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Research article

Global existence and stability of three species predator-prey system with prey-taxis

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Abstract: In this paper, we study the following initial-boundary value problem of a three species predator-prey system with prey-taxis which describes the indirect prey interactions through a shared predator, i.e.,

$$\begin{cases} u_t = d\Delta u + u(1-u) - \dfrac{a_1 uw}{1+a_2u+a_3v}, & \text{in } \Omega, t>0,\\ v_t = \eta d\Delta v + rv(1-v) - \dfrac{a_4 vw}{1+a_2u+a_3v}, & \text{in } \Omega, t>0,\\ w_t = \nabla\cdot(\nabla w - \chi_1 w\nabla u - \chi_2 w\nabla v) - \mu w + \dfrac{a_5 uw}{1+a_2u+a_3v} + \dfrac{a_6 vw}{1+a_2u+a_3v}, & \text{in } \Omega, t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n (n \geqslant 1)$ with smooth boundary, where the parameters $d, \eta, r, \mu, \chi_1, \chi_2, a_i > 0, i = 1, \ldots, 6$. We first establish the global existence and uniform-in-time boundedness of solutions in any dimensional bounded domain under certain conditions. Moreover, we prove the global stability of the prey-only state and coexistence steady state by using Lyapunov functionals and LaSalle's invariance principle.

Keywords: Predator-prey system; prey-taxis; local existence; global existence; global stabilization

1. Introduction

Predator-prey models were developed to describe the dynamics of interactions between prey and predator species. The relationship between prey and predator has been explored in recent years due to its importance in ecology. In addition to the differential operators in the predator-prey system, predators also move toward the higher prey density, which is so-called the prey-taxis, and it plays an important role in pest control and biological control [1–4]. The first predator-prey model with prey-taxis was derived by Kareiva and Odell [5] to describe the predator aggregation phenomenon:

$$\begin{cases} u_t = \nabla \cdot (d(w)\nabla u) - \nabla \cdot (u\chi(w)\nabla w) + F(u,w), \\ w_t = d\Delta w + G(u,w), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $w = w(x,t)$ denote the predator and prey densities, respectively, and the term $\nabla \cdot (d(w)\nabla u)$ denotes the diffusion of predators with diffusion coefficient $d(w)$. The term $-\nabla \cdot (u\chi(w)\nabla w)$ represents the prey-taxis with $\chi(w)$ as prey-taxis coefficient. The parameter $d > 0$ is the diffusion coefficient of prey. The typical form of the functions $F(u,w) = auf(w) + h(u)$ and $G(u,w) = g(w) - buf(w)$, where $f(w)$ represents the functional response, for numerous functional response functions (see [6–8]) and the parameters $a, b \in \mathbb{R}$ describe the inter-specific interactions between predator-preys. The intra-specific interactions of predators and prey are described by the functions $h(u)$ and $g(w)$, respectively. The results related to variants of the above prey-taxis system have been studied by many authors, as one can refer to [9–18], and nonlinear prey-taxis [19–24]. Moreover, the predator-prey system with prey-taxis and liquid surroundings was considered in [25], and proved global existence and large time behavior of solutions by using L^p estimates and Lyapunov functionals, respectively.

In this paper, we consider a PDE model of indirect interactions between two prey species and a shared predator with homogeneous Neumann boundary conditions:

$$\begin{cases} u_t = d_1\Delta u + \alpha_1 u\Big(1 - \dfrac{u}{K_u}\Big) - \dfrac{c_1uw}{1 + c_1T_1u + c_2T_2v}, & \text{in } \Omega, t > 0, \\ v_t = d_2\Delta v + \alpha_2 v\Big(1 - \dfrac{v}{K_v}\Big) - \dfrac{c_2vw}{1 + c_1T_1u + c_2T_2v}, & \text{in } \Omega, t > 0, \\ w_t = \nabla \cdot (d_3\nabla w - \chi_1 w\nabla u - \chi_2 w\nabla v) - dw & \\ +\dfrac{c_1\gamma_1uw}{1 + c_1T_1u + c_2T_2v} + \dfrac{c_2\gamma_2vw}{1 + c_1T_1u + c_2T_2v}, & \text{in } \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \text{ and } \ w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Where, $\Omega \subset \mathbb{R}^n (n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial \nu}$ represents the derivative with respect to outer normal of $\partial\Omega$, u is the native prey, v denotes the invasive prey and w is the predator species. The parameters d_1, d_2 denote the random diffusion rates of prey, and d_3 and d_4 denote the random diffusion rate of predators and the chemical concentration, respectively. K_u and K_v are the carrying capacities for these prey species. The constants $\alpha_1$and α_2 are intrinsic growth rate parameters. The parameters T_1 and T_2 usually represent the handling time required for catching and consuming a unit of prey type u and v, respectively. The constants c_1 and c_2 are capture rates per unit prey density while the predator is searching. In particular, c_1 is the capture rate of prey u and c_2 is the capture rate of prey v. In addition, d is an intrinsic growth (death) rate for the predator and η is a self-limiting or crowding coefficient for the predator. κ is the production rate of chemical signal per individual prey u and ξ is the decay rate of the chemical signal. The positive constants γ_1 and γ_2 denote the transformation rates of the predator.

The terms $-\nabla \cdot (\chi_1 w\nabla u)$ and $-\nabla \cdot (\chi_2 w\nabla v)$ denote the tendency of predators moves towards the high density of prey. The parameters χ_1 and χ_2 are the prey-taxis coefficients. The functions $\frac{c_1u}{1+c_1T_1u+c_2T_2v}$ and $\frac{c_2v}{1+c_1T_1u+c_2T_2v}$ these represent Holling type II functional responses for two preys that are consumed in a single habitat, so that handling one prey reduces the time available to capture the other.

Let $\tilde{u} = \frac{u}{K_u}, \tilde{v} = \frac{v}{K_v}, \tilde{w} = dw, d = \frac{d_1}{d_3}, \eta = \frac{d_2}{d_1}, L = \sqrt{\frac{d_3}{\alpha_1}}, T = \frac{L^2}{d_3} = \frac{1}{\alpha_1}, \tilde{t} = \frac{t}{T}, \tilde{x} = \frac{x}{L}, \tilde{y} = \frac{y}{L}, r = \alpha_2 T a_1 = c_1 T d, a_2 = c_1 T_1 K_u, a_3 = c_2 T_2 K_v, a_4 = c_2 d T, a_5 = c_1 \gamma_1 K_u d, a_6 = c_2 \gamma_2 K_v d$. Then, substituting these parameters into system (1.2) and dropping the tilde notation, we get a nondimensional system as follows:

$$\begin{cases} u_t = d\Delta u + u(1-u) - \dfrac{a_1 u w}{1 + a_2 u + a_3 v}, & \text{in } \Omega, t > 0, \\ v_t = \eta d \Delta v + r v(1-v) - \dfrac{a_4 v w}{1 + a_2 u + a_3 v}, & \text{in } \Omega, t > 0, \\ w_t = \nabla \cdot (\nabla w - \chi_1 w \nabla u - \chi_2 w \nabla v) - \mu w + \dfrac{a_5 u w}{1 + a_2 u + a_3 v} + \dfrac{a_6 v w}{1 + a_2 u + a_3 v}, & \text{in } \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \text{ and } w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

Let us recall some existing works on three species predator-prey systems with prey-taxis. Very recently, Haskel and Bell [26] proved the existence of positive classical solutions for the two-prey one-predator system with prey competitions and prey taxis. In addition, they also established the pattern formation by using bifurcation analysis. Further, they also studied the bifurcation analysis of two competing prey with one shared predator model by using the theories of Crandall-Rabinowitz and Hopf bifurcation in [27]. The steady-state bifurcation analysis of the two-prey one-predator model with two prey taxis was studied by Xu et al. [28]. Jin et al. [29] considered the three-species food chain model in a two-dimensional bounded domain, and they also proved the global existence of classical solutions and global stability of constant steady states. The traveling wave solutions for a nonlocal dispersal predator-prey system with one predator and two prey was studied in [30]. Amorim et al. [31] studied the boundedness and global well-posedness of the spatio-temporal evolution of two competitive prey, and one predator model with the intra-specific competition. The global existence and boundedness of classical solutions for the two-predators and one-prey with competition in a bounded domain with Neumann boundary conditions were proved by Min et al. in [32]. Recently, the global existence of weak solutions to the two-prey one-predator system with prey-taxis, and competition between prey was proved in any dimension in [33]. For the similar mathematical structure of (1.3), we refer to [34–36]. Throughout this paper, we assume the system parameters are positive. To the best of author's knowledge, there is no article that discusses the well-posedness of the considered system (1.3). The main purpose of this article is to discuss the global dynamics of the system (1.3) in any dimension ($n \geqslant 1$). In particular, we first prove the global existence of a classical solution in all dimensions, and then we investigate the global stability of steady states. Our main result regarding the global existence of classical solutions with uniform-in-time bound is stated below.

Theorem 1.1. (**Global existence**) *Let $\Omega \subset \mathbb{R}^n, n \geqslant 1$ be a bounded domain with smooth boundary and let $d, \eta > 0, h_1, h_2 > 1, k \geqslant 2, a_i > 0, i = 1, \ldots, 6, \mu > 0, K_0 = \max\{1, \|u_0\|_{L^\infty}\}$ and $K_1 = \max\{1, \|v_0\|_{L^\infty}\}$. For any $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ with $p > n$ and $u_0, v_0, w_0 \geqslant 0 (\not\equiv 0)$, if $d > \max\{5kK_0, \frac{5kK_1}{\eta}\}$ and χ_1 satisfies*

$$\chi_1 \leqslant \min\left\{ \frac{d}{5kK_0(d+1)}, \frac{d}{5kK_0} - 1, \frac{d}{5kK_0(d+1)\sqrt{h_2}}, \frac{2(\eta d + 1)\sqrt{dK_1}}{K_0(d+1)\sqrt{5k\eta h_2}} \right\} \tag{1.4}$$

and χ_2 *satisfies*

$$\chi_2 \leqslant \min\left\{\frac{\eta d}{5kK_1\sqrt{h_1}(\eta d+1)}, \frac{2\eta(d+1)\sqrt{dK_0}}{\sqrt{5kh_1}(\eta d+1)K_1}, \frac{\eta d}{5kK_1(\eta d+1)}, \frac{\eta d}{5kK_1}-1\right\}, \tag{1.5}$$

then there exists a unique global classical solution $(u,v,w) \in [C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty))]^3$ *solving the problem* (1.3)*. Moreover, the solution satisfies* $u,v,w>0$ *for all* $t>0$ *and*

$$\|u(x\cdot,t)\|_{L^\infty} + \|v(x\cdot,t)\|_{L^\infty} + \|w(x\cdot,t)\|_{L^\infty} \leqslant C \text{ for all } t>0,$$

where $C>0$ *is a constant independent of* t.

Next, we shall study the large time behaviour of the constant steady states (u_s,v_s,w_s) of the system (1.3) solving the following system

$$\begin{cases} u_s\left[1-u_s-\dfrac{a_1w_s}{1+a_2u_s+a_3v_s}\right]=0, \\ v_s\left[r(1-v_s)-\dfrac{a_4w_s}{1+a_2u_s+a_3v_s}\right]=0, \\ w_s\left[\dfrac{a_5u_s}{1+a_2u_s+a_3v_s}+\dfrac{a_6v_s}{1+a_2u_s+a_3v_s}-\mu\right]=0. \end{cases}$$

If we solve the above system, we will find the following steady states

$$(u_s,v_s,w_s) = \begin{cases} (0,0,0) \text{ or } (1,0,0) \text{ or } (0,1,0) \text{ or } (1,1,0), & \text{if } \mu > \frac{a_5+a_6}{1+a_2+a_3}, a_4 < \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}, \\ (0,0,0) \text{ or } (1,0,0) \text{ or } (0,1,0) \text{ or } (1,1,0) \text{ or } E_*^1, & \text{if } \mu > \frac{a_5+a_6}{1+a_2+a_3}, a_4 < \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}, \\ & \mu < \frac{a_5}{1+a_2}, \\ (0,0,0) \text{ or } (1,0,0), (0,1,0), (1,1,0), E_*^2, & \text{if } \mu > \frac{a_5+a_6}{1+a_2+a_3}, a_4 < \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}, \\ & \mu < \frac{a_6}{1+a_3}, \\ (0,0,0) \text{ or } (1,0,0) \text{ or } (0,1,0) \text{ or } (1,1,0) \text{ or } E_*^1 \text{ or } E_*^2 \text{ or } E_*=(u_*,v_*,w_*) & \\ & \mu < \frac{a_5+a_6}{1+a_2+a_3}, a_4 > \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}, \end{cases} \tag{1.6}$$

where

$$\begin{aligned} E_*^1 &= \left(\frac{\mu}{a_5-a_2\mu}, 0, \frac{a_5(a_5-(1+a_2)\mu)}{a_1(a_5-a_2\mu)^2}\right), \\ E_*^2 &= \left(0, \frac{\mu}{a_6-a_3\mu}, \frac{a_6r(a_6-(1+a_3)\mu)}{a_4(a_6-a_3\mu)^2}\right) \\ u_* &= \frac{a_4(a_6-a_3\mu)+a_1r(-a_6+\mu+a_3\mu)}{a_1r(a_5-a_2\mu)+a_4(a_6-a_3\mu)} \\ v_* &= \frac{a_1r(a_5-a_2\mu)+a_4(-a_5+\mu+a_2\mu)}{a_1r(a_5-a_2\mu)+a_4(a_6-a_3\mu)} \end{aligned}$$

$$w_* = \frac{-r[(1+a_2)a_4a_6 + a_1(a_5 - a_2a_6)r + a_3(-a_4a_5 + a_1a_5r)](-a_5 - a_6 + (1+a_2+a_3)\mu)}{(a_1r(a_5 - a_2\mu) + a_4(a_6 - a_3\mu))^2}$$

and $(0,0,0)$ is the extinction steady state, $(1,0,0)$ is the prey u only steady state, $(0,1,0)$ is the prey v only steady state. E_*^1 and E_*^2 denote the semi-coexistence steady state. Finally, E_* denotes the coexistence steady state. Next, we shall explore the following question: which of the above seven homogeneous steady states will be asymptotically stable? As we know that the global stability of the cross-diffusion system is difficult and many techniques are not available, we try to use the Lyapunov functionals to prove the global stability of the homogeneous steady states under some conditions. Next, we state our stability results as in the following theorem:

Theorem 1.2. (**Global stability**) *Assume the conditions in Theorem 1.1 hold. Let* (u, v, w) *be the solution of* (1.3) *obtained in Theorem 1.1 and let* $K_0 = \max\{1, \|u_0\|_{L^\infty}\}, K_1 = \max\{1, \|v_0\|_{L^\infty}\}, \Gamma_1 = \frac{a_5+(a_3a_5-a_2a_6)v_*}{a_1(1+a_2u_*+a_3v_*)}$ *and* $\Gamma_2 = \frac{a_5+(a_3a_5-a_2a_6)u_*}{a_1(1+a_2u_*+a_3v_*)}$. *Then the following results hold true.*

- *If* $\mu > \frac{a_5+a_6}{1+a_2+a_3}, a_4 \leqslant \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}$, *then the steady state* $(1,1,0)$ *is globally asymptotically stable.*
- *If* $\mu < \frac{a_5+a_6}{1+a_2+a_3}, a_4 > \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}$, *the steady state* E_* *defined by* (1.6) *is globally asymptotically stable provided*

$$\frac{\Gamma_1(2a_2+a_3)}{2} + \frac{\Gamma_2a_2r}{2} < \Gamma_1 + \frac{\Gamma_1(2a_2+a_3)u_*}{2} + \frac{\Gamma_2a_2rv_*}{2}, \tag{1.7}$$

$$\frac{\Gamma_2r(2a_3+a_2)}{2} + \frac{\Gamma_1a_3}{2} < r\Gamma_2 + \frac{\Gamma_2r(2a_3+a_2)v_*}{2} + \frac{\Gamma_1a_3u_*}{2}, \tag{1.8}$$

$$4d\Gamma_1\Gamma_2\eta u_*v_* > \Gamma_1\chi_2^2u_*w_*\|v\|_{L^\infty}^2 + \chi_1^2\eta\Gamma_2\|u\|_{L^\infty}^2v_*w_*. \tag{1.9}$$

where $\|u\|_{L^\infty}$ *and* $\|v\|_{L^\infty}$ *depends on* d, η, a_1, a_2, a_3 *but independent of* χ_1, χ_2.

The paper is organized as follows: In section 2, we first present some preliminary results, and then we state and prove the local existence of solutions of (1.3). Section 3 deals with the existence of globally bounded classical solutions as stated in Theorem 1.1. In Section 4, we establish the global stability results as stated in Theorem 1.2 by using the Lyapunov functionals and LaSalle's invariance principle.

2. Preliminaries and local existence

In what follows, we shall abbreviate $\int_\Omega f\mathrm{d}x$ as $\int_\Omega f$ for simplicity. In this section, we first prove the local existence of classical solutions to the system (1.3) in any dimension $\Omega \subset \mathbb{R}^n, n \geqslant 1$ using the Amann's approach (cf. [37, 38]). We use the following Gagliardo-Nirenberg interpolation inequality in the sequel.

Lemma 2.1. *([39]) There exists a constant* $C_4 > 0$*, such that for all* $u \in W^{1,q}(\Omega)$,

$$\|u\|_{L^p} \leqslant C_4\|u\|_{W_{1,q}}^a\|u\|_{L^m}^{1-a}, \tag{2.1}$$

where $p, q \geqslant 1$ *which satisfies* $p(n-q) < nq, m \in (0,p)$ *with* $a = \frac{\frac{n}{m}-\frac{n}{p}}{\frac{n}{m}+1-\frac{n}{q}} \in (0,1)$.

Lemma 2.2. *([39]) There exists a constant* $C_5 > 0$*, such that for all* $u \in W_{1,q}(\Omega)$*, we have*

$$\|u\|_{W^{1,p}} \leqslant C_5(\|\nabla u\|_{L^p} + \|u\|_{L^q}), \tag{2.2}$$

where $p > 1$ *and* $q > 0$.

Lemma 2.3. ([40]) *Let $T > 0, \tau \in (0, T), \sigma \geqslant 0, a > 0, b \geqslant 0$, and suppose that $f : [0, T) \to [0, \infty)$ is absolutely continuous, and satisfies*

$$f'(t) + af^{1+\sigma}(t) \leqslant h(t),\ t \in \mathbb{R}, \tag{2.3}$$

where $h \geqslant 0, h(t) \in L^1_{loc}([0, T))$ and

$$\int_{t-\tau}^{t} h(s)ds \leqslant b, \text{ for all } t \in [\tau, T). \tag{2.4}$$

Then,

$$\sup_{t\in(0,T)} f(t) + a \sup_{t\in(\tau,T)} \int_{t-\tau}^{t} f^{1+\sigma}(s)ds \leqslant b + 2\max\{f(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau\}. \tag{2.5}$$

Lemma 2.4. ([39]) *Assume that $m \in \{0, 1\}, p \in [1, \infty]$ and $q \in (1, \infty)$. Then there exists some positive constant C_1, such that*

$$\|\phi\|_{W^{m,p}} \leqslant C_1\|(A + 1)^{\theta}\phi\|_{L^q}, \tag{2.6}$$

for any $\phi \in D((A + 1)^{\theta})$, where $\theta \in (0, 1)$ satisfies

$$m - \frac{n}{p} < 2\theta - \frac{n}{q}.$$

If in addition $q \geqslant p$, then there exist constants $C_2 > 0$ and $\gamma > 0$, such that for any $\phi \in L^p(\Omega)$,

$$\|(A + 1)^{\theta}e^{-t(A+1)}\phi\|_{L^q} \leqslant C_2t^{-\theta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\mu t}\|\phi\|_{L^p}, \tag{2.7}$$

where the semigroup $\{e^{-t(A+1)}\}_{t\geqslant 0}$ maps $L^p(\Omega)$ into $D((A + 1)^{\theta})$. Moreover, for any $p \in (0, \infty)$ and $\epsilon > 0$, there exist constants $C_3 > 0$ and $\gamma > 0$, such that

$$\|(A + 1)^{\theta}e^{-tA}\nabla \cdot \phi\|_{L^p} \leqslant C_3t^{-\theta-\frac{1}{2}-\epsilon}e^{-\gamma t}\|\phi\|_{L^p} \tag{2.8}$$

this is valid for all $\mathbb{R}^n-$ valued $\phi \in L^p(\Omega)$.

Theorem 2.1. (**Local existence**) *Let the assumptions in Theorem 1.1 hold. Then, there exists a $T_{max} \in (0, \infty]$, such that the problem (1.3) has a unique classical solution*

$$(u, v, w) \in \left(C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max}))\right)^3,$$

which satisfies $(u, v, w) > 0$ for all $t > 0$. Further,

$$if\ \ T_{max} < \infty,\ then\ \lim_{t\nearrow T_{max}} \sup(\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} + \|w(\cdot, t)\|_{L^\infty}) = \infty. \tag{2.9}$$

Proof. Denote $\psi = (u, v, w)$. Then, the problem (1.3) can be written as

$$\begin{cases} \psi_t = \nabla \cdot (A(\psi)\nabla\psi) + F(\psi),\ x \in \Omega, t > 0, \\ \partial_\nu\psi = 0,\ x \in \partial\Omega, t > 0, \\ \psi(\cdot, 0) = (u_0, v_0, w_0),\ \ x \in \Omega, \end{cases} \tag{2.10}$$

where

$$A(\psi) = \begin{bmatrix} d & 0 & 0 \\ 0 & \eta d & 0 \\ -\chi_1 w & -\chi_2 w & 1 \end{bmatrix} \text{ and } F(\psi) = \begin{bmatrix} \dfrac{-a_1 uw}{1 + a_2 u + a_3 v} \\ \dfrac{-a_4 vw}{1 + a_2 u + a_3 v} \\ \dfrac{a_5 uw + a_6 vw}{1 + a_2 u + a_3 v} \end{bmatrix}. \tag{2.11}$$

Since the eigenvalues of $A(\psi)$ are positive, the system (1.3) is normally parabolic(cf. [37, 38]). Then, the application of Theorem 7.3 and Corollary 9.3 in [37] yields a $T_{max} > 0$, such that system (1.3) admits a unique solution $(u, v, w) \in [C^0(\overline{\times}[0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max}))]^3$. Next, we show the nonnegativity of (u, v, w) by using the maximum principle. To do so, we need to rewrite the third equation of the system (1.3) as follows:

$$\begin{cases} w_t = \Delta w + p_1(x,t) \cdot \nabla w + p_2(x,t) w = 0, & x \in \Omega, t \in (0, T_{max}), \\ \partial_\nu w = 0, \ x \in \Omega, t \in (0, T_{max}), \\ w(x, 0) = w_0 \geqslant 0 \ \text{ in } x \in \Omega, \end{cases} \tag{2.12}$$

where $p_1(x,t) = \chi_1 \nabla u + \chi_2 \nabla v$ and $p_2(x,t) = \chi_1 \Delta u + \chi_2 \Delta v - \frac{a_5 u + a_6 v}{1 + a_2 u + a_3 v}$. Hence, we apply the maximum principle for parabolic equation with Neumann boundary condition to (2.12) and we get $w \geqslant 0$ for all $(x,t) \in \Omega \times (0, T_{max})$. In addition, we also obtain $w > 0$ by strong maximum principle since the initial function $w_0 \not\equiv 0$. In the same way, we can obtain that $u, v > 0$ for all $(x,t) \in \Omega \times (0, T_{max})$. Because $A(\psi)$ is lower triangular, (2.9) follows from Theorem 5.2 in [41]. □

Lemma 2.5. *The solution* (u, v, w) *of the system* (1.3) *satisfies*

$$0 < u(x,t) \leqslant K_0 := \max\{\|u_0\|_{L^\infty(\Omega)}, 1\}, \ \limsup_{t\to\infty} u(x,t) \leqslant 1, \tag{2.13}$$

$$0 < v(x,t) \leqslant K_1 := \max\{\|v_0\|_{L^\infty(\Omega)}, 1\}, \ \limsup_{t\to\infty} v(x,t) \leqslant 1, \tag{2.14}$$

$$\|w(x,t)\|_{L^1(\Omega)} \leqslant K_2 := \frac{\delta}{a_1 a_4}, \tag{2.15}$$

where $K_0, K_1, K_2, \delta = \max\{a_4 a_5 \|u_0\|_{L^1} + a_1 a_6 \|v_0\|_{L^1} + a_1 a_4 \|w_0\|_{L^1}, \frac{c_2}{c_1}\}$ *are positive constants independent of* t.

Proof. The proof is similar to Lemma 2.2 in [21] but for reader's convenience, we provide the proof here. We have already proved that the solution (u, v, w) of the system (1.3) is non-negative. Using this fact, we have

$$\begin{cases} u_t - d\Delta u = u(1-u) - \dfrac{a_1 uw}{1 + a_2 u + a_3 v} \leqslant u(1-u), & x \in \Omega, t > 0, \\ \partial_\nu u = 0, \ x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), x \in \Omega. \end{cases} \tag{2.16}$$

Let $u^*(t)$ be a solution to the following ODE problem

$$\begin{cases} \frac{du^*}{dt} = u^*(1-u^*), \ t>0, \\ u^*(0) = \|u_0\|_{L^\infty}. \end{cases} \tag{2.17}$$

The solution of the above ODE satisfies $u^*(t) \leqslant K_0 = \max\{\|u_0\|_{L^\infty}, 1\}$, and in addition, $u^*(t)$ is a super solution of the following PDE problem

$$\begin{cases} U_t - d\Delta U = U(1-U) \ \ x \in \Omega, t>0, \\ \partial_\nu U = 0, \ x \in \partial\Omega, t>0, \\ U(x,0) = u_0(x), x \in \Omega. \end{cases} \tag{2.18}$$

Hence, we have $U(x,t) \leqslant u^*(t)$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. Using the strong maximum principle to the problem (2.18), we obtain $0 < U(x,t) \leqslant u^*(t)$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. From (2.16)–(2.18), and using the comparison principle, we conclude that

$$0 < u(x,t) \leqslant U(x,t) \leqslant u^*(t) \leqslant K_0, \text{for all } (x,t) \in \overline{\Omega} \times (0,\infty), \tag{2.19}$$

which yields (2.13). Similarly, we can also prove (2.14).
Multiplying the first, second and third equations of (1.3) by a_4a_5, a_1a_6 and a_1a_4, respectively, and adding the resulting equations and then integrating it over Ω, we get

$$\begin{aligned} \frac{d}{dt}\Big(\int_\Omega a_4a_5u + a_1a_6v + a_1a_4w\Big) &= \int_\Omega a_4a_5u(1-u) + a_1a_6rv(1-v) - a_1a_4\mu w \\ &= \int_\Omega a_4a_5u - \int_\Omega a_4a_5u^2 + a_1a_6rv - a_1a_6rv^2 - a_1a_4\mu w. \end{aligned} \tag{2.20}$$

Using Cauchy-Schwartz inequality, one has

$$\Big(\int_\Omega u\Big)^2 \leqslant \Big(\int_\Omega u^2\Big)|\Omega|$$

which implies

$$-\int_\Omega u^2 \leqslant -\frac{1}{|\Omega|}\Big(\int_\Omega u\Big)^2. \tag{2.21}$$

Using Young's inequality ($-a^2 - b^2 \leqslant -2ab, a, b > 0$) yields

$$-\int_\Omega u^2 \leqslant -2\int_\Omega u + |\Omega|. \tag{2.22}$$

Substituting (2.22) into (2.20), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega a_4a_5u + a_1a_6v + a_1a_4w\Big) \leqslant \int_\Omega a_4a_5u - 2\int_\Omega a_4a_5u + a_4a_5|\Omega| + \int_\Omega a_1a_6rv$$
$$- 2a_1a_6r\int_\Omega v + a_1a_6|\Omega| - a_1a_4\mu\int_\Omega w$$
$$\leqslant -\int_\Omega a_4a_5u - \int_\Omega a_1a_6rv - \int_\Omega a_1a_4\mu w + a_4a_5|\Omega| + a_1a_6|\Omega|. \quad (2.23)$$

Set $y(t) = \int_\Omega a_4a_5u + a_1a_6v + a_1a_4w$ and choose $c_1 = \min\{1, r, \mu\}$ then (2.23) can be written as

$$y'(t) + c_1y(t) \leqslant c_2, \quad (2.24)$$

where $c_2 = (a_4a_5 + a_1a_6)|\Omega|$ and which yields (2.15) with the help of Gronwall's inequality. We further have from (2.17) that $\lim_{t\to\infty}\sup u(x,t) \leqslant 1$. The proof of Lemma 2.5 is complete. □

3. Global existence and boundedness

In this subsection, we prove the global existence and boundedness of solutions. In order to prove the global existence, we first derive a uniform bound for w in L^{n+1} by using a weight function argument and the proof is inspired from [23], which also concerns the predator-prey taxis with a single prey population. It is worth mentioning that the method was initially developed in [42].

Lemma 3.1. *Assume that χ_1 and χ_2 satisfy* (1.4) *and* (1.5)*, respectively, and let* (u, v, w) *be the solution of* (1.3)*. Then, there exists a positive constant $c_0 > 0$, such that*

$$\|w(\cdot,t)\|_{L^{n+1}(\Omega)} \leqslant c_0 \ \textit{for} \ \ t \in (0, T_{max}). \quad (3.1)$$

Proof. Let us define the constants and weight functions

$$k := n+1, \beta_1 := \sqrt{\frac{(k-1)d}{10k}}\frac{1}{(d+1)K_0}, \beta_2 := \sqrt{\frac{(k-1)\eta d}{10k}}\frac{1}{(\eta d+1)K_1}, \quad (3.2)$$
$$1 \leqslant \varphi_1(u) \leqslant e^{(\beta_1K_0)^2} := h_1 > 1,\ 0 \leqslant u \leqslant K_0 \text{ and } 1 \leqslant \varphi_2(v) \leqslant e^{(\beta_2K_1)^2} := h_2 > 1,\ 0 \leqslant v \leqslant K_1. \quad (3.3)$$

Now, by using the weight function and the first and second equation of (1.3), we obtain

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_1(u) = \int_\Omega w^{k-1}\varphi_1(u)w_t + \frac{1}{k}\int_\Omega w^k\varphi_1'(u)u_t$$
$$= \int_\Omega w^{k-1}\varphi_1(u)\Delta w - \chi_1\int_\Omega w^{k-1}\varphi_1(u)\nabla\cdot(w\nabla u) - \chi_2\int_\Omega w^{k-1}\varphi_1(u)\nabla\cdot(w\nabla v)$$
$$+ \int_\Omega w^k\varphi_1(u)\frac{a_5u + a_6v}{1 + a_2u + a_3v} - \mu\int_\Omega w^k\varphi_1(u) + \frac{1}{k}\int_\Omega w^k\varphi_1'(u)\Delta u$$
$$+ \frac{1}{k}\int_\Omega w^k\varphi_1'(u)u(1-u) - \frac{1}{k}\int_\Omega w^{k+1}\varphi_1'(u)\frac{a_1u}{1 + a_2u + a_3v} \quad (3.4)$$
$$\leqslant -(k-1)\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2 + \chi_1(k-1)\int_\Omega w^{k-1}\nabla w\cdot\nabla u\varphi_1(u)$$

$$+\chi_1\int_\Omega w^k\varphi_1'(u)|\nabla u|^2+\chi_2(k-1)\int_\Omega w^{k-1}\varphi_1(u)\nabla w\cdot\nabla v+\chi_2\int_\Omega w^k\varphi_1'(u)\nabla u\cdot\nabla v$$
$$+C\int_\Omega w^k\varphi_1(u)-d\int_\Omega w^{k-1}\varphi_1'(u)\nabla w\cdot\nabla u-\frac{d}{k}\int_\Omega w^k\varphi_1''(u)|\nabla u|^2$$
$$+\frac{1}{k}\int_\Omega w^k u\varphi_1'(u)+\frac{2\beta_1^2}{k}\int_\Omega w^k\varphi_1(u)u^2, \tag{3.5}$$

where we used the boundedness of the functional response, $C > 0$. The above inequality can be written as

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_1(u)+(k-1)\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\frac{d}{k}+\int_\Omega w^k\varphi_1''(u)|\nabla u|^2$$
$$\leqslant-(d+1)\int_\Omega w^{k-1}\varphi_1'(u)\nabla u\cdot\nabla w+\chi_1(k-1)\int_\Omega w^{k-1}\nabla w\cdot\nabla u\varphi_1(u)+\chi_1\int_\Omega w^k\varphi_1'(u)|\nabla u|^2$$
$$+\chi_2(k-1)\int_\Omega w^{k-1}\varphi_1(u)\nabla w\cdot\nabla v+\chi_2\int_\Omega w^k\varphi_1'(u)\nabla u\cdot\nabla v+C\int_\Omega w^k\varphi_1(u)$$
$$+\frac{2\beta_1^2}{k}\int_\Omega w^k\varphi_1(u)u^2. \tag{3.6}$$

By using Young's inequality, we get

$$-(d+1)\int_\Omega w^{k-1}\varphi_1'(u)\nabla u\cdot\nabla w\leqslant\epsilon\frac{(d+1)}{2}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\frac{(d+1)}{2\epsilon}\int_\Omega w^k\frac{\varphi_1'^2(u)}{\varphi_1(u)}|\nabla u|^2$$
$$\leqslant\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\frac{(d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_1'^2(u)}{\varphi_1(u)}|\nabla u|^2, \tag{3.7}$$

and

$$\chi_1(k-1)\int_\Omega w^{k-1}\varphi_1(u)\nabla w\cdot\nabla u\leqslant\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla u|^2, \tag{3.8}$$

$$\chi_2(k-1)\int_\Omega w^{k-1}\varphi_1(u)\nabla w\cdot\nabla v\leqslant\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla v|^2. \tag{3.9}$$

Again,

$$\chi_2\int_\Omega w^k\varphi_1'(u)\nabla u\cdot\nabla v\leqslant\frac{\epsilon}{2}\chi_2\int_\Omega w^k\varphi_1'(u)|\nabla u|^2+\frac{\chi_2}{2\epsilon}\int_\Omega w^k\varphi_1'(u)|\nabla v|^2$$
$$\leqslant\int_\Omega w^k\varphi_1'(u)|\nabla u|^2+\frac{\chi_2^2}{4}\int_\Omega w^k\varphi_1'(u)|\nabla v|^2. \tag{3.10}$$

Substituting (3.7)–(3.10) into (3.6), one has

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_1(u)+\frac{(k-1)}{4}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\frac{d}{k}+\int_\Omega w^k\varphi_1''(u)|\nabla u|^2$$
$$\leqslant\frac{(d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_1'^2(u)}{\varphi_1(u)}|\nabla u|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla u|^2+\chi_1\int_\Omega w^k\varphi_1'(u)|\nabla u|^2$$

$$+\chi_1^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla v|^2+\chi_1\int_\Omega w^k\varphi_1'(u)|\nabla u|^2+\frac{\chi_2^2}{4\chi_1}\int_\Omega w^k\varphi_1'(u)|\nabla v|^2$$
$$+C\int_\Omega w^k\varphi_1(u)+\frac{2\beta_1^2}{k}\int_\Omega w^k\varphi_1(u)u^2. \tag{3.11}$$

Now, we multiply the third equation of (1.3) by $\varphi_2(v)$, we have

$$\begin{aligned}\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_2(v)=&\int_\Omega w^{k-1}\varphi_2(v)w_t+\frac{1}{k}\int_\Omega w^k\varphi_2'(v)v_t\\
\leqslant&\int_\Omega w^{k-1}\varphi_2(v)\Delta w-\chi_1\int_\Omega w^{k-1}\varphi_2(v)\nabla\cdot(w\nabla u)-\chi_2\int_\Omega w^{k-1}\varphi_2(v)\nabla\cdot(w\nabla v)\\
&+C\int_\Omega w^k\varphi_2(v)+\frac{\eta d}{k}\int_\Omega w^k\varphi_2'(v)\Delta v+\frac{r}{k}\int_\Omega w^k\varphi_2'(v)v\\
\leqslant&-(k-1)\int_\Omega w^{k-2}\varphi_2(u)|\nabla w|^2-(\eta d+1)\int_\Omega w^{k-1}\varphi_2'(v)\nabla w\cdot\nabla v\\
&+\chi_1(k-1)\int_\Omega w^{k-1}\varphi_2(v)\nabla w\cdot\nabla u+\chi_1\int_\Omega w^k\varphi_2'(v)\nabla v\cdot\nabla u\\
&+\chi_2(k-1)\int_\Omega w^{k-1}\varphi_2(v)\nabla w\cdot\nabla v+\chi_2\int_\Omega w^k\varphi_2'(v)|\nabla v|^2+B_3\int_\Omega w^k\varphi_2(v)\\
&+\frac{2r\beta_2^2}{k}\int_\Omega w^k\varphi_2(v)v^2.\end{aligned} \tag{3.12}$$

By using Young's inequality, we arrive at

$$-(\eta d+1)\int_\Omega w^{k-1}\varphi_2'(v)\nabla w\cdot\nabla v\leqslant\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_2(v)|\nabla w|^2+\frac{(\eta d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_2'^2(v)}{\varphi_2(v)}|\nabla v|^2, \tag{3.13}$$

and

$$\chi_1(k-1)\int_\Omega w^{k-1}\varphi_2(v)\nabla w\cdot\nabla u\leqslant\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_2(v)|\nabla w|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla u|^2 \tag{3.14}$$

$$\chi_2(k-1)\int_\Omega w^{k-1}\varphi_2(v)\nabla w\cdot\nabla v\leqslant\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_2(v)|\nabla w|^2+\chi_2^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla v|^2 \tag{3.15}$$

$$\begin{aligned}\chi_1\int_\Omega w^k\varphi_2'(v)\nabla u\cdot\nabla v\leqslant&\frac{\epsilon\chi_1}{2}\int_\Omega w^k\varphi_2'(v)|\nabla u|^2+\frac{\chi_1}{2\epsilon}w^k\varphi_2'(v)|\nabla v|^2\\
\leqslant&\int_\Omega w^k\varphi_2'(v)|\nabla v|^2+\frac{\chi_1^2}{4}\int_\Omega w^k\varphi_2'(v)|\nabla u|^2.\end{aligned} \tag{3.16}$$

Substituting (3.13)–(3.16) into (3.12), we derive that

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_2(v)+\frac{(k-1)}{4}\int_\Omega w^{k-2}\varphi_2(u)|\nabla w|^2+\frac{\eta d}{k}\int_\Omega w^k\varphi_2''(v)|\nabla v|^2$$
$$\leqslant\frac{(\eta d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_2'^2(v)}{\varphi_2(v)}|\nabla v|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla u|^2+\chi_2\int_\Omega w^k\varphi_2'(v)|\nabla v|^2$$
$$+\frac{\chi_1^2}{4\chi_2}\int_\Omega w^k\varphi_2'(v)|\nabla u|^2+\chi_2^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla v|^2+\chi_2\int_\Omega w^k\varphi_2'(v)|\nabla v|^2+C\int_\Omega w^k\varphi_2(v)$$
$$+\frac{2r\beta_2^2}{k}\int_\Omega w^k\varphi_2(v)v^2. \tag{3.17}$$

Now, adding (3.11) and (3.17), the resulting inequality becomes

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_1(u)+\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_2(v)+\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_2(v)|\nabla w|^2$$
$$+\frac{d}{k}\int_\Omega w^k\varphi_1''(u)|\nabla u|^2+\frac{d}{k}\int_\Omega w^k\varphi_2''(v)|\nabla v|^2$$
$$\leqslant\frac{(d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_1'^2(u)}{\varphi_1(u)}|\nabla u|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla u|^2+(1+\chi_1)\int_\Omega w^k\varphi_1'(u)|\nabla u|^2$$
$$+\chi_2^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla v|^2+\frac{\chi_2^2}{4}\int_\Omega w^k\varphi_1'(u)|\nabla v|^2+B_3\int_\Omega w^k\varphi_1(u)+\frac{2\beta_1^2}{k}\int_\Omega w^k\varphi_1(u)u^2$$
$$+\frac{(\eta d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_2'^2(v)}{\varphi_2(v)}|\nabla v|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla u|^2+\chi_2^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla v|^2$$
$$+(1+\chi_2)\int_\Omega w^k\varphi_2'(v)|\nabla v|^2+\frac{\chi_1^2}{4}\int_\Omega w^k\varphi_2'(v)|\nabla u|^2+C\int_\Omega w^k\varphi_2(v)+\frac{2r\beta_2^2}{k}\int_\Omega w^k\varphi_2(v)v^2. \tag{3.18}$$

Now, we do some computation to show that the terms involving $|\nabla u|^2$ and $|\nabla v|^2$ on the right-hand side of above inequality are dominated by $\int_\Omega w^k\varphi_1''|\nabla u|^2$ and $\int_\Omega w^k\varphi_2''|\nabla v|^2$, respectively. For $s\geqslant 0$, define

$$j_1(s)=\frac{(d+1)^2}{(k-1)}\frac{\varphi_1'^2(u)}{\varphi_1(u)}=\frac{4\beta_1^4s^2\varphi_1^2(s)}{k-1},\ j_2(s)=\chi_1^2(k-1)\varphi_1(s),j_3(s)=2(1+\chi_1)\beta_1^2s\varphi_1(s) \tag{3.19}$$
$$j_4(s)=\chi_1^2(k-1)\varphi_2(s),\ j_5(s)=\frac{\chi_1^2\beta_2^2s\varphi_2(s)}{2},j_6(s)=2\frac{d}{k}\beta_1^2\varphi_1(s)+4\frac{d}{k}\beta_1^4s^2\varphi_1(s) \tag{3.20}$$

and

$$i_1(s)=\frac{4(\eta d+1)^2\beta_2^4s^2\varphi_2(s)}{(k-1)},\ i_2(s)=\chi_2^2(k-1)\varphi_1(s),\ i_3(s)=\frac{\chi_2^2\beta_1^2s\varphi_1(s)}{2} \tag{3.21}$$
$$i_4(s)=\chi_2^2(k-1)\varphi_2(s),\ i_5(s)=2(1+\chi_2)\beta_2^2s\varphi_2(s). \tag{3.22}$$

Now, combining (3.19) and (3.20), one has

$$\frac{(d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_1'^2(u)}{\varphi_1(u)}|\nabla u|^2+\chi_1^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla u|^2+(1+\chi_1)\int_\Omega w^k\varphi_1'(u)|\nabla u|^2$$

$$+\chi_1^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla u|^2+\frac{\chi_1^2}{4}\int_\Omega w^k\varphi_2'(v)|\nabla u|^2$$
$$\leqslant \frac{d}{k}\int_\Omega w^k\varphi_1''(u)|\nabla u|^2. \tag{3.23}$$

Similarly, we combine (3.21) and (3.22), we have that

$$\chi_2^2(k-1)\int_\Omega w^k\varphi_1(u)|\nabla v|^2+\frac{\chi_2^2}{4}\int_\Omega w^k\varphi_1'(u)|\nabla v|^2+\frac{(\eta d+1)^2}{k-1}\int_\Omega w^k\frac{\varphi_2'^2(v)}{\varphi_2(v)}|\nabla v|^2$$
$$+\chi_2^2(k-1)\int_\Omega w^k\varphi_2(v)|\nabla v|^2+(1+\chi_2)\int_\Omega w^k\varphi_2'(v)|\nabla v|^2$$
$$\leqslant \frac{d}{k}\int_\Omega w^k\varphi_2''(v)|\nabla v|^2. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.18), one has

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_1(u)+\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_2(v)+\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2+\frac{k-1}{4}\int_\Omega w^{k-2}\varphi_1(v)|\nabla w|^2$$
$$\leqslant B_3\int_\Omega w^k\varphi_1(u)+\frac{2\beta_1^2}{k}\int_\Omega w^k\varphi_1(u)u^2+B_3\int_\Omega w^k\varphi_2(v)+\frac{2r\beta_2^2}{k}\int_\Omega w^k\varphi_2(v)v^2$$
$$\leqslant\left(C+\frac{2\beta_1^2K_0^2}{k}\right)\int_\Omega w^k\varphi_1(u)+\left(B_3+\frac{2r\beta_2^2K_1^2}{k}\right)\int_\Omega w^k\varphi_2(v)$$
$$\leqslant c_1\int_\Omega w^k\varphi_1(u)+c_2\int_\Omega w^k\varphi_2(v) \tag{3.25}$$

where $c_1=C+\frac{2\beta_1^2K_0^2}{k}$ and $c_2=C+\frac{2r\beta_2^2K_1^2}{k}$. Using Lemma 2.1, Lemma 2.2, and (3.3), we get the estimate

$$\begin{aligned}\int_\Omega w^k\varphi_1(u)\leqslant& h_1\int_\Omega w^k=h_1\|w^{\frac{k}{2}}\|_{L^2}^2\\ \leqslant& h_1C_4\|w^{\frac{k}{2}}\|_{W^{1,2}}^{2a}\|w^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{2(1-a)}\\ \leqslant& hC_4\left(C_5\|\nabla w^{\frac{k}{2}}\|_{L^2}+\|w^{\frac{k}{2}}\|_{L^2}\right)^{2a}\|w^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{2(1-a)}\\ \leqslant& h_1C_4C_5\left(\|\nabla w^{\frac{k}{2}}\|_{L^2}+\|w^{\frac{k}{2}}\|_{L^1}^{\frac{k}{2}}\right)^{2a}\|w\|_{L^1}^{k(1-a)}\\ \leqslant& h_1C_4C_5\left(\|\nabla u^{\frac{k}{2}}\|_{L^2}+K_2^{\frac{k}{2}}\right)^{2a}K_2^{k(1-a)}\\ \leqslant& C_6\left(\|\nabla u^{\frac{k}{2}}\|_{L^2}^2+1\right)^a,\end{aligned} \tag{3.26}$$

where $a=\dfrac{\frac{kn}{2}-\frac{n}{2}}{\frac{kn}{2}+1-\frac{n}{2}}\in(0,1)$. Now using the fact (3.3) and from (3.26), one can obtain

$$\begin{aligned}\int_\Omega w^{k-2}\varphi_1(u)|\nabla w|^2 \geqslant& \int_\Omega w^{k-2}|\nabla w|^2\\ \geqslant&\frac{4}{k^2}\int_\Omega |\nabla w^{\frac{k}{2}}|^2\\ \geqslant&\frac{4}{k^2}\Big[\frac{1}{C_6^{1/a}}\Big(\int_\Omega w^k\varphi_1(u)\Big)^{\frac{1}{a}}-1\Big]\\ \geqslant&\frac{4}{k^2C_6^{1/a}}\Big(\int_\Omega w^k\varphi_1(u)\Big)^{\frac{1}{a}}-\frac{4}{k^2}.\end{aligned}\tag{3.27}$$

Similarly, we get

$$\int_\Omega w^{k-2}\varphi_2(v)|\nabla w|^2 \geqslant \int_\Omega w^{k-2}|\nabla w|^2 \geqslant \frac{4}{k^2C_7^{1/a}}\Big(\int_\Omega w^k\varphi_2(v)\Big)^{\frac{1}{a}}-\frac{4}{k^2}.\tag{3.28}$$

From (3.27) and (3.28), we have

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_1(u)+\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^k\varphi_2(v)\leqslant -\frac{(k-1)}{k^2C_6^{\frac{1}{a}}}\Big(\int_\Omega w^k\varphi_1(u)\Big)^{\frac{1}{a}}-\frac{(k-1)}{k^2C_7^{\frac{1}{a}}}\Big(\int_\Omega w^k\varphi_2(v)\Big)^{\frac{1}{a}}+\frac{2(k-1)}{k^2}\tag{3.29}$$

for all $t\in(0,T_{max})$ and where $\frac{1}{a}>1$. Set $y(t):=\frac{1}{k}\int_\Omega w^k(\varphi_1(u)+\varphi_2(v))$. By using the inequality $x^p+y^p\geqslant n^{1-p}(x+y)^p, p\geqslant 1$, we get

$$y'(t)\leqslant -C_8y^{\frac{1}{a}}(t)+\frac{2(k-1)}{k^2}\quad\text{for all } t\in(0,T_{max}),\tag{3.30}$$

where $C_8:=\min\Big\{\frac{(k-1)}{k^2C_6^{\frac{1}{a}}},\frac{(k-1)}{k^2C_7^{\frac{1}{a}}}\Big\}n^{1-\frac{1}{a}}$ and $\frac{1}{a}>1$. By using Lemma 2.3 and the fact that (3.3), one has

$$\|w(\cdot,t)\|_{L^k}\leqslant\Big(\int_\Omega w^k(\varphi_1(u)+\varphi_2(v))\Big)^{\frac{1}{k}}\leqslant C\tag{3.31}$$

for all $t\in(0,T_{max})$, where $C(u_0,v_0,C_8,\tau)>0$. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Let* (u,v,w) *be a solution of the system* (1.3)*. Then, there exists a positive constant* $c>0$*, such that*

$$\|w(\cdot,t)\|_{L^\infty}\leqslant c \textit{ for all } t\in(0,T_{max}).\tag{3.32}$$

Proof. To obtain the $L^\infty-$ bound of w, we use the semigroup estimates. In order to do this, we first obtain that for any $\tau\in(0,T_{max})$, there exists a constant $C>0$, such that

$$\|(u(\cdot,t),v(\cdot,t))\|_{W^{1,\infty}}\leqslant C(\tau)\text{ for all } t\in(\tau,T_{max})\tag{3.33}$$

Let $\tau \in (0, T_{max})$ be given such that $\tau < 1$ and also let $q := n + 1$ and $\theta \in \left(\frac{1}{2}(1 + \frac{n}{q}), 1\right)$. To begin with, we rewrite the first equation of (1.3) as follows:

$$u_t = d\Delta u - u + g(x, t), \tag{3.34}$$

with $g(x, t) := u(1-u) - \dfrac{a_1 uw}{1 + a_2 u + a_3 v} + u$. Then, by Lemma 3.1 and the fact that $0 < u \leqslant K_0, 0 < v \leqslant K_1$ (see Lemma 2.5) and $\dfrac{a_1 u}{1 + a_2 u + a_3 v} \leqslant \tilde{K}$, $\tilde{K} > 0$, we have

$$\begin{aligned}
\|g(x, t)\|_{L^q} =& \|u(1 - u) - \frac{a_1 uw}{1 + a_2 u + a_3 v} + u\|_{L^q} \\
&\leqslant K_0(1 + K_0)|\Omega|^{\frac{1}{q}} + a_1 \tilde{K}\|w(\cdot, t)\|_{L^q} + K_0|\Omega|^{\frac{1}{q}} \\
&\leqslant [K_0(1 + K_0) + K_0]|\Omega|^{\frac{1}{q}} + a_1 \tilde{K}\|w(\cdot, t)\|_{L^q} \\
&\leqslant K_0[2 + K_0]|\Omega|^{\frac{1}{q}} + a_1 \tilde{K}\|w(\cdot, t)\|_{L^q} \\
&\leqslant K_0[2 + K_0]|\Omega|^{\frac{1}{q}} + a_1 \tilde{K} c_0.
\end{aligned} \tag{3.35}$$

We apply the variation-of-constants formula to (3.34) and obtain

$$u(\cdot, t) = e^{-t(A_d+1)}u_0 + \int_0^t e^{-(A_d+1)(t-s)} g(\cdot, s)\mathrm{d}s, \tag{3.36}$$

where $A_d = -d\Delta$. Then using (2.6), (2.7) and the estimate (3.35) in (3.36), one can derive

$$\begin{aligned}
\|u(\cdot, t)\|_{W^{1,\infty}} \leqslant& C_1\|(A_d + 1)^{\theta} u(\cdot, t)\|_{L^q} \\
\leqslant& C_1 t^{-\theta} e^{-\mu t}\|u_0\|_{L^q} + C_1 \int_0^t (t - s)^{-\theta} e^{-\mu(t-s)}\|g(\cdot, s)\|_{L^q}\mathrm{d}s \\
&\leqslant C_1 t^{-\theta} e^{-\mu t}\|u_0\|_{L^q(\Omega)} + C_1 \int_0^t (t - s)^{-\theta} e^{-\mu(t-s)}[K_0[2 + K_0]|\Omega|^{\frac{1}{q}} + a_1 \tilde{K} c_0]\mathrm{d}s \\
&\leqslant C_1 t^{-\theta} e^{-\mu t}\|u_0\|_{L^q(\Omega)} + C_1[K_0[2 + K_0]|\Omega|^{\frac{1}{q}} + a_1 \tilde{K} c_0] \int_0^\infty \sigma^{-\theta} e^{-\mu\sigma}\mathrm{d}\sigma \\
&\leqslant C_1 t^{-\theta}\|u_0\|_{L^q(\Omega)} + C_1[K_0[2 + K_0]|\Omega|^{\frac{1}{q}} + a_1 \tilde{K} c_0]\mu^{\theta}\Gamma(1 - \theta) \\
&\leqslant C_1 t^{-\theta} + C_1
\end{aligned} \tag{3.37}$$

for all $t \in (\tau, T_{max})$, where $\Gamma(1 - \theta) > 0$. From the last inequality (3.37), we get the desired estimate

$$\|u(\cdot, t)\|_{W^{1,\infty}} \leqslant C_1(\tau^{-\theta} + 1) := C(\tau) \text{ for all } t \in (\tau, T_{max}). \tag{3.38}$$

where C_1 is a generic constant which may vary line to line. Next, we obtain the bound for $\|v(\cdot, t)\|_{W^{1,\infty}}$. To this end, we rewrite the second equation of (1.3) as follows:

$$v_t = \eta d\Delta v - v + h(x, t), \tag{3.39}$$

with $h(x,t) := rv(1-v) - \dfrac{a_4vw}{1+a_2u+a_3v} + v$. Then, by Lemma 3.1 and the fact that $0 < u \leqslant K_0, 0 < v \leqslant K_1$ (see Lemma 2.5) and $\dfrac{a_4v}{1+a_2u+a_3v} \leqslant \overline{K}, \overline{K} > 0$, we have

$$\begin{aligned}\|h(x,t)\|_{L^q} =& \|rv(1-v) - \frac{a_4vw}{1+a_2u+a_3v} + v\|_{L^q} \\ \leqslant& K_1(r(1+K_1)+1)|\Omega|^{\frac{1}{q}} + a_4\overline{K}\|w(\cdot,t)\|_{L^q} \\ \leqslant& K_1(r(1+K_1)+1)|\Omega|^{\frac{1}{q}} + a_4\overline{K}c_0. \end{aligned} \tag{3.40}$$

We apply the variation-of-constants formula to (3.34) and obtain

$$v(\cdot,t) = e^{-t(A_{\eta d}+1)}v_0 + \int_0^t e^{-(A_{\eta d}+1)(t-s)}h(\cdot,s)\mathrm{d}s. \tag{3.41}$$

Then, using (2.6), (2.7) and the estimate (3.40) in (3.41), we find

$$\begin{aligned}\|v(\cdot,t)\|_{W^{1,\infty}} \leqslant& C_1\|(A_{\eta d}+1)^{\theta}v(\cdot,t)\|_{L^q} \\ \leqslant& C_1t^{-\theta}e^{-\mu t}\|v_0\|_{L^q} + C_1\int_0^t (t-s)^{-\theta}e^{-\mu(t-s)}\|h(\cdot,s)\|_{L^q}\mathrm{d}s \\ \leqslant& C_1t^{-\theta}e^{-\mu t}\|v_0\|_{L^q(\Omega)} + C_1\int_0^t (t-s)^{-\theta}e^{-\mu(t-s)}[K_1(r(1+K_1)+1)|\Omega|^{\frac{1}{q}} + a_4\overline{K}c_0]\mathrm{d}s \\ \leqslant& C_1t^{-\theta}e^{-\mu t}\|v_0\|_{L^q(\Omega)} + C_1[K_1(r(1+K_1)+1)|\Omega|^{\frac{1}{q}} + a_4\overline{K}c_0]\int_0^{\infty} \sigma^{-\theta}e^{-\mu\sigma}\mathrm{d}\sigma \\ \leqslant& C_1t^{-\theta}\|v_0\|_{L^q(\Omega)} + C_1[K_1(r(1+K_1)+1)|\Omega|^{\frac{1}{q}} + a_4\overline{K}c_0]\mu^{\theta}\Gamma(1-\theta) \\ \leqslant& C_1t^{-\theta} + C_1 \end{aligned} \tag{3.42}$$

for all $t \in (\tau, T_{max})$ where $\Gamma(1-\theta) > 0$. From the last inequality (3.42), we obtain

$$\|v(\cdot,t)\|_{W^{1,\infty}} \leqslant C_1(\tau^{-\theta}+1) := C(\tau) \text{ for all } \ t \in (\tau, T_{max}). \tag{3.43}$$

Next we derive the L^{∞}- bound of $w(\cdot,t)$. We rewrite the third equation of (1.3) as follows:

$$w_t = \Delta w - w - \nabla\cdot(\chi_1 w\nabla u + \chi_2 w\nabla v) + \frac{a_5uw}{1+a_2u+a_3v} + \frac{a_6vw}{1+a_2u+a_3v} + w - \mu w. \tag{3.44}$$

Then, applying the variation-of-constants formula to (3.44), one has

$$\begin{aligned} w(\cdot,t) =& e^{-t(A+1)}w_0 - \chi_1\int_0^t e^{-(t-s)(A+1)}\nabla\cdot(w\nabla u)\mathrm{d}s - \chi_2\int_0^t e^{-(t-s)(A+1)}\nabla\cdot(w\nabla v)\mathrm{d}s \\ &+ \int_0^t e^{-(t-s)(A+1)}f(u,v,w)\mathrm{d}s := I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.45}$$

where $f(u,v,w) = \dfrac{a_5uw + a_6vw}{1+a_2u+a_3v} + (1-\mu)w$. Now, we take the L^{∞}-norm on both sides of the above equation, we have

$$\|w(\cdot,t)\|_{L^{\infty}} \leqslant \|I_1\|_{L^{\infty}} + \|I_2\|_{L^{\infty}} + \|I_3\|_{L^{\infty}} + \|I_4\|_{L^{\infty}}. \tag{3.46}$$

First, we estimate the term I_1 as follows:

$$\|I_1\|_{L^\infty} \leqslant C_2\tau^{-\theta}e^{-\mu t}\|u_0\|_{L^\infty} \leqslant C_2\tau^{-\theta}\|u_0\|_{L^\infty} \tag{3.47}$$

for all $t \in (\tau, T_{max})$ and $\theta \in (\frac{n}{2q}, 1)$ and $\mu > 0$. In order to estimate the term I_2, we set $m = 0, q = n+1, p = \infty$ for (2.8) and we use (3.33) and (3.1), one has

$$\begin{aligned}
\|I_2\|_{L^\infty} &\leqslant C_3\chi_1 \int_0^t \|(A+1)^\theta e^{-(t-s)(A+1)}\nabla\cdot(w\nabla u)\|_{L^q}\mathrm{d}s \\
&\leqslant C_3\chi_1 \int_0^t e^{-(t-s)}\|(A+1)^\theta e^{-(t-s)A}\nabla\cdot(w\nabla u)\|_{L^q}\mathrm{d}s \\
&\leqslant C_4 \int_0^t (t-s)^{\theta-\frac{1}{2}-\epsilon}e^{-(\mu+1)(t-s)}\|w\nabla u\|_{L^q}\mathrm{d}s \\
&\leqslant C_0C_3C(\tau)\int_0^t (t-s)^{\theta-\frac{1}{2}-\epsilon}e^{-(\mu+1)(t-s)}\mathrm{d}s \\
&\leqslant C_4 \int_0^\infty \rho^{-\theta-\frac{1}{2}-\epsilon}e^{-(\mu+1)\rho}\mathrm{d}\rho \\
&\leqslant C_5\Gamma(\frac{1}{2}-\theta-\epsilon),
\end{aligned} \tag{3.48}$$

where $\Gamma(\frac{1}{2}-\theta-\epsilon)$ is a Gamma function which is positive since $\frac{1}{2}-\theta-\epsilon > 0$ and $\mu, C_5 > 0$.

Next, we obtain the bound for I_3. As in the estimate of I_2, we set $m = 0, q = n+1, p = \infty$ for (2.8) and we use (3.33) and (3.1), one has

$$\begin{aligned}
\|I_3\|_{L^\infty} &\leqslant C_3\chi_1 \int_0^t \|(A+1)^\theta e^{-(t-s)(A+1)}\nabla\cdot(w\nabla v)\|_{L^q}\mathrm{d}s \\
&\leqslant C_3\chi_2 \int_0^t e^{-(t-s)}\|(A+1)^\theta e^{-(t-s)A}\nabla\cdot(w\nabla v)\|_{L^q}\mathrm{d}s \\
&\leqslant C_6 \int_0^t (t-s)^{\theta-\frac{1}{2}-\epsilon}e^{-(\mu+1)(t-s)}\|w\nabla v\|_{L^q}\mathrm{d}s \\
&\leqslant C_0C_6C(\tau)\int_0^t (t-s)^{\theta-\frac{1}{2}-\epsilon}e^{-(\mu+1)(t-s)}\mathrm{d}s \\
&\leqslant C_7 \int_0^\infty \rho^{-\theta-\frac{1}{2}-\epsilon}e^{-(\mu+1)\rho}\mathrm{d}\rho \\
&\leqslant C_8\Gamma(\frac{1}{2}-\theta-\epsilon),
\end{aligned} \tag{3.49}$$

where $\Gamma(\frac{1}{2}-\theta-\epsilon)$ is a Gamma function which is positive since $\frac{1}{2}-\theta-\epsilon > 0$ and $\mu, C_8 > 0$. Finally, we obtain the bound for I_4. To this end, we use (2.6) and (2.7) and let $m = 1, p = (n, \infty]$ and $q = n+1$. Hence, we can choose $\theta \in (\frac{1}{2}(1-\frac{n}{p}+\frac{n}{q}), 1)$. Then one has

$$\begin{aligned}
\|I_4\|_{W^{1,p}} &\leqslant C_1\|(A+1)^\theta I_4\|_{L^q} \\
&\leqslant C_1C_2\int_0^t (t-s)^{-\theta}e^{-\nu t}\|f(u,v,w)\|_{L^q}\mathrm{d}s.
\end{aligned} \tag{3.50}$$

Using the fact that $0 < u \leqslant K_0, 0 < v \leqslant K_1$ and(3.1), we can get

$$\begin{aligned}\left\|\frac{a_5uw + a_6vw}{1 + a_2u + a_3v} + (1-\mu)w\right\|_{L^q} &\leqslant \tilde{K}\|w\|_{L^q} + (1+\mu)\|w\|_{L^q} \\ &\leqslant [\tilde{K} + (1+\mu)]\|w\|_{L^q} \\ &\leqslant [\tilde{K} + (1+\mu)]C(\tau) \end{aligned} \tag{3.51}$$

for all $t \in (\tau, T_{max})$. Hence, we have

$$\begin{aligned}\|I_4\|_{W^{1,p}} &\leqslant C_1C_2[\tilde{K} + (1+\mu)]C(\tau)\int_0^t (t-s)^{-\theta}e^{-\nu t}\mathrm{d}s \\ &\leqslant C_1C_2[\tilde{K} + (1+\mu)]C(\tau)\int_0^\infty \sigma^{-\theta}e^{-\nu t}\mathrm{d}\sigma \\ &\leqslant C_1C_2[\tilde{K} + (1+\mu)]C(\tau)\nu^{\theta}\Gamma(1-\theta) \end{aligned} \tag{3.52}$$

for all $t \in (\tau, T_{max})$ and where $\Gamma(1-\theta)$ is a Gamma function and it is positive since $1-\theta > 0$ and $\nu > 0$. Since $p > n$, Sobolev embedding theorem yields that

$$\|I_4\|_{L^\infty} \leqslant C_9 \text{ for all } t \in (\tau, T_{max}). \tag{3.53}$$

Substituting the estimates (3.47), (3.48), (3.49), (3.53) into (3.46) which yields (3.32). Hence, this completes the proof. □

Proof of Theorem 1.1. From Lemma 2.5, we obtain $\|(u(\cdot,t), v(\cdot,t))\|_{L^\infty(\Omega)} \leqslant C$. Further, we also obtain the bound for $\|w(\cdot,t)\|_{L^\infty}$ from Lemma 3.2. By noticing these results now, we can conclude that

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty} \leqslant c \text{ for all } t \in (0, T_{max}), \tag{3.54}$$

where c is a positive constant. From the criterion (2.9), we obtain that $T_{max} = \infty$ and hence $\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty} \leqslant c$ for all $t \in (0,\infty)$. The proof of Theorem 1.1 is complete.

4. Global stability of solutions

In this section, we shall prove the global stability of solutions of (1.3) by constructing some suitable Lyapunov functionals, and then we use the LaSalle's principle.

4.1. Global stability of Prey only state

Lemma 4.1. *Let* (u, v, w) *be the solution of* (1.3) *and let* $\Gamma_1 = \frac{(a_5(1+a_3-a_2a_6))}{a_1(1+a_2+a_3)}, \Gamma_2 = \frac{(a_6(1+a_2)-a_3a_5)}{a_4(1+a_2+a_3)}$. *Then, if* $\mu \geqslant \frac{a_5+a_6}{1+a_2+a_3}$ *and* $a_4 < \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}$, *it holds that*

$$\lim_{t\to\infty}(\|u(\cdot,t)-1\|_{L^\infty} + \|v(\cdot,t)-1\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty}) = 0. \tag{4.1}$$

Proof. Let us define the energy functional

$$\mathcal{E}(t) := \Gamma_1\int_\Omega(u - 1 - \ln u) + \Gamma_2\int_\Omega(v - 1 - \ln v) + \int_\Omega w. \tag{4.2}$$

First we need to prove that $\mathcal{E}(t) \geqslant 0$ and $\mathcal{E}(t) = 0$ iff $(u, v, w) = (1, 1, 0)$. To this end, let $\psi(x) = x - g_* \ln x$ and by Taylor's formula , one has

$$g - g_* - g_* \ln \frac{g}{g_*} = \psi(g) - \psi(g_*) = \psi'(g_*)(g - g_*) + \frac{1}{2}\psi''(\delta)(g - g_*)^2 \\ = \frac{g_*}{2\delta^2}(g - g_*),$$

where we choose δ in between g_* and g. Now putting $g = u$ and $g_* = 1$ in the last equation, we have

$$u - 1 - \ln u = \frac{1}{2\delta^2}(u - 1)^2 \geqslant 0.$$

Similarly, we can show that

$$v - 1 - \ln v = \frac{1}{2\delta^2}(v - 1)^2 \geqslant 0.$$

Therefore, we get $\mathcal{E}(u, v, w) = \mathcal{E}(1, 1, 0) = 0$ and $\mathcal{E}(u, v, w) \geqslant 0$ for $(u, v, w) \neq (1, 1, 0)$. Differentiating (4.2) with respect to t, and then substituting equations from (1.3), we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t} =& \Gamma_1 \int_\Omega \frac{u-1}{u} u_t + \Gamma_2 \int_\Omega \frac{v-1}{u} v_t + \int_\Omega w_t \\
=& - \Gamma_1 d \int_\Omega \frac{|\nabla u|^2}{u^2} + \Gamma_1 \int_\Omega \left[1 - u - \frac{a_1 w}{1 + a_2 u + a_3 v}\right](u - 1) + \Gamma_2 \eta d \int_\Omega \frac{|\nabla v|^2}{v^2} \\
&+ \Gamma_2 \int_\Omega \left[r(1 - v) - \frac{a_4 w}{1 + a_2 u + a_3 v}\right](v - 1) + \int_\Omega \left[\frac{a_5 u + a_6 v}{1 + a_2 u + a_3 v} - \mu\right] w \\
\leqslant& - \Gamma_1 d \int_\Omega \frac{|\nabla u|^2}{u^2} + \Gamma_1 \int_\Omega \left[1 - u - \frac{a_1 w}{1 + a_2 u + a_3 v}\right](u - 1) - \Gamma_2 \eta d \int_\Omega \frac{|\nabla v|^2}{v^2} \\
&+ \Gamma_2 \int_\Omega \left[r(1 - v) - \frac{a_4 w}{1 + a_2 u + a_3 v}\right](v - 1) + \int_\Omega \left[\frac{a_5 u + a_6 v}{1 + a_2 u + a_3 v} - \frac{a_5 + a_6}{1 + a_2 + a_3}\right] w \\
\leqslant& - \Gamma_1 d \int_\Omega \frac{|\nabla u|^2}{u^2} + \Gamma_1 \int_\Omega \left[1 - u - \frac{a_1 w}{1 + a_2 u + a_3 v}\right](u - 1) - \Gamma_2 \eta d \int_\Omega \frac{|\nabla v|^2}{v^2} \\
&+ \Gamma_2 \int_\Omega \left[r(1 - v) - \frac{a_4 w}{1 + a_2 u + a_3 v}\right](v - 1) \\
&+ \int_\Omega \left[\frac{a_5(u - 1) + a_3 a_5(u - v)}{(1 + a_2 u + a_3 v)(1 + a_2 + a_3)} + \frac{a_6(v - 1) + a_2 a_6(v - u)}{(1 + a_2 + a_3)(1 + a_2 u + a_3 v)}\right] w.
\end{aligned} \tag{4.3}$$

By using the assumptions of Γ_1 and Γ_2 in (4.3), one has

$$\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t} \leqslant -\Gamma_1 d \int_\Omega \frac{|\nabla u|^2}{u^2} - \Gamma_2 \eta d \int_\Omega \frac{|\nabla v|^2}{v^2} - \Gamma_1 \int_\Omega (u - 1)^2 - \Gamma_2 r \int_\Omega (v - 1)^2, \tag{4.4}$$

which yields

$$\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t} \leqslant 0, \tag{4.5}$$

for all (u, v, w) and also the equality holds when $(u, v, w) = (1, 1, 0)$. At last, the LaSalle's invariance principle (cf. [43], Theorem 3) yields that the solutions (u, v, w) converge to the constant steady state $(1, 1, 0)$ as time $t \to \infty$. □

4.2. Global stability of coexistence state

Lemma 4.2. *Let* $\Gamma_1 = \frac{a_5+(a_3a_5-a_2a_6)v_*}{a_1(1+a_2u_*+a_3v_*)}$ *and* $\Gamma_2 = \frac{a_6+(a_2a_6-a_3a_5)u_*}{a_4(1+a_2u_*+a_3v_*)}$ *be positive constants. Let* (u, v, w) *be the solution of* (1.3). *If* $\mu < \frac{a_5+a_6}{1+a_2+a_3}$, $a_4 > \frac{a_1r(a_5+a_3a_5-a_2a_6)}{a_3a_5-a_6-a_2a_6}$ *and*

$$\frac{\Gamma_1(2a_2+a_3)}{2} + \frac{\Gamma_2a_2r}{2} < \Gamma_1 + \frac{\Gamma_1(2a_2+a_3)u_*}{2} + \frac{\Gamma_2a_2rv_*}{2}, \tag{4.6}$$

$$\frac{\Gamma_2r(2a_3+a_2)}{2} + \frac{\Gamma_1a_3}{2} < r\Gamma_2 + \frac{\Gamma_2r(2a_3+a_2)v_*}{2} + \frac{\Gamma_1a_3u_*}{2}, \tag{4.7}$$

$$4d\Gamma_1\Gamma_2\eta u_*v_* > \Gamma_1\chi_2^2u_*w_*K_1^2 + \chi_1^2\eta\Gamma_2K_0^2v_*w_*, \tag{4.8}$$

then it holds that

$$\lim_{t\to\infty}(\|u(\cdot,t)-u_*\|_{L^\infty} + \|v(\cdot,t)-v_*\|_{L^\infty} + \|w(\cdot,t)-w_*\|_{L^\infty}) = 0. \tag{4.9}$$

Proof. The coexistence steady state (u_*, v_*, w_*) of (1.3) satisfies equations

$$\begin{aligned}(1-u_*) - \frac{a_1w_*}{1+a_2u_*+a_3v_*} &= 0,\\ r(1-v_*) - \frac{a_4w_*}{1+a_2u_*+a_3v_*} &= 0,\\ -\mu + \frac{a_5u_*}{1+a_2u_*+a_3v_*} + \frac{a_6v_*}{1+a_2u_*+a_3v_*} &= 0.\end{aligned}$$

Let us define the Lyapunov functional $\mathcal{E}(u, v, w)$ as

$$\mathcal{E}(u,v,w) := \Gamma_1\mathcal{F}_1(t) + \Gamma_2\mathcal{F}_2(t) + \mathcal{F}_3(t), \tag{4.10}$$

where

$$\mathcal{F}_1(t) = \int_\Omega u - u_* - u_*\log\left(\frac{u}{u_*}\right), \mathcal{F}_2(t) = \int_\Omega v - v_* - v_*\log\left(\frac{v}{v_*}\right), \mathcal{F}_3(t) = \int_\Omega w - w_* - w_*\log\left(\frac{w}{w_*}\right). \tag{4.11}$$

Next, we take the derivative of $\mathcal{E}(t)$ with respect to t along the trajectory of the system (1.3) and we obtain

$$\frac{d\mathcal{E}(t)}{dt} = \Gamma_1\frac{d\mathcal{F}_1(t)}{dt} + \Gamma_2\frac{d\mathcal{F}_2(t)}{dt} + \frac{d\mathcal{F}_3(t)}{dt} := \Gamma_1I_1 + \Gamma_2I_2 + I_3. \tag{4.12}$$

Using the definition of $\mathcal{F}_1(t)$, the first equation of (1.3) and the fact that $(1-u_*) - \frac{a_1w_*}{1+a_2u_*+a_3v_*} = 0$, we estimate the term I_1 as follows:

$$
\begin{aligned}
I_1 &= \int_\Omega \frac{u-u_*}{u}\left(d\Delta u + u(1-u) - \frac{a_1 uw}{1+a_2u+a_3v}\right)\\
&= -u_* \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega (u-u_*)\left(1-u-\frac{a_1 w}{1+a_2u+a_3v} - 1 + u_* + \frac{a_1 w_*}{1+a_2u_*+a_3v_*}\right)\\
&= -u_* d \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega (u-u_*)\left(-(u-u_*) - \frac{a_1 w(1+a_2u_*+a_3v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\right.\\
&\quad \left. + \frac{a_1 w_*(1+a_2u+a_3v)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\right)\\
&= -u_* d \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega (u-u_*)^2 + \int_\Omega \frac{[-a_1 w(1+a_2u_*+a_3v_*)](u-u_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\\
&\quad + \frac{[a_1 w_*(1+a_2u+a_3v)](u-u_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}.
\end{aligned}
\tag{4.13}
$$

Now, let us simplify the numerator in the integrand of the third integral on the R.H.S. of (4.13) as follows:

$$
\begin{aligned}
&[-a_1 w(1+a_2u_*+a_3v_*)](u-u_*) + [a_1 w_*(1+a_2u+a_3v)](u-u_*)\\
&= [-a_1(w-w_*) + a_1a_2(uw_* - wu_*) + a_1a_3(w_*v - wv_*)](u-u_*)\\
&= [-a_1(w-w_*) + a_1a_2(uw_* + u_*w_* - u_*w_* - wu_*)\\
&\quad + a_1a_3(w_*v + v_*w_* - v_*w_* - wv_*)](u-u_*)\\
&= -a_1(w-w_*)(u-u_*) + a_1a_2\Big(w_*(u-u_*) - u_*(w-w_*)\Big)(u-u_*)\\
&\quad + a_1a_3\Big(w_*(v-v_*) - v_*(w-w_*)\Big)(u-u_*).
\end{aligned}
\tag{4.14}
$$

Substituting (4.14) into the last integral on the R.H.S of (4.13), we obtain

$$
\int_\Omega \frac{[-a_1 w(1+a_2u_*+a_3v_*) + a_1 w_*(1+a_2u+a_3v)]\,(u-u_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} = \int_\Omega \frac{a_1a_2w_*(u-u_*)^2}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}
\tag{4.15}
$$

$$
- \int_\Omega \frac{a_1[1+a_2u_*+a_3v_*](u-u_*)(w-w_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} + \int_\Omega \frac{a_1a_3w_*(u-u_*)(v-v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}.
\tag{4.16}
$$

Again, inserting (4.16) into (4.13), we end up with

$$
\begin{aligned}
I_1 &= -u_* d \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega (u-u_*)^2 + \int_\Omega \frac{a_1a_2w_*(u-u_*)^2}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\\
&\quad - \int_\Omega \frac{a_1[1+a_2u_*+a_3v_*](u-u_*)(w-w_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} + \int_\Omega \frac{a_1a_3w_*(u-u_*)(v-v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\\
&= -u_* d \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega \left(\frac{a_1a_2w_*}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} - 1\right)(u-u_*)^2\\
&\quad - \int_\Omega \frac{a_1[1+a_2u_*+a_3v_*](u-u_*)(w-w_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} + \int_\Omega \frac{a_1a_3w_*(u-u_*)(v-v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}.
\end{aligned}
\tag{4.17}
$$

Similarly, we estimate the term I_2. Using the definition of $\mathcal{F}_2(t)$, the second equation of (1.3) and the fact that $r(1-v_*) - \frac{a_4 w_*}{1+a_2u_*+a_3v_*} = 0$, we estimate I_2 as follows:

$$
\begin{aligned}
I_2 =& \int_\Omega \frac{v-v_*}{v}\left(\eta d\Delta v + rv(1-v) - \frac{a_4vw}{1+a_2u+a_3v}\right)\\
=& - v_*\eta d\int_\Omega \frac{|\nabla v|^2}{v^2} + \int_\Omega (v-v_*)\left(-rv - \frac{a_4w}{1+a_2u+a_3v} + rv_* + \frac{a_4w_*}{1+a_2u_*+a_3v_*}\right)\\
=& - v_*\eta d\int_\Omega \frac{|\nabla v|^2}{v^2} + \int_\Omega (v-v_*)\left(-r(v-v_*) - \frac{a_4w(1+a_2u_*+a_3v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\right.\\
&\left.+\frac{a_4w_*(1+a_2u+a_3v)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\right)\\
=& - v_*\eta d\int_\Omega \frac{|\nabla v|^2}{v^2} - r\int_\Omega (v-v_*)^2 + \int_\Omega \frac{[-a_4w(1+a_2u_*+a_3v_*)](v-v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\\
&+ \frac{[a_4w_*(1+a_2u+a_3v)](v-v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}.
\end{aligned}
\tag{4.18}
$$

Now, let us simplify the numerator in the integrand of the third integral on the R.H.S. of (4.18) as follows:

$$
\begin{aligned}
[-a_4w(1+a_2u_*+a_3v_*) + a_4w_*(1+a_2u+a_3v)](u-u_*) =& a_3a_4w_*(v-v_*)^2\\
&- a_4[1+a_2u_*+a_3v_*](v-v_*)(w-w_*) + a_2a_4w_*(u-u_*)(v-v_*).
\end{aligned}
\tag{4.19}
$$

Substituting (4.19) into (4.18) and simplifying, we obtain

$$
\begin{aligned}
I_2 =& - v_*\eta d\int_\Omega \frac{|\nabla v|^2}{v^2} - r\int (v-v_*)^2 + \int_\Omega \frac{a_2a_3w_*(v-v_*)^2}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\\
&- \int_\Omega \frac{a_4[1+a_2u_*+a_3v_*](v-v_*)(w-w_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} + \int_\Omega \frac{a_2a_4w_*(u-u_*)(v-v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}\\
=& - v_*\eta d\int_\Omega \frac{|\nabla v|^2}{v^2} + \int_\Omega \left(\frac{a_2a_3w_*}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} - r\right)(v-v_*)^2\\
&- \int_\Omega \frac{a_4[1+a_2u_*+a_3v_*](v-v_*)(w-w_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)} + \int_\Omega \frac{a_2a_4w_*(u-u_*)(v-v_*)}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}.
\end{aligned}
\tag{4.20}
$$

Finally, we estimate the term I_3. Using the definition of $\mathcal{F}_3(t)$, the third equation of (1.3) and the fact that $-\mu + \frac{a_5u_*}{1+a_2u_*+a_3v_*} + \frac{a_6v_*}{1+a_2u_*+a_3v_*} = 0$, we find

$$
\begin{aligned}
I_3 =& \int_\Omega \frac{w-w_*}{w}(\Delta w - \chi_1\nabla\cdot(w\nabla u) - \chi_2\nabla\cdot(w\nabla v))\\
&+ \int_\Omega (w-w_*)\left(-\mu + \frac{a_5u+a_6v}{1+a_2u+a_3v}\right)\\
=& - w_*\int_\Omega \frac{|\nabla w|^2}{w^2} + \chi_1w_*\int_\Omega \frac{\nabla u\nabla w}{w} + \chi_2w_*\int_\Omega \frac{\nabla v\nabla w}{w}\\
&+ \int_\Omega (w-w_*)\left(\frac{a_5u+a_6v}{1+a_2u+a_3v} - \frac{a_5u_*+a_6v_*}{1+a_2u_*+a_3v_*}\right).
\end{aligned}
\tag{4.21}
$$

Further, we simplify the numerator in the integrand of the third integral on the R.H.S. of (4.21), one has that

$$
\begin{aligned}
&(a_5u + a_6v)(1 + a_2u_* + a_3v_*) - (a_5u_* + a_6v_*)(1 + a_2u + a_3v)\\
&\quad = a_5(u - u_*) + a_6(v - v_*) + a_3a_5[v_*(u - u_*) - u_*(v - v_*)] + a_2a_6[u_*(v - v_*) - v_*(u - u_*)]\\
&\quad = [a_5 + (a_3a_5 - a_2a_6)v_*](u - u_*) + [a_6 + (a_2a_6 - a_3a_5)u_*](v - v_*).
\end{aligned}
$$

Now, we rewrite the fourth term in the R.H.S of (4.21) using the above estimate, we obtain

$$
\begin{aligned}
\int_\Omega (w - w_*)\left(\frac{a_5u + a_6v}{1 + a_2u + a_3v} - \frac{a_5u_* + a_6v_*}{1 + a_2u_* + a_3v_*}\right) = &\int_\Omega \frac{[a_5 + (a_3a_5 - a_2a_6)v_*](u - u_*)(w - w_*)}{(1 + a_2u + a_3v)(1 + a_2u_* + a_3v_*)}\\
&+ \int_\Omega \frac{[a_6 + (a_2a_6 - a_3a_5)u_*](v - v_*)(w - w_*)}{(1 + a_2u + a_3v)(1 + a_2u_* + a_3v_*)}. \quad (4.22)
\end{aligned}
$$

Substituting (4.22) into (4.21), we have

$$
\begin{aligned}
I_3 = &- w_* \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi_1 w_* \int_\Omega \frac{\nabla u \cdot \nabla w}{w} + \chi_2 w_* \int_\Omega \frac{\nabla v \cdot \nabla w}{w} + \int_\Omega \frac{[a_5 + (a_3a_5 - a_2a_6)v_*](u - u_*)(w - w_*)}{(1 + a_2u + a_3v)(1 + a_2u_* + a_3v_*)}\\
&+ \int_\Omega \frac{[a_6 + (a_2a_6 - a_3a_5)u_*](v - v_*)(w - w_*)}{(1 + a_2u + a_3v)(1 + a_2u_* + a_3v_*)}. \quad (4.23)
\end{aligned}
$$

Furthermore, inserting (4.17), (4.20) and (4.23) into (4.12) and let $p(u, v) = \frac{1}{(1+a_2u+a_3v)(1+a_2u_*+a_3v_*)}$, we arrive at

$$
\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} = &- u_* d\Gamma_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - v_*\eta d\Gamma_2 \int_\Omega \frac{|\nabla v|^2}{v^2} - w_* \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi_1 w_* \int_\Omega \frac{\nabla u \cdot \nabla w}{w} + \chi_2 w_* \int_\Omega \frac{\nabla v \cdot \nabla w}{w}\\
&+ \Gamma 1 \int_\Omega \left(\frac{a_1a_2w_*}{p(u, v)} - 1\right)(u - u_*)^2 + w_*(\Gamma_1 a_1a_3 + \Gamma_2 a_2a_4) \int_\Omega \frac{(u - u_*)(v - v_*)w_*}{p(u, v)}\\
&+ \Gamma_2 \int_\Omega \left(\frac{a_4a_3w_*}{p(u, v)} - r\right)(v - v_*)^2. \quad (4.24)
\end{aligned}
$$

Applying the Cauchy's inequality, we get

$$
\int_\Omega \frac{(u - u_*)(v - v_*)}{p(u, v)} \leqslant \frac{1}{2}\int_\Omega \frac{(u - u_*)^2}{p(u, v)} + \frac{1}{2}\int_\Omega \frac{(v - v_*)^2}{p(u, v)}. \quad (4.25)
$$

Inserting the last inequality (4.25) into (4.24), and letting $Z = (\frac{|\nabla u|}{u}, \frac{|\nabla v|}{v}, \frac{|\nabla w|}{w})$, we obtain

$$
\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} \leqslant &\underbrace{-u_* d\Gamma_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - v_*\eta d\Gamma_2 \int_\Omega \frac{|\nabla v|^2}{v^2} - w_* \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi_1 w_* \int_\Omega \frac{\nabla u \cdot \nabla w}{w} + \chi_2 w_* \int_\Omega \frac{\nabla v \cdot \nabla w}{w}}_{I_4}\\
&+ \Gamma_1 \int_\Omega \left(\frac{a_1a_2w_*}{p(u, v)} - 1\right)(u - u_*)^2 + \frac{w_*(\Gamma_1 a_1a_3 + \Gamma_2 a_2a_4)}{2} \int_\Omega \frac{(u - u_*)^2}{p(u, v)}\\
&+ \frac{w_*(\Gamma_1 a_1a_3 + \Gamma_2 a_2a_4)}{2} \int_\Omega \frac{(v - v_*)^2}{p(u, v)} + \Gamma_2 \int_\Omega \left(\frac{a_3a_4w_*}{p(u, v)} - r\right)(v - v_*)^2\\
\leqslant &\, I_4 + \int_\Omega \left(\frac{2\Gamma_1 a_1a_2w_* + w_*(\Gamma_1 a_1a_3 + \Gamma_2 a_2a_4)}{2p(u, v)} - \Gamma_1\right)(u - u_*)^2
\end{aligned}
$$

$$+\int_\Omega\Big(\frac{2\Gamma_2 a_3 a_4 w_* + w_*(\Gamma_1 a_1 a_3 + \Gamma_2 a_2 a_4)}{2p(u,v)} - r\Gamma_2\Big)(v - v_*)^2, \tag{4.26}$$

where

$$I_4 = -\int_\Omega Z^T BZ,$$

and the symmetric matrix is denoted by

$$B = \begin{bmatrix} d\Gamma_1 u_* & 0 & -\frac{\chi_1 w_* u}{2} \\ 0 & \eta d\Gamma_2 v_* & -\frac{\chi_2 w_* v}{2} \\ -\frac{\chi_1 w_* u}{2} & -\frac{\chi_2 w_* v}{2} & w_* \end{bmatrix}. \tag{4.27}$$

The above matrix B is positive definite if (4.8) holds. Therefore, we check that

$$\begin{vmatrix} dd\Gamma_1 u_* & 0 \\ 0 & \eta d\Gamma_2 v_* \end{vmatrix} = d^2\eta^2\Gamma^2 u_* v_* > 0 \tag{4.28}$$

and

$$|B| = \frac{dw_*}{4}[4d\Gamma_1\Gamma_2\eta u_* v_* - \Gamma_1\chi_2^2 u_* w_* v^2 - \chi_1^2\eta\Gamma_2 u^2 v_* w_*] > 0. \tag{4.29}$$

Hence, there exists a positive constant α, such that

$$\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t} \leqslant -\alpha\int_\Omega\Big(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla w|^2}{w^2}\Big) + \int_\Omega\Big(\frac{2\Gamma_1 a_1 a_2 w_* + w_*(\Gamma_1 a_1 a_3 + \Gamma_2 a_2 a_4)}{2p(u,v)} - \Gamma_1\Big)(u - u_*)^2$$
$$+\int_\Omega\Big(\frac{2\Gamma_2 a_2 a_3 w_* + w_*(\Gamma_1 a_1 a_3 + \Gamma_2 a_2 a_4)}{2p(u,v)} - r\Gamma_2\Big)(v - v_*)^2. \tag{4.30}$$

Noting the facts that $1 - u_* - \frac{a_1 w_*}{1+a_2 u_* + a_3 v_*} = 0$ and $r(1 - v_*) - \frac{a_4 w_*}{1+a_2 u_* + a_3 v_*} = 0$, one has that

$$\begin{aligned} -\Gamma_1 + \frac{2\Gamma_1 a_1 a_2 w_* + w_*(\Gamma_1 a_1 a_3 + \Gamma_2 a_2 a_4)}{2(1 + a_2 u + a_3 v)(1 + a_2 u_* + a_3 v_*)} &\leqslant -\Gamma_1 + \frac{2\Gamma_1 a_1 a_2 w_* + w_*(\Gamma_1 a_1 a_3 + \Gamma_2 a_2 a_4)}{2(1 + a_2 u_* + a_3 v_*)} \\ &= -\Gamma_1 + \Gamma_1 a_2(1 - u_*) + \frac{\Gamma_2 a_2 r(1 - v_*)}{2} + \frac{\Gamma_1 a_3(1 - u_*)}{2} \\ &\leqslant 0, \end{aligned}$$

and

$$\begin{aligned} -r\Gamma_2 + \frac{2\Gamma_2 a_2 a_3 w_* + w_*(\Gamma_1 a_1 a_3 + \Gamma_2 a_2 a_4)}{2p(u,v)} &\leqslant -r\Gamma_2 + \frac{2\Gamma_2 a_2 a_3 w_* + w_*(\Gamma_1 a_1 a_3 + \Gamma_2 a_2 a_4)}{2(1 + a_2 u_* + a_3 v_*)} \\ &= -r\Gamma_2 + \Gamma_2 a_3 r(1 - v_*) + \frac{\Gamma_1 a_3(1 - u_*)}{2} + \frac{\Gamma_2 a_2 r(1 - v_*)}{2} \\ &\leqslant 0, \end{aligned}$$

where we used the assumptions (4.6) and (4.7). Hence, we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant -c\int_\Omega\Big(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla w|^2}{w^2}\Big), \tag{4.31}$$

which yields that $\frac{d}{dt}\mathcal{E}(t) \leqslant 0$ for all u, v, w and the equality holds if $\nabla u = \nabla v = \nabla w = 0$. Therefore by applying LaSalle's invariance principle (cf. [43], Theorem 3) we can say that the solutions of (1.3) converges to the coexistence steady state (u_*, v_*, w_*) as time $t \to \infty$. □

Proof of Theorem 1.2. Theorem 1.2 is a consequence of Lemma 4.1 and Lemma 4.2.

Remark 4.1. We note that the condition (4.8) is a strong requirement, which implies that χ_1 and χ_2 have to be small enough.

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Conflict of interest

The author declares that there are no conflicts of interest.

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