



Theory article

Stability and persistence analysis of a microorganism flocculation model with infinite delay

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Abstract: In this paper, we study the global stability and persistence of a microorganism flocculation model with infinite delay. First, we make a complete theoretical analysis on the local stability of the boundary equilibrium (microorganism-free equilibrium) and the positive equilibrium (microorganism co-existent equilibrium), and give a sufficient condition for the global stability of the boundary equilibrium (applicable to the forward bifurcation and the backward bifurcation). Then, for the persistence of the model, we present an explicit estimate of the eventual lower bound of any positive solution for which only the parameter threshold $R_0 > 1$ is required. The obtained results extend some of the conclusions of the existing literatures on the case of discrete time delay.

Keywords: microorganism flocculation; infinite delay; persistence; local stability; global stability

1. Introduction

In the research of microbial growth dynamics, a rich variety of differential equation models and theoretical research of chemostats have been greatly developed, and fruitful research results have been achieved [1–4]. Flocculants are widely used in the sedimentation/coagulation/collection of microorganisms in petri dishes/reaction tanks, such as the removal of harmful microorganisms in polluted water and the development and utilization of beneficial microorganisms (such as microbial algae health food and algae-based microbial clean energy). Therefore, the research and development of flocculants (organic, inorganic, microbial, etc.) with the advantages of high efficiency, no pollution and low cost remains one of the important research topics in the field of microbial application technology. In literature [5–9], a dynamics model of microbial continuous culture with flocculation, time delay and different functional reaction functions was established based on the classical chemostat model, and the existence, local and global stability of the forward and backward bifurcation of the equilibrium, as well as the consistent persistence of the model were studied by constructing an appropriate Lyapunov

functional and other methods.

Based on the microorganism flocculation models in [6–8] with discrete time delays, in this paper, we intend to further consider the following microorganism flocculation model with infinite delay:

$$\begin{cases} n'(t) = Dn^0 - Dn(t) - h_1n(t)x(t), \\ x'(t) = h \int_0^\infty e^{-\mu\theta} f(\theta)n(t-\theta)x(t-\theta)d\theta - Dx(t) - h_2x(t)s(t), \\ s'(t) = Ds^0 - Ds(t) - h_3x(t)s(t). \end{cases} \quad (1.1)$$

Here, $n(t)$, $x(t)$ and $s(t)$ represent the concentrations of nutrients, microorganisms and flocculant at time t , respectively. The positive constants n^0 and s^0 indicate the input concentrations of nutrients and flocculant, respectively. The positive constant D indicates the outflow and inflow rate of the substances in the chemostat. The non-negative constants h_1 and h indicate the consumption ratio of nutrients and the growth ratio of microorganisms, respectively. The non-negative constants h_2 and h_3 indicate the flocculating ratio of microorganisms and the consumption ratio of flocculant, respectively. The distribution function $f(\theta)$ is non-negative and satisfies $\int_0^\infty f(\theta)d\theta = 1$. μ is a non-negative constant, and $e^{-\mu\theta}$ is the survival rate of microorganisms.

As we know, infinite delay was introduced early into dynamics models with a single nutrient and single-species microorganism growth problems. It can describe the cumulative cycle of microbial decomposition into nutrients (from an early time in the past) and the accumulated time it takes for nutrients to be absorbed/stored by microorganisms until they are converted into actual biomass [10–16]. Further, its use can be extended to the models with multiple microorganisms, substitutable nutrients, non-substitutable nutrients (fully complementary nutrients), etc. [17–19]. As pointed out in [15], the differential equations with distributed infinite delay have been successfully applied in biological modeling since the work of Volterra, and they are regarded to be more realistic than the differential equations with discrete and finite delay.

Similarly, infinite delay is also widely used to characterize the dynamic modeling of practical issues in many fields, such as population growth [20–24], infectious diseases in population [25, 26], virus infection and immunity [27–29].

The main purpose of this paper is to analyze the local and global stability of the boundary equilibrium (microorganism-free equilibrium) and the positive equilibrium (microorganism co-existent equilibrium) of model (1.1) and provide the corresponding criteria by using the stability theory of infinite delay differential equations and other related methods. At the same time, for the persistence of model (1.1), an explicit estimate of the eventual lower bound of its arbitrary positive solutions is given.

2. Dissipativeness and classification of the equilibria

For simplicity, we define the following parameter transformation:

$$n = n^0N, \quad x = X, \quad s = s^0S, \quad t = \frac{\bar{t}}{D}, \quad \theta = \frac{\bar{\theta}}{D}, \quad \mu = \bar{\mu}D, \quad r = \frac{hn^0}{D^2}, \quad \bar{h}_1 = \frac{h_1}{D}, \quad \bar{h}_2 = \frac{h_2s^0}{D}, \quad \bar{h}_3 = \frac{h_3}{D}.$$

The dimensionless system for (1.1) is given by

$$\begin{cases} N'(t) = 1 - N(t) - h_1N(t)X(t), \\ X'(t) = r \int_0^\infty e^{-\mu\theta} f(\theta)N(t-\theta)X(t-\theta)d\theta - X(t) - h_2X(t)S(t), \\ S'(t) = 1 - S(t) - h_3X(t)S(t). \end{cases} \quad (2.1)$$

Here, for the convenience of writing, the corresponding symbols are still used after the “overline” is removed. We assume that $\int_0^\infty e^{-\mu\theta} f(\theta)d\theta = q \leq 1$ and $\int_0^\infty \theta e^{-\mu\theta} f(\theta)d\theta < \infty$.

According to [23, 30–34], the phase space of model (2.1) can be taken as

$$C = \left\{ \phi \in C((-\infty, 0], R^3) : \phi(\theta)e^{\alpha\theta} \text{ is uniformly continuous, for } \theta \in (-\infty, 0], \|\phi\| < \infty \right\},$$

where α is a positive constant; the norm $\|\phi\| = \sup_{\theta \leq 0} |\phi(\theta)|e^{\alpha\theta}$. The nonnegative cone of C is $C_+ = \{\phi \in C : \phi(\theta) \geq 0, \theta \in (-\infty, 0]\}$.

For $\phi \in C_+$, we define $\phi_t \in C_+$, $\phi_t(\theta) = \phi(t + \theta)$, $\theta \in (-\infty, 0]$.

Considering the biological significance of model (2.1), for any initial function $\phi = (\phi_1, \phi_2, \phi_3)^T$, we restrict $\phi = (\phi_1, \phi_2, \phi_3)^T \in BC_+ \subset C_+$, where

$$BC_+ = \left\{ \phi \in C((-\infty, 0], R^3) : \phi(\theta) \text{ is nonnegative and bounded, for } \theta \in (-\infty, 0] \right\}.$$

Thus, we have the following theorem.

Theorem 2.1. For $\phi = (\phi_1, \phi_2, \phi_3)^T \in BC_+$, the solution of model (2.1) with the initial condition $(N(\theta), X(\theta), S(\theta))^T = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T$ ($\theta \in (-\infty, 0]$) is nonnegative and unique, and it satisfies $N(t) > 0$ ($t > 0$), $X(t) \geq 0$ ($t \geq 0$), $S(t) > 0$ ($t > 0$), $\limsup_{t \rightarrow \infty} N(t) \leq 1$, $\limsup_{t \rightarrow \infty} X(t) \leq \xi$, $\limsup_{t \rightarrow \infty} S(t) \leq 1$, where ξ is a positive constant not depending on the initial function.

Proof. From the first and third equations in model (2.1), we can easily get $\limsup_{t \rightarrow \infty} N(t) \leq 1$ and $\limsup_{t \rightarrow \infty} S(t) \leq 1$. Define

$$V(\phi) = \frac{rq}{h_1} \phi_1(0) + \phi_2(0) + \phi_3(0) + r \int_0^\infty e^{-\mu\theta} f(\theta) \int_{-\theta}^0 \phi_1(u) \phi_2(u) du d\theta.$$

Here, the convergence of $\int_0^\infty \theta e^{-\mu\theta} f(\theta)d\theta$ should be noticed. Then, for $t \geq 0$, the derivative of V along any solution of model (2.1) is

$$\begin{aligned} V'(\phi)|_{(2.1)} &= \frac{rq}{h_1} N'(t) + X'(t) + S'(t) + r \left[\int_0^\infty e^{-\mu\theta} f(\theta) \int_{t-\theta}^t N(u) X(u) du d\theta \right]' \\ &= \frac{rq}{h_1} [1 - N(t) - h_1 N(t) X(t)] + \left[r \int_0^\infty e^{-\mu\theta} f(\theta) N(t - \theta) X(t - \theta) d\theta - X(t) - h_2 X(t) S(t) \right] \\ &\quad + [1 - S(t) - h_3 X(t) S(t)] + r \int_0^\infty e^{-\mu\theta} f(\theta) [N(t) X(t) - N(t - \theta) X(t - \theta)] d\theta \\ &= \frac{rq}{h_1} + 1 - \frac{rq}{h_1} N(t) - X(t) - S(t) - h_2 X(t) S(t) - h_3 X(t) S(t) \\ &\leq \frac{rq}{h_1} + 1 - \left[\frac{rq}{h_1} N(t) + X(t) + S(t) \right]. \end{aligned}$$

From Theorem A in [35], we obtain that the solution of model (2.1) is ultimately uniformly bounded. Furthermore, there is a positive constant ξ independent of the initial function such that $\limsup_{t \rightarrow \infty} X(t) \leq \xi$.

According to the discussion in [5, 6], the equilibria of model (2.1) can be classified as follows.

The model (2.1) always has a boundary equilibrium (microorganism-free equilibrium) $E_0 = (1, 0, 1)^T$.

To consider the existence of the positive equilibria (microorganism co-existent equilibrium) for $X > 0$, we consider the equation of the variable X :

$$h_1 h_3 X^2 + (b - aR_0)X + h_2(1 - R_0) = 0, \quad (2.2)$$

where $R_0 = \frac{rq-1}{h_2}$, $a = h_2 h_3$, $b = h_1 h_2 + h_1$. If $rq \leq 1$ ($R_0 \leq 0$), (2.1) has no positive equilibria. If $rq > 1$, the following conclusion holds:

1) $a \leq b$. If $R_0 > 1$, there exists a unique positive equilibrium $E_1^* = (N_1^*, X_1^*, S_1^*)^T$. If $R_0 \leq 1$, there are no positive equilibria.

2) $a > b$. If $R_0 > 1$, there exists a unique positive equilibrium $E_1^* = (N_1^*, X_1^*, S_1^*)^T$.

If $R_0 < 1$, let $F(z) = a^2 \left[\left(z - \frac{b}{a} \right)^2 + \frac{4h_1}{a} (z - 1) \right]$. By $F(z) = 0$, we have

$$z_1 = \frac{b - 2h_1}{a} + \frac{2}{a} \sqrt{h_1^2 + (a - b)h_1}, \quad z_2 = \frac{b - 2h_1}{a} - \frac{2}{a} \sqrt{h_1^2 + (a - b)h_1},$$

and we know that z_1 satisfies $\frac{b}{a} < z_1 \equiv \omega < 1$.

If $R_0 = \omega$, there exists a unique positive equilibrium $E_\omega^* = (N_\omega^*, X_\omega^*, S_\omega^*)^T$.

If $\omega < R_0 < 1$, there exist two positive equilibria $E_1^* = (N_1^*, X_1^*, S_1^*)^T$, $E_2^* = (N_2^*, X_2^*, S_2^*)^T$.

If $R_0 < \omega$, there are no positive equilibria.

3. Stability of the boundary equilibrium

This section mainly considers the local and global stability of the boundary equilibrium E_0 .

3.1. Local stability of the boundary equilibrium

The linearized system of model (2.1) at any equilibrium is

$$\begin{cases} N'(t) = -(1 + h_1 \bar{X})N(t) - h_1 \bar{N}X(t), \\ X'(t) = r \int_0^\infty e^{-\mu\theta} f(\theta) \bar{X}N(t - \theta) d\theta + r \int_0^\infty e^{-\mu\theta} f(\theta) \bar{N}X(t - \theta) d\theta \\ \quad - (1 + h_2 \bar{S})X(t) - h_2 \bar{X}S(t), \\ S'(t) = -h_3 \bar{S}X(t) - (1 + h_3 \bar{X})S(t). \end{cases} \quad (3.1)$$

Then, we have the following theorem.

Theorem 3.1. *If $R_0 < 1$, then the boundary equilibrium E_0 is locally asymptotically stable. If $R_0 > 1$, then the boundary equilibrium E_0 is unstable.*

Proof. The characteristic equation at the boundary equilibrium E_0 is

$$(\lambda + 1)^2 \left(\lambda + 1 + h_2 - r \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta \right) = 0.$$

There are always eigenvalues $\lambda_{1,2} = -1 < 0$. Consider the following transcendental equation:

$$\psi(\lambda) = \lambda + 1 + h_2 - r \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta = 0. \quad (3.2)$$

Let us assume that Eq (3.2) has a root λ with a non-negative real part, i.e., $\text{Re}\lambda \geq 0$; this then implies that

$$rq = r \int_0^\infty e^{-\mu\theta} f(\theta) d\theta \geq \left| r \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta \right| = |\lambda + 1 + h_2| \geq 1 + h_2. \quad (3.3)$$

Obviously, the inequality Eq (3.3) contradicts $R_0 < 1$. Therefore, all roots of Eq (3.2) have negative real parts when $R_0 < 1$.

If $R_0 > 1$, we see that $\psi(0) = h_2(1 - R_0) < 0$. For $\lambda \geq 0$, we have

$$\psi(\lambda) \geq \lambda + 1 + h_2 - r \int_0^\infty e^{-\mu\theta} f(\theta) d\theta = \lambda + 1 + h_2 - rq.$$

Therefore, $\psi(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$. According to the intermediate value theorem of continuous functions, the equation $\psi(\lambda) = 0$ has at least one positive eigenvalue λ .

3.2. Global stability of the boundary equilibrium

Theorem 3.2. *If $R_0 < \frac{1}{1+h_3\xi}$ (< 1), the boundary equilibrium E_0 is globally asymptotically stable.*

Proof. Noting Theorem 3.1, we only need to prove that the boundary equilibrium E_0 is globally attractive.

For the arbitrary bounded solution of model (2.1), we consider the following differentiable function

$$\Psi_0(t) = q(N(t) - 1 - \ln N(t)) + \frac{h_1}{r} X(t) + h_1 \int_0^\infty e^{-\mu\theta} f(\theta) \int_{t-\theta}^t N(s) X(s) ds d\theta, \text{ for } t \geq 0.$$

Since $\limsup_{t \rightarrow \infty} X(t) \leq \xi$, then the third equation from model (2.1) yields $\liminf_{t \rightarrow \infty} S(t) \geq \frac{1}{1+h_3\xi}$. There exists a sufficiently large $t_1 > 0$; we have $S(t) > \frac{1}{1+h_3\xi} - \varepsilon > 0$ for $t > t_1$. Let $R_0 < \frac{1}{1+h_3\xi} - \varepsilon > 0$ for sufficiently small $\varepsilon > 0$. Therefore, for $t > t_1$,

$$\begin{aligned} \Psi'_0(t) &= q \left(2 - \frac{1}{N(t)} - N(t) \right) + \frac{h_1 h_2}{r} (R_0 - S(t)) X(t) \\ &\leq q \left(2 - \frac{1}{N(t)} - N(t) \right) + \frac{h_1 h_2}{r} \left(R_0 - \frac{1}{1+h_3\xi} + \varepsilon \right) X(t). \end{aligned}$$

Integration at both sides of the above inequality yields

$$\Psi_0(t) - \int_{t_1}^t \left(q \left(2 - \frac{1}{N(u)} - N(u) \right) + \frac{h_1 h_2}{r} \left(R_0 - \frac{1}{1+h_3\xi} + \varepsilon \right) X(u) \right) du \leq \Psi_0(t_1) < +\infty.$$

We can obtain $X(t)$ as integrable on $[t_1, +\infty)$. In addition, considering the initial function $\phi = (\phi_1, \phi_2, \phi_3)^T \in BC_+$ and the boundedness of the solution, we have that the derivative function $X'(t)$ is bounded on $[t_1, +\infty)$ from the second equation of model (2.1). Therefore, $X(t)$ is uniformly continuous on $[t_1, +\infty)$. Using the familiar Barbalet's lemma, we have $\lim_{t \rightarrow \infty} X(t) = 0$. Further, from the first and third equations of model (2.1), we can get $\lim_{t \rightarrow \infty} N(t) = \lim_{t \rightarrow \infty} S(t) = 1$.

4. Stability of the positive equilibria

For the local asymptotic stability of the positive equilibrium E_1^* , there is the following theorem.

Theorem 4.1. *The following conclusions hold:*

1) Let $a \leq b$ (the forward bifurcation). If $R_0 > 1$, then the positive equilibrium E_1^* is locally asymptotically stable.

2) Let $a > b$ (the backward bifurcation).

(i) If $R_0 > 1$, then the positive equilibrium E_1^* is locally asymptotically stable.

(ii) If $\omega < R_0 < 1$, then the positive equilibrium E_1^* is locally asymptotically stable and the positive equilibrium E_2^* is unstable.

(iii) If $R_0 = \omega$, then the positive equilibrium E_ω^* is linearly stable.

Proof. The characteristic equation of the linearized system at any positive equilibrium E^* is

$$\begin{aligned} J(\lambda) &= \begin{vmatrix} \lambda + 1 + h_1 X^* & h_1 N^* & 0 \\ -r X^* \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta & \lambda + 1 + h_2 S^* - r N^* \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta & h_2 X^* \\ 0 & h_3 S^* & \lambda + 1 + h_3 X^* \end{vmatrix} \\ &= (\lambda + 1 + h_1 X^*) \left[\left(\lambda + 1 + h_2 S^* - r N^* \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta \right) (\lambda + 1 + h_3 X^*) - h_2 X^* h_3 S^* \right] \\ &\quad + r X^* h_1 N^* (\lambda + 1 + h_3 X^*) \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta \\ &= (\lambda + 1 + h_1 X^*) (\lambda + 1 + h_2 S^*) (\lambda + 1 + h_3 X^*) - h_2 X^* h_3 S^* (\lambda + 1 + h_1 X^*) \\ &\quad + r X^* h_1 N^* (\lambda + 1 + h_3 X^*) \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta \\ &\quad - r N^* (\lambda + 1 + h_1 X^*) (\lambda + 1 + h_3 X^*) \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta \\ &= (\lambda + 1 + h_1 X^*) (\lambda + 1) (\lambda + 1 + h_2 S^* + h_3 X^*) - r N^* (\lambda + 1 + h_3 X^*) (\lambda + 1) \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta \\ &\equiv (\lambda + 1) \tilde{J}(\lambda), \end{aligned}$$

where

$$\tilde{J}(\lambda) = (\lambda + 1 + h_1 X^*) (\lambda + 1 + h_2 S^* + h_3 X^*) - r N^* (\lambda + 1 + h_3 X^*) \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta.$$

First, we consider the local asymptotic stability of E_1^* . Here, we notice the equations $N^* = \frac{1}{1+h_1 X^*}$, $S^* = \frac{1}{1+h_3 X^*}$ and $1 + h_2 S^* = r q N^*$. Similar to the discussion of Theorem 3.2 in [5], it is easy to obtain $J(0) = \tilde{J}(0) = N^* X^* H(X^*)$, where $H(X^*) = h_1^2 h_3 X^{*2} + 2h_1 h_3 X^* + h_3 + r q h_1 - r q h_3 > 0$. Therefore, $J(\lambda) = 0$ does not have zero root $\lambda = 0$.

Rewrite the equation $\tilde{J}(\lambda) = 0$ as

$$(\lambda + 1 + h_1 X^*) (\lambda + 1 + h_2 S^* + h_3 X^*) = r N^* (\lambda + 1 + h_3 X^*) \int_0^\infty e^{-\mu\theta} f(\theta) e^{-\lambda\theta} d\theta. \quad (4.1)$$

Let us assume that Eq (4.1) has roots $\lambda = x \pm iy$ ($x \geq 0$, $y \geq 0$ and $|x| + |y| > 0$). Taking the modulus on both sides of (4.1), it follows that

$$F(x, y) \equiv |\lambda + 1 + h_1 X^*|^2 |\lambda + 1 + h_2 S^* + h_3 X^*|^2 - (rqN^*)^2 |\lambda + 1 + h_3 X^*|^2 \leq 0.$$

On the other hand, we have

$$\begin{aligned} F(x, y) &= |\lambda + 1 + h_1 X^*|^2 |\lambda + rqN^* + h_3 X^*|^2 - (rqN^*)^2 |\lambda + 1 + h_3 X^*|^2 \\ &= |x + iy + 1 + h_1 X^*|^2 |x + iy + rqN^* + h_3 X^*|^2 - (rqN^*)^2 |x + iy + 1 + h_3 X^*|^2 \\ &= [(x + 1 + h_1 X^*)^2 + y^2][(x + rqN^* + h_3 X^*)^2 + y^2] - (rqN^*)^2 [(x + 1 + h_3 X^*)^2 + y^2] \\ &= [(x + 1 + h_1 X^*)^2 (x + rqN^* + h_3 X^*)^2 - (rqN^*)^2 (x + 1 + h_3 X^*)^2] \\ &\quad + y^2 [(x + 1 + h_1 X^*)^2 + (x + rqN^* + h_3 X^*)^2 + y^2 - (rqN^*)^2] \\ &\geq [(x + 1 + h_1 X^*)(x + rqN^* + h_3 X^*) + (rqN^*)(x + 1 + h_3 X^*)]G(x), \end{aligned}$$

where $G(x) = (x + 1 + h_1 X^*)(x + rqN^* + h_3 X^*) - (rqN^*)(x + 1 + h_3 X^*)$. For $x > 0$, it has from $G'(x) = 2x + 1 + h_1 X^* + h_3 X^* > 0$ that $G(x) \geq G(0) = N^* X^* H(X^*) > 0$. Therefore, $F(x, y) > 0$ for $x \geq 0$, $y \geq 0$ and $|x| + |y| > 0$. Clearly, this is a contradiction. Thus, it is proved that all roots of Eq (4.1) have negative real parts.

Next, we consider the instability of E_2^* . Completely similar to the discussion of Theorem 3.3 in [5], at the positive equilibrium E_2^* , $\tilde{J}(\lambda)$ satisfies $\tilde{J}(0) < 0$.

Then for $\lambda \geq 0$,

$$\tilde{J}(\lambda) \geq \lambda^2 + (1 + h_1 X^* + h_3 X^*)\lambda + (1 + h_1 X^*)(1 + h_2 S^* + h_3 X^*) - rqN^*(1 + h_3 X^*).$$

Therefore, $\tilde{J}(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$. Furthermore, it can be easily obtained that the equation $\tilde{J}(\lambda) = 0$ has at least one positive eigenvalue, implying that the positive equilibrium E_2^* is unstable.

Finally, for the positive equilibrium E_ω^* , completely similar to the discussion of Theorem 3.3 in [5], one obtains that the equation $\tilde{J}(\lambda) = 0$ has a single root $\lambda = 0$, and all other roots have negative real parts. Therefore, the positive equilibrium E_ω^* is linearly stable.

5. Permanence of the model

For the initial function $\phi = (\phi_1, \phi_2, \phi_3)^T \in BC_+$ and $\phi_1 \leq 1$ and $\phi_3 \leq 1$, it is not difficult to show that the solution $(N(t), X(t), S(t))^T$ of model (2.1) satisfies $N(t) \leq 1$ and $S(t) \leq 1$ for $t \geq 0$. Therefore, in the discussion of persistence, the initial function $\phi = (\phi_1, \phi_2, \phi_3)^T \in BC_+$ can be defined to satisfy $\phi_1 \leq 1$ and $\phi_3 \leq 1$.

Theorem 5.1. *If $R_0 > 1$, for $\phi = (\phi_1, \phi_2, \phi_3)^T \in BC_+$, $\phi_1 \leq 1$, $\phi_3 \leq 1$ and $\phi_2(0) > 0$, model (2.1) is uniformly persistent and the solution $(N(t), X(t), S(t))^T$ of model (2.1) satisfies*

$$\liminf_{t \rightarrow \infty} N(t) \geq \frac{1}{1 + h_1 \xi} \equiv v_1, \quad \liminf_{t \rightarrow \infty} X(t) \geq X_1 e^{-(1+h_2)(T_0+\tau_1)} \equiv v_2, \quad \liminf_{t \rightarrow \infty} S(t) \geq \frac{1}{1 + h_3 \xi} \equiv v_3,$$

where τ_1 is a positive constant, and

$$X_1 = \frac{\sigma}{h_1(1 + h_2)}, \quad T_0 = -\frac{1 + h_2}{1 + h_2 + \sigma} \ln \frac{qr - 1 - h_2 - \sigma}{qr}, \quad \sigma \in (0, qr - 1 - h_2).$$

$$A = \frac{1 + h_2}{1 + h_2 + \sigma}, \quad A + \left[\frac{1}{2(1 + h_1\xi)} - A \right] e^{-\frac{t_0}{A}} \equiv \rho \quad \left(1 > \rho > A - Ae^{-\frac{t_0}{A}} = \frac{1 + h_2}{qr} \right),$$

$$0 < \varepsilon < \min \left\{ q, \frac{qr\rho - (1 + h_2)}{r\rho} \right\}, \quad \int_0^{\tau_1} e^{-\mu\theta} f(\theta) d\theta > q - \varepsilon (> 0).$$

Proof. Let $(N(t), X(t), S(t))^T$ ($t \geq 0$) be a solution of model (2.1). By the conclusion of Theorem 2.1, we have $\limsup X(t) \leq \xi$; then, it follows from the first and third equations of model (2.1) that $\liminf_{t \rightarrow \infty} N(t) \geq v_1$, $\liminf_{t \rightarrow \infty} S(t) \geq v_3$.

For $t \geq 0$, we define auxiliary functions $V(t)$:

$$V(t) = X(t) + r \int_0^\infty e^{-\mu\theta} f(\theta) \int_{t-\theta}^t N(s)X(s) ds d\theta.$$

Noticing that the boundedness of the solution $(N(t), X(t), S(t))^T$ ($t \geq 0$) and $\int_0^\infty \theta e^{-\mu\theta} f(\theta) d\theta < \infty$, we obtain that the function $V(t)$ is bounded and continuously differentiable. For $t \geq 0$, we have

$$V'(t) = (qrN(t) - 1 - h_2S(t))X(t) \geq (qrN(t) - 1 - h_2)X(t),$$

$S(t) \leq 1$ ($t \geq 0$) is used here.

There exists a sufficiently large $T > 0$ such that $N(t) \geq \frac{1}{2(1+h_1\xi)}$ for $t \geq T$.

For every $t_0 \geq T > 0$, it is shown below that the inequality $X(t) \leq X_1$ ($t \geq t_0$) cannot hold.

If not, then there exists some $t_0 \geq T$ such that $X(t) \leq X_1$ for $t \geq t_0$. From the first equation of model (2.1), we have $N'(t) \geq 1 - (1 + h_1X_1)N(t) = 1 - \frac{1}{A}N(t)$ for $t \geq t_0$. Noticing that $\frac{1}{2(1+h_1\xi)} - A < 0$, therefore, for $t \geq t_0 + T_0$, we obtain

$$\begin{aligned} N(t) &\geq A + (N(t_0) - A)e^{-\frac{t-t_0}{A}} \geq A + \left[\frac{1}{2(1 + h_1\xi)} - A \right] e^{-\frac{t-t_0}{A}} \\ &\geq A + \left[\frac{1}{2(1 + h_1\xi)} - A \right] e^{-\frac{T_0}{A}} = \rho. \end{aligned}$$

Then, for $t \geq t_0 + T_0$, we have $V'(t) \geq (qr\rho - 1 - h_2)X(t)$.

It follows from the conditions in Theorem 5.1 that $r \int_0^{\tau_1} e^{-\mu\theta} f(\theta) d\theta > r(q - \varepsilon) > r\rho(q - \varepsilon) > 1 + h_2$.

Let $X_m = \min_{-\tau_1 \leq \theta \leq 0} X(t_0 + T_0 + \tau_1 + \theta)$; the following proof shows that $X(t) \geq X_m$ for all $t \geq t_0 + T_0$.

If not, then there exists a $T_1 \geq 0$ such that $X(t) \geq X_m$ for $t_0 + T_0 \leq t \leq t_0 + T_0 + \tau_1 + T_1 \equiv \bar{t}$, $X(\bar{t}) = X_m$ and $X'(\bar{t}) \leq 0$. It follows immediately from the second equation of model (2.1) that

$$\begin{aligned} X'(\bar{t}) &= r \int_0^{\tau_1+T_1} e^{-\mu\theta} f(\theta) N(\bar{t} - \theta) X(\bar{t} - \theta) d\theta + r \int_{\tau_1+T_1}^\infty e^{-\mu\theta} f(\theta) N(\bar{t} - \theta) X(\bar{t} - \theta) d\theta \\ &\quad - X(\bar{t}) - h_2 X(\bar{t}) S(\bar{t}) \\ &\geq r \int_0^{\tau_1+T_1} e^{-\mu\theta} f(\theta) N(\bar{t} - \theta) X(\bar{t} - \theta) d\theta - X(\bar{t}) - h_2 X(\bar{t}) S(\bar{t}) \\ &\geq r\rho X_m \int_0^{\tau_1+T_1} e^{-\mu\theta} f(\theta) d\theta - X_m - h_2 X_m \\ &> [r\rho(q - \varepsilon) - 1 - h_2] X_m > 0; \end{aligned}$$

$S(\bar{t}) \leq 1$ is used here. Clearly, this is a contradiction. It indicates that $X(t) \geq X_m$ for all $t \geq t_0 + T_0$. Thus, for $t \geq t_0 + T_0$, we have

$$V'(t) \geq (qr\rho - 1 - h_2)X(t) \geq (qr\rho - 1 - h_2)X_m > 0.$$

Then, $V(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, this is a contradiction with the boundedness of $V(t)$.

The following two cases are discussed.

Case (I) The inequality $X(t) \geq X_1$ holds for all large t , implying that $\liminf_{t \rightarrow \infty} X(t) \geq X_1 \geq \nu_2$.

Case (II) The function $X(t)$ oscillates infinitely above and below $X = X_1$ for all large enough t . At this time, we only consider interval $[t_1, t_2]$, where $t_2 > t_1 \geq T$, $X(t_1) = X_1$, $X(t_2) = X_1$, $X(t) < X_1$ ($t_1 < t < t_2$).

If $t_2 - t_1 \leq T_0 + \tau_1$, it follows from the second equation of model (2.1) that $X'(t) \geq -(1 + h_2)X(t)$ for $t \in [t_1, t_2]$. Therefore, by integrating, we have

$$X(t) \geq X_1 e^{-(1+h_2)(t-t_1)} \geq X_1 e^{-(1+h_2)(T_0+\tau_1)} = \nu_2 \text{ for } t \in [t_1, t_2].$$

If $t_2 - t_1 > T_0 + \tau_1$, this case is exactly similar to the derivation of Case (I), which make it easy to obtain $X(t) \geq \nu_2$ for $t \in [t_1, t_1 + T_0 + \tau_1]$.

It is further proved that there is still $X(t) \geq \nu_2$ for $t \in [t_1 + T_0 + \tau_1, t_2]$.

If this were not true, then there exists a $T_2 \geq 0$ such that $X(t) \geq \nu_2$ for $t_1 \leq t \leq t_1 + T_0 + \tau_1 + T_2 \equiv \hat{t}$, $X(\hat{t}) = \nu_2$ and $X'(\hat{t}) \leq 0$. Similarly, it also follows from the second equation of model (2.1) that

$$\begin{aligned} X'(\hat{t}) &= r \int_0^{\tau_1+T_2} e^{-\mu\theta} f(\theta) N(\hat{t}-\theta) X(\hat{t}-\theta) d\theta + r \int_{\tau_1+T_2}^{\infty} e^{-\mu\theta} f(\theta) N(\hat{t}-\theta) X(\hat{t}-\theta) d\theta \\ &\quad - X(\hat{t}) - h_2 X(\hat{t}) S(\hat{t}) \\ &\geq r \int_0^{\tau_1+T_2} e^{-\mu\theta} f(\theta) N(\hat{t}-\theta) X(\hat{t}-\theta) d\theta - X(\hat{t}) - h_2 X(\hat{t}) S(\hat{t}) \\ &\geq r\rho\nu_2 \int_0^{\tau_1+T_2} e^{-\mu\theta} f(\theta) d\theta - (1 + h_2)\nu_2 \\ &> [r\rho(q - \varepsilon) - 1 - h_2]\nu_2 > 0, \end{aligned}$$

$S(\hat{t}) \leq 1$ is also used here. Therefore, a contradiction is obtained. This shows that for $t \in [t_1, t_2]$, $X(t) \geq \nu_2$ always holds. Since the interval $[t_1, t_2]$ is arbitrarily chosen, this proves that for all sufficiently large t , there must be $X(t) \geq \nu_2$. Thus, it follows that $\liminf_{t \rightarrow \infty} X(t) \geq \nu_2$.

6. Conclusions

In this paper, we have obtained the dissipation of the solution and the local asymptotic stability of the equilibrium as well as the uniform persistence of model (2.1) with infinite delay through fine analysis on the distribution of the roots of the characteristic equations in the complex plane and the asymptotic properties of the solutions, combined with the construction of appropriate Lyapunov functionals. Furthermore, a sufficient condition for the global stability of the boundary equilibrium is given. In particular, the conclusion of uniform persistence suggests that microorganism flocculation and collection are sustainable as long as the threshold parameter $R_0 > 1$. Also, the explicit estimation

expressions for the eventual lower bound of the evolution of biomass (nutrient- $N(t)$, microorganism- $X(t)$, flocculant- $S(t)$) with time are given. These results further extend the corresponding conclusions in [5, 6].

However, it should be pointed out that (2.1) is an infinite time-delay system, and the information of the initial function is always stored in the time-delay term, which makes it more difficult to obtain an explicit expression of ξ in Theorem 2.1 than in the case of bounded time-delay systems, and we will leave this issue for further discussion. Second, the sufficient condition $R_0 < \frac{1}{1+h_3\xi}$ (< 1) given in Theorem 3.2 for the global stability of the boundary equilibrium (microorganism-free equilibrium) E_0 is also conservative, at least for the case of the forward bifurcation. In addition, we have not considered the global stability of the positive equilibrium E^* (microorganism co-existent equilibrium) of the model (2.1), since new Lyapunov functionals with infinite delay may need to be constructed (see, for example, [10–15]). Furthermore, we see from Theorem 5.1 that the parameter ξ in Theorem 2.1 plays an important role in the values of the parameters ν_1 , ν_2 and ν_3 .

Finally, similar to [16], let us choose the following weak kernel and strong kernel:

$$f(\theta) = \alpha e^{-\alpha\theta} \equiv f_w(\theta), \quad f(\theta) = \alpha^2 \theta e^{-\alpha\theta} \equiv f_s(\theta), \quad \theta \geq 0, \alpha > 0.$$

The corresponding parameters q can be easily calculated as

$$q_w = \int_0^\infty e^{-\mu\theta} f_w(\theta) d\theta = \frac{\alpha}{\alpha + \mu}, \quad q_s = \int_0^\infty e^{-\mu\theta} f_s(\theta) d\theta = \frac{\alpha^2}{(\alpha + \mu)^2}.$$

Therefore, the condition “ $\int_0^{\tau_1} e^{-\mu\theta} f(\theta) d\theta > q - \varepsilon$ ” in Theorem 5.1 becomes the following simpler form:

$$\text{(weak kernel)} \quad \frac{\alpha}{\alpha + \mu} e^{-(\mu+\alpha)\tau_1} < \varepsilon, \quad \text{(strong kernel)} \quad \frac{\alpha^2}{\mu + \alpha} \left(\tau_1 + \frac{1}{\mu + \alpha} \right) e^{-(\mu+\alpha)\tau_1} < \varepsilon.$$

Especially, consider when (2.1) degenerates to the following one with bounded time delay:

$$\begin{cases} N'(t) = 1 - N(t) - h_1 N(t)X(t), \\ X'(t) = r \int_0^\tau e^{-\mu\theta} f(\theta) N(t-\theta) X(t-\theta) d\theta - X(t) - h_2 X(t)S(t), \\ S'(t) = 1 - S(t) - h_3 X(t)S(t), \end{cases}$$

where $\tau \geq 0$ is a constant. Then similar to [5, 6], for $t \geq 0$, let us consider the following differentiable function:

$$G(t) = \frac{r}{h_1} \int_0^\tau e^{-\mu\theta} f(\theta) N(t-\theta) d\theta + X(t).$$

It follows that for $t \geq \tau$,

$$\begin{aligned} G'(t) &= \frac{r}{h_1} \int_0^\tau e^{-\mu\theta} f(\theta) d\theta - \frac{r}{h_1} \int_0^\tau e^{-\mu\theta} f(\theta) N(t-\theta) d\theta - X(t) - h_2 X(t)S(t) \\ &\leq \frac{r}{h_1} \int_0^\tau e^{-\mu\theta} f(\theta) d\theta - G(t). \end{aligned}$$

Thus, it has

$$\lim_{t \rightarrow +\infty} X(t) \leq \lim_{t \rightarrow +\infty} G(t) \leq \frac{r}{h_1} \int_0^\tau e^{-\mu\theta} f(\theta) d\theta \equiv \xi.$$

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Conflict of interest

The authors declare there is no conflict of interest.

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