Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(7): 11568–11584.
DOI: 10.3934/mbe.2023513
Received: 16 March 2023
Revised: 17 April 2023
Accepted: 23 April 2023
Published: 05 May 2023

Research article

The differential on operator $\mathcal{S}(\Gamma)$

Jair Castro [1], Ludwin A. Basilio[2], Gerardo Reyna[2] and Omar Rosario[2,*]

[1] Universidad Autónoma de Guerrero, Facultad de Matemáticas, Campus Zumpango, Guerrero, México

[2] Universidad Autónoma de Guerrero, Facultad de Matemáticas, Campus Acapulco, Guerrero, México

* **Correspondence:** Email: orosario@uagro.mx.

Abstract: Consider a simple graph $\Gamma = (V(\Gamma), E(\Gamma))$ with n vertices and m edges. Let P be a subset of $V(\Gamma)$ and $B(P)$ the set of neighbors of P in $V(\Gamma)\backslash P$. In the study of graphs, the concept of *differential* refers to a measure of how much the number of edges leaving a set of vertices exceeds the size of that set. Specifically, given a subset P of vertices, the differential of P, denoted by $\partial(P)$, is defined as $|B(P)| - |P|$. The *differential* of Γ, denoted by $\partial(\Gamma)$, is then defined as the maximum differential over all possible subsets of $V(\Gamma)$. Additionally, the subdivision operator $\mathcal{S}(\Gamma)$ is defined as the graph obtained from Γ by inserting a new vertex on each edge of Γ. In this paper, we present results for the differential of graphs on the subdivision operator $\mathcal{S}(\Gamma)$ where some of these show exact values of $\partial(\mathcal{S}(\Gamma))$ if Γ belongs to a classical family of graphs. We obtain bounds for $\partial(\mathcal{S}(\Gamma))$ involving invariants of a graph such as order n, size m and maximum degree Δ, and we study the realizability of the graph Γ for any value of $\partial(\mathcal{S}(\Gamma))$ in the interval $\left[n-2, \frac{n(n-1)}{2} - n + 2\right]$. Moreover, we give a characterization for $\partial(\mathcal{S}(\Gamma))$ using the notion of edge star packing.

Keywords: subdivision graph; differential of graphs; independence number; matching number

1. Introduction

Parameter studies on graph operators are a relatively new and active area of research in graph theory, which has attracted the attention of several authors seeking to better understand how the properties of a graph and its operators can affect certain graph parameters.

If Γ is a graph, $\Phi(\Gamma)$ is a graph parameter of Γ and $O(\Gamma)$ is a graph operator of Γ, What information can we obtain about $\Phi(O(\Gamma))$ when we know certain properties of Γ and $\Phi(\Gamma)$?

In [1, 2] was studied the total domination number $\gamma_t(\Gamma)$ on the operators $\mathcal{S}(\Gamma)$, $\mathcal{R}(\Gamma)$ and $\mathcal{Q}(\Gamma)$ and in [3–8] the authors studied some topological indices on operators in graphs. Other works in which at least one graph operator $O(\Gamma)$ has been studied are [9–16].

Throughout this paper, we consider a simple graph $\Gamma = (V(\Gamma), E(\Gamma))$ with n vertices and m edges, where $n \geq 2$. The vertex set of Γ is denoted as $V(\Gamma)$ and the edge set is denoted as $E(\Gamma)$. If P is a subset of $V(\Gamma)$, the subgraph of Γ induced by P will be denoted by $\langle P \rangle_\Gamma$. Consider two graphs, Γ and Γ'. The graph Γ is said to be Γ'*-free* if it does not contain any induced subgraph that is isomorphic to Γ'. A graph is *unicyclic* if contains exactly one cycle. Let v be a vertex of $V(\Gamma)$. As usual, $N_\Gamma(v)$ is the set of neighbors that v has in $V(\Gamma)$, and $N_\Gamma[v] := N_\Gamma(v) \cup \{v\}$. Let $\delta_\Gamma(v)$ denote the degree of vertex v in the graph Γ, which is given by the cardinality of the set of neighbors $N_\Gamma(v)$. Additionally, let δ and Δ denote the minimum and maximum degree of Γ, respectively. When $\delta_\Gamma(v) = 1$ we will say that v is a *leaf* of Γ. If P is a subset of $V(\Gamma)$, then we denote by $N_P(v)$ the set of neighbors that v has in P, $N_\Gamma(P) := \bigcup_{v \in P} N_\Gamma(v)$ and $N_\Gamma[P] := N_\Gamma(P) \cup P$. If there is no ambiguity which graph is being considered we will just write $\delta(v)$ and $N[v]$ instead $\delta_\Gamma(v)$ and $N_\Gamma[v]$.

Let $B_\Gamma(P)$ be the set of vertices in $V(\Gamma) \backslash P$ that have a neighbor in P, it is called the border of P in Γ and let $C_\Gamma(P)$ be the set $V(\Gamma) \backslash (P \cup B_\Gamma(P))$. Then P, $B_\Gamma(P)$ and $C_\Gamma(P)$ are disjoint sets such that $V(\Gamma) = P \cup B_\Gamma(P) \cup C_\Gamma(P)$. The *differential* of a set P is defined as $\partial_\Gamma(P) = |B_\Gamma(P)| - |P|$ and the *differential* of a graph Γ is defined as $\partial(\Gamma) = max\{\partial_\Gamma(P) : P \subseteq V(\Gamma)\}$. If there is no ambiguity which graph is being considered we will just write $B(P)$ and $\partial(P)$ instead $B_\Gamma(P)$ and $\partial_\Gamma(P)$. A subset P of vertices in a graph Γ is called a *differential set*, if Γ attains its maximum differential in P. If P has the smallest or largest possible number of vertices among all differential sets, it is called a *minimum differential set* or a *maximum differential set*, respectively. If Γ is disconnected and is composed of k connected components $\Gamma_1, \ldots, \Gamma_k$, then

$$\partial(\Gamma) = \partial(\Gamma_1) + \cdots + \partial(\Gamma_k),$$

for this reason we will only work with connected graphs.

In 2006 J. L. Mashburn et al. first introduced the concept of a differential of a graph in their paper entitled *Differentials in graphs* [17]. Since then the topic has been extensively studied in several directions, in [18] is shown that this problem is NP-complete and in [19–21] the authors obtain tight bounds. In [22–25] the differential has been studied on certain graph operators. In addition, other versions of the differential have been studied in [26–28]. Other works on the differential of graphs can be found in [29–34].

In a graph Γ with at least one cycle, the length of the longest cycle is referred to as the circumference of the graph, and the length of the shortest cycle is known as its girth. The notation $\mathcal{L}_\Gamma$ is commonly used to represent the circumference of Γ, while $g(\Gamma)$ denotes the girth of Γ.

The *subdivision* graph $\mathcal{S}(\Gamma)$ is a new graph that is created by taking the original graph Γ and adding an additional vertex on each edge of Γ (see Figure 1).

Let us enumerate certain fundamental characteristics of $\mathcal{S}(\Gamma)$ that can be inferred from its definition. Before continuing, we need the following definition: A subset $I \subset V(\Gamma)$ is an *independent* set of Γ if any two distinct vertices in I are non-adjacent in Γ. The *independence number* of Γ refers to the maximum size of an independent set in Γ, this quantity is denoted by $\alpha(\Gamma)$. Due to the structure of $\mathcal{S}(\Gamma)$, we can partition the set $V(\mathcal{S}(\Gamma))$ into two independent sets.

Remark 1. Suppose Γ is a graph with n vertices and m edges. Then:

(i) the set of vertices of $\mathcal{S}(\Gamma)$ corresponding to the edges of Γ is denoted with U, and the remaining is V. Hence $|V| = |V(\Gamma)| = n$ and $|U| = |E(\Gamma)| = m$,

(ii) $|V(\mathcal{S}(\Gamma))| = n + m$ and $|E(\mathcal{S}(\Gamma))| = 2m$,
(iii) Γ is isomorphic to $\mathcal{S}(\Gamma)$ if and only if Γ is isomorphic to the empty graph E_n,
(iv) $\delta_{\mathcal{S}(\Gamma)}(e) = 2$, for all $e \in U$,
(v) $\delta_{\mathcal{S}(\Gamma)}(v) = \delta_{\Gamma}(v)$, for all $v \in V$,
(vi) the sets U and V are maximal independent sets,
(vii) $g(\mathcal{S}(\Gamma)) \geq 6$,
(viii) $\mathcal{S}(\Gamma)$ is C_{2k+1}-free, for all $k \in \mathbb{N}$,
(ix) $\mathcal{S}(\Gamma)$ is not isomorphic to P_{2k}, for all $k \in \mathbb{N}$,
(x) $\mathcal{S}(\Gamma)$ is a bipartite graph with bipartition $\{V, U\}$.

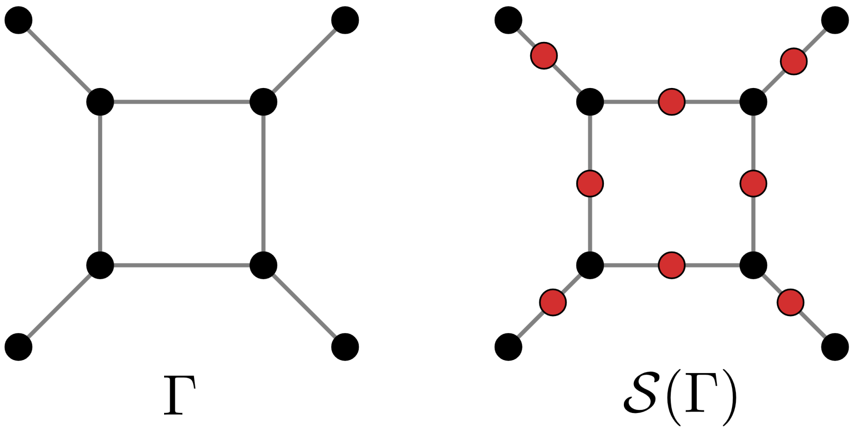

Figure 1. On the left, there is a simple graph Γ, on the right, the subdivision graph of Γ.

2. The differential on subdivision operator $\mathcal{S}(\Gamma)$

To start this section, we will begin with a characterization of the differential of graphs on the subdivision operator $\mathcal{S}(\Gamma)$ when it takes small values (see Proposition 2.2). Additionally, we will provide the exact values of the differential of graphs on the subdivision operator $\mathcal{S}(\Gamma)$ when Γ belongs to a classical family of graphs (see Proposition 2.4). Finally, we will present a bound on $\partial(\mathcal{S}(\Gamma))$ in relation to $\mathcal{L}_\Gamma$ (see Proposition 2.5).

The next result appears in [21].

Lemma 2.1. *For any graph Γ with maximum degree Δ, the following statements hold:*

(i) $\partial(\Gamma) = 1$ *if and only if* $\Gamma = C_3, C_4, C_5, P_3, P_4$ *or* P_5,
(ii) $\partial(\Gamma) = 2$ *if and only if* Γ *is a graph with either:*
  (a) $\Gamma = C_6, C_7, C_8, P_6, P_7$ *or* P_8 *or*
  (b) $\Delta = 3$ *and, for every vertex* $v \in V(\Gamma)$ *such that* $\delta(v) = 3$, *the subgraph induced by* $V(\Gamma)\backslash N[v]$ *has no graph isomorphic to* P_3, *and* Γ *has no* 3 *independent subgraphs isomorphic to* P_3.

A subset $M \subset E(\Gamma)$ is a *matching* of Γ if each any two distinct edges in M are not incident in Γ. The *matching number* of Γ is the cardinality of any largest matching of Γ and it is denoted by $\beta(\Gamma)$.

Proposition 2.2. *The assertions below are valid for any graph* Γ *that has a maximum degree of* Δ*:*

(1) $\partial(\mathcal{S}(\Gamma)) = 1$ *if and only if* $\Gamma = P_2, P_3$*,*
(2) $\partial(\mathcal{S}(\Gamma)) = 2$ *if and only if* $\Gamma = C_3, C_4, P_4, S_4$*,*
(3) $\partial(\mathcal{S}(\Gamma)) = 3$ *if and only if*
  (*a*) $\Gamma = C_5, P_5, P_6, S_5$ *or*
  (*b*) $\Delta = 3$ *and*

  (i) any two vertices of degree 3 *in* Γ *are adjacent,*
  (ii) Γ *contains at most two vertices of degree* 3*,*
  (iii) if v *is a vertex of* Γ *with* $\delta(v) = 3$*, then* $\beta(\Gamma\backslash\{v\}) = 1$.

Proof. (1) If $\partial(\mathcal{S}(\Gamma)) = 1$, by Lemma 2.1, the graph $\mathcal{S}(\Gamma)$ needs to have an isomorphism with one of the graphs : C_3, C_4, C_5, P_3, P_4 or P_5. However, among all of these graphs, the only possible candidates are P_3 and P_5 (see Remark 1 (*vii*), (*viii*), and (*ix*)). Thus, Γ is either P_2 or P_3.

(2) If $\Delta_{\mathcal{S}(\Gamma)} \neq 3$, once again, by Lemma 2.1, $\mathcal{S}(\Gamma)$ must be one of the following graphs: C_6, C_7, C_8, P_6, P_7 or P_8. Of all these the only ones that satisfy the condition of being a subdivision graph are C_6, C_8, P_7 (see again Remark 1 (*viii*) and (*ix*)), obtaining $\Gamma = C_3$, C_4 or P_4. If $\Delta_{\mathcal{S}(\Gamma)} = 3$ so that $\Delta_\Gamma = 3$ but, $\Gamma \neq S_4$, then Γ includes a subgraph that is isomorphic to one of those shown in Figure 2. In any case, by letting $P = \{v, e\}$ we obtain $\partial(P) = 3$, so that $\partial(\mathcal{S}(\Gamma)) \geq 3$.

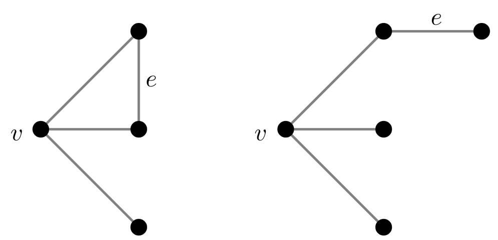

Figure 2. Graphs with differential equal to 3 on the subdivision operator.

(3) If $\partial(\mathcal{S}(\Gamma)) = 3$ and it also happens that $\partial(\Gamma) = 3$, then $\Gamma = S_5$. In another case, $\partial(\Gamma) = 1$ or $\partial(\Gamma) = 2$. Once again, using Lemma 2.1 we obtain that $\Gamma = C_5, P_5$ or P_6 provided that $\Delta_\Gamma \neq 3$.
In another case, if $\Delta_\Gamma = 3$, let us show that (*i*) − (*iii*) are satisfied.

If there are two vertices of degree three that are non-adjacent, they would produce in $\mathcal{S}(\Gamma)$ two independent stars whose centers form a set with differential ≥ 4, so (*i*) must be satisfied.
If Γ contains three vertices of degree three, by (*i*), these must be adjacent, so Γ contains a subgraph Γ' isomorphic to one of those shown in Figure 3. In any case, we can take vertices of $\mathcal{S}(\Gamma')$ with differential ≥ 4. This establishes (*ii*).

If a vertex v has degree three and $\Gamma\backslash\{v\}$ contains no edges, then Γ is isomorphic to S_4, and thus, $\partial(\mathcal{S}(\Gamma)) = \partial(\Gamma) = 2$, and this yields an absurd. If $\Gamma\backslash\{v\}$ contains two independent edges e_1 and e_2, then $\{v, e_1, e_2\}$ constitute a set of vertices in $\mathcal{S}(\Gamma)$ with a differential ≥ 4. This, once again, leads to a contradiction. Conversely, (1), (2) and (3) − (*a*) is a simple verification. If $\Delta = 3$ and Γ satisfies (*i*) − (*iii*), we have cases to consider:

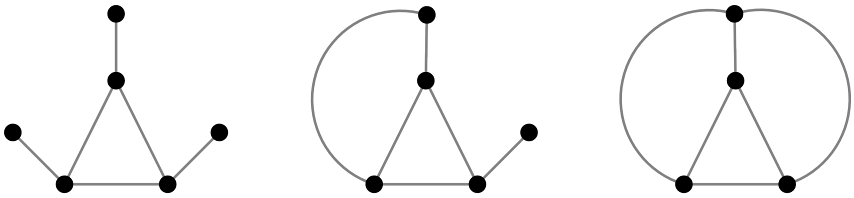

Figure 3. Graphs with at least three vertices of degree 3.

Case I *Within* Γ*, there exist precisely one vertex of degree three.*
By considering (*iii*) it can be observed that one of the graphs shown in Figure 4 must be isomorphic to Γ.

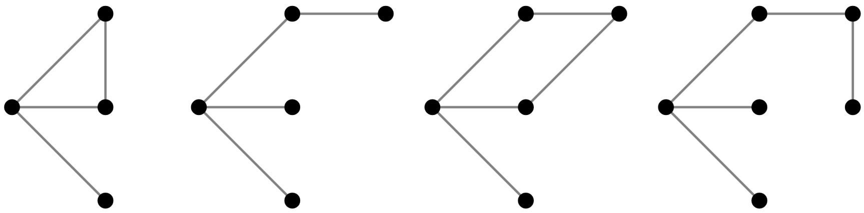

Figure 4. Graphs with exactly one vertex of degree 3.

Case II Γ *contains exactly two vertices of degree three.*
Here, taking into account condition (*iii*), it can also be observed that Γ can only be isomorphic to one of the graphs shown in Figure 5.

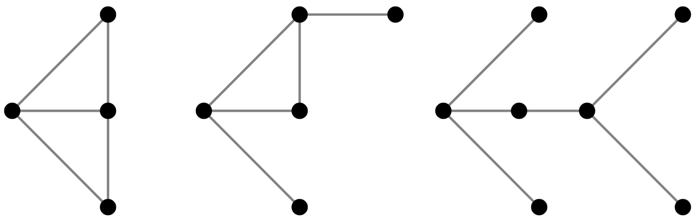

Figure 5. Graphs with exactly two vertices of degree 3.

In either cases, a simple calculation shows that $\partial(\mathcal{S}(\Gamma)) = 3$.

Part (*i*) in the next lemma appears in [17] and part (*ii*) appears in [21].

Lemma 2.3. *For paths P_n with $n \geq 1$, cycles C_n with $n \geq 3$, and star graphs of order n we have:*

(i) $\partial(P_n) = \partial(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$,
(ii) $\partial(S_n) = n - 2$.

Proposition 2.4. *Let P_n, C_n, S_n, $S_{p,q}$, $K_{p,q}$ and K_n be the path, cycle, star, double star, complete bipartite and complete graphs respectively, then:*

(i) $\partial(\mathcal{S}(P_n)) = \left\lfloor \frac{2n-1}{3} \right\rfloor$ *with* $n \geq 2$,
(ii) $\partial(\mathcal{S}(C_n)) = \left\lfloor \frac{2n}{3} \right\rfloor$ *with* $n \geq 3$,
(iii) $\partial(\mathcal{S}(S_n)) = n - 2$ *with* $n \geq 3$,
(iv) $\partial(\mathcal{S}(S_{p,q})) = n - 3$ *with* $p + q = n$,
(v) $\partial(\mathcal{S}(K_{p,q})) = pq - p$, *with* $p \leq q$,
(vi) $\partial(\mathcal{S}(K_n)) = \frac{n(n-1)}{2} + 2 - n$.

Proof. Notice that $\mathcal{S}(P_n) = P_{2n-1}$ and $\mathcal{S}(C_n) = C_{2n}$, by Lemma 2.3, we obtain (i) and (ii).

(iii) Let $P \subseteq V(\mathcal{S}(S_n))$ and let v be the vertex of S_n such that $\delta_{\mathcal{S}(S_n)}(v) = n-1$. If $P \subseteq U$, then $\partial(P) = 1$. If $P \subseteq V \setminus \{v\}$, then $\partial(P) = 0$. If $P = \{v\}$, then $\partial(P) = n - 2$. Let be $P = X \cup Y$ such that $X \subset V$ and $Y \subset U$ where X and Y are non-empty sets, then $\partial(P) \leq n - 1 - |P| \leq n - 3$. Having analysed all possible cases for P, it follows that $\partial(\mathcal{S}(S_n)) = n - 2$.

(iv) Let $P \subseteq V(\mathcal{S}(S_{p,q}))$ and let v and v' be vertices of $S_{p,q}$ such that $\delta_{\mathcal{S}(S_{p,q})}(v) = p-1$ and $\delta_{\mathcal{S}(S_{p,q})}(v') = q-1$. If we consider the different cases for the subsets P of U and V, we can obtain the following results:

- If P is a subset of U, then it follows that $\partial(P) \leq 2$.
- If P is a subset of $V\setminus\{v, v'\}$, then $\partial(P) = 0$.
- If $P = \{v\}$ or $P = \{v'\}$, then $\partial(P) = p - 1$ or $\partial(P) = q - 1$, respectively.
- If $P = \{v, v'\}$, then $\partial(P) = (p - 1) + (q - 1) + 1 - 2 = p + q - 3$.

Let P be the union of non-empty sets X and Y, where X is a subset of V and Y is a subset of U, then $\partial(P) \leq (p - 1) + (q - 1) + 1 - |P| = p + q - 1 - |P| \leq p + q - 1 - 3 = p + q - 4$. We have finished analysing all the cases for P, and thus we obtain $\partial(\mathcal{S}(S_{p,q})) = p + q - 3$.

(v) Let P be a subset of $V(\mathcal{S}(K_{p,q}))$, where $V(K_{p,q}) = X \cup Y$ and $|X| = p \leq q = |Y|$. If $P \subseteq X$, then $\partial(P) \leq pq-p$. If $P \subseteq Y$, then $\partial(P) \leq pq-q \leq pq-p$. Suppose that $P \subseteq U$, then $\partial(P) \leq 2p-p = p$. Therefore, $\partial(\mathcal{S}(K_{p,q})) = pq - p$.

(vi) Let K_n be the complete graph with vertex set $V(K_n) = \{v_1, \ldots, v_n\}$, and let $u \in U$ be the unique vertex whose neighborhood is $\{v_{n-1}, v_n\}$. If $P = \{v_1, v_2, \ldots, v_{n-2}, u\}$, is not very difficult to verify that $\partial(P) = \frac{n(n-1)}{2} - n + 2$. Now we will show that any non-empty subset $P' \subset V(\mathcal{S}(K_n))$ satisfies $\partial(P') \leq \partial(P)$. Suppose that $P' \subseteq U$, then $\partial(P') = |B(P')| - |P'| \leq n - |P'| \leq n - 1 \leq \partial(P)$.
Now let us suppose that $P' \subseteq V$ with cardinality $|P'| = n - k$ where $0 \leq k \leq n - 1$. Calculating the value of $|B(P')|$ we obtain

$$|B(P')| = \frac{n(n-1)}{2} - \frac{k(k-1)}{2},$$

and therefore $\partial(P') = \frac{n(n-1)}{2} - \frac{k(k-1)}{2} - (n - k) < \partial(P)$.
Suppose X and Y are non-empty and proper subsets of V and U, respectively. Let $P' = X \cup Y$. We can examine the following scenarios:

Case 1: $|P'| \leq n - 2$. Suppose that $|X| \leq n - 2 - i$, and $|Y| = i$ with $i = 1, \ldots, n - 2$, then $|B(P')| \leq |B(X)| + |B(Y)| \leq \frac{n(n-1)}{2} - i$, thus $\partial(P') \leq \frac{n(n-1)}{2} - n + i < \partial(P)$.

Case 2: $|P'| \geq n$. Suppose that $|X| \geq n - i$, and $|Y| = i$ with $i = 1, \ldots, n - 1$, then $\partial(P') \leq \frac{n(n-1)}{2} - n < \partial(P)$.

Case 3: $|P'| = n-1$. Suppose that $|X| \leq n-2-i$, and $|Y| = i$ with $i = 1, \ldots, n-2$, then $|B(P')| \leq \frac{n(n-1)}{2}$, therefore $\partial(P') = \frac{n(n-1)}{2} - n - 1 < \partial(P)$.

We have analysed all cases and therefore conclude that $\partial(\mathcal{S}(K_n)) = \frac{n(n-1)}{2} - n + 2$.

Proposition 2.5. *Suppose that* Γ *is a graph with a circumference* $\mathcal{L}_\Gamma$. *Then*

$$\partial(\mathcal{S}(\Gamma)) \geq \left\lfloor \frac{2\mathcal{L}_\Gamma}{3} \right\rfloor.$$

Proof. Let $\mathcal{L}_\Gamma$ be the circumference of Γ given by the cycle $C_\Gamma = \{v_1, v_2, \ldots, v_{\mathcal{L}}\}$, then the circumference $\mathcal{L}_{\mathcal{S}(\Gamma)}$ of $\mathcal{S}(\Gamma)$, given by the cycle $C_{\mathcal{S}(\Gamma)} = \{v_1, e_1, v_2, e_2, \ldots, v_{\mathcal{L}}, e_{\mathcal{L}}\}$, satisfies that $\mathcal{L}_{\mathcal{S}(\Gamma)} \geq 2\mathcal{L}_\Gamma$, therefore $\partial(\mathcal{S}(\Gamma)) \geq \partial(C_{\mathcal{S}(\Gamma)}) = \lfloor \frac{2\mathcal{L}_\Gamma}{3} \rfloor$.

In this part we study the properties of the independence number since it appears naturally in the study of differential sets (see Proposition 2.13).

Proposition 2.6. *Assuming* Γ *is a graph with n vertices and m edges, then:*

(i) $\beta(\mathcal{S}(\Gamma)) = min\{n, m\}$,
(ii) $\alpha(\mathcal{S}(\Gamma)) = max\{n, m\}$.

Proof. (*i*) If Γ is a tree, then we can arbitrarily select a vertex w in $V(\Gamma)$ as the root of the tree Γ. Since for each $v \in V(\Gamma) \setminus \{w\}$ there is only one unique edge $e_v \in E(\Gamma)$ that connects v with its respective parent vertex, we can define a bijection

$$f : V(\Gamma) \setminus \{w\} \to E(\Gamma)$$

as

$$f(v) = e_v.$$

If Γ has at least one cycle, then let us consider a spanning tree T of Γ. Let $e \in E(\Gamma) \setminus E(T)$, and let $x \in V(T)$ be one of the ends of e. By fixing x as the root of T, we can define a bijection similarly to how it was done previously

$$f : V(\Gamma) \to E(T) \cup \{e\}$$

as described below:

$$f(v) = \begin{cases} e, & \text{if } v = x \\ e_v, & \text{otherwise} \end{cases}$$

where e_v is the unique edge in $E(T)$ that connects v with its respective parent vertex.

(*ii*) Suppose there is an independent set I in the graph $\mathcal{S}(\Gamma)$ such that $|I| > max\{m, n\}$. Let us define $n_1 := |I \cap V|$, $m_1 := |I \cap U|$, $n_2 = n - n_1$ and $m_2 = m - m_1$. If $n \geq m$, then result (*i*) tells us that there exists a matching that covers V, and since I is independent, it follows that $m_2 \geq n_1$, therefore

$$m = m_1 + m_2 \geq m_1 + n_1 = |I| \geq m + 1.$$

The other case is similar.

A *vertex cover* of a graph Γ is a subset $C \subseteq V(\Gamma)$ such that every edge of Γ has at least one end vertex in C. The *vertex cover number* of Γ is the cardinality of any smallest vertex cover in Γ and it is denoted by $\tau(\Gamma)$. An *edge cover* of Γ is a subset $A \subseteq E(\Gamma)$ such that every vertex of Γ is incident to at least one edge of the set A. The *edge cover number* of Γ is the cardinality of any smallest edge cover in Γ and it is denoted by $\rho(\Gamma)$.

The next classical Gallai's theorems appear in [35].

Lemma 2.7. *The equation* $|V(\Gamma)| = \beta(\Gamma) + \rho(\Gamma)$ *holds true for any graph* Γ.

Lemma 2.8. *The equation* $|V(\Gamma)| = \alpha(\Gamma) + \tau(\Gamma)$ *holds true for any graph* Γ.

The next corollaries follows from Lemmas 2.7 and 2.8, and Proposition 2.6.

Corollary 2.9. *Given a graph* Γ *with n vertices and m edges, it follows that* $\tau(\mathcal{S}(\Gamma)) = min\{n, m\}$.

Corollary 2.10. *For a graph* Γ *with n vertices and m edges,* $\rho(\mathcal{S}(\Gamma)) = max\{n, m\}$.

The following propositions characterizes all graphs that have a matching number equal to n, and an independence number equal to m, on the subdivision operator $\mathcal{S}(\Gamma)$.

Proposition 2.11. *Consider a simple graph* $\Gamma = (V, E)$ *with n vertices and m edges. The statements that follow are equivalent:*

(i) Γ *contains at least one cycle,*
(ii) $\beta(\mathcal{S}(\Gamma)) = n$,
(iii) $\rho(\mathcal{S}(\Gamma)) = m$.

Proof. *(i)* ⇒ *(ii)*. Suppose that Γ contains at least one cycle, then $n \le m$. As stated in Proposition 2.6 (i), $\beta(\mathcal{S}(\Gamma)) = n$.

(ii) ⇒ *(i)*. Now, suppose that $\beta(\mathcal{S}(\Gamma)) = n$. By Proposition 2.6 ($i$), $n \le m$, therefore Γ contains at least one cycle.

The implications *(ii)* ⇒ *(iii)* and *(iii)* ⇒ *(ii)* follow by Lemma 2.7.

Proposition 2.12. *In any graph* $\Gamma = (V, E)$ *with n vertices and m edges, the statements that follow are equivalent:*

(i) Γ *contains at least one cycle,*
(ii) $\alpha(\mathcal{S}(\Gamma)) = m$,
(iii) $\tau(\mathcal{S}(\Gamma)) = n$.

Proof. *(i)* ⇒ *(ii)*. If Γ contains at least one cycle, then $n \le m$. According to Proposition 2.6 (ii), we have that $\alpha(\mathcal{S}(\Gamma)) = m$.

(ii) ⇒ *(i)*. Suppose that $\alpha(\mathcal{S}(\Gamma)) = m$. Then $n \le m$, therefore Γ contains at least one cycle.

The implications *(ii)* ⇒ *(iii)* and *(iii)* ⇒ *(ii)* follow by Lemma 2.8.

Proposition 2.13. *Any differential set P of vertices in the subdivision graph* $\mathcal{S}(\Gamma)$ *is also an independent set.*

Proof. Let $v, e \in P$ be adjacent vertices. By Remark 1 (*iv*), there exists another vertex $v' \in V$ adjacent to e, we note that this vertex is in P or $B(P)$. In both cases, it holds that

$$\partial(P\backslash\{e\}) = |B(P\backslash\{e\})| - |P\backslash\{e\}| \geq (|B(P)| - 1 + 1) - (|P| - 1) = \partial(P) + 1.$$

Theorem 2.14. *Let* Γ *be a graph and let* P *be a differential set of* Γ*, then*

$$\partial(\mathcal{S}(\Gamma)) \geq \partial(\Gamma) + |E(\langle P\rangle_\Gamma)| + \partial(\mathcal{S}(\langle V\backslash P\rangle_\Gamma)).$$

Proof. Let us define $M = P \cup B_{\mathcal{S}(\Gamma)}(P)$. Notice that $\langle M\rangle \cup \mathcal{S}(\langle V\backslash P\rangle_\Gamma)$ is a subgraph of $\mathcal{S}(\Gamma)$, therefore

$$\begin{aligned}\partial(\mathcal{S}(\Gamma)) &\geq \partial(\langle M\rangle \cup \mathcal{S}(\langle V\backslash P\rangle_\Gamma))\\ &= \partial(\langle M\rangle) + \partial(\mathcal{S}(\langle V\backslash P\rangle_\Gamma))\\ &\geq \partial(P) + |E(\langle P\rangle_\Gamma)| + \partial(\mathcal{S}(\langle V\backslash P\rangle_\Gamma))\\ &= \partial(\Gamma) + |E(\langle P\rangle_\Gamma| + \partial(\mathcal{S}(\langle V\backslash P\rangle_\Gamma)).\end{aligned}$$

Corollary 2.15. *Let* Γ *be a graph, then* $\partial(\Gamma) \leq \partial(\mathcal{S}(\Gamma))$.

Proposition 2.16. *For each graph* Γ *of order* $n \geq 3$, $\partial(\Gamma) = \partial(\mathcal{S}(\Gamma))$ *if and only if* $\Gamma \simeq S_n$.

Proof. Let P be a differential set of Γ. If $\partial(\mathcal{S}(\Gamma)) = \partial(\Gamma)$, then by Theorem 2.14 it follows that $|E(\langle P\rangle_\Gamma)| = \partial(\mathcal{S}(\langle V\backslash P\rangle_\Gamma)) = 0$, which means there are no edges in P nor in $\langle V\backslash P\rangle_\Gamma$, otherwise $\partial(\mathcal{S}(\Gamma)) > \partial(\Gamma)$, a contradiction. Therefore, Γ is a bipartite graph with bipartition $P \cup B_\Gamma(P)$.

Now we will prove that $|P| = 1$.

If $|P| \geq 2$, then since Γ is a connected bipartite graph, there must exist vertices $u, v, w \in V$ such that $u, v \in P$ and $w \in N(u) \cap N(v) \cap B(P)$. This implies that $\partial_{\mathcal{S}(\Gamma)}(P) = |B_{\mathcal{S}(\Gamma)}(P)| - |P| \geq |B_\Gamma(P)| + 1 - |P| = \partial(\Gamma) + 1$, which is a contradiction.

A *star* S_k (also known as S_k star or $S_{1,k}$) is a graph with a single central vertex, denoted by c, and k neighbors that only connect to c. These neighbors have no additional neighbors other than c. We denote an S_k star by $X = \{c, v_1, \ldots, v_k\}$ to indicate that the center is c and has k leaves, denoted by $v_1, \ldots, v_k$. If $k \geq 2$, we call it a big star.

Given a graph Γ, a *big star packing* is given by a vertex-disjoint collection $\mathcal{S} = \{X_i : 1 \leq i \leq k\}$ of (not necessarily induced) big stars $X_i \subseteq V(\Gamma)$ i.e, the graph induced by X_i contains some S_k with $k = |X_i| - 1 \geq 2$. By using the notation $\mathcal{S} \in SP(\Gamma)$, we are indicating that $\mathcal{S}$ is a member of the set of all possible star packings for a graph Γ.

The following lemma was proved in [20].

Lemma 2.17. *Given a graph* Γ, $\partial(\Gamma) = max\{\sum_{X\in\mathcal{S}}(|X| - 2) \mid \mathcal{S} \in SP(\Gamma)\}$.

One of the main results of the work is Theorem 2.19. In its proof we need the following definition.

Definition 2.18. Given a graph Γ, an *edge star packing* of Γ is a collection $\mathcal{T}$ of stars $Y \subseteq V$ ($|Y| \geq 2$) satisfying the following conditions:

(1) two different stars in $\mathcal{T}$ correspond to different centers,
(2) if $Y, Y' \in \mathcal{T}$ with $Y \neq Y'$, then $E(Y) \cap E(Y') = \emptyset$,
(3) the collection of stars in $\mathcal{T}$ isomorphic to P_2 form a matching of Γ and their vertices are not the centers of any other star.

We denote by $ESP(\Gamma)$ the collection of all edge star packings of Γ. If $\mathcal{T} \in ESP(\Gamma)$ and $\mathcal{T}' = \{Y \in \mathcal{T} \mid Y \simeq P_2\}$, let $\mathcal{X}_{\mathcal{T}'} : \mathcal{T} \longrightarrow \{0, 1\}$ be the characteristic function on $\mathcal{T}'$.

Theorem 2.19. *Given a graph* Γ,

$$\partial(\mathcal{S}(\Gamma)) = max\left\{\sum_{Y\in\mathcal{T}}(|Y| - 2 + \mathcal{X}_{\mathcal{T}'}(Y)) \mid \mathcal{T} \in ESP(\Gamma)\right\}.$$

Proof. Let $\mathcal{T} \in ESP(\Gamma)$ and $Y \in \mathcal{T}$. If $Y \simeq P_2$ let us make $X_Y = \mathcal{S}(Y)$; otherwise, if $Y = \{c, v_1, \ldots, v_k\}$, let us make $X_Y = \{c, e_1, \ldots, e_k\}$, where each e_i is that edge of Y that connects its center c with v_i. It can be verified that the collection $\mathcal{S}_{\mathcal{T}} = \{X_Y \mid Y \in \mathcal{T}\}$ is a big star packing of $\mathcal{S}(\Gamma)$ such that

$$\sum_{Y\in\mathcal{T}}(|Y| - 2 + \mathcal{X}_{\mathcal{T}'}(Y)) = \sum_{X_Y\in\mathcal{S}_{\mathcal{T}}}(|X_Y| - 2),$$

and hence

$$max\left\{\sum_{Y\in\mathcal{T}}(|Y| - 2 + \mathcal{X}_{\mathcal{T}'}(Y)) \mid \mathcal{T} \in ESP(\Gamma)\right\} \leq$$
$$max\left\{\sum_{X\in\mathcal{S}}(|X| - 2) \mid \mathcal{S} \in SP(\mathcal{S}(\Gamma))\right\}. \tag{2.1}$$

On the other hand, if $\mathcal{S} \in SP(\mathcal{S}(\Gamma))$ and $X \in \mathcal{S}$ is a star isomorphic to P_3 whose center is an edge e of Γ, let Y_X be the star of Γ formed precisely by the ends of e. In another case, if $X = \{c, e_1, \ldots, e_k\}$, where the center c is a vertex of Γ and each e_i is an edge of Γ; if v_i is the end vertex of e_i that is adjacent to c in Γ, let us denote $Y_X = \{c, v_1, \ldots, v_k\}$. Thus, it can be verified that $\mathcal{T}_{\mathcal{S}} = \{Y_X \mid X \in \mathcal{S}\}$ is an edge star packing of Γ such that

$$\sum_{X\in\mathcal{S}}(|X| - 2) = \sum_{Y_X\in\mathcal{T}_{\mathcal{S}}}(|Y_X| - 2 + \mathcal{X}_{\mathcal{T}'_{\mathcal{S}}}(Y_X)).$$

The aforementioned analysis indicates that the Inequality (2.1) is in fact an equality. Consequently, the result follows from Lemma 2.17.

Figure 6 illustrates two edge star packings of Γ defined as $\mathcal{T} = \{Y_1, Y_2, Y_3\}$ and $\mathcal{T}' = \{Y_4, Y_5\}$ where $Y_1 = \{v_1, v_2, v_3\}$, $Y_2 = \{v_4, v_3, v_2, v_5\}$, $Y_3 = \{v_2, v_5\}$, $Y_4 = \{v_2, v_1, v_4, v_5\}$ and $Y_5 = \{v_3, v_1, v_4\}$. Then $\mathcal{S}_{\mathcal{T}} = \{X_1, X_2, X_3\}$ and $\mathcal{S}_{\mathcal{T}'} = \{X_4, X_5\}$ where $X_1 = \{v_1, e_1, e_2\}$, $X_2 = \{v_4, e_5, e_3, e_6\}$, $X_3 = \{e_4, v_2, v_5\}$, $X_4 = \{v_2, e_1, e_3, e_4\}$ and $X_5 = \{v_3, e_2, e_5\}$ are two big star packings of $\mathcal{S}(\Gamma)$.

Proposition 2.20. *Let* Γ *be a graph of order n, then* $\partial(\mathcal{S}(\Gamma)) \leq \frac{n(n-1)}{2} + 2 - n$.

Proof. Since Γ is a subgraph of K_n, then each edge star packing of Γ is an edge star packing of K_n, thus

$$\partial(\mathcal{S}(\Gamma)) = max\left\{\sum_{Y\in\mathcal{T}}(|Y| - 2 + \mathcal{X}_{\mathcal{T}'}(Y)) \mid \mathcal{T} \in ESP(\Gamma)\right\} \leq$$

$$max\left\{\sum_{Y\in\mathcal{T}}(|Y| - 2 + \mathcal{X}_{\mathcal{T}'}(Y)) \mid \mathcal{T} \in ESP(K_n)\right\} = \partial(\mathcal{S}(K_n)).$$

By Proposition 2.4 (*vi*), $\partial(\mathcal{S}(\Gamma)) \leq \frac{n(n-1)}{2} + 2 - n$.

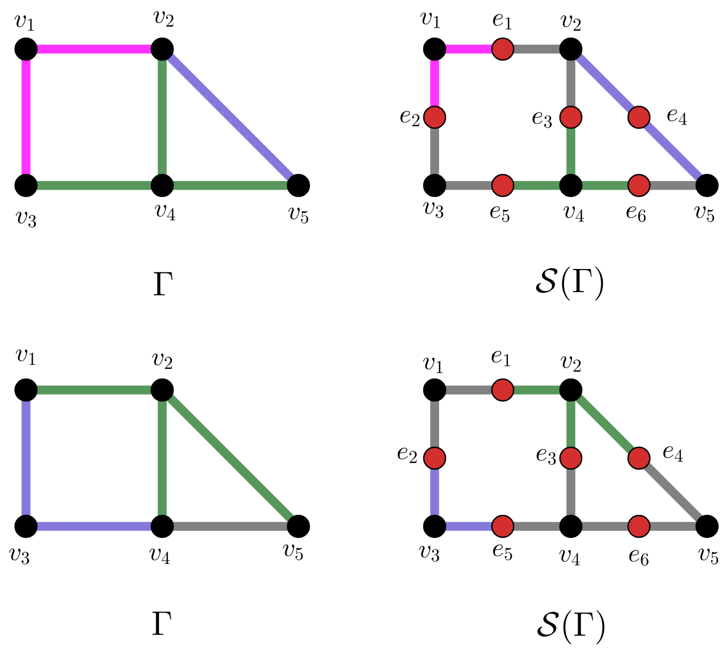

Figure 6. Two examples of distinct edge star packings of Γ that induce two distinct big star packings of $\mathcal{S}(\Gamma)$.

A subset $S \subseteq V(\Gamma)$ of vertices in graph Γ is called a *dominating* set if every vertex in Γ either belongs to S or is adjacent to at least one vertex in S. The *domination number* of Γ is the cardinality of any smallest dominating set in Γ and it is denoted by $\gamma(\Gamma)$.

Proposition 2.21. *Let Γ be a graph of order n. Then $\partial(\mathcal{S}(\Gamma)) = \frac{n(n-1)}{2} + 2 - n$ if and only if $\Gamma \simeq K_n$.*

Proof. If $\Gamma \simeq K_n$, by Proposition 2.4 (*vi*) we obtain $\partial(\mathcal{S}(\Gamma)) = \frac{n(n-1)}{2} + 2 - n$.

If Γ is not an isomorphic graph to K_n, then there exist at least two vertices $v, v' \in V(\Gamma)$ such that $vv' = e \notin E(\Gamma)$. Let P be a differential set of $\mathcal{S}(K_n)$. Since P is a dominating set of $\mathcal{S}(K_n)$, then we consider the following cases:

Case 1: $e \in P$. By Proposition 2.4 (*vi*), it follows that e is the only element such that $e \in P \cap U$. Additionally, there exist vertices $v, v' \in V$ such that $v, v' \in N(e)$. By Proposition 2.13, we know that $v, v' \in B(P)$. Thus, it follows that

$$\partial(\mathcal{S}(\Gamma)) \leq |B(P)| - 2 - (|P| - 1) = \partial(\mathcal{S}(K_n)) - 1.$$

Case 2: $e \in B(P)$. There exists $v, v' \in V$ such that $v, v' \in N(e)$. Notice that $v, v' \notin C(P)$, because P is a dominating set. Therefore

$$\partial(\mathcal{S}(\Gamma)) \leq |B(P)| - 1 - |P| = \partial(\mathcal{S}(K_n)) - 1.$$

Thus, $\partial(\mathcal{S}(\Gamma))$ and $\partial(\mathcal{S}(K_n))$ are different.

Proposition 2.22. *For each $n \geq 3$ and $n - 2 \leq r \leq \frac{n(n-1)}{2} + 2 - n$, there exists a graph Γ of order n such that $\partial(\mathcal{S}(\Gamma)) = r$.*

Proof. We proceed to prove the proposition using induction on r being the base case when $\Gamma \simeq S_n$.

Let H be a graph of order n with maximum number of edges such that $\partial(\mathcal{S}(H)) = r - 1$. Let $v, v' \in V$ be non-adjacent vertices in H and let $\Gamma = H \cup \{e\}$ where e has v and v' as ends. Let P be a differential set of $\mathcal{S}(\Gamma)$ and let us examine the possible location of v, e and v' with respect to this set, its border and its complement.

Notice that $e \notin C_{\mathcal{S}(\Gamma)}(P)$, otherwise $\partial_{\mathcal{S}(H)}(P) = \partial_{\mathcal{S}(\Gamma)}(P)$ and this contradicts the choice of H. Therefore $e \in P$ or $e \in B_{\mathcal{S}(\Gamma)}(P)$.

In the first case, by Proposition 2.13, P is an independent set of $\mathcal{S}(\Gamma)$, so v and v' are in $B_{\mathcal{S}(\Gamma)}(P)$. The set $P' = P \setminus \{e\}$ has as border to $B_{\mathcal{S}(H)}(P') = B_{\mathcal{S}(\Gamma)}(P) \setminus \{v, v'\}$, otherwise $\partial(\mathcal{S}(H)) \geq \partial(\mathcal{S}(\Gamma))$, a contradiction. Thus $\partial_{\mathcal{S}(H)}(P') = \partial(\mathcal{S}(\Gamma)) - 1$ and it follows that $\partial(\mathcal{S}(\Gamma)) = r$. Now suppose that $e \in B_{\mathcal{S}(\Gamma)}(P)$. If v and v' are in P, then $B_{\mathcal{S}(H)}(P) = B_{\mathcal{S}(\Gamma)}(P) \setminus \{e\}$, thus $\partial_{\mathcal{S}(H)}(P) = \partial(\mathcal{S}(\Gamma)) - 1$ again, it follows that $\partial(\mathcal{S}(\Gamma)) = r$.

Another case occurs when an end of e, say v, is in P and the other end v' is in $B_{\mathcal{S}(\Gamma)}(P)$. As in the previous case, it follows that $\partial_{\mathcal{S}(H)}(P) = \partial(\mathcal{S}(\Gamma)) - 1$ and therefore $\partial(\mathcal{S}(\Gamma)) = r$. Analogously, the desired equality is also obtained if we assume that $v \in P$ and $v' \in C(P)$.

In Figure 7 we show the subdivision of five graphs which satisfy the conditions of the Proposition 2.22, where $3 \leq r \leq 7$.

Remark 2. Let Γ be a graph with maximum degree Δ and P be a differential set of $\mathcal{S}(\Gamma)$. We denote the number of edges of P by E_P, the number of connecting vertices between P and $B(P)$ by η, the number of edges of $B(P)$ by $E_{B(P)}$, the number edges of connecting vertices between $B(P)$ and $C(P)$ by η' and the number of edges of $C(P)$ by $E_{C(P)}$. By Remark 1 (*ii*), we have:

$$2m = |E(\mathcal{S}(\Gamma))| = E_P + \eta + E_{B(P)} + \eta' + E_{C(P)}. \tag{2.2}$$

By Proposition 2.13, we have that P is an independent set, and hence $E_P = 0$. Notice that $2E_{B(P)} = \sum_{v \in B(P)} \delta_{B(P)}(v) \leq |B(P)|(\Delta - 1)$ and $\eta \leq |P|\Delta$. Additionally, if P is a maximum differential set of $\mathcal{S}(\Gamma)$, then $\{P, B(P), C(P)\}$ satisfies that $E_{C(P)} = 0$ and $\eta' \leq |B(P)|$.

Before we proceed to estimate a lower bound for $\partial(\mathcal{S}(\Gamma))$ involving $\gamma(\mathcal{S}(\Gamma))$, consider the following result that appears in [19].

Lemma 2.23. *If P is a differential set of a graph Γ, then $|P| \leq \gamma(\Gamma)$.*

Proposition 2.24. *Let Γ be a graph of size m and maximum degree Δ, then*

$$4m \leq (\Delta + 1)\partial(\mathcal{S}(\Gamma)) + (3\Delta + 1)\gamma(\mathcal{S}(\Gamma)).$$

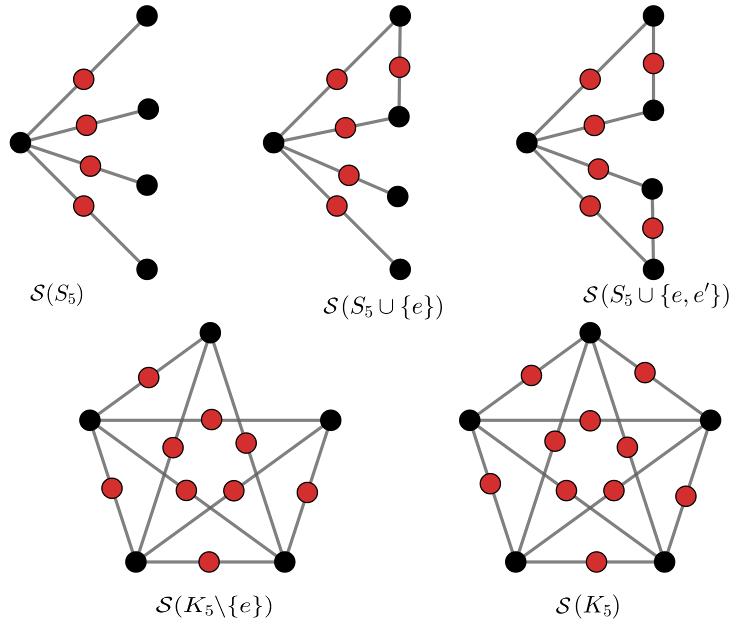

Figure 7. These graphs have differential 3, 4, 5, 6 and 7, respectively.

Proof. Let P be a maximum differential set of $\mathcal{S}(\Gamma)$. Using Eq (2.2) it follows that

$$2m = \eta + E_{B(P)} + \eta' \leq |P|\Delta + \frac{|B(P)|(\Delta - 1)}{2} + |B(P)|,$$

thus

$$\begin{aligned}4m \leq & 2|P|\Delta + |B(P)|(\Delta - 1) + 2|B(P)| \\ = & 2|P|\Delta + |B(P)|\Delta + |B(P)| \\ = & |B(P)|(\Delta + 1) - |P|(\Delta + 1) + (3\Delta + 1)|P| \\ \leq & (\Delta + 1)\partial(\mathcal{S}(\Gamma)) + (3\Delta + 1)\gamma(\mathcal{S}(\Gamma)).\end{aligned}$$

For the next two results, we will return to the definition of an edge star packing. To denote that set P is the set of vertices that are star centers of an edge star packing $\mathcal{T}$, we will use the notation $\mathcal{T}(P)$. We call an edge star packing $\mathcal{T} \in ESP(\Gamma)$ a *differential edge star packing* if it achieves the differential of the graph, i.e., if $\partial(\mathcal{T}) = \partial(\Gamma)$. A *maximum differential edge star packing* is a differential edge star packing of maximum cardinality, i.e., if it contains the highest possible number of stars among all sets of vertices in consideration. Let *max* $ESP(\Gamma)$ be the collection of all maximum differential edge star packings of Γ.

Proposition 2.25. *Let* Γ *be a graph of order n, then:*

(i) for every edge star packing $\mathcal{T}(P) \in$ *max* $ESP(\Gamma)$ *satisfies* $\delta_{C(P)}(v) \leq 1$ *for every* $v \in B(P)$*,*
(ii) there exists an edge star packing $\mathcal{T}(P) \in$ *max* $ESP(\Gamma)$ *such that* $B(P)$ *is a dominating set of* Γ.

Proof. (i) If there exists $Y = \{c, v_1, \dots, v_k\} \in \mathcal{T}(P)$ such that $\{c_1, c_2\} \subseteq N(v_1) \cap C(P)$, then we have the following cases: if $k = 1, 2$, $\partial((P \backslash \{c\}) \cup \{v_1\}) > \partial(P)$, a contradiction. If $k \geq 3$, we consider $Y' = \{c, v_2, \dots, v_k\}$ and $Y'' = \{v_1, c_1, c_2\}$, then $(\mathcal{T}(P) \backslash \{Y\}) \cup \{Y', Y''\}$ gives the same differential with an increased number of stars, a contradiction.

(ii) If there exists a vertex $v \in C(P)$ which is not dominated by vertices in $B(P)$, then there must exists another vertex $v' \in C(P)$ and $Y = \{c, v_1, \dots, v_k\} \in \mathcal{T}(P)$ such that v' is adjacent to v and v_1, so we have again the following cases: If $k = 1$, then $\partial((P \backslash \{c\}) \cup \{v_1\}) > \partial(P)$, a contradiction. If $k = 2$ we can take $P' = (P \backslash \{c\}) \cup \{v_1\}$, $Y' = \{v_1, c, v'\}$ and form $\mathcal{T}(P') = (\mathcal{T}(P) \backslash \{Y\}) \cup \{Y'\}$. Now, the vertex v is adjacent to an element of $B(P')$. If $k \geq 3$, we consider $Y' = \{c, v_2, \dots, v_k\}$ and $Y'' = \{v', v_1, v\}$, then $(\mathcal{T}(P) \backslash \{Y\}) \cup \{Y', Y''\}$ gives the same differential with a larger number of stars, which contradicts the assumption that $\mathcal{T}(P)$ has the maximum number of stars.

Theorem 2.26. *If* Γ *is a graph with size* $m \geq 2$ *and maximum degree* Δ*, then*

$$\left\lceil \frac{4m}{3\Delta + 2} \right\rceil \leq \partial(\mathcal{S}(\Gamma)) \leq m - 1.$$

Proof. If we take an edge star packing $\mathcal{T}(P) \in max\ ESP(\Gamma)$. By Proposition 2.25, we have that $\delta_{C(P)}(y) \leq 1$ for every $y \in B(P)$ and $\delta_{C(P)}(w) \leq 1$ for every $w \in C(P)$. Moreover, in the proof of Proposition 2.25 it was show that there is not any edge in $C(P)$ adjacent to a star $Y = \{c, v_1, \dots, v_k\}$ ($k \neq 2$) and, since $B(P)$ is a dominating set, an edge in $C(P)$ adjacent to a star $Y' = \{c', v'_1, v'_2\}$ would provide an induced subgraph isomorphic to C_5. In consequence, if Γ is a C_5-free graph, then we have $\sum_{w \in C(P)} \delta(w) \leq |B(P)|$, therefore

$$\begin{aligned} 2m &= \sum_{x \in P} \delta(x) + \sum_{y \in B(P)} \delta(y) + \sum_{w \in C(P)} \delta(w) \\ &\leq \Delta|P| + \Delta|B(P)| + |B(P)| \\ &= \Delta|P| + (\Delta + 1)|B(P)| \\ &= (2\Delta + 1)|P| + (\Delta + 1)|B(P)| - (\Delta + 1)|P| \\ &= (2\Delta + 1)|P| + (\Delta + 1)\partial(\Gamma). \end{aligned}$$

Since $|P| \leq \partial(\Gamma)$, we can derive that $2m \leq \partial(\Gamma)(3\Delta + 2)$, thus $\partial(\Gamma) \geq \frac{2m}{3\Delta+2}$. Now, since $g(\mathcal{S}(\Gamma)) \geq 6$, $\mathcal{S}(\Gamma)$ is a C_5-free graph, then $\partial(\mathcal{S}(\Gamma)) \geq \frac{4m}{3\Delta+2}$.

For the other inequality, let $P \subseteq V(\mathcal{S}(\Gamma))$. Being $\mathcal{S}(\Gamma)$ a bipartite graph with bipartition $\{V, U\}$, where $|V| = n$ and $|U| = m$, and together with the property that $\delta(e) = 2$ for all $e \in U$ we conclude that

$$\partial(P) = |B(P)| - |P| \leq m - |P| \leq m - 1.$$

Conclusions

In this paper, we have explored the differential of graphs with respect to the subdivision operator $\mathcal{S}(\Gamma)$, and we have derived upper and lower bounds for the differential of $\mathcal{S}(\Gamma)$ based on graph invariants such as order, size, and maximum degree. We have also investigated the realizability of graphs for different values of the differential of $\mathcal{S}(\Gamma)$ within the interval $\left[n-2, \frac{n(n-1)}{2} - n + 2\right]$, and proposed a closed formula based on the edge star packing notion (Definition 2.18). Our findings include exact values of the differential of $\mathcal{S}(\Gamma)$ for several families of graphs (Propositions 2.2 and 2.4).

Overall, our study has contributed to a better understanding of the differential of graphs on the subdivision operator $\mathcal{S}(\Gamma)$, and has provided useful tools for analyzing and characterizing different graph structures.

Final comments

The concept of differential has been extensively studied in graph theory ([17, 18, 22, 23, 27, 29, 31, 32, 34]), and our work has contributed new results regarding the differential of graphs on the subdivision operator $\mathcal{S}(\Gamma)$. Our findings emphasize the importance of investigating graph operators and their corresponding parameters, which allows us to better understand the behavior of graphs. Additionally, our results can be used to analyze various graph structures.

Acknowledgments

We express our gratitude to the reviewers for their diligent examination of the manuscript and valuable suggestions that have contributed to the enhancement of the quality of this work.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. J. M. Sigarreta, Total domination on some graph operators, *Mathematics*, **9** (2021), 241. https://doi.org/10.3390/math9030241
2. S. Bermudo, Total domination on tree operators, *Mediterr. J. Math.*, **20** (2023), 42. https://doi.org/10.1007/s00009-022-02236-7
3. A. Aslam, J. L. García-Guirao, S. Ahmad, W. Gao, Topological indices of the line graph of subdivision graph of complete bipartite graphs, *Appl. Math. Inf. Sci.*, **11** (2017), 1631–1636. http://doi.org/10.18576/amis/110610
4. A. R. Bindusree, I. N. Cangul, V. Lokesha, A. S. Cevik, Zagreb polynomials of three graph operators, *Filomat*, **30** (2016), 1979–1986. http://doi.org/10.2298/FIL1607979B
5. J. C. Hernández-Gómez, J. A. Méndez-Bermúdez, J. M. Rodríguez, J. M. Sigarreta, Harmonic index and harmonic polynomial on graph operations, *Symmetry*, **10** (2018), 456. https://doi.org/10.3390/sym10100456
6. V. Lokesha, R. Shruti, P. S. Ranjini, A. Sinan-Cevik, On certain topological indices of nanoestructures using $\mathcal{Q}(G)$ and $\mathcal{R}(G)$ operators, *Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat.*, **67** (2018), 178–187.

7. P. S. Ranjini, V. Lokesha, Smarandache-Zagreb index on three graph operators, *Int. J. Math. Comb.*, **3** (2010), 1. https://doi.org/10.5281/ZENODO.9051
8. W. Yan, B. Y. Yang, Y. N. Yeh, The behavior of Wiener indices and polynomials of graphs under five graph decorations, *Appl. Math. Lett.*, **20** (2007), 290–295. https://doi.org/10.1016/j.aml.2006.04.010
9. W. Carballosa, J. M. Rodríguez, J. M. Sigarreta, N. Vakhania, f-polynomial on some graph operations, *Mathematics*, **7** (2019), 1074. https://doi.org/10.3390/math7111074
10. W. Carballosa, J. M. Rodríguez, J. M. Sigarreta, M. Villeta, On the hyperbolicity constant of line graphs, *Electron. J. Comb.*, **18** (2011), 210. https://doi.org/10.37236/697
11. F. Harary, R. Z. Norman, Some properties of line digraphs, *Rend. Circ. Mat. Palermo*, **9** (1960), 161–168. https://doi.org/10.1007/BF02854581
12. J. A. Méndez-Bermúdez, R. Reyes, J. M. Rodríguez, J. M. Sigarreta, Hyperbolicity on graph operators, *Symmetry*, **10** (2018), 360. https://doi.org/10.3390/sym10090360
13. R. Nagarathinam, N. Parvathi, Study of chromatic parameters of line, total, middle graphs and graph operators of bipartite graph, *J. Phys. Conf. Ser.* **1000** (2018), 012036. https://doi.org/10.1088/1742-6596/1000/1/012036
14. C. Natarajan, S. K. Ayyaswamy, A note on the hop domination number of a subdivision graph, *Int. J. Appl. Math.*, **32** (2019), 381. https://doi.org/10.12732/ijam.v32i3.2
15. P. S. Ranjini, V. Lokesha, S. Kumar, Degree sequence of graph operator for some standard graphs, *Appl. Math. Nonlinear Sci.*, **5** (2020), 99–108. https://doi.org/10.2478/amns.2020.2.00018
16. J. M. Sigarreta, J. A. Rodríguez, On defensive alliances and line graphs, *Appl. Math. Lett.*, **19** (2006), 1345–1350. https://doi.org/10.1016/j.aml.2006.02.001
17. J. Mashburn, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, P. J. Slater, Differentials in graphs, *Util. Math.*, **69** (2006), 43–54.
18. S. Bermudo, H. Fernau, Computing the differential of a graph: hardness, approximability and exact algorithms, *Discrete Appl. Math.*, **165** (2014), 69–82. https://doi.org/10.1016/j.dam.2012.11.013
19. L. A. Basilio, S. Bermudo, J. M. Sigarreta, Bounds on the differential of a graph, *Util. Math.*, **103** (2017), 319–334.
20. S. Bermudo, H. Fernau, Lower bound on the differential of a graph, *Discrete Math.*, **312** (2012), 3236–3250. https://doi.org/10.1016/j.disc.2012.07.021
21. S. Bermudo, J. M. Rodriguez, J. M. Sigarreta, On the differential in graphs, *Util. Math.*, **97** (2015), 257–270.
22. L. A. Basilio, S. Bermudo, J. Leaños, J. M. Sigarreta, The differential of the line graph $\mathcal{L}(G)$, *Discrete Appl. Math.*, **321** (2022), 82–89. https://doi.org/10.1016/j.dam.2022.05.004
23. L. A. Basilio, J. Castro-Simón, J. Leaños, O. R. Cayetano, The differential on graph operator $Q(G)$, *Symmetry*, **12** (2020), 751. https://doi.org/10.3390/sym12050751
24. S. Bermudo, L. De la Torre, A. M. Martín-Caraballo, J. M. Sigarreta, The differential of the strong product graphs, *Int. J. Comput. Math.*, **92** (2015), 1124–1134. https://doi.org/10.1080/00207160.2014.941359

25. J. M. Sigarreta, Differential in cartesian product graphs, *Ars Comb.*, **126** (2016), 259–267.
26. C. Armada, J. S. Canoy, A-differential of graphs, *Int. J. Math. Anal.*, **9** (2015), 2171–2180. http://doi.org/10.12988/ijma.2015.54132
27. L. A. Basilio, S. Bermudo, J. Leaños, J. M. Sigarreta, β -differential of a graph, *Symmetry*, **9** (2017), 205. https://doi.org/10.3390/sym9100205
28. A. Cabrera-Martínez, J. A. Rodríguez-Velázquez, On the perfect differential of a graph, *Quaest. Math.*, **45** (2022), 327–345. https://doi.org/10.2989/16073606.2020.1858992
29. S. Bermudo, On the differential and roman domination number of a graph with minimum degree two, *Discrete Appl. Math.*, **232** (2017), 64–72. https://doi.org/10.1016/j.dam.2017.08.005
30. S. Bermudo, H. Fernau, Combinatorics for smaller kernels: the differential of a graph, *Theor. Comput. Sci.*, **562** (2015), 330–345. https://doi.org/10.1016/j.tcs.2014.10.007
31. S. Bermudo, H. Fernau, J. M. Sigarreta, The differential and the roman domination number of a graph, *Appl. Anal. Discrete Math.*, **8** (2014), 155–171. https://doi.org/10.2298/AADM140210003B
32. A. Kanli, Z. N. O. Berberler, Differential in infrastructure networks, *RAIRO Oper. Res.*, **55** (2021), S1249–S1259. https://doi.org/10.1051/ro/2020032
33. P. Pushpam, D. Yokesh, Differential in certain classes of graphs, *Tamkang J. Math.*, **41** (2010), 129–138. https://doi.org/10.5556/j.tkjm.41.2010.664
34. L. A. Basilio, W. Carballosa, J. Leaños, J. M. Sigarreta, On the differential polynomial of a graph, *Acta Math. Sin. Engl. Ser.*, **35** (2019), 338–354 . https://doi.org/10.1007/s10114-018-7307-3
35. T. Gallai, Uber extreme punkt-und kantenmengen, annales universitatis scientiarum budapestinensis de rolando eotvos nominatae, *Sect. Math.*, **2** (1959), 133–138.

AIMS Press

© 2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)