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*Research article*

# Linear barycentric rational collocation method to solve plane elasticity problems

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**Abstract:** A linear barycentric rational collocation method for equilibrium equations with polar coordinates is considered. The discrete linear equations is changed into the matrix forms. With the help of error of barycentrix polar coordinate interpolation, the convergence rate of the linear barycentric rational collocation method for equilibrium equations can be obtained. At last, some numerical examples are given to valid the proposed theorem.

**Keywords:** barycentric rational method; linear rational interpolation; barycentric matrix form; high order derivatives; equilibrium equations

## 1. Introduction

Equilibrium equations in elasticity are classical equations to solve plane problems. There are displacement and stress methods that are used to solve equilibrium equations. The displacement method takes displacement as an unknown quantity, as there is only the displacement component to deduce the equations and boundary conditions. For the stress method, there is only the stress component to deduce the equations and boundary conditions as the unknown quantities.

In the area of in-plane crack problems, heat transfer, nuclear reactor dynamics and so on, the impact of system memory is often dependent on the nonlinear fraction equation and nonlinear time-dependent Burgers' equations. These problems have been studied by using the Galerkin finite element method [1], localized collocation schemes [2] and singular boundary method (SBM) [3]. Lots of numerical methods such as finite element methods (FEM) [4–7], finite difference methods, spectral method [8, 9] and the differential quadrature method and so on are developed to solve plane elastic problems [10, 11].

In what follows, we consider the equilibrium equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + f_x = 0, \\ \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + f_y = 0 \end{cases} \tag{1.1}$$

where $\sigma_x, \sigma_y$ and $\tau_{yx}$ are the stress components.

Geometric equations:

$$\begin{cases} \epsilon_x = \dfrac{\partial u}{\partial x}, \\ \epsilon_y = \dfrac{\partial v}{\partial y}, \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}, \\ \epsilon_{ij} = \dfrac{1}{2}(u_{i,j} + u_{j,i}) \end{cases} \tag{1.2}$$

where $\epsilon_x, \epsilon_x$ and $\tau_{yx}$ are the strain components and $u$ and $v$ are displacement variables.

Constitutive relations of plane stress problem:

$$\begin{cases} \epsilon_x = \dfrac{1}{E}(\sigma_x - \mu\sigma_y), \\ \epsilon_y = \dfrac{1}{E}(\sigma_y - \mu\sigma_x), \\ \gamma_{xy} = \dfrac{2(1+\mu)}{E}\tau_{xy}. \end{cases} \tag{1.3}$$

Constitutive relations of the plane strain problem:

$$\begin{cases} \epsilon_x = \dfrac{1-\mu^2}{E}(\sigma_x - \dfrac{\mu}{1-\mu}\sigma_y), \\ \epsilon_y = \dfrac{1-\mu^2}{E}(\sigma_y - \dfrac{\mu}{1-\mu}\sigma_x), \\ \gamma_{xy} = \dfrac{2(1+\mu)}{E}\tau_{xy}. \end{cases} \tag{1.4}$$

Displacement boundary equation:

$$u|_{\Gamma_u} = \bar{u}, v|_{\Gamma_u} = \bar{v}, u_i|_{\Gamma_u} = \bar{u}_i \tag{1.5}$$

Stress boundary equations:

$$\begin{cases} (n_1\sigma_x + n_2\tau_{yx})_{\Gamma_\sigma} = \bar{t}_x, \\ (n_2\sigma_y + n_1\tau_{xy})_{\Gamma_\sigma} = \bar{t}_y, \\ n_j\sigma_{ij} = \bar{t}_i. \end{cases} \tag{1.6}$$

Combining Eqs (1.4) and (1.2), we have the stress components of displacement:

$$\begin{cases} \sigma_x = \dfrac{E}{1-\mu^2}\left(\dfrac{\partial u}{\partial x} + \mu\dfrac{\partial v}{\partial y}\right), \\ \sigma_y = \dfrac{E}{1-\mu^2}\left(\dfrac{\partial v}{\partial y} + \mu\dfrac{\partial u}{\partial x}\right), \\ \tau_{xy} = \dfrac{E}{2(1+\mu)}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \end{cases} \tag{1.7}$$

and the equilibrium equations:

$$\begin{cases} \dfrac{E}{1-\mu^2}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1-\mu}{2}\dfrac{\partial^2 v}{\partial y^2} + \dfrac{1+\mu}{2}\dfrac{\partial^2 v}{\partial x\partial y}\right) + f_x = 0, \\ \dfrac{E}{1-\mu^2}\left(\dfrac{\partial^2 v}{\partial y^2} + \dfrac{1-\mu}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1+\mu}{2}\dfrac{\partial^2 v}{\partial x\partial y}\right) + f_y = 0 \end{cases} \tag{1.8}$$

and the displacement boundary equations:

$$\begin{cases} \dfrac{E}{1-\mu^2}\left[\left(\dfrac{\partial u}{\partial x} + \mu\dfrac{\partial v}{\partial y}\right)n_1 + \dfrac{1-\mu}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right)n_2\right]_{\Gamma_\sigma} = \bar{t}_x, \\ \dfrac{E}{1-\mu^2}\left[\left(\dfrac{\partial v}{\partial y} + \mu\dfrac{\partial u}{\partial x}\right)n_2 + \dfrac{1-\mu}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right)n_1\right]_{\Gamma_\sigma} = \bar{t}_y. \end{cases} \tag{1.9}$$

Barycentric formulae have been studied by [12–18] to avoid the Runge phenomenon. Volterra equations (VE) and Volterra Integro-Differential equations (VIDE) [19–23] have been investigated through the use of linear barycentric collocation methods (LBCMs). The LBCM types include the linear barycentric Lagrange collocation method (LBLCM) and linear barycentric rational collocation method (LBRCM). By comparing the with LBLCM and LBRCM, we can get the error estimate of linear rational barycentric interpolation; then, the convergence rate of the LBRCM can be obtained. Initial value and boundary value problems [24], plane elasticity problems [25], incompressible plane problems [26] and non linear problems [27] have been the focus of barycentric interpolation and rational collocation method in recent years. In previous studies [28, 29], heat conduction and telegram equations were solved by LBRCM. In other studies [30,31], biharmonic equation and fractional differential equations were solved by using the LBRCM.

In this paper, first, the polar coordinates of the equilibrium equations are obtained via the transformation of $x = \rho\cos\theta, y = \rho\sin\theta$. Second, the LBRCM for equilibrium equations is constructed and the matrix equation of the LBRCM is also presented. Third, the convergence rate of LBRCM is proved for the equilibrium equations. At last, some numerical examples are given to validate the proposed theorem.

## 2. Polar coordinates of equilibrium equations

In order to get the polar coordinates of the equilibrium equations, let us take $x = \rho\cos\theta, y = \rho\sin\theta$ and $(\rho, \theta)$ at some point $P(\rho, \theta)$; the displacement components are $u_\rho$ and $u_\theta$, stress components are

$\sigma_\rho$, $\sigma_\theta$ and $\tau_{\theta r}$ and the physical components are $f_\rho$ and $f_\theta$. The equilibrium equations of the polar coordinates can be represented as:

$$\begin{cases} \dfrac{\partial \sigma_\rho}{\partial \rho} + \dfrac{\sigma_\rho - \sigma_\theta}{\rho} + \dfrac{1}{\rho}\dfrac{\partial \tau_{\rho\theta}}{\partial \theta} + f_\rho = 0, \\ \dfrac{1}{\rho}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{\rho\theta}}{\partial \rho} + 2\dfrac{\tau_{\rho\theta}}{\rho} + f_\theta = 0. \end{cases} \tag{2.1}$$

Geometric equations:

$$\begin{cases} \epsilon_\rho = \dfrac{\partial u_\rho}{\partial \rho}, \\ \epsilon_\theta = \dfrac{u_\rho}{\rho} + \dfrac{1}{\rho}\dfrac{\partial u_\theta}{\partial \theta}, \\ \gamma_{\rho\theta} = \gamma_{\theta\rho} = \dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \theta} + \dfrac{\partial u_\theta}{\partial \rho} - \dfrac{u_\theta}{\rho}. \end{cases} \tag{2.2}$$

Constitutive relations of plane stress problem:

$$\begin{cases} \epsilon_\rho = \dfrac{1}{E}(\sigma_\rho - \mu\sigma_\theta), \\ \epsilon_\theta = \dfrac{1}{E}(\sigma_\theta - \mu\sigma_\rho), \\ \gamma_{\rho\theta} = \dfrac{2(1+\mu)}{E}\tau_{\rho\theta}. \end{cases} \tag{2.3}$$

Combining Eqs (2.1)–(2.3), the displacement of the equilibrium equations for the plane stress problem is expressed as:

$$\begin{cases} \dfrac{\partial^2 u_\rho}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \rho} - \dfrac{u_\rho}{\rho^2} + \dfrac{1+\mu}{2\rho}\dfrac{\partial^2 u_\theta}{\partial \rho \partial \theta} - \dfrac{3-\mu}{2\rho^2}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{1-\mu}{2\rho^2}\dfrac{\partial^2 u_\rho}{\partial \theta^2} + f_\rho = 0, \\ \dfrac{1+\mu}{2\rho}\dfrac{\partial^2 u_\rho}{\partial \rho \partial \theta} + \dfrac{3-\mu}{2\rho^2}\dfrac{\partial u_\rho}{\partial \theta} + \dfrac{1-\mu}{2}\left(\dfrac{\partial^2 u_\theta}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u_\theta}{\partial \rho} - \dfrac{u_\theta}{\rho^2}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u_\theta}{\partial \theta^2} + f_\theta = 0. \end{cases} \tag{2.4}$$

The displacement of th stress components is expressed as follows:

$$\begin{cases} \sigma_\rho = \dfrac{E}{1-\mu^2}\left[\dfrac{\partial u_\rho}{\partial \rho} + \mu\left(\dfrac{u_\rho}{\rho} + \dfrac{1}{\rho}\dfrac{\partial u_\theta}{\partial \theta}\right)\right], \\ \sigma_\theta = \dfrac{E}{1-\mu^2}\left(\mu\dfrac{\partial u_\rho}{\partial \rho} + \dfrac{u_\rho}{\rho} + \dfrac{1}{\rho}\dfrac{\partial u_\theta}{\partial \theta}\right), \\ \tau_{\rho\theta} = \dfrac{E}{2(1+\mu)}\left(\dfrac{\partial u_\theta}{\partial \rho} - \dfrac{u_\theta}{\rho} + \dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \theta}\right) \end{cases} \tag{2.5}$$

where we have used

$$x = \rho\cos\theta, y = \rho\sin\theta; \tag{2.6}$$

then we have

$$\begin{cases} \sigma_\rho = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta, \\ \sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta, \\ \tau_{\rho\theta} = (\sigma_x - \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{cases} \tag{2.7}$$

and

$$\begin{cases} \sigma_x = \sigma_\rho \cos^2\theta + \sigma_\theta \sin^2\theta + 2\tau_{\rho\theta}\sin\theta\cos\theta, \\ \sigma_y = \sigma_\rho \sin^2\theta + \sigma_\theta \cos^2\theta - 2\tau_{\rho\theta}\sin\theta\cos\theta, \\ \tau_{xy} = (\sigma_\rho - \sigma_\theta)\cos\theta\sin\theta + \tau_{\rho\theta}(\cos^2\theta - \sin^2\theta) \end{cases} \tag{2.8}$$

with the equilibrium condition, we get $\phi(r, \theta)$ below:

$$\begin{cases} \sigma_\rho = \dfrac{1}{\rho}\dfrac{\partial\phi}{\partial\rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2\phi}{\partial\theta^2}, \\ \sigma_\theta = \dfrac{\partial^2\phi}{\partial\rho^2}, \\ \tau_{\rho\theta} = \dfrac{1}{\rho^2}\dfrac{\partial\phi}{\partial\theta} - \dfrac{1}{\rho}\dfrac{\partial^2\phi}{\partial\rho\partial\theta} \end{cases} \tag{2.9}$$

and

$$\nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \phi = 0 \tag{2.10}$$

where $\nabla^2\phi = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)$ is the Laplace operator.

## 3. Collocation method for equilibrium equations

We partition the area $[\rho_a, \rho_b] \times [\theta_0, \theta_{2\pi}]$ into $\rho_a = \rho_0 < \rho_1 < \cdots < \rho_m = \rho_b, h = \frac{\rho_b - \rho_a}{m}$ and $[\theta_a, \theta_b]$ into $\theta_0 = \theta_0 < \theta_1 < \cdots < \theta_n = \theta_{2\pi}, \tau = \frac{\theta_{2\pi} - \theta_0}{n}$ with $[\rho_a, \rho_b] \times [\theta_0, \theta_{2\pi}]$ and $\left(\rho_i, \theta_j\right), \quad i = 0, 1, \cdots, m; j = 0, 1, , \cdots, n$.

$$\phi(\rho, \theta) := r_{m,n}(\rho, \theta) = \sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j(\theta) \phi_{ij} \tag{3.1}$$

where $r_i(\rho)$ and $r_j(\theta)$ are the barycentric rational interpolation basis functions [24] for $\rho$ and $\theta$, respectively, which can be given as

$$r_i(\rho) = \frac{\dfrac{w_i}{\rho - \rho_i}}{\displaystyle\sum_{j=0}^{n} \frac{w_j}{\rho - \rho_j}}, \; w_i = \sum_{k \in J_i} (-1)^k \prod_{j=k, j \neq i}^{k+d_1} \frac{1}{\rho_k - \rho_j}, \quad i = 0, 1, 2, \cdots, n \tag{3.2}$$

$J_i = \{k \in I_m : i - d_1 \le k \le i\}, I_m = \{0, \cdots, m - d_1\}$, and

$$r_j(\theta) = \frac{\dfrac{w_j}{\theta - \theta_j}}{\displaystyle\sum_{j=0}^{n} \frac{w_j}{\theta - \theta_j}},\ w_j = \sum_{k \in J_j} (-1)^k \prod_{i=k, j \ne i}^{k+d_2} \frac{1}{\theta_k - \theta_i}, \quad j = 0, 1, 2, \cdots, n, \tag{3.3}$$

$J_j = \{k \in I_n : j - d_2 \le k \le j\}, I_n = \{0, \cdots, n - d_2\}$.

Combining Eqs (3.1), (2.4) and (2.5), we get the discrete equilibrium equations, which be expressed as

$$\begin{cases} \displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n} r_i''(\rho) r_j(\theta)\phi_{\rho ij} + \frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i'(\rho) r_j(\theta)\phi_{\rho ij} \\ \displaystyle -\frac{1}{\rho^2}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j(\theta)\phi_{\rho ij} + \frac{1+\mu}{2r}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i'(\rho) r_j'(\theta)\phi_{\theta ij} \\ \displaystyle +\frac{3-\mu}{2\rho^2}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j'(\theta)\phi_{\theta ij} + \frac{1-\mu}{2\rho^2}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j''(\theta)\phi_{\theta ij} + f_{\rho ij}(\rho_i, \theta_j) = 0, \\ \displaystyle \frac{1+\mu}{2\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j'(\theta)\phi_{\rho ij} + \frac{3-\mu}{2\rho^2}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j'(\theta)\phi_{\rho ij} + \frac{1-\mu}{2} \\ \displaystyle \cdot\left(\sum_{i=0}^{m}\sum_{j=0}^{n} r_i''(\rho) r_j(\theta)\phi_{\theta ij} + \frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i'(\rho) r_j(\theta)\phi_{\theta ij} - \frac{1}{\rho^2}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j(\theta)\phi_{\theta ij}\right) \\ \displaystyle +\frac{1}{\rho^2}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j''(\theta)\phi_{\theta ij} + f_{\theta ij}(\rho_i, \theta_j) = 0 \end{cases} \tag{3.4}$$

and

$$\begin{cases} \displaystyle \sigma_\rho = \frac{E}{1-\mu^2}\left[\sum_{i=0}^{m}\sum_{j=0}^{n} r_i'(\rho) r_j(\theta)\phi_{\rho ij} + \mu\left(\frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j''(\theta)\phi_{\rho ij} + \frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j'(\theta)\phi_{\theta ij}\right)\right], \\ \displaystyle \sigma_\theta = \frac{E}{1-\mu^2}\left(\mu\sum_{i=0}^{m}\sum_{j=0}^{n} r_i'(\rho) r_j(\theta)\phi_{\rho ij} + \frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j(\theta)\phi_{\theta ij} + \frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j'(\theta)\phi_{\rho ij}\right), \\ \displaystyle \tau_{\rho\theta} = \frac{E}{2(1+\mu)}\left(\sum_{i=0}^{m}\sum_{j=0}^{n} r_i'(\rho) r_j(\theta)\phi_{\rho ij} - \frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j(\theta)\phi_{\theta ij} + \frac{1}{\rho}\sum_{i=0}^{m}\sum_{j=0}^{n} r_i(\rho) r_j'(\theta)\phi_{\theta ij}\right). \end{cases} \tag{3.5}$$

Equations (3.4) and (3.5) can be written in matrix form

$$\begin{cases} \left[(R^{(2,0)} \otimes I_n) + \mathrm{diag}(\frac{1}{\rho})(R^{(1,0)} \otimes I_n) - \mathrm{diag}(\frac{1}{\rho^2})(I_m \otimes I_n)\right] U_\rho \\ + \left[\mathrm{diag}(\frac{1+\mu}{2r})(R^{(1,0)} \otimes R^{(0,1)}) + \mathrm{diag}(\frac{3-\mu}{2\rho^2})(I_m \otimes R^{(0,1)}) + \mathrm{diag}(\frac{1-\mu}{2\rho^2})(I_n \otimes R^{(0,2)})\right] \\ U_\theta + F_\rho = 0, \\ \\ \left[\mathrm{diag}(\frac{1+\mu}{2\rho})(R^{(1,0)} \otimes R^{(0,1)}) + \mathrm{diag}(\frac{3-\mu}{2\rho^2})(I_m \otimes R^{(0,1)})\right] U_\rho \\ + \left[\frac{1-\mu}{2}\left((R^{(2,0)} \otimes I_n) + \mathrm{diag}\frac{1}{\rho}(R^{(1,0)} \otimes I_n) - \mathrm{diag}\frac{1}{\rho^2}(I_m \otimes I_n)\right) + \mathrm{diag}\frac{1}{\rho^2}(R^{(2,0)} \otimes I_n)\right] \\ U_\theta + F_\theta = 0 \end{cases} \tag{3.6}$$

and

$$\begin{cases} \sigma_\rho = \frac{E}{1-\mu^2}\left[\{(R^{(1,0)} \otimes I_n) + \mu\mathrm{diag}(\frac{1}{\rho})(I_m \otimes I_n)\}U_\rho + \mu\mathrm{diag}\frac{1}{\rho}(I_m \otimes R^{(0,1)})U_\theta\right], \\ \sigma_\theta = \frac{E}{1-\mu^2}\left[\{\mu(R^{(1,0)} \otimes I_n) + \mathrm{diag}(\frac{1}{\rho})(I_m \otimes I_n)\}U_\rho + \mathrm{diag}\frac{1}{\rho}(I_m \otimes R^{(0,1)})U_\theta\right], \\ \tau_{\rho\theta} = \frac{E}{2(1+\mu)}\left[\mathrm{diag}(\frac{1}{\rho})(I_m \otimes R^{(0,1)})U_\rho + [\mathrm{diag}(\frac{1}{\rho})(I_m \otimes I_n) + (R^{(1,0)} \otimes I_n)]U_\theta\right] \end{cases} \tag{3.7}$$

where $\otimes$ is the Kronecher product of the matrix and $R^{(0,k)} = (R_{ij}^{(0,k)})_{m\times m}, R^{(k,0)} = (R_{ij}^{(k,0)})_{n\times n}, k = 1,2,\ U = [u_{00}, u_{01}, \cdots, u_{0n}, u_{10}, u_{11}, \cdots, u_{1n}, \cdots, u_{m0}, u_{m1}, \cdots, u_{mn}]^{\mathrm{T}},\ F_\rho = [f_{00}, f_{01}, \cdots, f_{0n}, f_{10}, f_{11}, \cdots, f_{1n}, \cdots, f_{m0}, f_{m1}, \cdots, f_{mn}]^{\mathrm{T}},\ f_{ij} = \rho_i^2 f(\rho_i, \theta_j)$ and

$$R_{ij}^{(0,1)} = r_i'(\theta_j), R_{ij}^{(0,2)} = r_i''(\theta_j), \quad R_{ij}^{(1,0)} = r_i'(\rho_j), \quad R_{ij}^{(2,0)} = r_i''(\rho_j). \tag{3.8}$$

Taking the notations as

$$\begin{cases} A_{11} = (R^{(2,0)} \otimes I_n) + \mathrm{diag}(\frac{1}{\rho})(R^{(1,0)} \otimes I_n) - \mathrm{diag}(\frac{1}{\rho^2})(I_m \otimes I_n) \\ A_{12} = \mathrm{diag}(\frac{1+\mu}{2r})(R^{(1,0)} \otimes R^{(0,1)}) + \mathrm{diag}(\frac{3-\mu}{2\rho^2})(I_m \otimes R^{(0,1)}) \\ +\mathrm{diag}(\frac{1-\mu}{2\rho^2})(I_n \otimes R^{(0,2)}), \\ \\ A_{21} = \mathrm{diag}(\frac{1+\mu}{2\rho})(R^{(1,0)} \otimes R^{(0,1)}) + \mathrm{diag}(\frac{3-\mu}{2\rho^2})(I_m \otimes R^{(0,1)}) \\ A_{22} = \frac{1-\mu}{2}\left((R^{(2,0)} \otimes I_n) + \mathrm{diag}\frac{1}{\rho}(R^{(1,0)} \otimes I_n) - \mathrm{diag}\frac{1}{\rho^2}(I_m \otimes I_n)\right) \\ +\mathrm{diag}\frac{1}{\rho^2}(R^{(2,0)} \otimes I_n), \end{cases} \tag{3.9}$$

then we have

$$\begin{cases} A_{11}U_\rho + A_{21}U_\theta + F_\rho = 0, \\ A_{21}U_\rho + A_{22}U_\theta + F_\rho = 0. \end{cases} \tag{3.10}$$

## 4. Convergence and error analysis

Replacing the barycentric rational interpolants of the function $u(\rho,\theta)$ with $r_{m,n}(\rho,\theta)$ in Eq (3.1), we have

$$r_{m,n}(\rho,\theta)=\frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{w_{i,j}}{(\rho-\rho_i)(\theta-\theta_j)}u_{i,j}}{\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{w_{i,j}}{(\rho-\rho_i)(\theta-\theta_j)}}, \tag{4.1}$$

where

$$w_{i,j}=(-1)^{i-d_1+j-d_2}\sum_{k_1\in J_i}\sum_{k_2\in J_j}\prod_{h_1=k_1,h_1\neq j}^{k_1+d_1}\frac{1}{\left|\rho_i-\rho_{h_1}\right|}\prod_{h_2=k_2,h_2\neq j}^{k_2+d_2}\frac{1}{\left|\theta_j-\theta_{h_2}\right|}. \tag{4.2}$$

Then the error function is defined as

$$\begin{aligned} e(\rho,\theta):\ &= u(\rho,\theta)-r_{m,n}(\rho,\theta) \\ &= (\rho-\rho_i)\cdots(\rho-\rho_{i+d_1})u[\rho_i,\rho_{i+1},\ldots,\rho_{i+d_1},\rho] \\ &+ (\theta-\theta_j)\cdots(\theta-\theta_{j+d_2})u[\theta_j,\theta_{j+1},\ldots,\theta_{j+d_2},\theta]. \end{aligned} \tag{4.3}$$

Now we give the theorem as below

**Theorem 1.** *For $e(\rho,\theta)$ defined in Eq(4.3) and $u(\rho,\theta)\in C^{d_1+2}[0,\rho]\times C^{d_2+2}[0,\theta]$, we have*

$$|e(\rho,\theta)|\leq C(h^{d_1+1}+\tau^{d_2+1}). \tag{4.4}$$

Proof. For $(\rho,\theta)$, the function $w_{i,j}(\rho,\theta)$ is defined as Eq (4.2); then, we get

$$u(\rho,\theta)-r_{m,n}(\rho,\theta)=\frac{\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}\lambda_i(\rho)\lambda_j(\theta)(u(\rho,\theta)-r_n(\rho,\theta))}{\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}\lambda_i(\rho)\lambda_j(\theta)}, \tag{4.5}$$

where

$$\lambda_i(\rho)=\frac{(-1)^i}{(\rho-\rho_i)\cdots(\rho-\rho_{i+d_1})},\lambda_j(\theta)=\frac{(-1)^j}{(\theta-\theta_j)\cdots(\theta-\theta_{j+d_2})}$$

see [24].

By the error formula

$$\begin{aligned} u(\rho,\theta)-r_{m,n}(\rho,\theta) &= u(\rho,\theta)-u_1(\rho,\theta)+u_1(\rho,\theta)-r_{m,n}(\rho,\theta) \\ &= (\rho-\rho_i)\cdots(\rho-\rho_{i+d_1})u\left[\rho_i,\rho_{i+1},\ldots,\rho_{i+d_1},\rho,\theta\right] \\ &+(\theta-\theta_j)\cdots(\theta-\theta_{j+d_2})u\left[\theta_j,\theta_{j+1},\cdots,\theta_{j+d_2},\rho,\theta\right]. \end{aligned} \tag{4.6}$$

it follows that

$$u(\rho,\theta)-r_{m,n}(\rho,\theta)=\frac{\sum_{i=0}^{m-d_1}(-1)^i u\left[\rho_i,\rho_{i+1},\ldots,\rho_{i+d_1},\rho,\theta\right]}{\sum_{i=0}^{m-d_1}\lambda_i(\rho)} + \frac{\sum_{j=0}^{n-d_2}(-1)^j u\left[\theta_j,\theta_{j+1},\cdots,\theta_{j+d_2},\rho,\theta\right]}{\sum_{j=0}^{n-d_2}\lambda_j(\theta)}. \tag{4.7}$$

By a similar method of analysis as that of Floater and Kai [15], we have

$$\left|\sum_{i=0}^{m-d_1}\lambda_i(\rho)\right| \geq \frac{1}{d_1!h^{d_1+1}} \tag{4.8}$$

and

$$\left|\sum_{j=0}^{n-d_2}\lambda_j(\theta)\right| \geq \frac{1}{d_2!\tau^{d_2+1}}. \tag{4.9}$$

Combining Eqs (4.7)–(4.9) together, the proof of Theorem 1 is completed.

**Corollary 1.** *For $e(\rho,\theta)$ defined in (4.3), we have*

$$\begin{cases} \left|e_\rho(\rho,\theta)\right| \leq C(h^{d_1}+\tau^{d_2+1}), & u(\rho,\theta)\in C^{d_1+3}[\rho_a,\rho_b]\times C^{d_2+2}[\theta_0,\theta_{2\pi}], \\ |e_\theta(\rho,\theta)| \leq C(h^{d_1+1}+\tau^{d_2}), & u(\rho,\theta)\in C^{d_1+2}[\rho_a,\rho_b]\times C^{d_2+3}[\theta_0,\theta_{2\pi}], \\ \left|e_{\rho\rho}(\rho,\theta)\right| \leq C(h^{d_1-1}+\tau^{d_2+1}), & u(\rho,\theta)\in C^{d_1+4}[\rho_a,\rho_b]\times C^{d_2+2}[\theta_0,\theta_{2\pi}], d_1\geq 2. \\ |e_{\theta\theta}(\rho,\theta)| \leq C(h^{d_1+1}+\tau^{d_2-1}), & u(\rho,\theta)\in C^{d_1+2}[\rho_a,\rho_b]\times C^{d_2+4}[\theta_0,\theta_{2\pi}], d_1\geq 2. \end{cases} \tag{4.10}$$

This corollary can be obtained similarly as Theorem 1, so it is omitted here.

**Theorem 2.** *Let*

$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2}=0, (\rho,\theta)\in\Omega \tag{4.11}$$

*and*

$$\phi(\rho,\theta)=g(\rho,\theta), (\rho,\theta)\in\partial\Omega \tag{4.12}$$

*where $\Omega=[\rho_a,\rho_b]\times[\theta_0,\theta_{2\pi}]$ and $g(\rho,\theta)$ is consistent. Then we get*

$$\max_{\Omega_k}|\phi_{i,j}-\phi(\rho_i,\theta_j)| \leq C(h^{d_1-1}+\tau^{d_2-1}) \tag{4.13}$$

*where $u(\rho,\theta)\in C^{d_1+4}[\rho_a,\rho_b]\times C^{d_2+4}[\theta_0,\theta_{2\pi}], d_1\geq 2, d_2\geq 2$.*

**Theorem 3.** *Let*

$$\nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \phi = 0, (\rho,\theta) \in \Omega \tag{4.14}$$

*and*

$$\phi(\rho,\theta) = g(\rho,\theta), (\rho,\theta) \in \partial\Omega \tag{4.15}$$

*where* $\Omega = [\rho_a, \rho_b] \times [\theta_0, \theta_{2\pi}]$ $g(\rho,\theta)$ *is consistent and*

$$\max_{\Omega_{kl}} |\phi_{i,j} - \phi(\rho_i, \theta_j)| \le C(h^{d_1-3} + \tau^{d_2-3}) \tag{4.16}$$

*also,* $\Omega_{kl} = [\rho_k, \rho_{k+1}] \times [\theta_l$ *and* $\theta_{l+1}], \phi(\rho,\theta) \in C^{d_1+6}[\rho_a,\rho_b] \times C^{d_2+6}[\theta_0,\theta_{2\pi}], d_1 \ge 4, d_2 \ge 4.$

Proof. Let $\phi(\rho,\theta)$ and $\phi_{i,j}$ be the analysis solution and numerical solution of Eq (4.14) respectively:

$$\begin{aligned}
&\nabla^2\nabla^2\phi(\rho,\theta) - \nabla^2\nabla^2\phi(\rho_i,\theta_j)\\
&= \frac{\partial^4\phi(\rho,\theta)}{\partial\rho^4} + \frac{\partial^2\phi(\rho,\theta)}{\rho^2\partial\rho^2} + \frac{2}{\rho^2}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho^3} + \frac{1}{\rho^4}\frac{\partial^4\phi(\rho,\theta)}{\partial\theta^4} + \frac{2}{\rho^2}\frac{\partial^4\phi(\rho,\theta)}{\partial\rho^2\partial\theta^2} + \frac{2}{\rho^3}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho\partial\theta^2}\\
&-\left[\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^4} + \frac{\partial^2\phi(\rho_i,\theta_j)}{\rho^2\partial\rho^2} + \frac{2}{\rho^2}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho^3} + \frac{1}{\rho^4}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\theta^4} + \frac{2}{\rho^2}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^2\partial\theta^2} + \frac{2}{\rho^3}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho\partial\theta^2}\right]\\
&= \left[\frac{\partial^4\phi(\rho,\theta)}{\partial\rho^4} - \frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^4}\right] + \left[\frac{\partial^2\phi(\rho,\theta)}{\rho^2\partial\rho^2} - \frac{\partial^2\phi(\rho_i,\theta_j)}{\rho^2\partial\rho^2}\right]\\
&+\left[\frac{1}{\rho^4}\frac{\partial^4\phi(\rho,\theta)}{\partial\rho^4} - \frac{1}{\rho^4}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^4}\right] + \left[\frac{2}{\rho^2}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho^3} - \frac{2}{\rho^2}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho^3}\right]\\
&+\left[\frac{2}{\rho^2}\frac{\partial^4\phi(\rho,\theta)}{\partial\rho^2\partial\theta^2} - \frac{2}{\rho^2}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^2\partial\theta^2}\right] + \left[\frac{2}{\rho^3}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho\partial\theta^2} - \frac{2}{\rho^3}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho\partial\theta^2}\right]\\
&:= R_1(\rho,\theta) + R_2(\rho,\theta) + R_3(\rho,\theta)\\
&+R_4(\rho,\theta) + R_5(\rho,\theta) + R_6(\rho,\theta)
\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}
R_1(\rho,\theta) &= \frac{\partial^4\phi(\rho,\theta)}{\partial\rho^4} - \frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^4},\\
R_2(\rho,\theta) &= \frac{\partial^2\phi(\rho,\theta)}{\rho^2\partial\rho^2} - \frac{\partial^2\phi(\rho_i,\theta_j)}{\rho^2\partial\rho^2},\\
R_3(\rho,\theta) &= \frac{1}{\rho^4}\frac{\partial^4\phi(\rho,\theta)}{\partial\theta^4} - \frac{1}{\rho^4}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\theta^4}\\
R_4(\rho,\theta) &= \frac{2}{\rho^2}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho^3} - \frac{2}{\rho^2}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho^3},\\
R_5(\rho,\theta) &= \frac{2}{\rho^2}\frac{\partial^4\phi(\rho,\theta)}{\partial\rho^2\partial\theta^2} - \frac{2}{\rho^2}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^2\partial\theta^2},\\
R_6(\rho,\theta) &= \frac{2}{\rho^3}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho\partial\theta^2} - \frac{2}{\rho^3}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho\partial\theta^2}
\end{aligned}$$

for $R_1(\rho,\theta)$, we have

$$\begin{aligned}
R_1(\rho,\theta) &= \frac{\partial^4\phi(\rho,\theta)}{\partial\rho^4} - \frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^4}\\
&= \frac{\partial^4\phi(\rho,\theta)}{\partial\rho^4} - \frac{\partial^4\phi(\rho,\theta_j)}{\partial\rho^4} + \frac{\partial^4\phi(\rho,\theta_j)}{\partial\rho^4} - \frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^4}\\
&= \frac{\sum_{i=0}^{m-d_1}(-1)^i\frac{\partial^4\phi}{\partial\rho^4}[\rho_i,\rho_{i+1},\ldots,\rho_{i+d_1},\rho,\theta]}{\sum_{i=0}^{m-d_1}\lambda_i(\rho)}\\
&+\frac{\sum_{j=0}^{n-d_2}(-1)^j\frac{\partial^4\phi}{\partial\rho^4}[\theta_j,\theta_{j+1},\ldots,\theta_{j+d_2},\rho_i,\theta]}{\sum_{j=0}^{n-d_2}\lambda_j(\theta)}\\
&= \frac{\partial^4 e(\rho,\theta_j)}{\partial\rho^4} + \frac{\partial^4 e(\rho_i,\theta_j)}{\partial\rho^4},
\end{aligned}$$

where

$$|R_1(\rho,\theta)| \le \left|\frac{\partial^4 e(\rho,\theta_j)}{\partial\rho^4} + \frac{\partial^4 e(\rho_i,\theta_j)}{\partial\rho^4}\right| \le C(h^{d_1-3}+\tau^{d_2-3}). \tag{4.18}$$

For $R_2(\rho,\theta)$, we have

$$\begin{aligned}
R_2(\rho,\theta) &= \frac{\partial^2\phi(\rho,\theta)}{\rho^2\partial\rho^2} - \frac{\partial^2\phi(\rho_i,\theta_j)}{\rho^2\partial\rho^2}\\
&= \frac{\partial^2\phi(\rho,\theta)}{\rho^2\partial\rho^2} - \frac{\partial^2\phi(\rho,\theta_j)}{\rho^2\partial\rho^2} + \frac{\partial^2\phi(\rho,\theta_j)}{\rho^2\partial\rho^2} - \frac{\partial^2\phi(\rho,\theta_j)}{\rho^2\partial\rho^2}\\
&= e_{\rho\rho}(\theta,\theta_n) + e_{\rho\rho}(\rho_m,\theta_n)
\end{aligned} \tag{4.19}$$

and

$$|R_2(\rho,\theta)| \le |e_{\rho\rho}(\theta,\theta_i) + e_{\rho\rho}(\rho_i,\theta_j)| \le C(h^{d_1-1}+\tau^{d_2-1}). \tag{4.20}$$

For $R_3(\rho,\theta)$ we have

$$R_3(\rho,\theta) = \frac{1}{\rho^4}\frac{\partial^4\phi(\rho,\theta)}{\partial\theta^4} - \frac{1}{\rho^4}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\theta^4} = \frac{\partial^4 e(\rho,\theta_i)}{\partial\theta^4} + \frac{\partial^4 e(\rho_i,\theta_j)}{\partial\theta^4} \tag{4.21}$$

and

$$|R_3(\rho,\theta)| \le \left|\frac{\partial^4 e(\rho,\theta_i)}{\partial\theta^4} + \frac{\partial^4 e(\rho_i,\theta_j)}{\partial\theta^4}\right| \le C(h^{d_1-3}+\tau^{d_2-3}). \tag{4.22}$$

Similarly, for $R_4(\rho,\theta)$, $R_5(\rho,\theta)$ and $R_6(\rho,\theta)$, we also get

$$|R_4(\rho,\theta)| \le \left|\frac{2}{\rho^2}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho^3} - \frac{2}{\rho^2}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho^3}\right| \le C(h^{d_1-2}+\tau^{d_2-2}), \tag{4.23}$$

$$|R_5(\rho,\theta)| \le \left|\frac{2}{\rho^2}\frac{\partial^4\phi(\rho,\theta)}{\partial\rho^2\partial\theta^2} - \frac{2}{\rho^2}\frac{\partial^4\phi(\rho_i,\theta_j)}{\partial\rho^2\partial\theta^2}\right| \le C(h^{d_1-3} + \tau^{d_2-3}), \tag{4.24}$$

$$|R_6(\rho,\theta)| \le \left|\frac{2}{\rho^3}\frac{\partial^3\phi(\rho,\theta)}{\partial\rho\partial\theta^2} - \frac{2}{\rho^3}\frac{\partial^3\phi(\rho_i,\theta_j)}{\partial\rho\partial\theta^2}\right| \le C(h^{d_1-2} + \tau^{d_2-2}). \tag{4.25}$$

Combining Eqs (4.18) and (4.19)–(4.25), the proof of Theorem 3 is completed.

## 5. Numerical examples

In the following part, we present some examples to illustrate our numerical scheme analysis. We define the absolute error estimate and relative error estimate as

$$Er = \max\{u_e - u_a\}$$

and

$$Err = \frac{\max\{u_e - u_a\}}{u_e}.$$

**Example 1.** *Consider the following elastic polar curved bar bending*

$$\begin{cases} u_\rho = \dfrac{\sin\theta}{E}\left[D(1-\mu)\ln\rho + A(1-3\mu)\rho^2 + \dfrac{B(1+\mu)}{\rho^2}\right] \\ -\dfrac{2D}{E}\theta\cos\theta + K\sin\theta + L\cos\theta, \\ u_\theta = -\dfrac{\cos\theta}{E}\left[-D(1-\mu)\ln\rho + A(5+\mu) + \dfrac{B(1+\mu)}{\rho^2}\right] \\ +\dfrac{2D}{E}\theta\sin\theta + \left[\dfrac{D(1+\mu)}{E} + K\right]\cos\theta - L\sin\theta \end{cases} \tag{5.1}$$

*and*

$$\begin{cases} \sigma_\rho = \left(2A\rho - \dfrac{2B}{\rho^2} + \dfrac{D}{\rho}\right)\sin\theta, \\ \sigma_\theta = \left(6A\rho + \dfrac{2B}{\rho^2} + \dfrac{D}{\rho}\right)\sin\theta, \\ \tau_{\rho\theta} = -\left(-2A\rho - \dfrac{2B}{\rho^2} + \dfrac{D}{\rho}\right)\cos\theta, \end{cases} \tag{5.2}$$

*where* $A = \frac{P}{2N}, B = -\frac{Pa^2b^2}{2N}, D = -\frac{P}{N}(a^2+b^2), L = \frac{D\pi}{E}, N = a^2 - b^2 + (a^2+b^2)\ln\frac{a}{n}$, $K = -\frac{1}{E}\left[D(1-\mu)\ln\rho_0 + A(1-3\mu)\rho_0^2 + \frac{B(1+\mu)}{\rho_0^2}\right], \rho_0 = \frac{a+b}{2}, a < \theta < b$ *and* $0 < \theta < \frac{\pi}{2}$ *and the boundary conditions are given as* $\sigma_\rho|_{\rho=a} = 0, \sigma_\rho|_{\rho=b} = 0, \tau_{\rho\theta}|_{\rho=a} = 0, \tau_{\rho\theta}|_{\rho=b} = 0, \sigma_\theta = 0, \int_a^b \tau_{\rho\theta}d\rho = P, \theta = 0, u_\rho = 0, u_\theta = 0, \theta = \pi/2$ *and*

$$\sigma_\theta = 0, \tau_{\rho\theta} = -\left(-2A\rho - \frac{2B}{\rho^2} + \frac{D}{\rho}\right), \theta = 0,$$

$$u_\rho = 0, \tau_{\rho\theta} = \frac{1}{E}\left[D(1-\mu)\ln\rho + A(1-3\mu)\rho^2 + \frac{B(1+\mu)}{\rho^2}\right] + K, \theta = \frac{\pi}{2}$$

$$u_\theta = \frac{2D}{E}\sin\theta + L.$$

**Table 1.** Error estimate of barycentric rational interpolation collocation methods with $d = 5$.

| $n$ | *Function* | Equidistant nodes | | Quasi-equidistant nodes | |
|---|---|---|---|---|---|
| | | Absolute error | Relative error | Absolute error | Relative error |
| | $u_\rho$ | 1.0843e-07 | 5.2465e-04 | 3.9398e-10 | 1.9063e-06 |
| 11 | $\sigma_\rho$ | 4.9419e-01 | 2.4710e-04 | 1.0401e-01 | 5.2007e-05 |
| | $\sigma_\theta$ | 2.1686e+00 | 5.9143e-04 | 2.9761e-02 | 8.1168e-06 |
| | $u_\rho$ | 1.0468e-08 | 5.0653e-05 | 5.5667e-12 | 2.6935e-08 |
| 19 | $\sigma_\rho$ | 5.5156e-02 | 2.7578e-05 | 1.0824e-02 | 5.4119e-06 |
| | $\sigma_\theta$ | 2.0936e-01 | 5.7099e-05 | 3.2952e-03 | 8.9868e-07 |

**Table 2.** Error estimate of Lagrange interpolation collocation methods.

| $n$ | *Function* | Equidistant nodes | | Quasi-equidistant nodes | |
|---|---|---|---|---|---|
| | | Absolute error | Relative error | Absolute error | Relative error |
| | $u_\rho$ | 1.4060e-09 | 6.8033e-06 | 2.0507e-11 | 9.9229e-08 |
| 11 | $\sigma_\rho$ | 6.1776e-03 | 3.0888e-06 | 2.4225e-04 | 1.2112e-07 |
| | $\sigma_\theta$ | 2.8120e-02 | 7.6692e-06 | 4.1015e-04 | 1.1186e-07 |
| | $u_\rho$ | 2.3788e-13 | 1.1510e-09 | 1.4206e-15 | 6.8738e-12 |
| 19 | $\sigma_\rho$ | 3.2498e-06 | 1.6249e-09 | 1.8951e-08 | 9.4753e-12 |
| | $\sigma_\theta$ | 3.5000e-06 | 9.5454e-10 | 2.8475e-08 | 7.7659e-12 |

In Tables 1 and 2, the error estimates of displacement and stress are presented for the barycentric rational interpolation collocation methods (BRICMs) with $d = 5$ and barycentric Lagrange interpolation collocation methods with $n = 11$ and $n = 19$. From the table, the displacement and stress have higher accuracy for the Lagrange interpolation collocation methods than for the BRICMs.

**Table 3.** Errors of equidistant nodes with $d_1$ for $\sigma_\rho$.

| $n$ | $d_1 = 2$ | | $d_1 = 3$ | | $d_1 = 4$ | | $d_1 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 3.8722e+00 | | 1.8512e+00 | | 6.7278e-02 | | 3.9574e-02 | |
| 16 | 1.4909e+00 | 1.3770 | 2.5799e-01 | 2.8430 | 6.2873e-03 | 3.4196 | 1.3624e-03 | 4.8604 |
| 32 | 6.0715e-01 | 1.2961 | 3.3272e-02 | 2.9549 | 4.5096e-04 | 3.8014 | 4.4118e-05 | 4.9486 |
| 64 | 2.0902e-01 | 1.5384 | 4.2086e-03 | 2.9829 | 2.9866e-05 | 3.9164 | 1.3797e-06 | 4.9990 |

**Table 4.** Errors of equidistant nodes with $d_2$ for $\sigma_\theta$.

| $n$ | $d_2 = 2$ | | $d_2 = 3$ | | $d_2 = 4$ | | $d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 2.4659e+02 | | 1.2345e+02 | | 5.0487e+00 | | 3.1449e+00 | |
| 16 | 1.4900e+02 | 7.2685e-01 | 3.1157e+01 | 1.9863 | 8.4074e-01 | 2.5862 | 1.9203e-01 | 4.0336 |
| 32 | 8.3557e+01 | 8.3445e-01 | 7.7866e+00 | 2.0005 | 1.1597e-01 | 2.8578 | 1.1921e-02 | 4.0097 |
| 64 | 4.4684e+01 | 9.0301e-01 | 1.9452e+00 | 2.0011 | 1.5115e-02 | 2.9398 | 7.4421e-04 | 4.0016 |

In Tables 3 and 4, the error estimates of $\sigma_\rho$ and $\sigma_\theta$ are presented for barycentric rational interpolation with $d_1 = d_2 = 2, 3, 4, 5$ for equidistant nodes.

**Table 5.** Errors of quasi-equidistant nodes with $\sigma_\rho$.

| $n$ | $d_1 = 2$ | | $d_1 = 3$ | | $d_1 = 4$ | | $d_1 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 7.6984e+00 | | 3.0925e+00 | | 2.7111e-02 | | 7.8893e-03 | |
| 16 | 2.5097e+00 | 1.6170 | 1.2113e-01 | 4.6742 | 1.6515e-03 | 4.0370 | 3.3223e-04 | 4.5696 |
| 32 | 7.5343e-01 | 1.7360 | 1.3255e-02 | 3.1919 | 7.3991e-05 | 4.4803 | 4.6671e-06 | 6.1535 |
| 64 | 2.0590e-01 | 1.8715 | 1.0018e-03 | 3.7259 | 6.6256e-06 | 3.4812 | 3.8948e-06 | |

**Table 6.** Errors of quasi-equidistant nodes with $d_2$ for $\sigma_\theta$.

| $n$ | $d_2 = 2$ | | $d_2 = 3$ | | $d_2 = 4$ | | $d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.6510e+02 | | 1.1570e+02 | | 2.1174e+00 | | 1.0292e+00 | |
| 16 | 1.0058e+02 | 7.1503e-01 | 3.4378e+00 | 5.0727 | 5.6098e-02 | 5.2382 | 1.3486e-02 | 6.2539 |
| 32 | 5.1539e+01 | 9.6454e-01 | 5.7939e-01 | 2.5689 | 1.6043e-03 | 5.1280 | 8.7592e-05 | 7.2665 |
| 64 | 2.0298e+01 | 1.3443e+00 | 6.4441e-02 | 3.1685 | 1.2028e-04 | 3.7375 | 8.0433e-05 | |

In Tables 5 and 6, the error estimates of $\sigma_\rho$ and $\sigma_\theta$ are presented for barycentric rational interpolation with $d_1 = d_2 = 2, 3, 4, 5$ for quasi-equidistant nodes.

**Table 7.** Errors of equidistant nodes with $d_1 = d_2$ for $u_\rho$.

| $m \times n$ | $d_1 = d_2 = 2$ | | $d_1 = d_2 = 3$ | | $d_1 = d_2 = 4$ | | $d_1 = d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| $8 \times 8$ | 4.6386e-01 | | 2.3538e-01 | | 7.3939e-02 | | 4.3353e-02 | |
| $16 \times 16$ | 2.2488e-01 | 1.0445 | 3.7954e-02 | 2.6326 | 5.0343e-03 | 3.8765 | 8.9588e-04 | 5.5967 |
| $32 \times 32$ | 7.1140e-02 | 1.6604 | 4.5791e-03 | 3.0511 | 3.0493e-04 | 4.0452 | 2.1490e-05 | 5.3816 |
| $64 \times 64$ | 1.8606e-02 | 1.9349 | 5.4216e-04 | 3.0783 | 1.8514e-05 | 4.0418 | 5.8452e-07 | 5.2003 |

**Table 8.** Errors of equidistant nodes with $d_1 = d_2$ for $u_\theta$.

| $m \times n$ | $d_1 = d_2 = 2$ | | $d_1 = d_2 = 3$ | | $d_1 = d_2 = 4$ | | $d_1 = d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| $8 \times 8$ | 3.8459e-01 | | 2.0080e-01 | | 6.4882e-02 | | 3.7789e-02 | |
| $16 \times 16$ | 1.9199e-01 | 1.0023 | 3.3038e-02 | 2.6036 | 4.3719e-03 | 3.8915 | 7.7682e-04 | 5.6042 |
| $32 \times 32$ | 6.1740e-02 | 1.6368 | 4.0051e-03 | 3.0442 | 2.6542e-04 | 4.0419 | 1.8712e-05 | 5.3756 |
| $64 \times 64$ | 1.6255e-02 | 1.9253 | 4.7537e-04 | 3.0747 | 1.6149e-05 | 4.0387 | 5.1045e-07 | 5.1960 |

**Table 9.** Errors of equidistant nodes with $d$ for $\sigma_\rho$.

| $m \times n$ | $d_1 = d_2 = 2$ | | $d_1 = d_2 = 3$ | | $d_1 = d_2 = 4$ | | $d_1 = d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| $8 \times 8$ | 2.4305e+04 | | 9.1855e+03 | | 3.3907e+03 | | 1.9465e+03 | |
| $16 \times 16$ | 9.0087e+03 | 1.4319 | 1.6141e+03 | 2.5087 | 2.2601e+02 | 3.9071 | 4.0143e+01 | 5.5996 |
| $32 \times 32$ | 3.1667e+03 | 1.5084 | 2.0801e+02 | 2.9560 | 1.4637e+01 | 3.9487 | 1.3872e+00 | 4.8549 |
| $64 \times 64$ | 9.2345e+02 | 1.7778 | 2.8337e+01 | 2.8759 | 1.0162e+00 | 3.8484 | 7.2075e-02 | 4.2665 |

**Table 10.** Errors of equidistant nodes with $d$ for $\sigma_\theta$.

| $m \times n$ | $d_1 = d_2 = 2$ | | $d_1 = d_2 = 3$ | | $d_1 = d_2 = 4$ | | $d_1 = d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| $8 \times 8$ | 6.7329e+04 | | 3.6992e+04 | | 1.3543e+04 | | 7.7757e+03 | |
| $16 \times 16$ | 3.6203e+04 | 0.8951 | 6.4833e+03 | 2.5124 | 9.0279e+02 | 3.9070 | 1.6079e+02 | 5.5957 |
| $32 \times 32$ | 1.2708e+04 | 1.5104 | 8.3482e+02 | 2.9572 | 5.8440e+01 | 3.9494 | 4.1882e+00 | 5.2627 |
| $64 \times 64$ | 3.7039e+03 | 1.7786 | 1.1076e+02 | 2.9140 | 3.9820e+00 | 3.8754 | 1.4185e-01 | 4.8839 |

**Table 11.** Errors of equidistant nodes with $d$ for $\tau_{\rho\theta}$.

| $m \times n$ | $d_1 = d_2 = 2$ | | $d_1 = d_2 = 3$ | | $d_1 = d_2 = 4$ | | $d_1 = d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| $8 \times 8$ | 4.7320e+03 | | 2.8512e+03 | | 1.0721e+03 | | 6.4853e+02 | |
| $16 \times 16$ | 3.8711e+03 | 0.2897 | 7.7859e+02 | 1.8726 | 1.1139e+02 | 3.2667 | 2.0975e+01 | 4.9504 |
| $32 \times 32$ | 1.8997e+03 | 1.0270 | 1.3801e+02 | 2.4961 | 9.7341e+00 | 3.5165 | 7.2237e-01 | 4.8598 |
| $64 \times 64$ | 6.7700e+02 | 1.4886 | 2.1544e+01 | 2.6795 | 7.6752e-01 | 3.6648 | 2.5826e-02 | 4.8058 |

In Tables 7–11, the errors of $u_\rho$, $u_\theta$, $\sigma_\rho$, $\sigma_\theta$ and $\tau_{r\theta}$ are shown for barycentric rational interpolation with $d = 2, 3, 4, 5$ for equidistant nodes. The convergence rate is $O(h^d)$ for $u_\rho$, $u_\theta$, $\sigma_\rho$ and $\sigma_\theta$ , and $O(h^{d-1})$ for $\tau_{\rho\theta}$ which agrees with our theorem analysis.

**Example 2.** *Consider the the following elastic thick circular:*

$$\begin{cases} \sigma_\rho = \dfrac{a^2 P_a - b^2 P_b}{b^2 - a^2} + \dfrac{a^2 b^2 (P_b - P_a)}{b^2 - a^2} \dfrac{1}{\rho^2} \\ \sigma_\rho = \dfrac{a^2 P_a - b^2 P_b}{b^2 - a^2} - \dfrac{a^2 b^2 (P_b - P_a)}{b^2 - a^2} \dfrac{1}{\rho^2} \end{cases} \tag{5.3}$$

*with*

$$u_\rho = \frac{(1-\mu)(a^2P_a - b^2P_b)}{E(b^2-a^2)} + \frac{(1+\mu)a^2b^2(P_b - P_a)}{E(b^2-a^2)\rho}. \tag{5.4}$$

*Then we get the displacement equation as*

$$\frac{d^2u_\rho}{d\rho^2} + \frac{1}{\rho}\frac{du_\rho}{d\rho} - \frac{1}{\rho^2}u_\rho = 0, a < \rho < b \tag{5.5}$$

*and the boundary conditions can be given as*

$$\sigma_\rho(a) = -P_a, \sigma_\rho(b) = -P_b,$$

*which means that*

$$\frac{E}{1-\mu^2}\left(\frac{\partial u_\rho}{\partial \rho} + \mu\frac{u_\rho}{\rho}\right)_{\rho=a} = -P_a, \frac{E}{1-\mu^2}\left(\frac{\partial u_\rho}{\mu\partial \rho} + \frac{u_\rho}{\rho}\right)_{\rho=b} = -P_b, \tag{5.6}$$

*and the matrix equations can be given as*

$$\left[R^{(2,0)} + \mathrm{diag}(\frac{1}{\rho})R^{(1,0)} + \mathrm{diag}(\frac{1}{\rho^2})\right]U_\rho = 0 \tag{5.7}$$

*and*

$$\begin{cases} \sigma_\rho = \dfrac{E}{1-\mu^2}\left(R^{(1,0)} + \mu\mathrm{diag}(\dfrac{1}{\rho})\right)U_\rho, \\ \sigma_\theta = \dfrac{E}{1-\mu^2}\left(\mu R^{(1,0)} + \mathrm{diag}(\dfrac{1}{\rho})\right)U_\rho \end{cases} \tag{5.8}$$

*with* $a = 0.5$ *m*, $b = 1$ *m*, $P_a = 1000$ *Pa*, $P_b = 2000$ *Pa*, $E = 10^7$ *Pa*, $\mu = 0.3$.

*In Tables 12 and 13, the error estimates of the BRICM with* $d = 5$ *and Lagrange interpolation collocation methods with* $n = 11$ *and* $n = 19$ *for displacement and stress are given.*

**Table 12.** Error estimates of the BRICM with different $d$ with $d = 5$.

| $n$ | *Function* | Equidistant nodes | | Quasi-equidistant nodes | |
|---|---|---|---|---|---|
| | | Absolute error | Relative error | Absolute error | Relative error |
| | $u_\rho$ | 2.9925e-02 | 9.2382e-03 | 1.7356e-04 | 4.9117e-05 |
| | $u_\theta$ | 1.7418e-02 | 9.9419e-03 | 9.7870e-05 | 4.7756e-05 |
| 11 | $\sigma_\rho$ | 4.7749e+02 | 2.1424e-02 | 8.8496e+00 | 4.7244e-04 |
| | $\sigma_\theta$ | 2.8427e+03 | 9.4039e-03 | 2.3377e+01 | 6.5748e-05 |
| | $\tau_{\rho\theta}$ | 2.7594e+02 | 1.2381e-02 | 4.3209e+00 | 2.3067e-04 |
| | $u_\rho$ | 2.9642e-03 | 5.5736e-04 | 6.5569e-07 | 1.1227e-07 |
| | $u_\theta$ | 1.6923e-03 | 6.0049e-04 | 2.1714e-07 | 6.4625e-08 |
| 19 | $\sigma_\rho$ | 3.8746e+01 | 1.0246e-03 | 2.2016e-01 | 6.9270e-06 |
| | $\sigma_\theta$ | 2.6988e+02 | 5.5473e-04 | 1.7981e+00 | 3.0837e-06 |
| | $\tau_{\rho\theta}$ | 2.8528e+01 | 7.5436e-04 | 4.6880e-01 | 1.4750e-05 |

**Table 13.** Error estimates of Lagrange interpolation collocation methods.

| $n$ | *Function* | Equidistant nodes | | Quasi-equidistant nodes | |
|---|---|---|---|---|---|
| | | Absolute error | Relative error | Absolute error | Relative error |
| | $u_\rho$ | 5.7368e-05 | 1.7710e-05 | 2.1383e-09 | 6.0515e-10 |
| | $u_\theta$ | 3.1838e-05 | 1.8173e-05 | 1.3071e-09 | 6.3780e-10 |
| 11 | $\sigma_\rho$ | 4.4327e+00 | 1.9889e-04 | 3.2005e-03 | 1.7086e-07 |
| | $\sigma_\theta$ | 5.4486e+00 | 1.8024e-05 | 3.7800e-04 | 1.0631e-09 |
| | $\tau_{\rho\theta}$ | 4.0339e-01 | 1.8099e-05 | 5.6771e-05 | 3.0307e-09 |
| | $u_\rho$ | 5.3104e+00 | 9.9853e-01 | 9.0233e-09 | 1.5450e-09 |
| | $u_\theta$ | 2.8335e+00 | 1.0054e+00 | 5.5352e-09 | 1.6474e-09 |
| 19 | $\sigma_\rho$ | 6.3820e+05 | 1.6876e+01 | 5.9085e-03 | 1.8591e-07 |
| | $\sigma_\theta$ | 4.8026e+05 | 9.8716e-01 | 1.9612e-03 | 3.3635e-09 |
| | $\tau_{\rho\theta}$ | 4.7011e+04 | 1.2431e+00 | 1.2233e-03 | 3.8490e-08 |

*In Tables 14–16, the errors of barycentric rational interpolation, i.e., $u_\rho$, $\sigma_\rho$ and $\sigma_\theta$ are shown with $d = 2, 3, 4, 5$ for equidistant nodes and $O(h^d)$ for $u_\rho$, $\sigma_\rho$ and $\sigma_\theta$ which agrees with our theorem analysis.*

**Table 14.** Errors of equidistant nodes with $d_1$ for $u_\rho$.

| $n$ | $d_1 = 2$ | | $d_1 = 3$ | | $d_1 = 4$ | | $d_1 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 6.9993e-06 | | 3.3694e-06 | | 1.0401e-06 | | 6.6611e-07 | |
| 16 | 1.7434e-06 | 2.0053 | 4.4148e-07 | 2.9320 | 9.2244e-08 | 3.4952 | 2.9354e-08 | 4.5041 |
| 32 | 4.3685e-07 | 1.9967 | 5.7223e-08 | 2.9477 | 6.7371e-09 | 3.7753 | 1.1405e-09 | 4.6858 |
| 64 | 1.0946e-07 | 1.9967 | 7.2943e-09 | 2.9717 | 4.5402e-10 | 3.8913 | 4.0016e-11 | 4.8330 |

**Table 15.** Errors of equidistant nodes with $d_1$ for $\sigma_\rho$.

| $n$ | $d_1 = 2$ | | $d_1 = 3$ | | $d_1 = 4$ | | $d_1 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 2.8959e+01 | | 1.3156e+01 | | 3.9232e+00 | | 2.4593e+00 | |
| 16 | 9.5402e+00 | 1.6019 | 2.3041e+00 | 2.5135 | 4.6863e-01 | 3.0655 | 1.4678e-01 | 4.0665 |
| 32 | 2.6910e+00 | 1.8258 | 3.3753e-01 | 2.7711 | 3.8783e-02 | 3.5949 | 6.4731e-03 | 4.5031 |
| 64 | 7.1232e-01 | 1.9176 | 4.5531e-02 | 2.8901 | 2.7689e-03 | 3.8080 | 2.4068e-04 | 4.7492 |

**Table 16.** Errors of equidistant nodes with $d_2$ for $\sigma_\theta$.

| $n$ | $d_2 = 2$ | | $d_2 = 3$ | | $d_2 = 4$ | | $d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 6.7387e+01 | | 1.3999e+02 | | 2.0802e+01 | | 1.3322e+01 | |
| 16 | 8.8297e+00 | 2.0053 | 3.4869e+01 | 2.9320 | 1.8449e+00 | 3.4952 | 5.8709e-01 | 4.5041 |
| 32 | 1.1445e+00 | 1.9967 | 8.7371e+00 | 2.9477 | 1.3474e-01 | 3.7753 | 2.2810e-02 | 4.6858 |
| 64 | 1.4589e-01 | 1.9967 | 2.1892e+00 | 2.9717 | 9.0804e-03 | 3.8913 | 8.0031e-04 | 4.8330 |

*In Tables 17–19, for quasi-equidistant nodes, the errors of $u_\rho$, $\sigma_\rho$ and $\sigma_\theta$ are given for the barycen-*

*tric rational interpolation with $d_1 = d_2 = 2, 3, 4, 5$ . The convergence rate is $O(h^{2d-2})$ for $u_\rho$ and $O(h^d)$ for $\sigma_\rho$ and $\sigma_\theta$ which coincides with our theorem analysis.*

**Table 17.** Errors of quasi-equidistant nodes with $d_1$ for $u_\rho$.

| $n$ | $d_1 = 2$ | | $d_1 = 3$ | | $d_1 = 4$ | | $d_1 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 5.1242e-07 | | 4.7200e-08 | | 5.6060e-08 | | 4.4966e-08 | |
| 16 | 5.7993e-08 | 3.1434 | 4.6281e-09 | 3.3503 | 5.2101e-10 | 6.7495 | 5.7807e-11 | 9.6034 |
| 32 | 7.9035e-09 | 2.8753 | 1.8128e-10 | 4.6741 | 2.8497e-12 | 7.5144 | 1.5231e-13 | 8.5681 |
| 64 | 7.2603e-10 | 3.4444 | 5.1410e-13 | 8.4620 | 1.3194e-12 | 1.1110 | 6.6684e-12 | - |

**Table 18.** Errors of quasi-equidistant nodes with $d_1$ for $\sigma_\rho$.

| $n$ | $d_1 = 2$ | | $d_1 = 3$ | | $d_1 = 4$ | | $d_1 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 2.3800e+01 | | 6.5238e+00 | | 1.0394e+00 | | 5.2459e-01 | |
| 16 | 4.1561e+00 | 2.5176 | 7.9468e-01 | 3.0373 | 1.3660e-01 | 2.9276 | 3.1504e-02 | 4.0576 |
| 32 | 8.5118e-01 | 2.2877 | 8.1116e-02 | 3.2923 | 7.5261e-03 | 4.1819 | 9.8814e-04 | 4.9947 |
| 64 | 1.9227e-01 | 2.1464 | 8.8700e-03 | 3.1930 | 5.2740e-04 | 3.8349 | 5.4116e-04 | 0.8686 |

**Table 19.** Errors of quasi-equidistant nodes with $d_2$ for $\sigma_\theta$.

| $n$ | $d_2 = 2$ | | $d_2 = 3$ | | $d_2 = 4$ | | $d_2 = 5$ | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.0685e+01 | | 2.6150e+00 | | 1.1212e+00 | | 8.9933e-01 | |
| 16 | 1.9955e+00 | 2.4207 | 2.8961e-01 | 3.1746 | 4.3436e-02 | 4.6900 | 1.0093e-02 | 6.4774 |
| 32 | 3.5851e-01 | 2.4767 | 2.6615e-02 | 3.4438 | 2.2886e-03 | 4.2463 | 2.9545e-04 | 5.0943 |
| 64 | 6.6816e-02 | 2.4237 | 2.6669e-03 | 3.3190 | 1.5324e-04 | 3.9007 | 2.3041e-04 | - |

## 6. Concluding remarks

In this paper, first, the equilibrium equations were transformed into polar coordinates; then, the LBRCM was presented to solve the equilibrium equations. Third, the matrix equations of the equilibrium equations were obtained and the convergence rate of the LBRCM was also proved. At last, some numerical examples were given to validate the proposed theorem. The plane elasticity problems under the irregular domain can also be solved by using the LBRCM, as will be discussed it in the near future.

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## Conflict of interest

The author declares that there are no conflicts of interest.

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