Mathematical Biosciences and Engineering

MBE, 20(5): 9470–9488.
DOI: 10.3934/mbe.2023416
Received: 19 January 2023
Revised: 27 February 2023
Accepted: 08 March 2023
Published: 20 March 2023

http://www.aimspress.com/journal/MBE

Research article

Reliability analysis of s-out-of-k multicomponent stress-strength system with dependent strength elements based on copula function

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Abstract: This paper considers the reliability analysis of a multicomponent stress-strength system which has k statistically independent and identically distributed strength components, and each component is constructed by a pair of statistically dependent elements. These elements are exposed to a common random stress, and the dependence among lifetimes of elements is generated by Clayton copula with unknown copula parameter. The system is regarded to be operating only if at least s ($1 \le s \le k$) strength variables in the system exceed the random stress. The maximum likelihood estimates (MLE) of unknown parameters and system reliability is established and associated asymptotic confidence interval is constructed using the asymptotic normality property and delta method, and the bootstrap confidence intervals are obtained using the sampling theory. Finally, Monte Carlo simulation is conducted to support the proposed model and methods, and one real data set is analyzed to demonstrate the applicability of our study.

Keywords: reliability analysis; s-out-of-k system; multicomponent; stress-strength model; copula

1. Introduction

In the field of reliability engineering, stress-strength model is frequently used to measure the reliability of system which has a random strength Y and is subject to a random stress X. If the stress exceeds the strength, the system will fail. The stress-strength model was introduced by Birnbaum [1] and the estimation of the reliability $R = P(Y > X)$ has been extensively discussed by many authors when the stress variable X and the strength variable Y follow a specified distribution, see Srinivasa et al. [2], Kohansal [3], Bai et al. [4,5] and Sharma [6].

With the development of technology, the multicomponent stress-strength system is common in daily life. A typical multicomponent system is s-out-of-k system which appears in industrial and military applications [7]. Such system functions when s $(1 \le s \le k)$ or more components simultaneously survive. Recently, many authors contributed their work to study the reliability analysis of the multicomponent stress-strength system. Srinivasa et al. [8] studied the estimation of the reliability when X and Y are independent random variables following exponentiated Weibull distribution with different shape parameters, and common shape and scale parameters, respectively. Liu et al. [9] proposed the reliability estimation of a N-M-cold-standby redundancy system when underlying distribution is generalized half-logistic distribution. Kızılaslan [10] discussed the classical and Bayesian estimation of reliability in a multicomponent stress-strength system for proportional reversed hazard rate distribution. Zhang et al. [11] proposed the Bayesian inference of reliability in a multicomponent stress-strength system when X and Y follow Marshall-Olkin bivariate Weibull distribution. Wang et al. [12] discussed the reliability analysis in a multicomponent stress-strength system when the latent strength and stress variables follow Kumaraswamy distributions with common shape parameter. Other related work can be seen in [13–17] and the references therein.

The literature aforementioned are all based on the assumption that the strength variable is constructed by one element. However, in some practical situation, it is more realistic to assume that the strength variable is constructed by a pair of dependent elements. For example, in a suspension bridge, the number of vertical cable pairs, which support the bridge deck is considered as dependent strength elements[18,19]. Therefore, it is meaningful to discuss the case when the strength variable is conducted by dependent elements. Actually, Nadar and Kızılaslan [18], Kızılaslan and Nadar [19] have discussed the estimation of reliability in a multicomponent stress-strength system when the strength elements are dependent, and the dependent relationship is described by a bivariate distribution. Nevertheless, a bivariate distribution needs the marginal distributions are the same type. To overcome this limitation, copula function is used, which is a link function between the joint cumulative distribution and the marginal distribution, and it has no limitation on the type and family of the marginal distributions. In our article, copula function is used to describe the dependent relationship of strength elements and the reliability analysis is discussed.

The main objective of our study is to discuss the reliability analysis of a s-out-of-k multicomponent stress-strength system when the strength variable is constructed by dependent elements, which is described by a copula function. The rest of the paper is organized as follows. Section 2 introduces some copula theory. In Section 3, the model description is provided and the reliability of s-out-of-k system is derived. Point and interval estimates are presented in Sections 4 and 5, respectively. Section 6 provides simulation studies and a real data analysis. Finally, some concluding remarks are given in Section 7.

2. Copula theory

Copula is a very convenient way to model the dependence of the random variables. A probabilistic way to define the copula is provided by Sklar [20]. More details about copulas can be found in [21]. In the following, we introduce some basic theory.

Let $S(x_1, x_2)$ be a two-dimensional joint survival function with marginal function R_1, R_2 and let R_1^{-1}, R_2^{-1} be quasi-inverses of R_1, R_2. For $\forall\, u_1, u_2 \in [0,1]$, there is a copula C as

$$C(u_1,u_2)=S(R_1^{-1}(u_1),R_2^{-1}(u_2)),\ \ S(x_1,x_2)=C\left(R_1(x_1),R_2(x_2)\right). \tag{1}$$

Then $C(\cdot)$ is called a survival copula.

Let $H(x_1,x_2)$ be a two-dimensional joint failure function with marginal function F_1,F_2 and let F_1^{-1},F_2^{-1} be quasi-inverses of F_1, F_2, respectively. For $\forall\, u_1,u_2\in[0,1]$, there is a copula $\widetilde{C}$ as

$$\widetilde{C}(u_1,u_2)=H(F_1^{-1}(u_1),F_2^{-1}(u_2)), \tag{2}$$

$$H(x_1,x_2)=\widetilde{C}\left(F_1(x_1),F_2(x_2)\right). \tag{3}$$

Then $\widetilde{C}(\cdot)$ is called a failure distribution copula.

The relationship between the failure copula $\widetilde{C}$ and the survival copula C, is

$$C\left(R_1(x_1),R_2(x_2)\right)=1-F_1(x_1)-F_2(x_2)+\widetilde{C}\left(F_1(x_1),F_2(x_2)\right),$$

$$\widetilde{C}\left(F_1(x_1),F_2(x_2)\right)=1-R_1(x_1)-R_2(x_2)+C\left(R_1(x_1),R_2(x_2)\right). \tag{4}$$

Let $f(x_1,x_2)$ be the joint probability density function (PDF) of X_1,X_2, then

$$f(x_1,x_2)=\frac{\partial^2 H(x_1,x_2)}{\partial x_1\partial x_2}=\frac{\partial^2\widetilde{C}\left(F_1(x_1),F_2(x_2)\right)}{\partial F_1(x_1)\partial F_2(x_2)}f_1(x_1)f_2(x_2), \tag{5}$$

where $\dfrac{\partial^2\widetilde{C}\left(F_1(x_1),F_2(x_2)\right)}{\partial F_1(x_1)\partial F_2(x_2)}$ is defined to be the PDF of $\widetilde{C}\left(F_1(x_1),F_2(x_2)\right)$.

In our study, a 2-dimensional Clayton copula is used to depict the dependence relationship of strength elements, which is a kind of Archimedean copula and widely used because of its nice properties such as its simple form, symmetry and the ability of combining [21]. Its mathematical form is given as

$$C(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta},\ \ \theta\in[-1,\infty)\setminus\{0\}, \tag{6}$$

where the parameter θ measures the dependence. It becomes an independent copula as θ approaches to zero.

3. Model description and reliability of s-out-of-k system

Assume X follows Weibull distribution with shape parameter λ and scale parameter α, denoted by $WE(\lambda,\alpha)$. Then the PDF and the cumulative distribution function (CDF) of X are, respectively,

$$g_X(x)=\lambda\alpha x^{\alpha-1}e^{-\lambda x^{\alpha}},\ \ x>0,\lambda>0,\alpha>0, \tag{7}$$

and

$$G_X(x)=1-e^{-\lambda x^{\alpha}}, x>0,\lambda>0,\alpha>0. \tag{8}$$

Let T, $Z_1,Z_2,\cdots,Z_k$ be s-independent, $G(t)$ be the CDF of stress variable T, and $F(z)$ be the common CDF of strength variables $Z_1,Z_2,\cdots,Z_k$. For the general case, the reliability of s-out-of-k system in a multicomponent stress-strength model developed by Bhattacharyya and Johnson [22] is given by

$$\begin{aligned} R_{s,k} &= P(at\ \ least\ s\ of\ the\ (Z_1, Z_2, \cdots, Z_k)\ exceed\ T) \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{+\infty} (1 - F(t))^i (F(t))^{k-i} dG(t). \end{aligned} \tag{9}$$

Suppose that the dependence between $X_1 \sim WE(\lambda_1, \alpha)$ and $X_2 \sim WE(\lambda_2, \alpha)$ is represented by a 2-dimensional Clayton copula. According to Eqs (6)–(8), the joint survival function of (X_1, X_2) is given by

$$S(x_1, x_2) = C\left(R(x_1), R(x_2)\right) = \left(e^{\lambda_1 \theta x_1^\alpha} + e^{\lambda_2 \theta x_2^\alpha} - 1\right)^{-\frac{1}{\theta}},$$

and according to Eqs (4)–(6), the joint PDF of (X_1, X_2) can be written as

$$f(x_1, x_2) = \left(e^{\lambda \theta x_1^\alpha} + e^{\lambda_2 \theta x_2^\alpha} - 1\right)^{-\frac{1}{\theta} - 1} \left(e^{\lambda_1 \theta x_1^\alpha} \lambda_1 \alpha x_1^{\alpha - 1} + e^{\lambda_2 \theta x_2^\alpha} \lambda_2 \alpha x_2^{\alpha - 1}\right). \tag{10}$$

We consider a system which has k statistically independent and identically distributed strength components and each component is constructed by a pair of statistically dependent elements. The system is subjected to a common random stress and it works when s or more components simultaneously survive, and a component is alive only if the weakest elements is operating. Assume that the marginal distribution of strength vectors $(X_{11}, X_{21}), (X_{12}, X_{22}), \cdots, (X_{1k}, X_{2k})$ and the stress variable T are Weibull distribution. Let $Z_i = \min(X_{1i}, X_{2i}), i = 1, 2, \cdots, k$. The survival function and the PDF of $Z = \min(X_1, X_2)$ are given by, respectively,

$$\begin{aligned} R_Z(t) &= P\left(Z > t\right) = P\left(X_1 > t, X_2 > t\right) \\ &= S\left(t, t\right) = \left(e^{\lambda_1 t^\alpha \theta} + e^{\lambda_2 t^\alpha \theta} - 1\right)^{-\frac{1}{\theta}}, \quad t > 0, \end{aligned} \tag{11}$$

and

$$f_Z(t) = \frac{dS(t,t)}{dt} = \left(e^{\lambda_1 t^\alpha \theta} + e^{\lambda_2 t^\alpha \theta} - 1\right)^{-\frac{1}{\theta} - 1} \left(\lambda_1 e^{\lambda_1 t^\alpha \theta} + \lambda_2 e^{\lambda_2 t^\alpha \theta}\right) \alpha t^{\alpha - 1}, \quad t > 0. \tag{12}$$

Let $T \sim WE(\lambda_3, \alpha)$ be the stress variable. Using Eqs (8) and (9), $R_{s,k}$ is given as

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_0^{+\infty} \left(e^{\lambda_1 t^\alpha \theta} + e^{\lambda_2 t^\alpha \theta} - 1\right)^{-\frac{i}{\theta}} \left(1 - \left(e^{\lambda_1 t^\alpha \theta} + e^{\lambda_2 t^\alpha \theta} - 1\right)^{-\frac{1}{\theta}}\right)^{k-i} \lambda_3 \alpha t^{\alpha - 1} e^{-\lambda_3 t^\alpha} dt \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_0^{+\infty} \left(e^{\lambda_1 \theta u} + e^{\lambda_2 \theta u} - 1\right)^{-\frac{i}{\theta}} \left(1 - \left(e^{\lambda_1 \theta u} + e^{\lambda_2 \theta u} - 1\right)^{-\frac{1}{\theta}}\right)^{k-i} \lambda_3 e^{-\lambda_3 u} du \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} C_k^i \cdot C_{k-i}^j (-1)^j \lambda_3 \int_0^{+\infty} \left(e^{\lambda_1 \theta u} + e^{\lambda_2 \theta u} - 1\right)^{-\frac{i+j}{\theta}} e^{-\lambda_3 u} du \end{aligned} \tag{13}$$

where $u = t^\alpha$.

4. Maximum likelihood estimation of $R_{s,k}$

Suppose that n systems are put on a life experiment. The potential data are $(X_{1i1}, X_{2i1}), (X_{1i2}, X_{2i2}), ..., (X_{1ik}, X_{2ik})$ and $T_i, i = 1, 2, \cdots, n$, the observed data are $Z_{i1}, Z_{i2}, \cdots, Z_{ik}$ and T_i, where $Z_{ij} = \min(X_{i1j}, X_{i2j})$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, k$. The likelihood function of these observed samples $\underset{\sim}{z} = \left\{z_{ij}, i = 1, 2, \cdots, n, j = 1, 2, \cdots, k\right\}$ and $\underset{\sim}{t} = (t_1, t_2, \cdots, t_n)$ is expressed as

$$\begin{aligned} L(\lambda_1, \lambda_2, \lambda_3, \alpha, \theta; \underset{\sim}{z}, \underset{\sim}{t}) &= \prod_{i=1}^{n}\left(\prod_{j=1}^{k} f_Z\left(z_{ij}\right)\right) g\left(t_i\right) \\ &= \prod_{i=1}^{n}\left(\prod_{j=1}^{k}\left(e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1\right)^{-\frac{1}{\theta}-1}\left(\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}\right)\alpha z_{ij}^{\alpha-1}\right)\lambda_3\alpha t_i^{\alpha-1} e^{-\lambda_3 t_i^{\alpha}} \\ &= \alpha^{nk+n}\lambda_3^{n}\prod_{i=1}^{n}\left(\prod_{j=1}^{k}\left(e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1\right)^{-\frac{1}{\theta}-1}\left(\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}\right) z_{ij}^{\alpha-1}\right)\cdot\prod_{i=1}^{n} t_i^{\alpha-1} e^{-\lambda_3 t_i^{\alpha}}, \end{aligned} \quad (14)$$

The log-likelihood function ignoring the additive constant is given as

$$\begin{aligned} \log L = &\sum_{i=1}^{n}\sum_{j=1}^{k}\left\{-\frac{1+\theta}{\theta}\log\left(e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1\right) + \log\left(\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}\right) + (\alpha-1)\log z_{ij}\right\} \\ &+ n(k+1)\log\alpha + n\log\lambda_3 + (\alpha-1)\sum_{i=1}^{n}\log t_i - \lambda_3\sum_{i=1}^{n} t_i^{\alpha}. \end{aligned} \quad (15)$$

Taking derivatives with respect to $\lambda_1, \lambda_2, \lambda_3, \alpha, \theta$ and equating them to zero, the likelihood equations are obtained as

$$\frac{\partial \log L}{\partial \lambda_1} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left\{-\frac{1+\theta}{\theta}\frac{e^{\lambda_1\theta z_{ij}^{\alpha}}\theta z_{ij}^{\alpha}}{e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1} + \frac{e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}}\theta z_{ij}^{\alpha}}{\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}}\right\} = 0, \quad (16)$$

$$\frac{\partial \log L}{\partial \lambda_2} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left\{-\frac{1+\theta}{\theta}\frac{e^{\lambda_2\theta z_{ij}^{\alpha}}\theta z_{ij}^{\alpha}}{e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1} + \frac{e^{\lambda_2\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}\theta z_{ij}^{\alpha}}{\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}}\right\} = 0, \quad (17)$$

$$\frac{\partial \log L}{\partial \lambda_3} = \frac{n}{\lambda_3} - \sum_{i=1}^{n} t_i^{\alpha} = 0, \quad (18)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} = &\sum_{i=1}^{n}\sum_{j=1}^{k}\left\{-\frac{1+\theta}{\theta}\frac{(\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}})\theta z_{ij}^{\alpha}\log z_{ij}}{e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1} + \frac{(\lambda_1^2 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2^2 e^{\lambda_2\theta z_{ij}^{\alpha}})\theta z_{ij}^{\alpha}\log z_{ij}}{\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}} + \log z_{ij}\right\} \\ &+ \frac{n(k+1)}{\alpha} + \sum_{i=1}^{n}\log t_i - \lambda_3\sum_{i=1}^{n} t_i^{\alpha}\log t_i, \end{aligned} \quad (19)$$

$$\frac{\partial \log L}{\partial \theta} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left\{\frac{1}{\theta^2}\log\left(e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1\right) - \left(\frac{1}{\theta}+1\right)\frac{(\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}})z_{ij}^{\alpha}}{e^{\lambda_1\theta z_{ij}^{\alpha}} + e^{\lambda_2\theta z_{ij}^{\alpha}} - 1} + \frac{(\lambda_1^2 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2^2 e^{\lambda_2\theta z_{ij}^{\alpha}})z_{ij}^{\alpha}}{\lambda_1 e^{\lambda_1\theta z_{ij}^{\alpha}} + \lambda_2 e^{\lambda_2\theta z_{ij}^{\alpha}}}\right\}. \quad (20)$$

Due to the complex form, we cannot find the analytical solutions of the likelihood equations. The numerical methods such as Newton-Raphson iteration algorithm and asymptotic methods [23–25] can be applied to get the MLEs $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\alpha}$ and $\hat{\theta}$.

Hence, using the invariance property of MLE, the MLE of $R_{s,k}$ is obtained from Eq (13) as

$$\widehat{R}_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k-i} C_k^i C_{k-i}^j (-1)^j \hat{\lambda}_3 \int_0^{+\infty} \left(e^{\hat{\lambda}_1\hat{\theta}u} + e^{\hat{\lambda}_2\hat{\theta}u} - 1\right)^{-\frac{i+j}{\hat{\theta}}} e^{-\hat{\lambda}_3 u} du,$$

where $u = t^{\hat{\alpha}}$.

5. Confidence interval

In this section, we propose two different methods to construct confidence intervals for unknown parameters and stress-strength model reliability $R_{s,k}$.

5.1. Asymptotic confidence intervals

The asymptotic confidence intervals (ACIs) are developed based on the asymptotic normality of MLE. Let $\underline{\boldsymbol{\eta}} = (\lambda_1, \lambda_2, \lambda_3, \alpha, \theta)$, the observed Fisher information matrix of parameter $\underline{\boldsymbol{\eta}}$ can be written as

$$I(\underline{\boldsymbol{\eta}}) = \begin{pmatrix} I_{11}(\underline{\boldsymbol{\eta}}) & I_{12}(\underline{\boldsymbol{\eta}}) & \cdots & I_{15}(\underline{\boldsymbol{\eta}}) \\ I_{21}(\underline{\boldsymbol{\eta}}) & I_{22}(\underline{\boldsymbol{\eta}}) & \cdots & I_{25}(\underline{\boldsymbol{\eta}}) \\ \vdots & \vdots & \cdots & \vdots \\ I_{51}(\underline{\boldsymbol{\eta}}) & I_{52}(\underline{\boldsymbol{\eta}}) & \cdots & I_{55}(\underline{\boldsymbol{\eta}}) \end{pmatrix}_{\boldsymbol{\eta}=(\lambda_1,\lambda_2,\lambda_3,\alpha,\theta)}, \tag{21}$$

where $I_{ij}(\underline{\boldsymbol{\eta}}) = -\dfrac{\partial^2 \ln L(\underline{\boldsymbol{\eta}})}{\partial\eta_i \partial\eta_j}$, $\eta_1 = \lambda_1, \eta_2 = \lambda_2, \eta_3 = \lambda_3, \eta_4 = \alpha$ and $\eta_5 = \theta$.

Therefore, the asymptotic variance-covariance matrix of $\underline{\boldsymbol{\eta}}$ can be given by

$$\hat{V} = I^{-1}\left(\hat{\underline{\boldsymbol{\eta}}}\right) = \begin{pmatrix} I_{11} & I_{12} & \cdots & I_{15} \\ I_{21} & I_{22} & \cdots & I_{25} \\ \vdots & \vdots & \vdots & \vdots \\ I_{51} & I_{52} & \cdots & I_{55} \end{pmatrix}^{-1}_{(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)=(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3,\hat{\alpha},\hat{\theta})} \triangleq \begin{pmatrix} \hat{v}_{11} & \hat{v}_{12} & \cdots & \hat{v}_{15} \\ \hat{v}_{21} & \hat{v}_{22} & \cdots & \hat{v}_{25} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{v}_{51} & \hat{v}_{52} & \cdots & \hat{v}_{55} \end{pmatrix}. \tag{22}$$

The asymptotic distribution of the pivotal quantities $\left(\hat{\eta}_i - \eta_i\right)/\sqrt{\hat{v}_{ii}}$, $i = 1, 2, \cdots, 5$ can be used to construct confidence intervals for η_i. A two-side $100(1-\gamma)$ % ACIs for η_i can be constructed by

$$\left(\hat{\eta}_i - z_{\gamma/2}\sqrt{\hat{v}_{ii}},\ \hat{\eta}_i + z_{\gamma/2}\sqrt{\hat{v}_{ii}}\right), i = 1, 2, \cdots, 5. \tag{23}$$

where $z_{\gamma/2}$ is the upper $z_{\gamma/2}$-th percentile point of standard normal distribution.

Furthermore, from Eq (13) we know that $R_{s,k}$ is a continuous function of $\lambda_1, \lambda_2, \lambda_3, \alpha$ and θ. Let $R_{s,k} = h(\lambda_1, \lambda_2, \lambda_3, \alpha, \theta)$. Then $h(\cdot)$ is a continuous function of $\lambda_1, \lambda_2, \lambda_3, \alpha$ and θ. Hence, $\hat{R}_{s,k} = h(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\alpha}, \hat{\theta})$ is a consistent estimator of $R_{s,k}$. Furthermore, $h(\cdot)$ has continuous first-order partial derivatives. Thus, using the Delta method, we have

$$\frac{\hat{R}_{s,k}-R_{s,k}}{\sqrt{Var(\hat{R}_{s,k})}}\to N(0,1)\,, \tag{24}$$

where $Var(\hat{R}_{s,k})=\sigma^2_{R_{s,k}}=\sum_{i=1}^{5}\sum_{j=1}^{5}\frac{\partial R_{s,k}}{\partial\eta_i}\frac{\partial R_{s,k}}{\partial\eta_j}I_{ij}^{-1}$.

Then, the two-side $100(1-\gamma)$ % ACI for $R_{s,k}$ can be written as

$$\left(\hat{R}_{s,k}-z_{\gamma/2}\sqrt{Var(\hat{R}_{s,k})},\hat{R}_{s,k}+z_{\gamma/2}\sqrt{Var(\hat{R}_{s,k})}\right). \tag{25}$$

Note that the ACI of $R_{s,k}$ may not be within the interval (0,1). Using logarithmic trans-formation and delta method, the asymptotic normality distribution of $\log\left(\widehat{R}_{s,k}\right)$ can be arrived as

$$\left(\log\left(\widehat{R}_{s,k}\right)-\log\left(\widehat{R}_{s,k}\right)\right)/\sqrt{Var\left(\log\left(\widehat{R}_{s,k}\right)\right)}\sim N(0,1)\,. \tag{26}$$

Therefore, using the inverse logarithmic transformation, the log-normal $100(1-\gamma)$% ACI of the reliability $R_{s,k}$ becomes

$$\left(\widehat{R}_{s,k}\exp\left(-z_{\gamma/2}\sqrt{Var(\widehat{R}_{s,k})}\,/\,\widehat{R}_{s,k}\right),\quad \widehat{R}_{s,k}\exp\left(z_{\gamma/2}\sqrt{Var(\widehat{R}_{s,k})}\,/\,\widehat{R}_{s,k}\right)\right). \tag{27}$$

5.2. Bootstrap confidence intervals

The bootstrap method is used to construct confidence interval for the unknown parameters [26,27] when the sample size is small. Compared to the ordinary bootstrap confidence interval (BCI), the bias-corrected percentile BCI is considered to perform better. The steps to construct the bias-corrected percentile BCI are as follow.

Step 1: Based on the observed sample $\underline{z}$ and $\underline{t}$, we compute the MLEs $\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3,\,\hat{\alpha},\hat{\theta}$ and $\widehat{R}_{s,k}$.

Step 2: Use the Clayton copula function, $\hat{\lambda}_1,\hat{\lambda}_2,\hat{\alpha}$ and $\hat{\theta}$ to generate a dependent bootstrap sample of strength element, and $\hat{\alpha}$ and $\hat{\lambda}_3$ to generate a bootstrap stress sample.

Step 3: Based on the bootstrap sample in step 2, we get the bootstrap estimate of $\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3,\hat{\alpha},\hat{\theta}$, say $\hat{\lambda}_1^*,\hat{\lambda}_2^*,\hat{\lambda}_3^*,\hat{\alpha}^*,\hat{\theta}^*$.

Step 4: Repeat Steps 2–3 N times to obtain $\underset{\rightarrow}{\hat{\vartheta}}^{*(1)},\underset{\rightarrow}{\hat{\vartheta}}^{*(2)},...,\underset{\rightarrow}{\hat{\vartheta}}^{*(N)}$, where $\underset{\rightarrow}{\hat{\vartheta}}^{*(k)}=(\hat{\eta}_1^{*(k)},...,\hat{\eta}_6^{*(k)})=(\hat{\lambda}_1^{*(k)},\hat{\lambda}_2^{*(k)},\hat{\lambda}_3^{*(k)},\hat{\alpha}^{*(k)},\hat{\theta}^{*(k)},\hat{R}_{s,k}^{*(k)})$ and

$$\hat{R}_{s,k}^{*(k)}=\sum_{i=s}^{k}\sum_{j=0}^{k-i}C_k^iC_{k-i}^j(-1)^j\hat{\lambda}_3^{*(k)}\int_0^{+\infty}\left(e^{\hat{\lambda}_1^{*(k)}\hat{\theta}^{*(k)}u}+e^{\hat{\lambda}_2^{*(k)}\hat{\theta}^{*(k)}u}-1\right)^{-\frac{i+j}{\hat{\theta}^{*(k)}}}e^{-\hat{\lambda}_3^{*(k)}u}du\,.$$

Step 5: For each variable η_i, arrange its bootstrap estimate in an ascending order to obtain $\hat{\eta}_i^{*[1]},\hat{\eta}_i^{*[2]},...,\hat{\eta}_i^{*[N]},i=1,2,\cdots,6$.

Then, a two-sided $100(1-\gamma)$ % bias-corrected percentile BCI of η_i is given by

$$(\hat{\eta}^*_{iL}, \hat{\eta}^*_{iU}) = (\hat{\eta}_i^{*[N\alpha_{1i}]}, \hat{\eta}_i^{*[N\alpha_{2i}]}),$$

where $\alpha_{1i} = \Phi(2z_{0i} + z_{\alpha/2})$ and $\alpha_{2i} = \Phi(2z_{0i} + z_{1-\alpha/2})$, Φ is the standard normal cumulative distribution function with $z_\alpha = \Phi^{-1}(\alpha)$, and the value of bias correction z_{0i} is $z_{0i} = \Phi^{-1}\left(\frac{number \text{ of } \{\hat{\eta}_i^{*[j]} < \hat{\eta}_i\}}{N}\right)$, $i = 1, 2, \cdots, 6$, $j = 1, 2, \cdots, N$.

6. Simulation study and data analysis

6.1. Simulation study

For illustration, a simulation study is performed to compare the performance of the estimates of unknown parameters and reliability $R_{s,k}$ in a multicomponent stress-strength system, which are obtained for different sample sizes, different model parameters and dependence parameters. The performances of the point estimates are compared by using estimated risks (ERs). We also compare the ACIs and BCIs in terms of the average interval lengths. The ER of δ, when δ is estimated by $\hat{\delta}_i$, is given by

$$ER(\delta) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\delta}_i - \delta)^2,$$

where n is the sample size.

We simulate different strength and stress populations corresponding to the parameters $(\lambda_1, \lambda_2, \lambda_3, \alpha) = \{(7,4,4,3),(1,2,3,4,)\}$ and $\theta = 1, 2$ with different sample sizes $n = 20\ (30)\ 80$. Without loss of generality, the 1-out-of-3 multicomponent system and the 2-out-of-4 multicomponent system are studied, i.e. $(s,k) = (1,3)$ and (2,4). The true value of $R_{1,3}$ with the given parameter $(\lambda_1, \lambda_2, \lambda_3, \theta, \alpha) = (7,4,4,1,3), (7,4,4,2,3), (1,2,4,1,3)$ and $(1,2,4,2,3)$ are 0.5529, 0.5587, 0.8626 and 0.8792, respectively. The true value of $R_{2,4}$ with the given parameter $(\lambda_1, \lambda_2, \lambda_3, \theta, \alpha) = (7,4,4,1,3), (7,4,4,2,3), (1,2,4,1,3)$ and $(1,2,4,2,3)$ are 0.5639, 0.6420, 0.9727 and 0.9905, respectively. The MLEs, ERs and the 95% ACIs, BCIs, and the lengths of ACI and BCI based on 5000 replications are listed in Tables 1–4, where $\hat{R}_{s,k}$ and $\tilde{R}_{s,k}$ represent the estimated results when the dependence of the strength elements is considered and the dependence of the strength elements is ignored, respectively. The MLEs and ERs of θ for different model parameters are reported in Table 5. All of the computations are performed using R software and run on LAPTOP with 1.80 and 2.30 GHz CPU processor, 12.0 GB RAM memory, and windows 10 operating system. Newton- Raphson procedure is adopted in the calculation process, and the starting values of unknown parameters are randomly chosen around their true values. We have chosen different initial values, and the estimated results are stable.

To study the effect of the dependence between the strength elements on the reliability $R_{s,k}$ in a multicomponent stress-strength system, we draw graph of $\hat{R}_{s,k}$ versus the dependency parameter θ for different pairs of $(\lambda_1, \lambda_2, \lambda_3, \alpha)$. Variations in $\hat{R}_{s,k}$ with respect to θ are displayed in Figure 1 for different model parameters and $n = 50$.

Table 1. MLEs, ERs and 95% CIs for parameters and $R_{s,k}$ when $(\lambda_1,\lambda_2,\lambda_3,\alpha,\theta)=(7,4,4,3,1)$.

n		λ_1	λ_2	λ_3	α	θ	$R_{1,3}$	$R_{2,4}$	$\widetilde{R}_{1,3}$	$\widetilde{R}_{2,4}$
20	MLE	7.3170	3.8023	4.1631	3.1722	0.8779	0.5313	0.5435	0.3906	0.4055
	ER	0.6994	0.4423	0.4135	0.0639	0.2638	0.0519	0.0329	0.0544	0.0595
	ACI_Lower	6.5132	3.5625	3.7228	2.7825	0.5095	0.4403	0.4301	0.2883	0.3009
	ACI_Upper	7.8855	4.2988	4.4224	3.5217	1.0902	0.6192	0.6241	0.4942	0.5085
	ACI_Length	1.3723	0.7364	0.6996	0.7392	0.5807	0.1789	0.1940	0.2059	0.2076
	BCI_Lower	6.6649	3.4171	3.7933	2.7740	0.5578	0.4477	0.4411	0.1752	0.1749
	BCI_Upper	7.9421	4.1874	4.5329	3.5704	1.1980	0.6323	0.6459	0.6598	0.6517
	BCI_Length	1.2772	0.7703	0.7396	0.7964	0.6402	0.1846	0.2048	0.4845	0.4768
50	MLE	7.2733	3.8240	4.1305	3.1028	0.9147	0.5344	0.5438	0.3808	0.4296
	ER	0.4590	0.3450	0.3386	0.0492	0.1878	0.0421	0.0637	0.0634	0.0635
	ACI_Lower	6.7840	3.7792	3.5933	2.8160	0.7622	0.4602	0.4491	0.4517	0.3697
	ACI_Upper	7.8365	4.1925	4.4927	3.2982	1.3162	0.6187	0.6385	0.5235	0.7011
	ACI_Length	1.0525	0.4132	0.8994	0.4823	0.5540	0.1585	0.1894	0.0719	0.3314
	BCI_Lower	6.6890	3.4738	3.6804	2.8466	0.6384	0.4516	0.4371	0.2647	0.3879
	BCI_Upper	7.8576	4.1742	4.5806	3.3590	1.1909	0.6173	0.6265	0.5190	0.6226
	BCI_Length	1.1686	0.7004	0.9002	0.5124	0.5525	0.1658	0.1894	0.2543	0.2347
80	MLE	7.2216	3.8627	3.9569	3.0746	0.9363	0.5410	0.5532	0.3764	0.3271
	ER	0.4451	0.3267	0.3147	0.0364	0.1421	0.0289	0.0459	0.0350	0.0341
	ACI_Lower	6.7946	3.5681	3.6579	2.8912	0.7808	0.4701	0.4637	0.5807	0.5661
	ACI_Upper	7.8106	4.3031	4.5964	3.2230	1.2548	0.6081	0.6426	0.6476	0.8229
	ACI_Length	1.0160	0.7350	0.9385	0.3319	0.4740	0.1380	0.1789	0.0669	0.2567
	BCI_Lower	6.7015	3.5423	3.4737	2.8005	0.6762	0.4558	0.4422	0.2608	0.2791
	BCI_Upper	7.7417	4.1831	4.4401	3.3487	1.1964	0.6263	0.6210	0.5049	0.6340
	BCI_Length	1.0402	0.6408	0.9664	0.5482	0.5202	0.1705	0.1789	0.2441	0.3549

Table 2. MLEs, ERs and 95% CIs for parameters and $R_{s,k}$ when $(\lambda_1, \lambda_2, \lambda_3, \alpha, \theta) = (7, 4, 4, 3, 2)$.

n		λ_1	λ_2	λ_3	α	θ	$R_{1,3}$	$R_{2,4}$	$\widetilde{R}_{1,3}$	$\widetilde{R}_{2,4}$
20	MLE	7.5126	4.4792	4.4516	3.1571	1.7615	0.5461	0.5549	0.4195	0.4188
	ER	0.8864	0.8293	0.9219	0.3424	0.4729	0.0348	0.0945	0.0387	0.0460
	ACI_Lower	6.7231	3.7272	3.7538	2.8649	1.4560	0.4230	0.4266	0.1824	0.1678
	ACI_Upper	7.9214	4.9292	4.8672	3.4931	2.3191	0.6212	0.6783	0.7259	0.6698
	ACI_Length	1.1983	1.2020	1.1134	0.6282	0.8631	0.1982	0.2517	0.5435	0.5020
	BCI_Lower	6.8125	3.8535	3.8818	2.8369	1.2013	0.4636	0.4043	0.5344	0.3342
	BCI_Upper	8.2127	5.1049	5.0214	3.4773	2.3217	0.6286	0.6449	0.6848	0.6741
	BCI_Length	1.4002	1.2514	1.1396	0.6404	1.1204	0.1650	0.2406	0.1305	0.3399
50	MLE	7.3853	4.2408	4.2863	3.1043	1.7820	0.5514	0.5684	0.4078	0.4186
	ER	0.6216	0.7029	0.7903	0.1751	0.4002	0.0211	0.0771	0.0279	0.0421
	ACI_Lower	6.9271	3.8161	3.8278	2.7834	1.3922	0.4812	0.4439	0.2592	0.1976
	ACI_Upper	7.9522	4.8798	4.9973	3.3308	2.3916	0.6387	0.6930	0.5865	0.6396
	ACI_Length	1.0252	1.0638	1.1695	0.5474	0.9994	0.1575	0.2491	0.3273	0.4420
	BCI_Lower	6.8415	3.7187	3.7011	2.8180	1.2808	0.4594	0.4326	0.3538	0.3787
	BCI_Upper	7.9292	4.7629	4.8715	3.3906	2.2832	0.6434	0.6642	0.6187	0.6553
	BCI_Length	1.0877	1.0442	1.1704	0.5726	1.0024	0.1840	0.2316	0.2649	0.2766
80	MLE	7.2852	3.9432	3.9673	3.0752	1.8959	0.5549	0.6216	0.4076	0.4178
	ER	0.5421	0.6416	0.6117	0.1370	0.3880	0.0175	0.0676	0.0252	0.0405
	ACI_Lower	6.8238	3.5884	3.5435	2.8338	1.5015	0.4621	0.4439	0.2861	0.2148
	ACI_Upper	7.8814	4.5192	4.5687	3.2804	2.3985	0.6186	0.6930	0.5450	0.6208
	ACI_Length	1.0575	0.9308	1.0251	0.4466	0.8970	0.1564	0.2491	0.2589	0.4060
	BCI_Lower	6.7750	3.4496	3.4667	2.8429	1.4090	0.4708	0.4728	0.3976	0.3587
	BCI_Upper	7.7954	4.4368	4.4679	3.3075	2.3828	0.6390	0.6970	0.6191	0.6379
	BCI_Length	1.0204	0.9872	1.0012	0.4646	0.9738	0.1682	0.2242	0.2215	0.2792

Table 3. MLEs, ERs and 95% CIs for parameters and $R_{s,k}$ when $(\lambda_1, \lambda_2, \lambda_3, \alpha, \theta) = (1, 2, 4, 3, 1)$.

n		λ_1	λ_2	λ_3	α	θ	$R_{1,3}$	$R_{2,4}$	$\widetilde{R}_{1,3}$	$\widetilde{R}_{2,4}$
20	MLE	0.8704	2.2951	3.8924	3.2294	0.8718	0.7680	0.8270	0.6777	0.7504
	ER	0.1532	0.3900	0.4775	0.1119	0.3649	0.0383	0.1020	0.1341	0.1428
	ACI_Lower	0.5787	1.7522	3.3125	2.7813	0.6052	0.5378	0.6738	0.4756	0.5301
	ACI_Upper	1.2411	2.7380	5.2672	3.8409	1.2156	0.9982	0.9802	0.8798	0.9707
	ACI_Length	0.6624	0.9858	1.9548	1.0596	0.6105	0.4604	0.3064	0.4042	0.4406
	BCI_Lower	0.5246	1.7830	2.8667	2.3394	0.5514	0.4499	0.6738	0.6234	0.6195
	BCI_Upper	1.2162	2.8172	4.9181	3.7183	1.1922	0.9103	0.9802	0.9384	0.9631
	BCI_Length	0.6916	1.0342	2.0514	1.3789	0.6408	0.4604	0.3064	0.3150	0.3436
50	MLE	0.8843	2.1736	4.1870	3.1676	0.8748	0.7441	0.8827	0.6460	0.7453
	ER	0.1457	0.2458	0.3909	0.0919	0.2339	0.0319	0.0831	0.1486	0.1562
	ACI_Lower	0.6777	1.6930	3.3846	2.8009	0.4330	0.5716	0.6752	0.4459	3.4105
	ACI_Upper	1.2654	2.4176	5.2489	3.7343	1.1634	0.9766	0.9999	0.8461	3.5467
	ACI_Length	0.5878	0.7247	1.8644	0.9334	0.7304	0.4050	0.3247	0.4002	0.1362
	BCI_Lower	0.5821	1.8504	3.2635	2.7118	0.4850	0.5382	0.7523	0.5707	0.6717
	BCI_Upper	1.1865	2.4968	5.1105	3.6234	1.2646	0.9432	0.9940	0.9896	0.9428
	BCI_Length	0.6044	0.6464	1.8470	0.9116	0.7796	0.4050	0.2417	0.4189	0.2711
80	MLE	0.9149	2.1299	3.9192	3.1293	0.8840	0.8373	0.9733	0.7821	0.8024
	ER	0.1407	0.1667	0.3068	0.0896	0.2191	0.0307	0.0771	0.0765	0.1074
	ACI_Lower	0.7294	1.8963	2.7490	2.7951	0.7467	0.6262	0.8421	0.5985	0.5298
	ACI_Upper	1.2383	2.4338	4.4348	3.6241	1.3511	1.0484	1.0445	0.9657	0.9693
	ACI_Length	0.5090	0.5375	1.6858	0.8290	0.6044	0.4222	0.2024	0.3672	0.4395
	BCI_Lower	0.7023	1.8467	3.0978	2.5967	0.5616	0.6059	0.8121	0.5657	0.7649
	BCI_Upper	1.1275	2.4131	4.7406	3.6619	1.1964	0.9988	0.9877	0.9377	0.9656
	BCI_Length	0.4252	0.5664	1.6428	1.0652	0.6348	0.3929	0.1756	0.3721	0.2007

Table 4. MLEs, ERs and 95% CIs for parameters and $R_{s,k}$ when $(\lambda_1, \lambda_2, \lambda_3, \alpha, \theta) = (1, 2, 4, 3, 2)$

n		λ_1	λ_2	λ_3	α	θ	$R_{1,3}$	$R_{2,4}$	$\widetilde{R}_{1,3}$	$\widetilde{R}_{2,4}$
20	MLE	1.2792	2.3766	4.3199	3.2593	2.3293	0.7946	0.8345	0.5658	0.6629
	ER	0.3905	0.4561	0.7625	0.1128	0.4498	0.1996	0.1702	0.1503	0.1842
	ACI_Lower	0.9152	1.6528	3.7554	2.9289	1.8456	0.5311	0.6518	0.3353	0.3786
	ACI_Upper	1.4212	2.9281	4.9722	3.6493	2.7519	0.9581	0.9627	0.7963	0.9472
	ACI_Length	0.5060	1.2753	1.2168	0.7204	0.9063	0.4270	0.3109	0.4610	0.5686
	BCI_Lower	0.9578	1.7776	3.6985	2.9272	1.8169	0.4311	0.6741	0.3685	0.3139
	BCI_Upper	1.5666	2.9756	4.9413	3.5915	2.8417	0.9581	0.9949	0.9753	0.9707
	BCI_Length	0.6088	1.1979	1.2428	0.6643	1.0248	0.5270	0.3208	0.6068	0.6568
50	MLE	1.2357	2.2266	4.1305	3.2452	2.2998	0.8174	0.8981	0.7563	0.6126
	ER	0.3602	0.3852	0.3386	0.0741	0.3213	0.1444	0.1370	0.2775	0.2738
	ACI_Lower	0.8732	1.6254	3.6278	2.7983	1.8392	0.5343	0.7523	0.5131	0.3505
	ACI_Upper	1.4121	2.8985	4.8997	3.3308	2.7819	1.1004	0.9824	0.9995	0.8747
	ACI_Length	0.5389	1.2731	1.2719	0.5325	0.9427	0.5661	0.2301	0.4864	0.5242
	BCI_Lower	0.9810	1.5997	3.5251	2.9720	1.8426	0.4522	0.7760	0.3809	0.3733
	BCI_Upper	1.4904	2.8535	4.7359	3.5184	2.7571	0.9882	0.9720	0.9763	0.9832
	BCI_Length	0.5094	1.2537	1.2108	0.5464	0.9144	0.5360	0.1960	0.5953	0.6099
80	MLE	1.1497	2.2147	3.9586	3.1028	2.1064	0.8668	0.9675	0.7012	0.7589
	ER	0.2870	0.2790	0.2685	0.0292	0.2721	0.0878	0.0948	0.1277	0.1622
	ACI_Lower	0.9224	1.6588	3.4544	2.9134	1.5015	0.7812	0.8143	0.5807	0.5572
	ACI_Upper	1.4814	2.7192	4.5687	3.3128	2.3985	0.9524	1.0799	0.8217	0.9606
	ACI_Length	0.5590	1.0604	1.1143	0.3994	0.8970	0.1712	0.2656	0.2410	0.4034
	BCI_Lower	0.8906	1.5826	3.3977	2.8876	1.0650	0.6299	0.8143	0.8676	0.5779
	BCI_Upper	1.4948	2.8468	4.5194	3.3180	2.5429	1.1037	1.0407	1.3989	0.9162
	BCI_Length	0.6043	1.2642	1.1217	0.4304	1.4779	0.4738	0.2264	0.5312	0.3383

From Tables 1–4, it is observed that the MLEs for unknown parameters and system reliability

$R_{1,3}$, $R_{2,4}$ are close to the true value in most cases and the ERs are considerably small for all cases. As the sample size increases, the ERs, ACI lengths, BCI lengths for unknown parameters, and system reliability $R_{1,3}$, $R_{2,4}$ are decrease as expected. The ACIs are wider than the BCIs in most cases, and all the interval estimates cover the true value of the corresponding parameter. The ERs, ACI lengths and BCI lengths of $R_{1,3}$ and $R_{2,4}$ considering the dependence of strength elements perform better than those ignoring dependence of the strength elements. From Table 5, we can observed that the MLEs of θ are close to the true value for $\theta = 2$, rationally close for $\theta = 1,4$ and move away from the true value for $\theta = 6,8$. The ERs for $\hat{\theta}$ are considerably small in Table 5. From Figure 1, it is observed that as the increase of the dependence parameter θ, the stress-strength reliability $\hat{R}_{s,k}$ is increasing.

Table 5. MLEs, ERs of θ under different parameter when n = 50.

$(\lambda_1, \lambda_2, \lambda_3, \alpha)$		$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 6$	$\theta = 8$
(2, 3, 4, 3)	MLE	1.1984	2.0886	3.8748	6.3712	8.2351
	ER	0.2037	0.1356	0.3884	0.3862	0.3765
(1, 2, 4, 3)	MLE	1.2413	2.1189	4.0413	5.6773	7.7763
	ER	0.2173	0.1461	0.1754	0.3780	0. 5906
(7, 4, 4, 3)	MLE	1.2081	1.9498	4.2337	6.1694	7.7575
	ER	0.2028	0.1527	0.3838	0.3957	0.5667

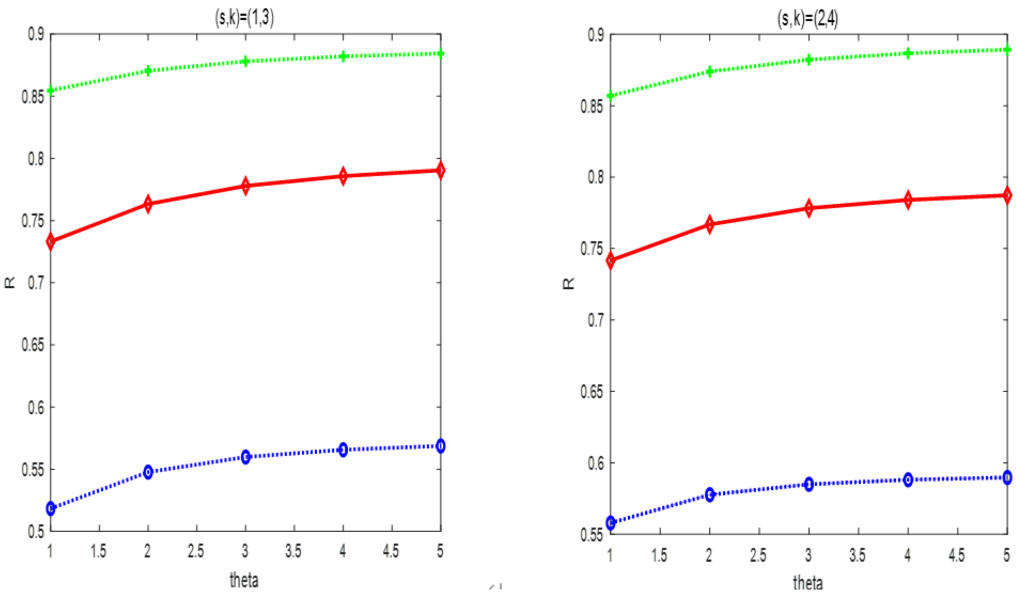

Figure 1. Variation in $R_{s,k}$ with respect to θ for different parameters of $(\lambda_1, \lambda_2, \lambda_3, \alpha) = \{(7,4,4,3), (2,3,4,3)\diamond, (1,2,4,3)+\}$.

6.2. Data analysis

In this section, a real data set is analyzed to investigate scenarios of excessive drought. It can be found in http://cdec.water.ca.gov/cgi-progs/queryMonthly? SHA, and the data has been studied by Wang et al. [12], Kohansal [13], Zhu [16], Kızılaslan and Nadar [18], and kohansal and Shoaee et al. [28]. If the water capacity of a reservoir on December of the previous year is over roughly half of the maximum capacity, and the minimum water level of August and September is more than the amount of water achieved on December at least two years out of the next 5 years, it is claimed that there will be no excessive drought afterward. Let $T_1, T_2, \cdots, T_6$ denote the capacity of December 1980, 1986, 1992, …, 2010, and $X_{1k}, Y_{1k}, k = 1, \cdots, 5$ be the capacities of August and September in $1980 \sim 1985$, respectively. Let X_{2k} and $Y_{2k}, k = 1, \cdots, 5$ be the capacities of August and September in $1987 \sim 1991$, respectively. The data are proceeded up to 2015. We convert each data between 0 and 1 by dividing the total capacity of Shasta reservoir 4,552,000 acre-foot and then the transformed data are obtained as:

$$X_1 = \begin{pmatrix} X_{111} & X_{112} & \cdots & X_{115} \\ X_{121} & X_{122} & \cdots & X_{125} \\ \vdots & & \cdots & \vdots \\ X_{161} & X_{162} & \cdots & X_{165} \end{pmatrix} = \begin{pmatrix} 0.5597 & 0.8112 & 0.8296 & 0.7262 & 0.4238 \\ 0.4637 & 0.3634 & 0.4637 & 0.3719 & 0.2912 \\ 0.7540 & 0.5381 & 0.7449 & 0.7226 & 0.5612 \\ 0.7552 & 0.6686 & 0.5249 & 0.6060 & 0.7159 \\ 0.7188 & 0.7420 & 0.4688 & 0.3451 & 0.4253 \\ 0.7951 & 0.6139 & 0.4616 & 0.2948 & 0.3929 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} X_{211} & X_{212} & \cdots & X_{215} \\ X_{221} & X_{222} & \cdots & X_{225} \\ \vdots & & \cdots & \vdots \\ X_{261} & X_{262} & \cdots & X_{265} \end{pmatrix} = \begin{pmatrix} 0.5449 & 0.7659 & 0.7946 & 0.7118 & 0.4345 \\ 0.4631 & 0.3484 & 0.4605 & 0.3597 & 0.2943 \\ 0.6814 & 0.4617 & 0.6890 & 0.6786 & 0.5071 \\ 0.7310 & 0.6558 & 0.4832 & 0.5620 & 0.6941 \\ 0.6667 & 0.7041 & 0.4128 & 0.3041 & 0.3897 \\ 0.7340 & 0.5693 & 0.4187 & 0.2542 & 0.3520 \end{pmatrix},$$

$$T' = \begin{pmatrix} 0.7009 & 0.6532 & 0.4589 & 0.7183 & 0.531 & 0.7665 \end{pmatrix}.$$

Let $Z_{ik} = \min(X_{1ik}, X_{2ik})$, $Z = \{Z_{ik}, i = 1, \cdots, 6, k = 1, \cdots, 5\}$, $T = \{T_1, T_2, \cdots, T_6\}$, then the observed data (Z, T) can be viewed as the observation from a 2-out-of-5 system.

Before progressing further, we first check whether Weibull distribution in Eq (8) could be used to analyze these real-life data. For X_1, the MLEs of parameters (λ_1, α), Kolmogorov-Smirnov (K-S) statistic and the corresponding p-value are (6.2289,4.0025), 0.1717 and 0.3037, respectively. For X_2, the MLEs of parameters (λ_2, α), the K-S statistic and the corresponding p-value are (7.5507, 3.9070), 0.1660 and 0.3417, respectively. For T, the MLEs of parameters (λ_3, α), the K-S statistic and the corresponding p-value are (17.8408,7.5439), 0.2047 and 0.9212, respectively. It is observed that Weibull distribution is considered as an appropriate model for X_1, X_2 and T. Moreover, for further illustration, the empirical cumulative distributions plot and overlay the theoretical Weibull distribution are shown in Figure 2, and the probability-probability (P-P) plots are shown in Figure 3, which also imply that the Weibull distribution could be considered as an appropriate model. To check the correlation, we compute the correlation coefficient of X_1 and X_2 using the Pearson's method, it is 0.9918 and the p-value is 0.0000, so the data (X_1, X_2) can be considered to be dependent.

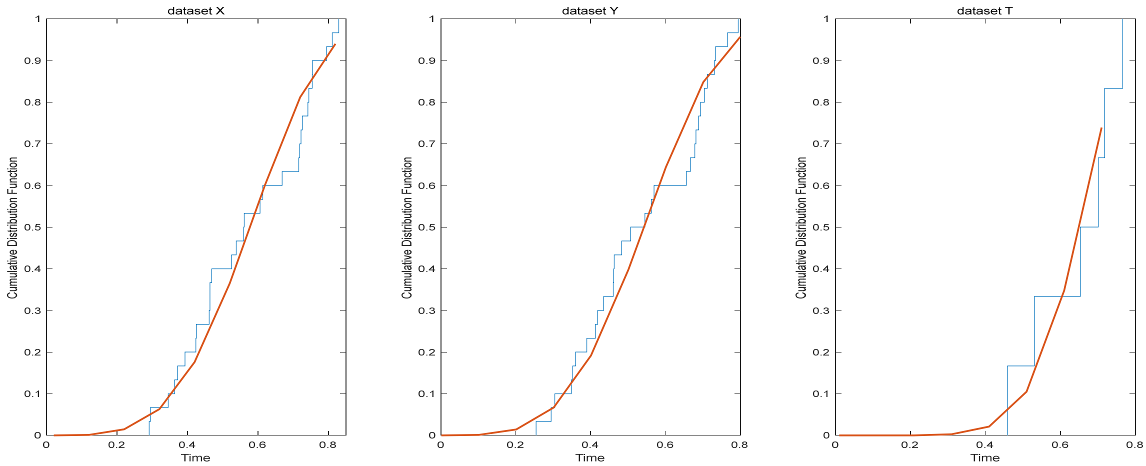

Figure 2. Empirical distribution under real data.

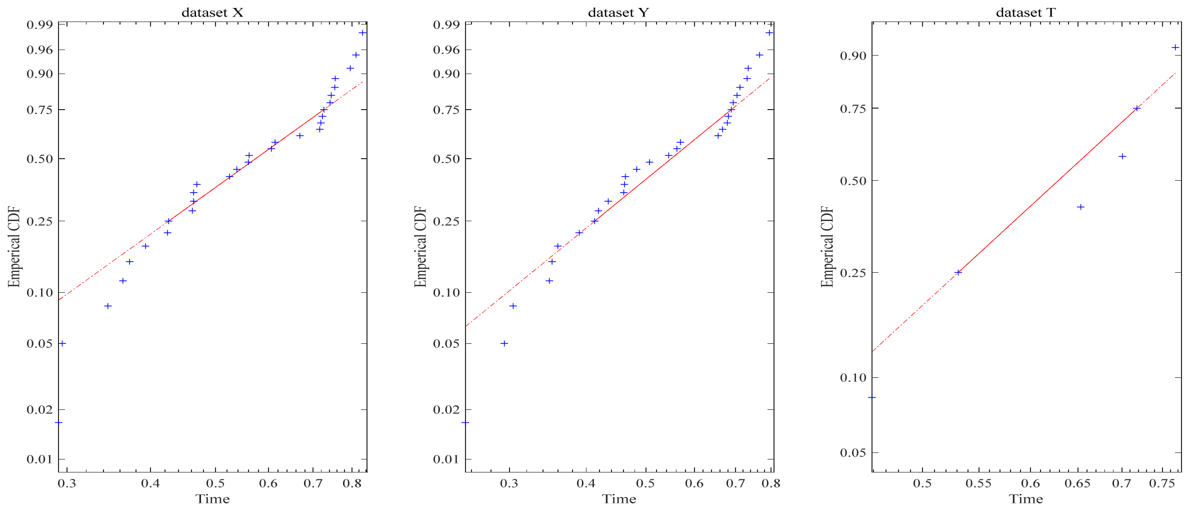

Figure 3. Fitted Weibull models P-P plots under real data.

Regard X_1 and X_2 as the dependent elements of strength variable and T as the stress variable. The probability P (at least s of the ($Z_1, Z_2, \cdots, Z_k$) exceed T) can be viewed as the measure of no excessive drought. Based on the proposed methods, the estimates and 95% confidence intervals of the model parameters and reliability are listed in Table 6.

Table 6. Estimates and 95% CIs for data (Z, T).

	λ_1	λ_2	λ_3	α	θ	$R_{2,5}$	$\widetilde{R}_{2,5}$
MLEs	4.0798	5.1597	3.5496	4.0182	4.4405	0.5227	0.5707
ACI_Lower	3.0553	4.0073	2.6484	3.2071	3.5417	0.4242	0.4799
ACI_Upper	5.1043	6.3121	4.4508	5.0345	5.3393	0.6212	0.6786
ACI_Length	2.0490	2.3048	1.8024	1.8274	1.7976	0.1970	0.1987
BCI_Lower	3.0634	3.9896	2.4534	3.3251	3.4979	0.4294	0.3329
BCI_Upper	5.0213	6.2142	4.2781	5.1210	5.3015	0.6215	0.6201
BCI_Length	1.9579	2.2246	1.8247	1.7959	1.8036	0.1921	0.2872

7. Conclusions

In this paper, we have studied the reliability analysis of multicomponent stress-strength model for the s-out-of-k system when the strength variable is constructed by a pair of s-dependent elements, which is described by a Clayton copula function. Based on the observed sample and the copula theory, the MLEs, ACIs as well as the BCIs for unknown parameters and $R_{s,k}$ are obtained using the asymptotic normality property, delta method and the sampling theory. The simulation study indicates that the ERs, ACI lengths and BCI lengths for the unknown parameters and $R_{s,k}$ are decreasing as the sample size increases. The BCIs are more attractive than the associated ACIs in terms of the average confidence interval lengths, and all the confidence intervals cover the true value of the corresponding parameter. The ERs, ACI lengths and BCI lengths of $R_{1,3}$ and $R_{2,4}$ for the case of considering the dependence perform better than those for the case of ignoring the dependence. The MLEs of θ are close to the true value for $\theta = 2$, rationally close for $\theta = 1,4$ and move away from the true value for $\theta = 6,8$. The variables in $R_{s,k}$ with respect to θ is moderate, and $R_{s,k}$ increases with respect to θ for different parameters.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11901134, 71901078 and 62162012), the Science and Technology Support Program of Guizhou (QKHZC2021 YB531), the Natural Science Research Project of Department of Education of Guizhou Province (QJJ2022015, QJJ2022047), the Scientific Research Platform Project of Guizhou Minzu University (GZMUSYS [2021]04), and the Natural Science Foundation of the Higher Education Institutions of Anhui Province (KJ2021A0386).

Conflict of interest

The authors declare no conflict of interest.

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Nomenclature

X_{1i}, X_{2i}	strength variable
$Z_i = \min(X_{1i}, X_{2i})$	minimum of the strength variables
T	stress variable
k	number of components
$R_{s,k}$	reliability of s -out-of- k system
PDF	probability density function
CDF	cumulative distribution function
$F(\cdot)$	CDF of strength variable
$G(\cdot)$	CDF of stress variable
$f_Z(\cdot)$	PDF of Z
$f(x_1, x_2)$	joint PDF of X_1 and X_2
$C(\cdot)$	survival copula
$\widetilde{C}(\cdot)$	failure distribution copula
$WE(\lambda, \alpha)$	Weibull distribution with shape parameter λ and scale parameter α
MLE	maximum likelihood estimate
$\widehat{R}_{s,k}$	MLE of $R_{s,k}$ when the dependence is considered
$\widetilde{R}_{s,k}$	MLE of $R_{s,k}$ when the dependence is ignored
ER	estimated risk
ACI	asymptotic confidence interval
BCI	bootstrap confidence interval