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Survey

Survey of semi-tensor product method in robustness analysis on finite systems

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Abstract: Recently, the theory of semi-tensor product (STP) method of matrices has received much attention from variety communities covering engineering, economics and industries, etc. This paper describes a detailed survey on some recent applications of the STP method in finite systems. First, some useful mathematical tools on the STP method are provided. Second, many recent developments about robustness analysis on the given finite systems are delineated, such as robust stable analysis of switched logical networks with time-delayed, robust set stabilization of Boolean control networks, event-triggered controller design for robust set stabilization of logical networks, stability analysis in distribution of probabilistic Boolean networks, and how to solve a disturbance decoupling problem by event triggered control for logical control networks. Finally, several research problems in future works are predicted.

Keywords: semi-tensor product method of matrices; finite system; robustness; logical networks; algebraic formulation

1. Introduction

Nowadays, Boolean networks and logical networks, which are two classical finite systems, have been extensively investigated in both theory and applications. The concept of Boolean networks is initiated by Kauffman [1] to model gene regulatory networks. Especially, Boolean network is one kind of logical networks.

Recently, Cheng [2] proposed a new powerful mathematical tool, which is the STP method. From then, many scholars have applied the STP method to model and analyze Boolean networks and logical networks. After the algebraic expressions for Boolean networks and logical networks had been established, Boolean networks and logical networks have been commonly used mathematica models in a variety of communities. Some typical communities includes game theory, networked evolutionary games [3, 4], cyber-physical system [5] and gene regulatory networks [6–8].

What is noteworthy is that, many great results have been obtained after scholars applied the STP method to solve all kinds of classical control problems, such as stable analysis, stabilization controller design, optimize control, pinning control, etc. Readers can see more details in [2, 6–36].

Robustness is a system property. It describes the ability of a system to function correctly when unforeseen events appears. Robustness analysis is also a hot research topic for scholars in the control community. This paper focuses on the recent developments on the applications of the STP method in robustness analysis for finite systems and aims to give a comprehensive survey on these results.

The content of this survey covers many recent developments about robustness analysis on finite systems, such as robust stable analysis of switched logical networks with time delays, under impulsive effects of a robust set stabilization of Boolean control networks, event-triggered control for robust set stabilization of logical networks with control inputs, stability analysis in distribution of probabilistic Boolean networks under function perturbation impact, and how to solve a disturbance decoupling problem by event triggered control for of logical control networks. Furthermore, this survey forecasts some research works in the future.

The rest of this manuscript is organized as follows. Section 2 contains some necessary preliminaries on STP and game theory. Section 3.1 introduces the recent developments about robust stable analysis of switched logical networks with time delays. Section 3.2 delineates the idea for under impulsive effects robust set stabilization of Boolean control networks. Section 3.3 describes event-triggered control for robust set stabilization of logical networks with control inputs. Section 3.4 gives stability analysis in distribution of probabilistic Boolean networks under function perturbation impacts. Section 3.5 recalls how to solve disturbance decoupling problems by event triggered control for logical control networks, which is followed by a brief conclusion in Section 4.

Notations: $\mathbb{R}_{m\times n}$ denotes the set of $m\times n$ real matrices. $\mathbb{R}^+_{m\times n}$ denotes the set of $m\times n$ nonnegative real matrices. $\Delta_n := \left\{\delta_n^i \,|i = 1, 2, \cdots, n\right\}$, where δ_n^i is the i-th column of the identity matrix I_n. An $n\times t$ matrix M is called a logical matrix, if $M = [\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_t}]$, which is briefly denoted by $M = \delta_n[i_1\ i_2\ \cdots\ i_t]$. Define the set of $n \times t$ logical matrices as $\mathcal{L}_{n\times t}$. $Col_i(L)$ ($Row_i(L)$) is the i-th column (row) of matrix L. For a set E, $|E|$ denotes the number of elements in E. $r = (r_1, \cdots, r_k)^T \in \mathbb{R}_k$ is called a probabilistic vector, if $r_i \geq 0$, $i = 1, \cdots, k$, and $\sum_{i=1}^k r_i = 1$. The set of k dimensional probabilistic vectors is denoted by Υ_k. If $M \in \mathbb{R}^+_{m\times n}$ and $Col(M) \subset \Upsilon_m$, M is called a probabilistic matrix. The set of $m\times n$ probabilistic matrices is denoted by $\Upsilon_{m\times n}$.

2. Materials and methods

In the beginning, we give the definition of the STP method.

Given two matrices $A \in \mathbb{R}_{m\times n}$ and $B \in \mathbb{R}_{p\times t}$. The STP of them is defined as $A \ltimes B = (A\otimes I_{\frac{\alpha}{n}})(B\otimes I_{\frac{\alpha}{p}})$, where α is the least common multiple of n and p and $\otimes$ is the Kronecker product ($M * N$, as $M * N = [Col_1(M) \ltimes Col_1(N) \quad Col_2(M) \ltimes Col_2(N) \quad \cdots \quad Col_s(M) \ltimes Col_s(N)] \in \mathbb{R}_{pq\times s}$.).

Note that, if $n = p$ holds, the STP method is considered as the ordinary matrix product. In the rest of this paper, we simply call it "product" and omit the symbol "$\ltimes$" without confusion.

The theory of the STP method has many useful mathematic tools. We list four of them in the following. They will be useful throughout this paper.

Given $X \in \mathbb{R}_m$, $Y \in \mathbb{R}_n$, and define

$$D_f^{p,q} = \delta_p[\underbrace{1 \cdots 1}_{q} \ \underbrace{2 \cdots 2}_{q} \cdots \underbrace{p \cdots p}_{q}],$$

and

$$D_r^{p,q} = \delta_q[\underbrace{\underbrace{1\ 2 \cdots q}\ \underbrace{1\ 2 \cdots q} \cdots \underbrace{1\ 2 \cdots q}}_{p}],$$

we have

- $W_{[m,n]}XY = YX$ holds,
- $XA = (I_t \otimes A)X$ holds,
- $D_f^{p,q}XY = X$ holds,
- $D_r^{p,q}XY = Y$ holds,

where $W_{[m,n]}$ denotes the swap matrix (especially $W_{[n,n]} := W_{[n]}$).

In additional, we have a method to express a pseudo-logical $f(x_1, x_2, \cdots, x_n)$ is a mapping from Δ_k^n to $\mathbb{R}$ function into its algebraic form.

Lemma 2.1. *[11] Let $f : \Delta_k^n \to \mathbb{R}$ (or $f : \Delta_k^n \to \Delta_m$) be a pseudo-logical function. Then there exists a unique structural matrix M_f, called the structural matrix of f, such that*

$$f(x_1, x_2, \cdots, x_n) = M_f \ltimes_{i=1}^n x_i,$$

where $x_i \in \Delta_k$, $i = 1, 2, \cdots, n$, $Col_j(M_f) = f(\delta_{k^n}^j)$, and $j = 1, 2, \cdots, k^n$.

3. Results

3.1. Robust stable analysis of switched logical networks with time-delayed

This section delineates robust analysis of switched logical networks with time delays (SDLNs) with all unstable modes. In the beginning, we introduce the description of SDLNs. Then, some new results about them are presented. Future works could generalize these results to switched logical networks with time delays under state-dependent delay and state constraints.

3.1.1. Description of SDLNs

There is a classical SDLN with n state nodes, m disturbance inputs, and s subnetworks:

$$\begin{cases} x_1(t+1) = g_1^{\varrho(t)}(X(t-\tau), \cdots, X(t), \Upsilon(t)), \\ \quad \vdots \\ x_n(t+1) = g_n^{\varrho(t)}(X(t-\tau), \cdots, X(t), \Upsilon(t)), \end{cases} \tag{3.1}$$

where $\tau \in \mathbb{Z}_+$ is the state time delay, the switching signal is denoted by $\varrho : \mathbb{N} \to S := \{1, 2, \cdots, s\}$, $X(t) := (x_1(t), x_2(t), \cdots, x_n(t))$ is the states, $\Upsilon(t) := (\gamma_1(t), \gamma_2(t), \cdots, \gamma_m(t)) \in \mathcal{D}_k^m$ represents the disturbance input, and $g_j^i : \mathcal{D}_k^{n(\tau+1)+m} \to \mathcal{D}_k$, $i = 1, 2, \cdots, s$ denote k-valued logical functions.

Especially, the initial state trajectory $Y_0 := (X(-\tau), X(-\tau+1), \cdots, X(0)) \in \mathcal{D}_k^{n(\tau+1)}$, a switching sequence $\{\varrho(t) : t \in \mathbb{N}\} \subseteq S$, the disturbance signal $\{\Upsilon(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}_k^m$ are given.

With the STP method, one can obtain the algebraic express of the above systems. Let $x(t) = \ltimes_{j=1}^{n} x_j(t) \in \Delta_{k^n}$, $\gamma(t) = \ltimes_{j=1}^{m} \gamma_j(t) \in \Delta_{k^m}$ and $y(t) = \ltimes_{j=t-\tau}^{t} x(j) \in \Delta_{k^{n(\tau+1)}}$. SDLN (3.1) is converted into the following equivalent algebraic expression:

$$x(t+1) = K_{\varrho(t)}\gamma(t)y(t), \tag{3.2}$$

where $K_{\varrho(t)} \in \mathcal{L}_{k^n \times k^{n(\tau+1)+m}}$.

Then, we rewrite (3.2) as the following form:

$$y(t+1) = \widehat{K}_{\varrho(t)}\gamma(t)y(t),$$

where

$$\widehat{K}_{\varrho(t)} = D_r[k^n, k^n](I_{k^{n(\tau+1)}} \otimes K_{\varrho(t)})W_{[k^m, k^{n(\tau+1)}]}(I_{k^m} \otimes M_{r,k^{n(\tau+1)}}) \in \mathcal{L}_{k^{n(\tau+1)} \times k^{n(\tau+1)+m}}.$$

System (3.1) is equivalent to the above system. Given an equilibrium point $X_e = (x_1^e, \cdots, x_n^e) \in \mathcal{D}_k^n$, the vector form of X_e is $x_e = \ltimes_{j=1}^{n} x_j^e := \delta_{k^n}^q$. Let

$$y_e = (x_e)^{\tau+1} := \delta_{k^{n(\tau+1)}}^{\alpha}.$$

Letting $\varrho(t) = i \sim \delta_s^i$ and using logical variables as in the vector form, we rewrite (3.2) into the following form:

$$x(t+1) = \mathcal{K}\gamma(t)\varrho(t)y(t), \tag{3.3}$$

where

$$\mathcal{K} := [K_1 \ K_2 \cdots K_s]W_{[k^m,s]} \in \mathcal{L}_{k^n \times sk^{n(\tau+1)+m}},$$

and $K_i \in \mathcal{L}_{k^n \times k^{n(\tau+1)+m}}$, are obtained from (3.2), and $i = 1, 2, \cdots, s$.

The two following assumptions are the fundamental bases in this subsection. Further, these two assumptions always hold.

Assumption 1. *Assume that:*

1) *all the modes of (3.1) are not robustly stable, i.e., all the modes of system (3.3) do not satisfy*

$$Row_\alpha(Q_i^\chi) = k^{m\chi}\mathbf{1}_{k^{n(\tau+1)}};$$

2) *the i^*-th mode of (3.3) satisfies $y_e = \widehat{K}_{i^*}\gamma y_e$, $\forall\ \gamma \in \Delta_{k^m}$.*

Note that, it should be pointed out that the above assumptions are necessary for stability analysis.

Similarly, one rewrite (3.3) into the following form:

$$y(t+1) = \bar{\mathcal{K}}\gamma(t)\varrho(t)y(t),$$

where

$$\bar{\mathcal{K}} := D_r[k^n, k^n](I_{k^{n(\tau+1)}} \otimes \overline{K})W_{[sk^m, k^{n(\tau+1)}]}(I_{sk^m} \otimes M_{r,k^{n(\tau+1)}}) \in \mathcal{L}_{k^{n(\tau+1)} \times sk^{n(\tau+1)+m}}.$$

3.1.2. Robust stable analysis of SDLN

Via (3.4), we can analyze the robust stability for SDLN (3.1) with all unstable modes under by the following controller:

$$\varrho(t) = f(X(t-\tau), X(t-\tau+1), \cdots, X(t)), \tag{3.4}$$

where controller $f : \mathcal{D}_k^{n(\tau+1)} \to S$ is to be constructed.

Firstly, there is the definition of robustly stable for SDLN (3.1) in the following.

Definition 3.1. *System (3.1) is said to be robustly stable at X_e, if, there exists a state feedback controller (3.4) and $\chi \in \mathbb{Z}_+$ such that $X(t; Y_0, \varrho, \Upsilon) = X_e$ holds, $\forall$ initial trajectory $Y_0 \in \mathcal{D}_k^{n(\tau+1)}$, disturbance input sequence $\{\Upsilon(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}_k^m$, and integer $t \geq \chi$.*

Similarly, we convert switching signal (3.4) into the following form:

$$\varrho(t) = Fy(t),$$

where $F \in \mathcal{L}_{s \times k^{n(\tau+1)}}$ is state feedback gain.

Given $\mu \in \mathbb{Z}_+$, there is the concept of μ-th step robustly stable for system (3.4).

Definition 3.2. *Part with respect to y_e, a set $D_\mu(y_e) \subseteq \Delta_{k^{n(\tau+1)}}$ is the μ-th step robustly stable, if there exists $\{\varrho(t) : t \in \mathbb{N}\}$ such that $y(t; y(0), \varrho, \gamma) = y_e$ holds, $\forall$ $y(0) \in D_\mu(y_e)$, $\forall$ $\{\gamma(t) : t \in \mathbb{N}\} \subseteq \Delta_{k^m}$, and $\forall$ integer $t \geq \mu$.*

Definition 3.3. *([38]) Consider system (3.4). $D \subseteq \Delta_{k^{n(\tau+1)}}$, which is a nonempty set, is said to be one step robustly reachable from $y \in \Delta_{k^{n(\tau+1)}}$, if there exist a switching signal $\varrho \in \Delta_s$ and $y_j \in D$, such that $y_j = \widehat{\widehat{K}} \delta_{k^m}^j \varrho y$ holds for any $j = 1, 2, \cdots, k^m$, where $\widehat{\widehat{K}}$ is structural matrix for controller and defined in [38].*

In the following, there are five main results for robustly stability for SDLN (3.1) with all unstable modes.

Under Assumption 1, we get that

1) With respect to y_e, $D_\mu(y_e) = \{\delta_{k^{n(\tau+1)}}^{\psi_1}, \cdots, \delta_{k^{n(\tau+1)}}^{\psi_{m_\mu}}\}$ is the μ-th step robustly stable part of system (3.4), if and only if

$$\prod_{i=1}^{m_\mu} \max_{(h_0, \cdots, h_{\mu-1}) \in S^\mu} \{(Q_{h_{\mu-1}} \cdots Q_{h_0})_{\alpha, \psi_i}\} = k^{\mu m_\mu m};$$

2) With respect to y_e, $D_\mu(y_e) = \{\delta_{k^{n(\tau+1)}}^{\psi_1}, \cdots, \delta_{k^{n(\tau+1)}}^{\psi_{m_\mu}}\}$ is the μ-th step robustly stable part of system (3.4), if

$$\prod_{i=1}^{m_\mu} (Q^\mu)_{\alpha, \psi_i} > 0;$$

3) System (3.4) is robustly stable at $y_e = \delta_{k^{n(\tau+1)}}^\alpha$ under state feedback controller (3.4), if and only if there exists a positive integer $\chi \leq k^{n(\tau+1)}$ such that

$$D_\chi(y_e) = \Delta_{k^{n(\tau+1)}}. \tag{3.5}$$

Furthermore, based on the above results, we have more general results in the following:

1) System (3.1) is robustly stable at X_e under the state feedback controller (3.4), if and only if condition (3.5) holds.
2) System (3.1) without disturbances is stable at X_e under the state feedback controller (3.4), if and only if there exists an integer $1 \leq \chi \leq k^{n(\tau+1)}$ such that

$$D_\chi(y_e) = \Delta_{k^{n(\tau+1)}}.$$

For the complete proofs for the aforementioned results in this subsection, readers can see more details in [39].

3.2. *Robust stable analysis of Boolean control networks under impulsive effects*

This subsection introduces some recent works on, under impulsive effects, the robust set stabilization problem of Boolean control networks (BCNs).

For the complete proofs for the results in this section, readers can see more details in [40].

3.2.1. Robust set stabilization of BCNs with impulsive effects

The classical definition for a Boolean control network under impulsive effects is

$$\begin{cases} x_i(t+1) = f_{1i}(X(t), U(t), \Xi(t)),\ i = 1, \cdots, n, t_{k-1} \leq t < t_k - 1; \\ x_i(t_k) = f_{2i}(X(t_k - 1), \Xi(t_k - 1)),\ i = 1, \cdots, n, k \in \mathbb{Z}; \\ y_j(t) = h_j(X(t)),\ j = 1, \cdots, p, \end{cases} \tag{3.6}$$

where $t_0 = 0$, $\{t_k : k \in \mathbb{Z}_+\} \subseteq \mathbb{Z}_+$ satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ is the impulsive time sequence, $X(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathcal{D}^n$, $U(t) = (u_1(t), \cdots, u_m(t)) \in \mathcal{D}^m$, $\Xi(t) = (\xi_1(t), \cdots, \xi_q(t)) \in \mathcal{D}^q$ and $Y(t) = (y_1(t), \cdots, y_p(t)) \in \mathcal{D}^p$ denote the state variables, the control inputs, the disturbance inputs and the outputs of the system (3.6), respectively, and $f_{1i} : \mathcal{D}^{n+m+q} \to \mathcal{D}$, $f_{2i} : \mathcal{D}^{n+q} \to \mathcal{D}, i = 1, \cdots, n$ and $h_j : \mathcal{D}^n \to \mathcal{D}, j = 1, \cdots, p$ are logical functions.

Now, there is the basic definition of robustly stabilizable to the set.

Definition 3.4. *Consider a set $A \subseteq \mathcal{D}^n$, which is not empty. To the set A, BCN (3.6) is robustly stabilizable, if, one can find a suitable control sequence $\{U(t) : t \in \mathbb{N}\}$ and a integer $\tau > 0$ such that*

$$X(t; X(0), U, \Xi) \in A$$

holds for $forallt \geq \tau$, $\forall X(0) \in \mathcal{D}^n$, and $\forall$ disturbance $\{\Xi(t) : t \in \mathbb{N}\} \subseteq D^q$.

Normally, the state feedback control for the above systems can be written as

$$u(t) = Gx(t),$$

where $G \in \mathcal{L}_{2^m \times 2^n}$ is called the state feedback gain matrix.

The following definition is about robust L-invariant set.

Definition 3.5. *Consider $x(t+1) = L\xi(t)x(t)$, where $x(t) \in \Delta_{2^n}$, $\xi(t) \in \Delta_{2^q}$, and $L \in \mathcal{L}_{2^n \times 2^{n+q}}$ hold. Let $S \subseteq \Delta_{2^n}$ be a nonempty set. S is a robust L-invariant set, if, $L\xi x \in S$ holds for $\forall x \in S$ and $\forall \xi \in \Delta_{2^q}$.*

Then, we present two results. These two results reveal that how to design suitable control sequences under difference situations.

1) Consider a set $A \subseteq \Delta_{2^n}$, which is not empty. We assume that A is both a robust $\bar{L}_1$ and a robust L_2-invariant set. For A, system (3.6) is said to be robustly stabilizable via controller $u(t) = Gx(t)$, if and only if, there exists a positive integer τ s.t.

$$Col(\tilde{L}_\tau) \subseteq A$$

holds, where $\tilde{L}_\tau$ is defined as

$$\tilde{L}_\tau = \begin{cases} \bar{L}_1(I_{2^q} \otimes \tilde{L}_{\tau-1}), \text{when } t_k < \tau < t_{k+1}, \\ L_2(I_{2^q} \otimes \tilde{L}_{\tau-1}), \text{when } \tau = t_{k+1}. \end{cases}$$

2) Consider a set $A \subseteq \Delta_{2^n}$, which is not empty. We assume that there exists a positive integer $\alpha \le 2^m$ s.t. A is a both robust $Blk_\alpha(\hat{L}_1)$-invariant set and L_2-invariant set. For set A, (3.6) is said to be robustly stabilizable under a free-form control sequence, if and only if, there exist two integers $\tau > 0$ and $\beta > 0$ such that

$$Col(Blk_\beta(\hat{L}_\tau)) \subseteq A. \tag{3.7}$$

Furthermore, if (3.7) holds, then the control sequence is constructed as

$$u(t) = \begin{cases} u^*(t), \ t \in ([0, \tau-1] \cap \mathbb{N}) \setminus \Lambda(t), \\ \delta_{2^m}^\alpha, \ t \in ([\tau, +\infty) \cap \mathbb{N}) \setminus \Lambda(t), \end{cases}$$

where u^* is described by

$$\begin{cases} \ltimes_{i=\tau-1, i\notin\Lambda(\tau)}^0 u^*(i) = \delta_{2^{(\tau-k+1)m}}^\beta, \text{ when } t_{k-1}+1 \le \tau < t_k; \\ \ltimes_{i=\tau-2, i\notin\Lambda(\tau)}^0 u^*(i) = \delta_{2^{(\tau-k)m}}^\beta, \text{ when } \tau = t_k. \end{cases}$$

3.2.2. Robust partial stable analysis of BCNs

This subsection focus on robust partial stabilization problem of (3.6). Letting $(x_1^*, \cdots, x_r^*) \in \mathcal{D}^r$ with $r \le n$, one can obtain $x^r = \ltimes_{i=1}^r x_i^* = \delta_{2^r}^\theta$.

The following is the definition of robustly partial stabilizable.

Definition 3.6. *To x^r, system (3.6) is robustly partial stabilizable, if, one can find a control sequence $\{u(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}^m$ and an integer $\tau >$ such that*

$$x_i(t; x(0), u, \xi) = x_i^*,$$

holds for any integer $t \ge \tau$, $\forall x(0) \in \mathcal{D}^n$, $\forall \{\xi(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}^q$, and $i = 1, \cdots, r$.

According to the above definition, we present a necessary assumption.

Letting

$$A = \{\delta_{2^r}^\theta \ltimes \delta_{2^{n-r}}^\eta : \eta = 1, \cdots, 2^{n-r}\},$$

we assume that A is a robust both $\bar{L}_1$-invariant and robust L_2-invariant set. We would get some interesting results.

Theorem 3.7. *For the system (3.6), the following statements are equivalent:*

1) *To x^r, (3.6) is robustly partial stabilizable under the state feedback controller $u(t) = Gx(t)$;*

2) *To the set A, (3.6) is robustly stabilizable under the state feedback controller $u(t) = Gx(t)$;*

3) *One can find an integer $\tau > 0$ such that $Col(\tilde{L}_\tau) \subseteq A$ holds.*

3.2.3. output tracking problems of BCNs under impulsive effects

This subsection investigates the output tracking problems of (3.6). Firstly, we give the basic definition of robust output tracking.

Definition 3.8. *Given a reference trajectory* $Y_r = (y_1^r, \cdots, y_p^r) \in \mathcal{D}^p$*. The trajectory of (3.6) is said to robustly track trajectory* Y_r*, if, one can find a control sequence* $\{U(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}^m$ *and* $\tau > 0$ *such that*

$$Y(t; X(0), U, \Xi) = Y_r$$

holds for $\forall X(0) \in \mathcal{D}^n$*,* $\forall t \geq \tau$*, and* $\forall \{\Xi(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}^q$*.*

The corresponding result about robust output tracking is in the following.

Theorem 3.9. *Assume that* $O(\beta)$ *is a robust both* $\bar{L}_1$*-invariant and* L_2*-invariant set. The output trajectory of (3.6) robustly track* $y_r = \delta_{2^p}^{\beta}$ *under the controller* $u(t) = Gx(t)$*, if and only if, one can find an integer* $\tau > 0$ *such that*

$$Col(H\tilde{L}_\tau) = \{\delta_{2^p}^{\beta}\}.$$

3.3. Event-triggered control for logical control networks

This subsection addresses some new developments about the event-triggered control problem for k-valued logical control networks (KVLCNs), and proposes an event-triggered control method.

The first part addresses some results about robust set stabilization of KVLCNs. The second part introduces some recent works about event-triggered control of KVLCNs. For the complete proofs for the results in this section, readers can see more details in [41].

3.3.1. Robust set stabilization of k-valued logical control networks

There is a classical definition about k-valued logical control networks. A k-valued logical control network is described as follows:

$$\begin{cases} x_i(t+1) = f_i(X(t), U(t), \Xi(t)), \; i = 1, \cdots, n; \\ y_j(t) = h_j(X(t)), \; j = 1, \cdots, p, \end{cases} \tag{3.8}$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathcal{D}_k^n$, $U(t) = (u_1(t), \cdots, u_m(t)) \in \mathcal{D}_k^m$, $\Xi(t) = (\xi_1(t), \cdots, \xi_q(t)) \in \mathcal{D}_k^q$ and $Y(t) = (y_1(t), \cdots, y_p(t)) \in \mathcal{D}_k^p$ are states, control inputs, disturbance inputs and outputs at time t, respectively, and $f_i : \mathcal{D}_k^{n+m+q} \to \mathcal{D}_k$, $i = 1, \cdots, n$ and $h_j : \mathcal{D}_k^n \to \mathcal{D}_k$ are logical functions. Given $X(0) \in \mathcal{D}_k^n$, a control $\{U(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}_k^m$ and a disturbance inputs $\{\Xi(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}_k^q$, denote the trajectory of system (3.8) by $X(t; X(0), U, \Xi)$, $j = 1, \cdots, p$, the numbers of nodes, control inputs, outputs, and disturbances are n, m, p and q.

According to the above definition, we propose the definition of robust set stabilization for k-valued logical control networks (3.8) in the following.

Definition 3.10. *Define a set* $A \subseteq \mathcal{D}_k^n$*, which is nonempty, and let* $X(0) \in \mathcal{D}_k^n$*. To A, (3.8) is robustly stabilizable, if one can find a control sequence* $\{U(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}_k^m$ *and an integer* $\tau > 0$ *such that*

$$X(t; X(0), U, \Xi) \in A$$

holds, for any $t \geq \tau$ *and any* $\{\Xi(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}_k^q$*.*

Then, we rewrite k-valued logical control networks (3.8) into an algebraic form step by step.

Using the vector form of logical variables and setting $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{k^n}$, $u(t) = \ltimes_{i=1}^{m} u_i(t) \in \Delta_{k^m}$, $\xi(t) = \ltimes_{i=1}^{q} \xi_i(t) \in \Delta_{k^q}$ and $y(t) = \ltimes_{i=1}^{p} y_i(t) \in \Delta_{k^p}$, by STP method, k-valued logical control networks (3.8) can be rewritten into the following equivalent algebraic form:

$$\begin{cases} x(t+1) = Lu(t)x(t)\xi(t), \\ y(t) = Hx(t), \end{cases} \tag{3.9}$$

where $L \in \mathcal{L}_{k^n \times k^{n+m+q}}$ is the state transition matrix, and $H \in \mathcal{L}_{k^p \times k^n}$ is the output matrix.

This subsection considers the following state feedback control:

$$u_i(t) = \psi_i(t, X(t)), i = 1, \cdots, m, \tag{3.10}$$

where $\psi_i : \mathbb{N} \times \mathcal{D}_k^n \to \mathcal{D}_k$ are logical functions and $i = 1, 2 \cdots, m$.

For $\forall \psi_i$, we can construct a unique structural matrix $\Psi_i(t, x(0)) \in \mathcal{L}_{k \times k^n}$ such that $u_i(t) = \Psi_i(t, x(0))x(t)$. By the Khatri-Rao product of matrices, we construct a time-variant structural matrix $\Psi(t, x(0)) \in \mathcal{L}_{k^m \times k^n}$ such that

$$u(t) = \Psi(t, x(0))x(t),$$

where $\Psi(t, x(0)) = \Psi_1(t, x(0)) * \cdots * \Psi_m(t, x(0))$ and $i = 1, 2, \cdots, m$.

A sufficient and necessary criterion for the problem of robust set stabilization for (3.8) under the controller (3.10) is given in the following.

Theorem 3.11. *Define a set $A \subseteq \mathcal{D}_k^n$, which is nonempty, and let $x(0) = \delta_{k^n}^{\alpha}$ and $A \subseteq \Upsilon_1(A)$. k-valued logical control networks (3.8) is said to be robustly stabilizable under the controller (3.10), if and only if, one can find an integer $T > 0$ such that $x(0) \in \Upsilon_T(A)$ holds.*

3.3.2. Event-triggered control of k-valued logical control networks

We still consider system (3.9). Define a set $A \subseteq \mathcal{D}_k^n$, which is nonempty, and let $x(0) = \delta_{k^n}^{\alpha}$ be initial state. For a given controller $u(t) = \Psi(t, x(0))x(t)$, one has

$$\begin{aligned} & x(t+1) = Lu(t)x(t)\xi(t) \\ = \;& \ltimes_{i=t}^{0}(L\Psi(i, x(0))M_{r,k^n})x(0) \ltimes_{j=0}^{t} \xi(j) \\ = \;& Blk_{\alpha}\Big(\ltimes_{i=t}^{0} (L\Psi(i, x(0))M_{r,k^n})\Big) \ltimes_{j=0}^{t} \xi(j), \end{aligned}$$

where $t \in \mathbb{N}$, and $M_{r,k^n} = \text{Diag}\{\delta_{k^n}^1, \cdots, \delta_{k^n}^{k^n}\} \in \mathcal{L}_{k^{2n} \times k^n}$.

From the arbitrariness of $\ltimes_{j=0}^{t}\xi(j)$, $x(t+1)$ forms a set

$$\Omega(t+1) = Col\Big(Blk_{\alpha}\Big(\ltimes_{i=t}^{0} (L\Psi(i, x(0))M_{r,k^n})\Big)\Big).$$

Then, we give the event-triggered condition as

$$d_H(\Omega(t+1), A) > 0, \tag{3.11}$$

where $d_H(\Omega(t+1), A)$ denotes the typical Hausdorff distance.

For system (3.9), split L into k^m equal blocks as

$$L = [L_1 \;\; L_2 \;\; \cdots \;\; L_{k^m}],$$

where $L_i \in \mathcal{L}_{k^n \times k^{n+q}}$, $i = 1, 2, \cdots, k^m$. For any $i \in \{1, 2, \cdots, k^m\}$, split L_i into k^n equal blocks as

$$L_i = [L_{i,1} \;\; L_{i,2} \;\; \cdots \;\; L_{i,k^n}],$$

where $L_{i,j} \in \mathcal{L}_{k^n \times k^q}$, $j = 1, 2, \cdots, k^n$.

In the following, there exist a sufficient condition for the existence of event-triggered controller, and an method to construct the corresponding controller.

Theorem 3.12. *Define a set $A \subseteq \mathcal{D}_k^n$, which is nonempty, and let $x(0) = \delta_{k^n}^{\alpha}$. k-valued logical control networks (3.8) is said to be robustly stabilizable with the event-triggered condition (3.11), if $A \subseteq \Upsilon_1(A)$ and $x(0) \in \Upsilon_1(A)$ hold.*

3.4. Stability analysis in distribution of probabilistic Boolean networks under function perturbation impact

This subsection introduces some results about robust analysis of probabilistic Boolean networks (PBNs).

For the complete proofs for the results in this subsection, readers can see more details in [42].

3.4.1. The description of probabilistic Boolean networks

There is a classical definition for PBNs with n nodes:

$$X(t + 1) = f(X(t)), \tag{3.12}$$

where $X(t) = (x_1(t), \cdots, x_n(t)) \in \mathcal{D}^n$ is the state vector of PBN (3.12), and $f : \mathcal{D}^n \to \mathcal{D}^n$ is a logical mapping which is chosen from the set $\{f_1, f_2, \cdots, f_r\}$ with $\mathbb{P}\{f = f_i\} = p_i$, $\sum_{i=1}^{r} p_i = 1$.

Using the vector form and setting $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n}$, PBN (3.12) can be converted to the following algebraic expression:

$$x(t + 1) = Lx(t),$$

where $L \in \mathcal{L}_{2^n \times 2^n}$ is the structural matrix of f, which is chosen from the set $\{L_1, \cdots, L_r\}$ with $\mathbb{P}\{L = L_i\} = p_i$, and $L_i = \delta_{2^n}[\alpha_{i,1} \;\; \cdots \;\; \alpha_{i,2^n}] \in \mathcal{L}_{2^n \times 2^n}$ is the structural matrix of f_i.

There are two common definitions. Readers can see more details in [43].

We give the basic definitions of stability and set stability for (3.12) in the following.

Definition 3.13. *PBN (3.12) is stable at $x_e = \delta_{2^n}^{\theta}$ in distribution, if, $\lim_{t\to\infty} \mathbb{P}\{x(t; x_0) = x_e\} = 1$ holds for $\forall x_0 \in \Delta_{2^n}$.*

Definition 3.14. *PBN (3.12) is stable at a given nonempty set $\mathcal{M} \subseteq \Delta_{2^n}$ in distribution, if $\lim_{t\to\infty} \mathbb{P}\{x(t; x_0) \in \mathcal{M}\} = 1$ holds for any $x_0 \in \Delta_{2^n}$.*

To make this subsection more readable, we give an example to explain the difference between one-bit function and multi-bit function perturbation.

Table 1. Truth table of system (3.13).

x_1	x_2	f_1	f_2
1	1	0	1
1	0	1	1
0	1	0	0
0	0	0	0

Example 3.15. *Consider the following BN:*

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t)), \end{cases}$$

where $f_1 = x_1 \wedge \neg x_2$ and $f_2 = x_1$. The truth table of system (3.13) is given in Table 1.

The algebraic equation of PBN (3.13) is $x(t+1) = Fx(t)$, in which $F := \delta_4[3\ 1\ 4\ 4]$.

First, we define a one-bit perturbation, that is, $f_1(0,1)$ is changed from 0 to 1. Then, F is flipped to $\delta_4[3\ 1\ 2\ 4]$, i.e., $Col_3(F)$ is flipped from δ_4^4 to δ_4^2.

Second, consider a multi-bit perturbation, e.g., $f_1(0,1)$ and $f_2(0,1)$ are changed from 0 to 1, separately. Thus, F is flipped to $\delta_4[3\ 1\ 1\ 4]$. We can find that $Col_3(F)$ is flipped from δ_4^4 to δ_4^1. □

The objective of this section is to propose some criteria. These is very useful in guaranteeing the robustness of (3.12), i.e., PBN (3.12) can be still stable at x_e or $\mathcal{M}$ after function perturbation.

3.4.2. Stability of probabilistic Boolean networks

We assume that PBN (3.12) is stable at $x_e = \delta_{2^n}^{\theta}$ in distribution. There is a common assumption for the following results in this subsection.

Assumption 2. *For $\forall i \in \{1, \cdots, r\}$, the $\mathcal{B}$-th column of L_i is perturbed with $\mathcal{B} \neq \theta$. We assume that $\alpha_{i,\mathcal{B}}$ flipped to some $\gamma \in \{1, \cdots, 2^n\}$, where $\gamma \neq \alpha_{i,\mathcal{B}}$.*

According to Assumption 2, we construct a set

$$\Omega = \{x : \ x \text{ reach } x_e \text{ with positive probability, meanwhile any path from } x \text{ to } x_e \text{ must cover } \delta_{2^n}^{\mathcal{B}}\}.$$

Thus, one has the following result.

Theorem 3.16. *Given PBN (3.12) is stable at x_e. Under Assumption 2, PBN (3.12) is said to be stable at x_e after function perturbation, if and only if, $\delta_{2^n}^{\gamma} \notin \Omega$ holds.*

One can see from Theorem 3.16 that $\delta_{2^n}^{\gamma} \notin \Omega$ is very important for the stability in distribution of system (3.12). Then, by the transition probability matrix M, we construct

$$\varphi(\mathcal{B},\gamma,\theta) = \sum_{k=2}^{2^n} (M^k)_{\theta,\gamma} - \sum_{k=2}^{2^n} \sum_{s=1}^{k-1} (M^{k-s})_{\theta,\mathcal{B}} (M^s)_{\mathcal{B},\gamma}.$$

We have the following result on the verification of $\delta_{2^n}^{\gamma} \notin \Omega$.

Theorem 3.17. $\delta_{2^n}^{\gamma} \notin \Omega$*, if and only if*

$$\varphi(\mathcal{B},\gamma,\theta) > 0.$$

3.4.3. Set stability of probabilistic Boolean networks

This subsection assumes that system (3.12) is stable at a given nonempty set $\mathcal{M} \in \Delta_{2^n}$. There is a common used assumption in the following.

Assumption 3. *For $\forall i \in \{1, \cdots, r\}$, $Col_{\mathcal{B}}(L_i)$ doesn't belong to the set $I(\mathcal{M})$. We assume that $\alpha_{i,\mathcal{B}}$ changes to some $\gamma \in \{1, \cdots, 2^n\}$, where $\gamma \neq \alpha_{i,\mathcal{B}}$.*

Define

$$\begin{aligned} \varphi(\mathcal{B}, \gamma, \mathcal{M}) &= \sum_{\delta_{2^n}^{\theta} \in I(\mathcal{M})} \varphi(\mathcal{B}, \gamma, \theta) \\ &= \sum_{\delta_{2^n}^{\theta} \in I(\mathcal{M})} \Big[\sum_{k=2}^{2^n} (M^k)_{\theta,\gamma} - \sum_{k=2}^{2^n} \sum_{s=1}^{k-1} (M^{k-s})_{\theta,\mathcal{B}} (M^s)_{\mathcal{B},\gamma} \Big]. \end{aligned}$$

One has:

Theorem 3.18. *Consider PBN (3.12) is said to be stable at a given set $\mathcal{M}$, which is nonempty, in distribution. Under Assumption 3, PBN (3.12) is still said to be stable at $\mathcal{M}$ after function perturbation, if and only if, $\varphi(\mathcal{B}, \gamma, \mathcal{M}) > 0$ holds.*

3.5. Disturbance decoupling of logical networks via event-triggered control

This subsection introduces some recent development about event-triggered control for disturbance decoupling problem of mix-valued logical networks (MVLCNs).

3.5.1. Description of MVLCNs

There is a classical definition of MVLCNs:

$$\begin{cases} x_1(t+1) = f_1(X(t), U(t), \Xi(t)), \\ \vdots \\ x_n(t+1) = f_n(X(t), U(t), \Xi(t)); \\ y_j(t) = g_j(X(t)), \quad j = 1, \cdots, p, \end{cases} \tag{3.13}$$

where $X(t) = (x_1(t), \cdots, x_n(t))$ with $x_i(t) \in \mathcal{D}_{k_i}$ denotes states, $U(t) = (u_1(t), \cdots, u_m(t))$ with $u_i(t) \in \mathcal{D}_{l_i}$ denotes controls, $\Xi(t) = (\xi_1(t), \cdots, \xi_r(t))$ with $\xi_i(t) \in \mathcal{D}_{v_i}$ denoting disturbance inputs, and $y_j(t) \in \mathcal{D}_{w_j}$, denotes outputs, and $j = 1, 2, \cdots, p$. Define $k := k_1 \cdots k_n$, $l := l_1 \cdots l_m$, $v := v_1 \cdots v_r$ and $w := w_1 \cdots w_p$.

Define an n-ary logical function $h : \mathcal{D}_{k_1} \times \cdots \times \mathcal{D}_{k_n} \to \mathcal{D}_{k_0}$. To convert h into an equivalent algebraic form, we identify $\frac{k-i}{k-1} \sim \delta_k^i$, $i = 1, \cdots, k$. Then, we have $\mathcal{D}_k \sim \Delta_k$. δ_k^i is called the vector form of logical value $\frac{k-i}{k-1} \in \mathcal{D}_k$.

Based on the vector form of logical values and defining $x(t) = \ltimes_{i=1}^n x_i(t)$, $u(t) = \ltimes_{i=1}^m u_i(t)$, $\xi(t) = \ltimes_{i=1}^r \xi_i(t)$ and $y(t) = \ltimes_{j=1}^p y_j(t)$, one gets the following expression of (3.13):

$$\begin{cases} x(t+1) = Lu(t)x(t)\xi(t), \\ y(t) = Gx(t), \end{cases} \tag{3.14}$$

where $L \in \mathcal{L}_{k \times (klv)}$ and $G \in \mathcal{L}_{w \times k}$.

In the next, use the method in [44] for coordinate transformation.

We assume that there is a logical coordinate transformation

$$\{x_i : i = 1, \cdots, n\} \to \{z_i : i = 1, \cdots, n\},\ z_i \in \mathcal{D}_{k_{\alpha_i}}, \tag{3.15}$$

under which system (3.13) becomes

$$\begin{cases} z_i(t+1) = \hat{f}_i^1(Z(t), U(t), \Xi(t)),\ i = 1, \cdots, s, \\ z_i(t+1) = \hat{f}_i^2(Z(t), U(t), \Xi(t)),\ i = s+1, \cdots, n; \\ y_j(t) = \hat{g}_j(z_1(t), \cdots, z_s(t)), \qquad j = 1, \cdots, p, \end{cases} \tag{3.16}$$

where $\hat{f}_i^1$, $\hat{f}_i^2$, $i = s+1, s+2, \cdots, n$, $\hat{g}_j$, $j = 1, \cdots, p$ are logical functions, and $i = 1, 2 \cdots, s$

Let $z(t) = \ltimes_{i=1}^n z_i(t)$. For any $\mu \in \{1, \cdots, n\}$, set $\mathbf{z}_\mu(t) = \ltimes_{i=1}^\mu z_i(t)$, $\mathbf{k}_\mu = \ltimes_{i=1}^\mu k_{\alpha_i}$. For any $\mu \in \{1, \cdots, n-1\}$, set $\mathbf{z}_{-\mu}(t) = \ltimes_{i=\mu+1}^n z_i(t)$, and $\mathbf{k}_{-\mu} = \ltimes_{i=\mu+1}^n k_{\alpha_i}$. Next, set $\mathbf{k}_{-n} = 1$. Thus, we can find that $\mathbf{z}_n(t) = z(t)$ and $\mathbf{k}_n = k$. System (3.16) can be rewritten as in the following algebraic form:

$$\begin{cases} \mathbf{z}_s(t+1) = \hat{L}_s u(t) z(t) \xi(t), \\ \mathbf{z}_{-s}(t+1) = \hat{L}_{-s} u(t) z(t) \xi(t); \\ y(t) = \hat{G} \mathbf{z}_s(t), \end{cases} \tag{3.17}$$

where $\hat{L}_s \in \mathcal{L}_{\mathbf{k}_s \times (klv)}$, $\hat{L}_{-s} \in \mathcal{L}_{\mathbf{k}_{-s} \times (klv)}$ and $\hat{G} \in \mathcal{L}_{w \times \mathbf{k}_s}$.

When $s = n$ holds, MVLCN (3.17) is

$$\begin{cases} z(t) = \hat{L} u(t) z(t) \xi(t); \\ y(t) = \hat{G} z(t), \end{cases}$$

in which $\hat{L} \in \mathcal{L}_{k \times (klv)}$ and $\hat{G} \in \mathcal{L}_{w \times k}$.

Consider MVLCN (3.16) with $z(0) \in \mathcal{D}_{k_{\alpha_1}} \times \cdots \times \mathcal{D}_{k_{\alpha_n}}$, we can construct the following controller with respect to $z(0)$ as:

$$u_i(t) = k_t^i(z_1(t), \cdots, z_n(t)),\ i = 1, \cdots, m. \tag{3.18}$$

For $\forall t \in \mathbb{N}$, under controller (3.18), one can find all possible invariant subspaces as $\mathcal{Z}_{\sigma_h(t)} = F_l\{z_1, \cdots, z_{\sigma_h(t)}\} \supseteq F_l\{z_1, \cdots, z_s\}$, that is, system (3.16) becomes

$$\begin{cases} z_i(t+1) = \tilde{f}_i^1(z_1(t), \cdots, z_{\sigma_h(t)}(t)), \quad i = 1, \cdots, \sigma_h(t), \\ z_i(t+1) = \tilde{f}_i^2(z_1(t), \cdots, z_n(t), \Xi(t)),\ i = \sigma_h(t)+1, \cdots, n; \\ y_j(t) = \tilde{g}_j(z_1(t), \cdots, z_s(t)), \qquad j = 1, \cdots, p, \end{cases}$$

where $h \in \{1, \cdots, \lambda_t\}$, $\lambda_t \in \{1, \cdots, n-s\}$, and $\tilde{f}_i^1$, $i = 1, \cdots, \sigma_h(t)$, $\tilde{f}_i^2$, $i = \sigma_h(t)+1, \cdots, n$, $\tilde{g}_j$, $j = 1, 2, \cdots, p$ are mix-valued logical functions. For $\forall t \in \mathbb{N}$, denote

$$\Gamma_t = \{\sigma_h(t) : h = 1, \cdots, \lambda_t\},$$

and set

$$\sigma(t) = \max\{\sigma_h(t) : h = 1, \cdots, \lambda_t\}.$$

It is obvious that $\sigma_h(t) \in \{s, \cdots, \sigma(t-1)\}$ with $\sigma(-1) := n$.

In this subsection, we propose the following concept of disturbance decoupling of logical networks.

Definition 3.19. *Given the transformation (3.15) be given. With respect to initial state $z(0) \in \Delta_k$, the disturbance decoupling problem is solvable, if there exists a controller (3.18) corresponding to $z(0)$ such that $\Gamma_t \neq \emptyset$ holds, $\forall t \in \mathbb{N}$. The disturbance decoupling problem is solvable, if, with respect to $\forall z(0) \in \Delta_k$, it is solvable.*

3.5.2. New method to construct the output-friendly subspace

We consider (3.14) and assume

$$G = \delta_w[\gamma_1 \ \ \gamma_2 \ \ \cdots \ \ \gamma_k].$$

Denote

$$\eta_j = \Big|\{i : \gamma_i = j, 1 \le i \le k\}\Big|, \quad j = 1, 2, \cdots, w,$$

where $|\cdot|$ is the number of sets. Then, we have the following definition. For details, please refer to [44].

Definition 3.20. *Let* $H = (h_1, \cdots, h_p) : \mathcal{D}_{k_1} \times \cdots \times \mathcal{D}_{k_n} \to \mathcal{D}_{w_1} \times \cdots \times \mathcal{D}_{w_p}$ *be a mix-valued logical mapping. The variable* x_i *is said to be redundant, if* $H(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) = H(x_1, \cdots, x_{i-1}, \frac{\zeta}{k_i - 1}, x_{i+1}, \cdots, x_n)$ *holds for any* $\zeta = 1, \cdots, k_i - 1$.

Then, we have

Theorem 3.21. *Consider a logical mapping* $H = (h_1, \cdots, h_p) : \mathcal{D}_{k_1} \times \cdots \times \mathcal{D}_{k_n} \to \mathcal{D}_{w_1} \times \cdots \times \mathcal{D}_{w_p}$*. Let* $M_H \in \mathcal{L}_{w \times k}$ *be the structural matrix of H, and let an integer* $s \le n$ *be given. Split* M_H *into* $k' = k_1 \cdots k_s$ *equal blocks as* $M_H = [M_H^1 \ \ M_H^2 \ \ \cdots \ \ M_H^{k'}]$*, where* $M_H^1, M_H^2, \cdots, M_H^{k'} \in \mathcal{L}_{w \times \frac{k}{k'}}$*. Then,* $(x_{s+1}, \cdots, x_n)$ *are redundant variables if and only if* $\operatorname{rank}(M_H^i) = 1$ *holds for any* $i = 1, \cdots, k'$.

After that, it is easy to obtain that

$$\mathbf{z}_s(t) = F[\mathbf{k}_s, \mathbf{k}_{-s}] T x(t) := T_0 x(t). \tag{3.19}$$

4. Conclusions

This paper has described a comprehensive survey on some recent applications of the STP method on the theory of finite systems. After we introduced some useful mathematical tools on the STP method, some recent developments about robustness analysis on finite systems are delineated, such as robust stable analysis of switched logical networks with time-delayed, under impulsive effects robust set stabilization of Boolean control networks, event-triggered control for robust set stabilization of logical networks with control inputs, stability analysis in distribution of probabilistic Boolean networks under functional perturbation impact and how to solve disturbance decoupling problems by event triggered control of logical control networks have been presented.

Furthermore, the STP method is a generalization of ordinal products of matrices. It is inevitable to keep some shortcomings in ordinal product of matrices. One of them is that the dimensions of the structural matrices increase too rapidly. Thus, it leads to the calculation complexity's exponential growth. There are only a few results [45, 46] about that. Our future plan is to solve it. In addition, we can extend the existing results about logical control networks to the other systems. It also is a great research area for scholars, and readers can see [47–51], such as finite-time stability and settling-time estimation of nonlinear impulsive systems, nonlinear systems with delayed impulses and so on.

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Conflict of interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled, "survey of semi-tensor product method in robustness analysis on finite systems".

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