

$$\lambda^2 + T_k\lambda + D_k + \gamma\beta(a - b\beta)e^{-\lambda\tau} = 0, \quad k = 1, 2, \cdots , \tag{2.6}$$

where

$$\begin{cases} T_k = (1 + \varepsilon)\mu_k + b\beta, \\ D_k = \varepsilon\mu_k(\mu_k + b\beta), \\ \qquad k = 1, 2, \cdots . \end{cases}$$

Clearly, $T_k > 0$ and $D_k > 0$ when $k \geq 1$. In the case of $\tau = 0$, all the roots of (2.6) have negative real parts. Therefore, the system has a locally stable steady-state solution. Let $\pm i\omega$ ($\omega > 0$) be the roots of Eq (2.6). Then we have

$$-\omega^2 + i\omega T_k + D_k + \gamma\beta(a - b\beta)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts, we obtain

$$\begin{cases} -\omega^2 + D_k + \gamma\beta(a - b\beta)\cos\omega\tau = 0, \\ \omega T_k - \gamma\beta(a - b\beta)\sin\omega\tau = 0. \end{cases} \tag{2.7}$$

It follows from (2.7) that

$$\omega^4 + (T_k^2 - 2D_k)\omega^2 + D_k^2 - (\gamma\beta(a - b\beta))^2 = 0. \tag{2.8}$$

From

$$T_k^2 - 2D_k = \varepsilon^2\mu_k^2 + (\mu_k + b\beta)^2 > 0,$$

the roots of the Eq (2.8) are given by

$$\omega^2 = \frac{1}{2}[-(T_k^2 - 2D_k) + \sqrt{(T_k^2 - 2D_k)^2 - 4(D_k^2 - (\gamma\beta(a - b\beta))^2)}]. \tag{2.9}$$

Clearly, (2.9) does not make sense when $D_k \geq \gamma\beta(a - b\beta), k \geq 1$(i.e., $D_1 \geq \gamma\beta(a - b\beta)$). That is, (2.6) has no purely imaginary roots when $a \leq b\beta + \frac{D_1}{\gamma\beta}$. Meanwhile, it follows from $\lim_{k\to\infty} D_k = \infty$ that, if $a > b\beta + \frac{D_1}{\gamma\beta}$, there exists an integer $k_0 > 1$ such that (2.9) makes sense for $1 \leq k < k_0$, and does not when $k \geq k_0$. Combining with that all the roots of Eq (2.6) with $\tau = 0$ have negative real parts, and the zero is not a root of Eq (2.6), we have the following conclusions.

Lemma 1. *Suppose* (H_1) *is satisfied.*
(i) If $a \in (b\beta, b\beta + \frac{D_1}{\gamma\beta}]$*, then all the roots of Eq (2.6) have negative real parts for all* $\tau \geq 0$*;*
(ii) If $a > b\beta + \frac{D_1}{\gamma\beta}$*, then there exists an integer* $k_0 > 1$ *such that (2.9) makes sense for* $1 \leq k < k_0$*, and does not for* $k \geq k_0$*.*

We make the following assumption:

$$(H_2) \quad \begin{cases} a > b\beta + \frac{D_1}{\gamma\beta}, \ k_0 \text{ is the ingeger so that (2.9) makes sense} \\ \text{for } 1 \le k < k_0, \text{ and does not when } k \ge k_0. \end{cases}$$

Under the hypothesis (H_2) , we define

$$\tau_k^{(j)} = \frac{1}{\omega_k}[\arcsin \frac{\omega_k T_k}{\gamma\beta(a - b\beta)} + 2j\pi], k = 1, 2, \cdots, k_0 - 1; j = 0, 1, \cdots. \tag{2.10}$$

In fact, from the first equation in (2.7), we have $\omega_k \tau_k^{(0)} \in (0, \frac{\pi}{2}]$ when $\omega_k^2 - D_k \ge 0$, and $\omega_k \tau_k^{(0)} \in (\frac{\pi}{2}, \pi)$ when $\omega_k^2 - D_k < 0$.

So far, we have know that $\pm \mathrm{i}\omega_k$ are a pair of imaginary roots of Eq (2.6) with

$$\tau = \tau_k^{(j)}, k = 1, 2, \cdots, k_0 - 1; j = 0, 1, \cdots.$$

Set $\lambda(\tau)$ as the root of Eq (2.6) satisfying $\mathrm{Re}\lambda(\tau_k^{(j)}) = 0$ and $\mathrm{Im}\lambda(\tau_k^{(j)}) = \omega_k$.

Lemma 2. *Suppose (H_1) and (H_2) are established. Then* $\frac{d}{d\tau}\mathrm{Re}\lambda(\tau_k^{(j)}) > 0$.

The proof of the lemma can be found in [30].

Denote $\tau_0 = \min_{1 \le k \le k_0 - 1}\{\tau_k^{(0)}\}$ and let the corresponding purely imaginary roots be $\pm \mathrm{i}\omega_0$. Then we provide the following results for the distribution of the roots of Eq (2.6).

Lemma 3. *Suppose (H_1) and (H_2) are established. Then all the roots of Eq (2.6) have negative real parts when $\tau \in [0, \tau_0)$, the other roots of Eq (2.6) with $\tau = \tau_0$, except the imaginary roots $\pm \mathrm{i}\omega_0$, have negative real parts, and Eq (2.6) has at least a couple of roots with positive real parts when $\tau > \tau_0$.*

Applying Lemmas 1–3, we have the following conclusions on the dynamics of the model (1.1).

Theorem 1. *(i) If $a \in (b\beta, b\beta + \frac{D_1}{\gamma\beta}]$, then $(\beta, \frac{a-b\beta}{c})$ is locally asymptotically stable when $\tau \ge 0$.*
(ii) If (H_1) and (H_2) are established, then $(\beta, \frac{a-b\beta}{c})$ is locally asymptotically stable when $\tau \in [0, \tau_0)$, and unstable for $\tau > \tau_0$; meanwhile, the model (1.1) undergoes the Hopf bifurcation at $(\beta, \frac{a-b\beta}{c})$ when $\tau = \tau_k^{(m)}$, $k = 1, 2, \cdots, k_0 - 1; m = 0, 1, 2, \cdots$.

3. Stability and direction of bifurcating periodic solutions

We shall investigate the direction of the Hopf bifurcation and the stability of the periodic solutions bifurcating from τ_0. We introduce the variables changes:

$$y_1(x,t) = u(x, \tau t), y_2(x,t) = v(x, \tau t) \ \text{ and } \ \tau = \tau_0 + \nu.$$

Then system (2.1) can be rewritten as

$$\begin{cases} \frac{\partial y_1(x,t)}{\partial t} = (\tau_0 + \nu)[d\frac{\partial^2 y_1(x,t)}{\partial x^2} - \alpha\frac{\partial y_1(x,t)}{\partial x} - b\beta y_1(x,t) \\ \qquad - c\beta y_2(x,t) - b y_1^2(x,t) - c y_1(x,t) y_2(x,t)], x \in (0, L), t > 0, \\ \frac{\partial y_2(x,t)}{\partial t} = (\tau_0 + \nu)[\varepsilon d\frac{\partial^2 y_2(x,t)}{\partial x^2} - \varepsilon\alpha\frac{\partial y_2(x,t)}{\partial x} \\ \qquad + \frac{\gamma(a-b\beta)}{c} y_1(x, t-1) + \gamma y_1(x, t-1) y_2(x,t)], x \in (0, L), t > 0, \\ d\frac{\partial y_1(0,t)}{\partial x} - \alpha y_1(0,t) = 0, \ y_1(L,t) = 0, t \ge 0, \\ d\frac{\partial y_2(0,t)}{\partial x} - \alpha y_2(0,t) = 0, \ y_2(L,t) = 0, t \ge 0. \end{cases} \tag{3.1}$$

By the results obtained in the previous section, we know that the system (3.1) undergoes a Hopf bifurcation at the origin when $\nu = 0$. Meanwhile, when $\nu = 0$, the characteristic equation of (3.1) has a pair of simple purely imaginary roots $\pm i\tau_0\omega_0$ and all the other roots, except the pure imaginary roots, have negative real parts.

Denote

$$\mathcal{C} := C([-1,0];X),\ \ C := C([-1,0];\mathbb{R}^2),\ \ C^* := C([0,1];\mathbb{R}^2),$$

and $U_t \in \mathcal{C}$ for $U_t(\theta) = U(t+\theta) \in X$, $-1 \le \theta \le 0$. Let

$$\phi = (\phi^{(1)}, \phi^{(2)})^T \in \mathcal{C},\ \ D(\nu) = (\tau_0+\nu)\begin{pmatrix} d & 0 \\ 0 & \varepsilon d \end{pmatrix},\ \ M(\nu) = (\tau_0+\nu)\begin{pmatrix} \alpha & 0 \\ 0 & \varepsilon\alpha \end{pmatrix},$$

and denote

$$L_\nu(\phi) = (\tau_0+\nu)[\begin{pmatrix} -b\beta & -c\beta \\ 0 & 0 \end{pmatrix}\phi(0) + \begin{pmatrix} 0 & 0 \\ \frac{\gamma(a-b\beta)}{c} & 0 \end{pmatrix}\phi(-1)],$$

$$F(\nu,\phi) = (\tau_0+\nu)\begin{pmatrix} -b(\phi^{(1)}(0))^2 - c\phi^{(1)}(0)\phi^{(2)}(0) \\ \gamma\phi^{(1)}(-1)\phi^{(2)}(0) \end{pmatrix}.$$

Then in $\mathcal{C}$, the system (3.1) has the form

$$\frac{d}{dt}U(t) = D(\nu)\Delta U(t) - M(\nu)\nabla U(t) + L_\nu(U_t) + F(\nu, U_t), \tag{3.2}$$

where

$$U(t) = (y_1(x,t), y_2(x,t))^T,\ \ D(\nu)\Delta = (\tau_0+\nu)\begin{pmatrix} d\Delta & 0 \\ 0 & \varepsilon d\Delta \end{pmatrix} \text{ and } M(\nu)\nabla = (\tau_0+\nu)\begin{pmatrix} \alpha\nabla & 0 \\ 0 & \varepsilon\alpha\nabla \end{pmatrix}.$$

We also know that $\pm i\tau_0\omega_0$ are pure imaginary eigenvalues of linear equations of (3.2) at $(0,0)$, the linear equation is as follows.

$$\frac{d}{dt}U(t) = D(\nu)\Delta U(t) - M(\nu)\nabla U(t) + L_\nu(U_t). \tag{3.3}$$

The infinitesimal generator A_ν is given by

$$A_\nu\phi = \begin{cases} \dot{\phi}(\theta) & \theta \in [-1,0), \\ D(\nu)\Delta\phi(0) - M(\nu)\nabla\phi(0) + L_\nu(\phi) & \theta = 0. \end{cases} \tag{3.4}$$

Denote $\beta_k = \{(b_k,0)^T, (0,b_k)^T\}$, where b_k is defined by (2.5). Then $\{\beta_k\}_{k=1}^\infty$ form an orthogonal basis for X. For $\phi = (\phi^{(1)}, \phi^{(2)})^T \in \mathcal{C}$, let

$$\phi_k = \langle\phi, \beta_k\rangle = (\langle\phi^{(1)}, b_k\rangle, \langle\phi^{(2)}, b_k\rangle)^T \in C([-1,0];\mathbb{R}^2).$$

Then $\phi = \sum_{k=1}^\infty \phi_k b_k$. For $\phi_k b_k$, we have

$$L_\nu(\phi_k b_k) = K_1\phi_k(0)b_k + K_2\phi_k(-1)b_k,\quad k = 1,2,\cdots,$$

and

$$D(\nu)\Delta\phi_k(0)b_k - M(\nu)\nabla\phi_k(0)b_k = -\mu_k K_3\phi_k(0)b_k\quad k = 1,2,\cdots,$$

where

$$K_1 = (\tau_0 + \nu)\begin{pmatrix} -b\beta & -c\beta \\ 0 & 0 \end{pmatrix}, \; K_2 = (\tau_0 + \nu)\begin{pmatrix} 0 & 0 \\ \frac{\gamma(a-b\beta)}{c} & 0 \end{pmatrix}, \; K_3 = (\tau_0 + \nu)\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}. \tag{3.5}$$

Then it follows that

$$D(\nu)\Delta\phi_k(0)b_k - M(\nu)\nabla\phi_k(0)b_k + L_\nu(\phi_k(-1)b_k) = (-\mu_k K_3\phi_k(0) + K_1\phi_k(0) + K_2\phi_k(-1))b_k, \; k = 1, 2, \cdots .$$

Define

$$\mathcal{L}_{k,\nu}(\phi) = -\mu_k K_3\phi(0) + K_1\phi(0) + K_2\phi(-1), \;\; \text{for} \;\; \phi \in C.$$

According to the Riesz representation theorem, there exists a matrix $\eta_k(\nu, \theta)$ in $\theta \in [-1, 0]$ that satisfies the following expression:

$$\mathcal{L}_{k,\nu}(\phi) = \int_{-1}^{0} d\eta_k(\nu, \theta)\phi(\theta), \quad k = 1, 2, \cdots . \tag{3.6}$$

Let

$$\eta_k(\nu, \theta) = \begin{cases} -K_2 & \theta = -1, \\ 0 & \theta \in (-1, 0), \qquad k = 1, 2, \cdots . \\ K_1 - \mu_k K_3 & \theta = 0, \end{cases} \tag{3.7}$$

Then (3.6) is satisfied. Hence, (3.4) can be rewritten in the following form:

$$A_\nu\phi(\theta) = \begin{cases} \sum_{k=1}^{\infty} \dot{\phi}_k(\theta)b_k, & \theta \in [-1, 0), \\ \sum_{k=1}^{\infty} \int_{-1}^{0} d\eta_k^T(\nu, s)\phi_k(s)b_k, & \theta = 0. \end{cases} \tag{3.8}$$

Since

$$\begin{cases} A\varphi = d\varphi_{xx} - \alpha\varphi(x), \\ d\varphi_x(0) - \alpha\varphi(0) = 0, \;\; \varphi(L) = 0, \end{cases}$$

let A^* be the adjoint operator of A, we have

$$\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle.$$

Hence, A^* satisfies

$$\begin{cases} A^*\psi = d\psi_{xx} + \alpha\psi_x, \\ \psi_x(0) = 0, \psi(L) = 0. \end{cases} \tag{3.9}$$

Similar to the method used to solve Eq (2.3), we provide the solution for (3.9):

$$\psi_k = e^{-\frac{\alpha}{2d}x}\Big(\cos\frac{\sqrt{d\mu_k - \frac{\alpha^2}{4}}}{d}x + \frac{\alpha}{2\sqrt{d\mu_k - \frac{\alpha^2}{4}}}\sin\frac{\sqrt{d\mu_k - \frac{\alpha^2}{4}}}{d}x\Big), k = 1, 2, \cdots ,$$

The associated eigenfunction is

$$\tilde{b}_k = \frac{\psi_k}{\|\psi_k\|}, \tag{3.10}$$

Define $C^* = C([0,1];X)$ and a bilinear form $(\cdot,\cdot)$ on $C^* \times C$:

$$(\psi, \phi) = \sum_{k,j=1}^{\infty} (\psi_k, \phi_j)_c \int_0^L \tilde{b}_k b_j dx, \tag{3.11}$$

where $\psi = \sum\limits_{k=1}^{\infty} \psi_k \tilde{b}_k \in C^*$, $\phi = \sum\limits_{k=1}^{\infty} \phi_k b_k \in C$, and $\phi_k \in C$, $\psi_k \in C^*$ $(k = 1, 2, \cdots)$ are the "coordinate" of the component of ϕ, ψ. From the definition of b_k, we have

$$\langle \tilde{b}_k, b_j \rangle = \int_0^L \tilde{b}_k b_j dx = 0, k \neq j.$$

Thus the bilinear form is

$$(\psi, \phi) = \sum_{k=1}^{\infty} (\psi_k, \phi_k)_c \int_0^L \tilde{b}_k b_k dx,, \tag{3.12}$$

where $(\cdot,\cdot)_c$ is defined on $C^* \times C$:

$$(\psi_k, \phi_k)_c = \psi_k(0)\phi_k(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \psi_k(\xi - \theta) d\eta_k(\nu, \theta) \phi_k(\xi) d\xi, \quad k = 1, 2, \cdots, \tag{3.13}$$

where $\eta_k(\nu, \theta), k = 1, 2, \cdots$ are defined as (3.7). Therefore, we get the adjoint operator A_ν^* of A_ν as $(A_\nu^* \psi, \phi) = (\psi, A_\nu \phi)$, such that

$$A_\nu^* \psi(s) = \begin{cases} -\sum_{k=1}^{\infty} \dot{\psi}_k(s) \tilde{b}_k = -\dot{\psi}(s), & s \in (0, 1]; \\ \sum_{k=1}^{\infty} \int_{-1}^{0} d\eta_k^T(\nu, s) \psi_k(-s) \tilde{b}_k, & s = 0. \end{cases}$$

In the following, for a detailed calculation process, please refer to the Appendix. Here we give the main conclusions. The two key values μ_2 and β_2 are calculated as follows.

$$\begin{aligned} c_2(0) &= \frac{\mathrm{i}}{2\omega_0 \tau_0} \Big(G_{11} G_{20} - 2|G_{11}|^2 - \frac{|G_{02}|^2}{3} \Big) + \frac{G_{21}}{2}, \\ \nu_2 &= -\frac{\mathrm{Re}(c_2(0))}{\mathrm{Re}(\lambda'(\tau_0))}, \\ \beta_2 &= 2\mathrm{Re}(c_2(0)). \end{aligned}$$

As is well known, if $\nu_2 > 0 (< 0)$, the Hopf bifurcation is forward (backward) and the bifurcating periodic solutions are orbitally stable (unstable) if $\beta_2 < 0 (> 0)$. When $\tau > \tau_0 (< \tau_0)$, the bifurcating periodic solutions appear.

4. Numerical simulations

In order to better explain the theoretical results, we give some numerical calculation results.

4.1. The effect of time delay

The parameters in model (1.1) are selected as follows.

$$(\mathbf{D}) \quad d = 0.1, \varepsilon = 2, \alpha = 0.2, a = 0.7, b = 1, c = 1.2, \gamma = 2.8, \beta = 0.35, L = 3.$$

According to (2.4), we get

$$\mu_1 \approx 0.1670, \mu_2 \approx 0.4043, \mu_3 \approx 0.8479, \mu_4 \approx 1.5078, \ldots.$$

In addition, we can calculate that

$$a - b\beta - \frac{D_1}{\gamma\beta} \approx 0.2682, a - b\beta - \frac{D_2}{\gamma\beta} \approx -0.2723,$$

which implies that the condition (H_2) is satisfied, where $k_0 = 2$. Then from (2.9) and (2.10) we have

$$\omega_1 \approx 0.4029, \ \tau_1^{(0)} \approx 3.9742.$$

Clearly, $\tau_0 = \tau_1^{(0)} \approx 3.9742$. Furthermore, by using the formula given in the (3.14), we compute

$$c_2(0) \approx -2.167 - 0.2811\mathrm{i}, \nu_2 \approx 104.3416, \beta_2 \approx -4.334.$$

We first presented a bifurcation diagram of advection rate and time delay, with shaded areas representing stable regions, see Figure 1. We found that as α increases, the area of stable regions also increases. By Theorem 1, we have the followings: Under the data (**D**), the positive steady state $(0.35, 0.2917)$ of model (1.1) is asymptotically stable for $\tau \in [0, 3.9742)$, see Figure 2. Meanwhile, the model (1.1) undergoes a Hopf bifurcation at the positive steady state $(0.35, 0.2917)$ when $\tau = \tau_0 \approx 3.9742$. Since $\nu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is forward, that is, the periodic solutions exist for $\tau > \tau_0$, and the bifurcating periodic solutions are orbitally asymptotically stable, see Figure 3.

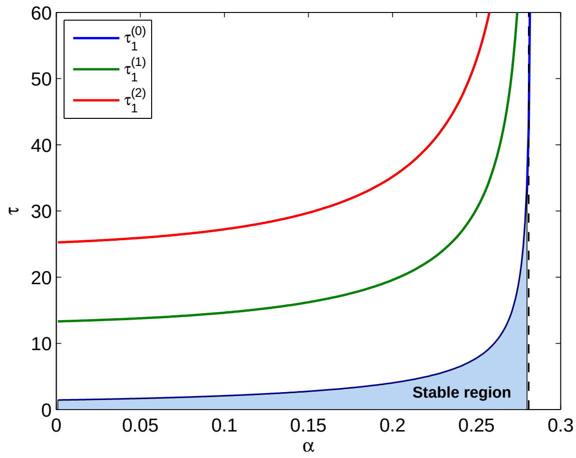

Figure 1. The bifurcation diagram of α and τ.

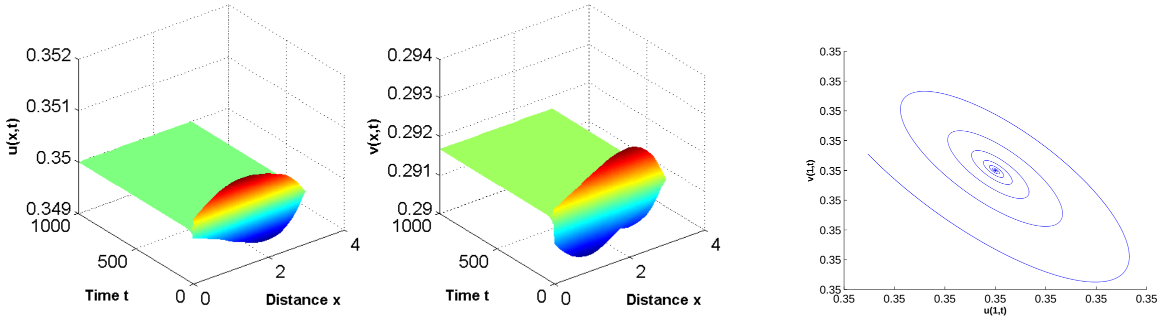

Figure 2. For model (0.35, 0.2917) is asymptotically stable when $\tau \in [0, 3.9742)$, where $\tau = 2.4 < \tau_0$, and the initial functions are $(0.35 - 0.001 \cos x, 0.2917 - 0.001 \cos x)$.

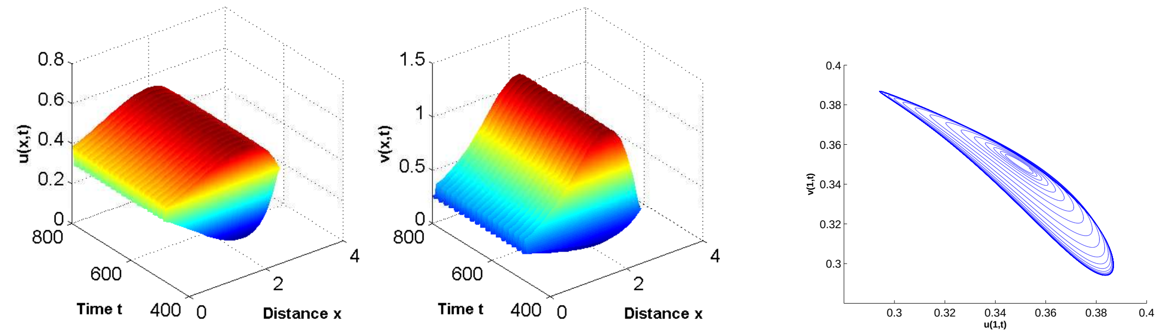

Figure 3. For model (1.1) with the data (**D**), the spatially nonhomogeneous periodic solution bifurcated from the positive steady state (0.35, 0.2917) is orbitally asymptotically stable, where $\tau = 4.8 > \tau_0 \approx 3.9742$, and the initial functions are $(0.35 - 0.001 \cos x, 0.2917 - 0.001 \cos x)$.

4.2. The effect of advection

In this section, we will discuss the impact of the advection rate on population size and explore the rules of population change when the rate changes. When the time delay τ is constant, the stable region of the system expands as the advection rate increases. When we fix τ to 3.8, other parameters are still selected from (**D**). Starting from 0, α increases, causing the gathering position of the predator to move downstream. At this point, the system has a spatially nonhomogeneous periodic solution. When $\alpha = 0.28$, it helps to maintain the stability of the coexistence state of the system. Let's take the predator as an example, and use Figure 4 to illustrate the change.

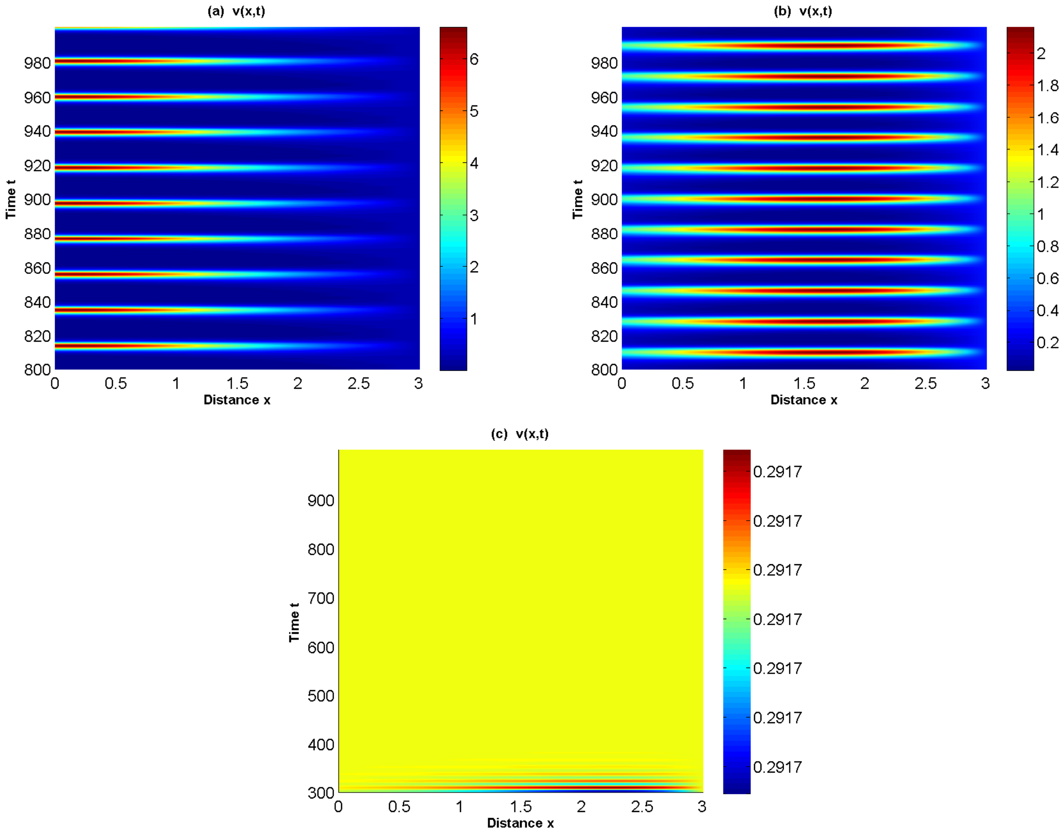

Figure 4. The change rule of predator quantity when α increases. (a) $\alpha = 0$; (b) $\alpha = 0.1$; (c) $\alpha = 0.28$.

5. Conclusions

We generalize the normal form theory of the reaction-diffusion equation. Under Neumann boundary conditions, we find that the eigenvalue mu cannot directly give a specific form. For a delayed predator-prey model with diffusion and advection, mode-n starts at least from mode-1. Taking time delay as the bifurcation parameter, when τ exceeds the critical value, we discover spatial inhomogeneous periodic oscillations induced by time delay. In addition, we also found that advection can cause oscillations in the system that are influenced by both time and space.

Acknowledgments

This research is supported by National Natural Science Foundation of China (No. 11031002) and Natural Science Foundation of Shandong Province (No. ZR2022QA075).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. S. Chen, J. Shi, J. Wei, Global stability and Hopf bifurcation in a delayed diffusive Leslie-Gower predator-prey system, *Int. J. Bifurcation Chaos*, **22** (2012), 1250061. https://doi.org/10.1142/S0218127412500617
2. T. Faria, Stability and bifurcation for a delayed predator-prey model and the effect of diffusion, *J. Math. Anal. Appl.*, **254** (2001), 433–463. https://doi.org/10.1006/jmaa.2000.7182
3. P. K. Hadeler, *Topics in Mathematical Biology*, Heidelberg, Springer, 2018. https://doi.org/10.1007/978-3-319-65621-2
4. G. Hu, W. Li, Hopf bifurcation analysis for a delayed predator-prey system with diffusion effects, *Nonlinear Anal. Real World Appl.*, **11** (2010), 819–826. https://doi.org/10.1016/j.nonrwa.2009.01.027
5. M. Wang, Stability and Hopf bifurcation for a prey-predator model with prey-stage structure and diffusion, *Math. Biosci.*, **212** (2008), 149–160. https://doi.org/10.1016/j.mbs.2007.08.008
6. X. Yan, Stability and Hopf bifurcation for a delayed prey-predator system with diffusion effects, *Appl. Math. Comput.*, **192** (2007), 552–566. https://doi.org/10.1016/j.amc.2007.03.033
7. X. Yan, C. Zhang, Asymptotic stability of positive equilibrium solution for a delayed prey-predator diffusion system, *Appl. Math. Modell.*, **34** (2010), 184–199. https://doi.org/10.1016/j.apm.2009.03.040
8. R. Yang, C. Zhang, Y. Zhang, A delayed diffusive predator-prey system with Michaelis-Menten type predator harvesting, *Int. J. Bifurcation Chaos*, **28** (2018), 1850099. https://doi.org/10.1142/S0218127418500992
9. F. Liu, R. Yang, L. Tang, Hopf bifurcation in a diffusive predator-prey model with competitive interference, *Chaos, Solitons Fractals*, **120** (2019), 250–258. https://doi.org/10.1016/j.chaos.2019.01.029
10. W. Zuo, J. Wei, Stability and Hopf bifurcation in a diffusive predator-prey system with delay effect, *Nonlinear Anal. Real World Appl.*, **12** (2011), 1998–2011. https://doi.org/10.1016/j.nonrwa.2010.12.016
11. G. Hu, W. Li, X. Yan, Hopf bifurcation and stability of periodic solutions in the delayed Lienard equaiton, *Int. J. Bifurcation Chaos*, **18** (2008), 3147–3157. https://doi.org/10.1142/S0218127408022317
12. Y. Su, J. Wei, J. Shi, Hopf bifurcations in a reaction-diffusion population model with delay effect, *J. Differ. Equations*, **247** (2009), 1156–1184. https://doi.org/10.1016/j.jde.2009.04.017
13. Y. Su, J. Wei, J. Shi, Bifurcation analysis in a delayed diffusive Nicholson's blowflies equation, *Nonlinear Anal. Real World Appl.*, **12** (2010), 1692–1703. https://doi.org/10.1016/j.nonrwa.2009.03.024
14. Y. Su, A. Wan, J. Wei, Bifurcation analysis in a diffusive food-limited model with time delay, *Appl. Anal.*, **89** (2010), 1161–1181. https://doi.org/10.1080/00036810903116010
15. X. Yan, W. Li, Stability and Hopf bifurcation for a delayed cooperative system with diffusion effects, *Int. J. Bifurcation Chaos*, **18** (2008), 441–453. https://doi.org/10.1142/S0218127408020434

16. W. Zuo, J. Wei, Stability and bifurcation analysis in a diffusive Brusselator system with delayed feedback control, *Int. J. Bifurcation Chaos*, **22** (2011), 1250037. https://doi.org/10.1142/S021812741250037X
17. W. Zuo, Y. Song, Stability and bifurcation analysis of a reaction–diffusion equation with spatio-temporal delay, *J. Math. Anal. Appl.*, **430** (2015), 243–261. https://doi.org/10.1016/j.jmaa.2015.04.089
18. F. Lutscher, E. McCauley, M. A. Lewis, Spatial patterns and coexistence mechanisms in systems with unidirectional flow, *Theor. Popul. Biol.*, **71** (2007), 267–277. https://doi.org/10.1016/j.tpb.2006.11.006
19. O. Vasilyeva, F. Lutscher, How flow speed alters competitive outcome in advective environments, *Bull. Math. Biol.*, **74** (2012), 2935–2958. https://doi.org/10.1007/s11538-012-9792-3
20. Y. Lou, H. Nie, Y. E. Wang, Coexistence and bistability of a competition model in open advective environments, *Math. Biosci.*, **306** (2018), 10–19. https://doi.org/10.1016/j.mbs.2018.09.013
21. Y. Lou, H. Nie, Global dynamics of a generalist predator-prey model in open advective environments, *J. Math. Biol.*, **84** (2022), 1–40. https://doi.org/10.1007/s00285-022-01756-w
22. J. Liu, J. Wei, On Hopf bifurcation of a delayed predator-prey system with diffusion, *Int. J. Bifurcation Chaos*, **23** (2013), 1350023. https://doi.org/10.1142/S0218127413500235
23. E. Beretta, Y. Kuang, Convergence results in a well-known delayed predator-prey system, *J. Math. Anal. Appl.*, **204** (1996), 840–853. https://doi.org/10.1006/jmaa.1996.0471
24. K. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993. https://doi.org/10.1016/0378-4754(93)90045-V
25. S. Ruan, On nonlinear dynamics of predator-prey models with discrete delay, *Math. Model. Nat. Phenom.*, **4** (2009), 140–188. https://doi.org/10.1051/mmnp/20094207
26. J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996. https://doi.org/10.1007/978-1-4612-4050-1
27. T. Faria, L. T. Magalhães, Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation, *J. Differ. Equations*, **122** (1995), 181–200. https://doi.org/10.1006/jdeq.1995.1144
28. T. Faria, Normal forms and Hopf bifurcation for partial differential equations with delays, *Trans. Amer. Math. Soc.*, **352** (2000), 2217–2238. https://doi.org/10.1090/S0002-9947-00-02280-7
29. B. D. Hassard, N. D. Kazarinoff, Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981. http://dx.doi.org/10.1090/conm/445
30. K. L. Cooke, Z. Grossman, Discrete delay, distributed delay and stability switches, *J. Math. Anal. Appl.*, **86** (1982), 592–627. https://doi.org/10.1016/0022-247X(82)90243-8

Appendix

Let $k = k_0$ and $q(\theta)b_k$ be the eigenfunction of $A(= A_0)$ corresponds to $\mathrm{i}\omega_0\tau_0$ and $\hat{q}^*(s)\tilde{b}_k$ be the eigenfunction of A^* corresponds to $\mathrm{i}\tau_0\omega_0$, that is,

$$Aq(\theta)b_k = \mathrm{i}\tau_0\omega_0 q(\theta)b_k,$$

and

$$A^*\hat{q}^*(s)b_k = \mathrm{i}\tau_0\omega_0\hat{q}^*(s)\tilde{b}_k.$$

Obviously,

$$\begin{aligned} Aq(0)b_k &= \int_{-1}^{0} d\eta_k(0,\theta)q(\theta)b_k, \\ A^*\hat{q}^*(0)\tilde{b}_k &= \int_{-1}^{0} d\eta_k^T(0,s)\hat{q}^*(-s)\tilde{b}_k. \end{aligned} \tag{A.1}$$

By calculation, we get

$$q(0) = (1, q_1)^T,\ \hat{q}^*(0) = M(1, q_2^*),$$

where

$$\begin{aligned} q_1 &= -\frac{\mathrm{i}\omega_0 + \mu_k + b\beta}{c\beta},\ q_2^* = -\frac{c\beta}{\mathrm{i}\omega_0 + \varepsilon\mu_k}, \\ M &= \Big[1 + q_2^* q_1 + \tau_0 q_2^* e^{-\mathrm{i}\tau_0\omega_0}\frac{\gamma(a - b\beta)}{c}\Big]^{-1}. \end{aligned}$$

We choose $\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)_{n_0}^{-1}\Psi^*$ such that $(\Psi^*, \Phi)_{n_0} = I_2$, where I_2 is a 2×2 identity matrix. Then the center subspace of the linear equation (3.3) with $\nu = 0$ is given by $P_{CN}C$, where

$$P_{CN}\varphi = \varphi(\Psi, \langle\varphi, \beta_{k_0}\rangle)\cdot\beta_{k_0}$$

for $\Psi \in C$, here $\beta_k = (\beta_k^1, \beta_k^2)$ and $c\cdot\beta_k = c_1\beta_k^1 + c_2\beta_k^2$ for any $c = (c_1, c_2)^T \in C$. According to [26], when $\nu = 0$, the flow of system (3.2) is as follows.

$$(x_1(t), x_2(t))^T = (\Psi, \langle U_t, \beta_{k_0}\rangle)_{k_0},$$

$$U_t = \Phi(x_1(t), x_2(t))^T\cdot\beta_{k_0} + h(x_1, x_2, 0), \tag{A.2}$$

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & \omega_0\tau_0\\ -\omega_0\tau_0 & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \Psi(0)\langle F(U_t, 0), \beta_{k_0}\rangle, \tag{A.3}$$

with $h(0,0,0) = 0$ and $Dh(0,0,0) = 0$. Let $z = x_1 - \mathrm{i}x_2$ and $\Psi(0) = (\Psi_1(0), \Psi_2(0))^T$. We know that $q = \Phi_1 + \mathrm{i}\Phi_2$, then (A.2) can be transformed into

$$U_t = \frac{1}{2}(qz + \bar{q}\bar{z})\cdot\beta_{k_0} + W(z, \bar{z}), \tag{A.4}$$

with $W(z,\bar{z}) = h(\frac{z+\bar{z}}{2}, \frac{\mathrm{i}(z-\bar{z})}{2}, 0)$. Combining (A.3) and formula (A.4), we know that z should satisfy

$$\dot{z} = \mathrm{i}\omega_0\tau_0 z + g(z, \bar{z}), \tag{A.5}$$

where

$$g(z,\bar{z}) = (\Psi_1(0) - \mathrm{i}\Psi_2(0))\langle F(U_t, 0), \beta_{k_0}\rangle = (\Psi_1(0) - \mathrm{i}\Psi_2(0))\langle F(U_t, 0), \beta_{k_0}\rangle \tag{A.6}$$

Let

$$
\begin{aligned}
g(z,\bar{z}) &= G_{20}\frac{z^2}{2} + G_{11}z\bar{z} + G_{02}\frac{\bar{z}^2}{2} + G_{21}\frac{z^2\bar{z}}{2} + \cdots, \\
W(z,\bar{z}) &= W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots.
\end{aligned}
\tag{A.7}
$$

From (A.2), (A.4) and (A.5), we have

$$
\begin{aligned}
G_{20} &= 2\tau_0 M(-b - cq_1 + \gamma q_1 q_2^* e^{-\mathrm{i}\omega_0\tau_0}) \int_0^L b_{k_0}^2 \tilde{b}_{k_0} dx, \\
G_{02} &= \overline{G}_{20}, \\
G_{11} &= \tau_0 M[-2b - c(q_1 + \bar{q}_1) + q_2^*\gamma(q_1 e^{\mathrm{i}\omega_0\tau_0} + \bar{q}_1 e^{-\mathrm{i}\omega_0\tau_0})] \int_0^L b_{k_0}^2 \tilde{b}_{k_0} dx, \\
G_{21} &= 2\tau_0 M \int_0^L Q b_{k_0} \tilde{b}_{k_0} dx
\end{aligned}
$$

where

$$
\begin{aligned}
Q &= -b(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) - c\Big(W_{11}^{(2)}(0) + \frac{W_{20}^{(2)}(0)}{2} + \frac{W_{20}^{(1)}(0)}{2}\bar{q}_1 + q_1 W_{11}^{(1)}(0)\Big) \\
&+ q_2^*\gamma\Big(q_1 W_{11}^{(1)}(-1) + \frac{W_{20}^{(1)}(-1)}{2}\bar{q}_1 + \frac{W_{20}^{(2)}(0)}{2} e^{\mathrm{i}\omega_0\tau_0} + W_{11}^{(2)}(0) e^{-\mathrm{i}\omega_0\tau_0}\Big)
\end{aligned}
$$

To obtain G_{21}, we need to calculate $W_{20}(\theta)$ and $W_{11}(\theta)$ ($\theta \in [-1, 0]$). A_U is the generator of the semigroup, which is generated by the linear system (3.3) with $\nu = 0$. According to (A.4) and (A.5), we have

$$
\begin{aligned}
\dot{W} &= \dot{U}_t - \frac{1}{2}(q\dot{z} + \bar{q}\dot{\bar{z}}) \cdot \beta_{k_0} \\
&= \begin{cases} A_U W - \frac{1}{2}(q(\theta)g(z,\bar{z}) + \bar{q}(\theta)\bar{g}(z,\bar{z})) \cdot \beta_{k_0}, & \theta \in [-1, 0), \\ A_U W - \frac{1}{2}(q(\theta)g(z,\bar{z}) + \bar{q}(\theta)\bar{g}(z,\bar{z})) \cdot \beta_{k_0} + F\big(0, \frac{1}{2}(qz + \bar{q}\bar{z}) \cdot \beta_{k_0} + W(z,\bar{z})\big), & \theta = 0, \end{cases} \\
&= A_U W + H(z,\bar{z},\theta),
\end{aligned}
\tag{A.8}
$$

where

$$
H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.
$$

Denote

$$
F(0, \frac{1}{2}(qz + \bar{q}\bar{z}) \cdot \beta_{k_0} + W(z,\bar{z})) = f_{z^2}\frac{z^2}{2} + f_{z\bar{z}}z\bar{z} + f_{\bar{z}^2}\frac{\bar{z}^2}{2} + \cdots.
$$

Therefore,

$$
H_{20}(\theta) = \begin{cases} -\frac{1}{2}(q(\theta)g_{20} + \bar{q}(\theta)\bar{g}_{02}) \cdot \beta_{k_0}, & \theta \in [-1, 0), \\ -\frac{1}{2}(q(\theta)g_{20} + \bar{q}(\theta)\bar{g}_{02}) \cdot \beta_{k_0} + f_{z^2}, & \theta = 0, \end{cases}
$$

$$H_{11}(\theta) = \begin{cases} -\frac{1}{2}(q(\theta)g_{11} + \bar{q}(\theta)\bar{g}_{11}) \cdot \beta_{k_0}, & \theta \in [-1, 0), \\ -\frac{1}{2}(q(\theta)g_{11} + \bar{q}(\theta)\bar{g}_{11}) \cdot \beta_{k_0} + f_{z\bar{z}}, & \theta = 0, \end{cases}$$

Notice that

$$\dot{W} = \frac{\partial W(z, \bar{z})}{\partial z}\dot{z} + \frac{\partial W(z, \bar{z})}{\partial \bar{z}}\dot{\bar{z}}.$$

From (A.7) and (A.8), we have

$$\begin{cases} H_{20} = (2\mathrm{i}\omega_0\tau_0 - A_U)W_{20}, \\ H_{11} = -A_U W_{11}. \end{cases} \tag{A.9}$$

Since $2i\omega_0\tau_0$ and 0 are not eigenvalues of (3.3), the system (A.9) has unique solutions W_{20} and W_{11} in $P_S C$, which are given by

$$\begin{cases} W_{20} = (2\mathrm{i}\omega_0\tau_0 - A_U)^{-1}H_{20}, \\ W_{11} = -A_U^{-1}H_{11}. \end{cases} \tag{A.10}$$

By (A.10), we obtain

$$\begin{aligned} W_{20}(\theta) &= \frac{-G_{20}}{\mathrm{i}\omega_0\tau_0}q(0)e^{\mathrm{i}\omega_0\tau_0\theta}b_k - \frac{\bar{G}_{02}}{3\mathrm{i}\omega_0\tau_0}\bar{q}(0)e^{-\mathrm{i}\omega_0\tau_0\theta}b_k + E_1 e^{2\mathrm{i}\omega_0\tau_0\theta}, \\ W_{11}(\theta) &= \frac{G_{11}}{\mathrm{i}\omega_0\tau_0}q(0)e^{\mathrm{i}\omega_0\tau_0\theta}b_k - \frac{\bar{G}_{11}}{i\omega_0\tau_0}\bar{q}(0)e^{-\mathrm{i}\omega_0\tau_0\theta}b_k + E_2, \end{aligned}$$

Denote

$$E_1 = \sum_{k=1}^{\infty} E_1^k b_k, \; E_2 = \sum_{n=1}^{\infty} E_2^k b_k,$$

E_1^k and E_2^k can be calculated by

$$\begin{aligned} E_1^k &= \left(2\mathrm{i}\omega_0\tau_0 I - \int_{-1}^{0} e^{2\mathrm{i}\omega_0\tau_0\theta}\mathrm{d}\eta_k(0, \theta)\right)^{-1} \langle f_{z^2}, \beta_k \rangle, \\ &= \tau_0^{-1}\begin{pmatrix} 2\mathrm{i}\omega_0 + \mu_k + b\beta & c\beta \\ -\frac{\gamma(a-b\beta)}{c}e^{-2\mathrm{i}\omega_0\tau_0} & 2\mathrm{i}\omega_0 + \varepsilon\mu_k \end{pmatrix}^{-1} \langle f_{z^2}, \beta_k \rangle, \end{aligned}$$

with

$$\langle f_{z^2}, \beta_k \rangle = 2\begin{pmatrix} -b - cq_1 \\ \gamma q_1 e^{-\mathrm{i}\omega_0\tau_0} \end{pmatrix} \int_0^L b_{k_0}^2 b_k dx.$$

$$
\begin{aligned}
E_2^k &= -\Big(\int_{-1}^{0} \mathrm{d}\eta_k(0,\theta)\Big)^{-1}\langle f_{z\bar{z}},\beta_k\rangle,\\
&= \tau_0^{-1}\begin{pmatrix}\mu_k + b\beta & c\beta\\ -\frac{\gamma(a-b\beta)}{c} & \varepsilon\mu_k\end{pmatrix}^{-1}\langle f_{z\bar{z}},\beta_k\rangle,\ k=1,2,\cdots.
\end{aligned}
$$

with

$$
\langle f_{z\bar{z}},\beta_k\rangle = \begin{pmatrix}-2b - c(q_1+\bar{q}_1)\\ \gamma(q_1 e^{\mathrm{i}\omega_0\tau_0} + \bar{q}_1 e^{-\mathrm{i}\omega_0\tau_0})\end{pmatrix}\int_0^L b_{k_0}^2 b_k dx.
$$

Therefore, $W_{20}(\theta)$ and $W_{11}(\theta)$ can be obtained, the expression of G_{21} is also obtained.

AIMS Press