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Research article

Adaptive fixed-time stabilization for a class of nonlinear uncertain systems

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Abstract: Finite-time stability (FTS) has attained great interest in nonlinear control systems in recent two decades. Fixed-time stability (FxTS) is an improved version of FTS in consideration of its settling time independent of the initial values. In this article, the adaptive fixed-time stabilization issue is studied for a kind of nonlinear systems with nonlinear parametric uncertainties and uncertain control coefficients. Using the adaptive estimate and the adding one power integrator (AOPI) design tool, we propose a two-phase control strategy, which makes that the system states tend to the origin in fixed-time, and other signals are bounded on $[0, +\infty)$. We prove the main results by means of the recently developed fixed-time Lyapunov stability theory. Finally, we apply the proposed adaptive fixed-time stabilizing control strategy into the pendulum system, and the simulation results verify the efficacy of the presented method.

Keywords: finite-time control; adaptive control; fixed-time stabilization; parametric uncertainty

1. Introduction

The fixed-time stabilization studied in this article for the family of nonlinear systems is described by

$$\begin{cases} \dot{\xi}_1 &= \lambda_1(t,\xi)\,\xi_2 + g_1(t,\xi,\theta) \\ \dot{\xi}_j &= \lambda_j(t,\xi)\,\xi_{j+1} + g_j(t,\xi,\theta),\ 1 \le j \le n-1 \\ \dot{\xi}_n &= \lambda_n(t,\xi)\,u + g_n(t,\xi,\theta) \end{cases} \tag{1.1}$$

with the system state $\xi = (\xi_1, \cdots, \xi_n)^{\mathrm{T}}$ and the control input $u \in R$. The uncertain functions $g_j : R^+ \times R^n \times R^{n_\theta} \to R$, $j = 1, \cdots, n$, are continuously differentiable with $g_j(t, 0, \theta) = 0$, $\forall\, t \ge 0$, and $\theta \in R^{n_\theta}$ represents the unknown constant vector. The continuous functions $\lambda_j(t, \xi) \ne 0$ denote the control coefficients, and specifically, $\lambda_n(t, \xi)$ is also named as the control direction.

The objective of this article is to answer the following questions:

(i) In the presence of nonlinear parametric uncertainties, is there a globally stabilizing controller that renders the equilibrium $x = 0$ of (1.1) fixed-time globally stable (i.e., global Lyapunov stability with additional properties such as fixed-time convergence)?

(ii) How to design a fixed-time controller which could render the closed-loop system fixed-time stable without residual sets allowing the nonlinear parametric uncertainties?

The stabilizing control of nonlinear uncertain systems is one of the most significant tasks in system control. It is noted that the tracking control, output regulation, or disturbance rejection, could be transformed into a stabilization issue by constructing some sort of "error" variable [1]. Asymptotic stabilization together with its enhanced version-exponential stabilization plays a key role in stabilization control for linear or nonlinear systems [2]. The settling time in asymptotic or exponential stabilization is infinite in theory. That is, the system states or tracking error approach to its equilibrium point when the time evolves to infinity. In practice, the finite settling time is obviously more appealing, considering the operation is done in a finite time. It is on this background that the finite-time stability (FTS) is proposed in [7] with the time-optimal control [3]. FTS characterizes that a control system is firstly Lyapunov stable and then exhibits finite-time convergent property of its equilibrium. As noted in [4–6], finite-time stability has better performances, such as rapid response, high precision, and robust properties. Currently, the finite-time stabilizing control is still one of the most active topics in the community of nonlinear control systems, see [8–22] and the references therein.

In last several years, a novel variant of FTS, i.e., fixed-time stability (FxTS), is proposed in [23], which is an improved version of finite-time stability. In finite-time stability, the settling time is depending on the system initial values, and it will become large when the initial values are far away from the origin. Fixed-time stability overcomes this drawback, and its settling time is bounded by one fixed constant independent of the initial values [24]. Fixed-time control has some advantages in comparison with finite-time control because the controller could be designed in a manner that its control performance is obtained in a predefined time and regardless to its initial conditions. As a result, fixed-time control displays a better behaviour in the sense that it does not require that the design parameters are re-tuned in order to to maintain the settling time. In recent years, fixed-time control is intensively studied in nonlinear control community, and many interesting results have been reported in this area. For example, the works in [25] and [26] develop the implicit Lyapunov function methodology to investigate the robust FTS and FxTS of nonlinear systems. The work [27] develops a fixed-time terminal sliding mode (TSM) controller for second-order nonlinear perturbed systems. The work in [28] proposes a finite/fixed-time stabilizing control scheme for nonlinear strict-feedback systems using the AOPI technique. The fixed-time control has found its applications in the multi-agent systems (MAS) [29], path-following of automatic vehicles [30, 31], nonlinear parametrization [32], nonholonomic systems [33], and robot systems [34], etc.

Due to the measurement limitations or the modelling errors in practice, the uncertainties are widespread in control systems. As is well known, the adaptive control plays a key role in dealing with parametric uncertainty [35–37]. In literature, there have been many results reported in adaptive fixed-time stabilizing control for nonlinear systems with unknown parameters, such as [38–44], etc. It is pointed out that the existing adaptive control results could not realize the zero error stabilization, that is, the system states does only converge to a small neighborhood of the origin in a given finite time. In fact, a weaker version of fixed-time stability, that is, the practical fixed-time stability is achieved in [40–44]. In view of the unknown parameters in the controlled plants and the finite convergent time regardless to initial values, it brings some residual terms when the control Lyapunov function method is used in control design. This results in that the fixed-time stabilizing without residual sets become extremely difficult. The related methods which effectively handle the adaptive finite-time stability such as [6] and [22] are inapplicable here.

In this work, we will address the fixed-time stabilizing control with zero residual set for system (1.1). Our aim is to propose a robust adaptive stabilizing controller keeping all the system states globally convergent to zero in some fixed-time. In comparison with the reported results on adaptive fixed-time stabilizing control for uncertain nonlinear systems, the novelties of this article lie in that: 1) This article provides a systematic design of a robust adaptive fixed-time stabilizer to deal with nonlinear parametric uncertainties. The problem of global adaptive fixed-time stabilizing control without residual sets for nonlinear uncertain systems (1.1) is well addressed. 2) We present a constructive procedure to carry out the analysis, since the widely used Barbalat's lemma may fail in nonsmooth feedback control, which brings severe difficulties to the stability analysis in fixed-time control. The constructive analysis gets around this burden, and realizes the zero-error stabilizing control in the presence of unknown parameters in this paper.

The article is structured as follows. Some preliminaries are provided in Section 2. Section 3 gives the adaptive fixed-time stabilizing control design. Main results are presented in Section 4. Section 5 illustrates the proposed control scheme via a pendulum system. Section 6 concludes the article.

Notations The following notations will be used in the paper: R denotes the set of real numbers, and R^n denotes the n-dimensional Euclidean space; $|a|$ denotes the absolute value of scalar $a \in R$; ξ^{T} stands for the transposition of a vector $\xi \in R^n$. For a vector $\xi = (\xi_1, \cdots, \xi_n)^{\mathrm{T}} \in R^n$, we denote $\bar{\xi}_i = (\xi_1, \cdots, \xi_i)^{\mathrm{T}}$ when $i = 1, \cdots, n$, and we let $\bar{\xi}_1 = \xi_1$ and $\bar{\xi}_n = \xi$. The arguments of functions will be sometimes omitted or simplified, for example, a function $f(\xi(t))$ is denoted by $f(\xi)$ or $f(\cdot)$ whenever no confusion can arise from the context.

2. Preliminaries

Take into account the following nonlinear system

$$\dot{\xi} = f(t, \xi), \ \xi(0) = \xi_0, \ \xi \in R^n \tag{2.1}$$

with $f : [0, \infty) \times S \to R^n$, $h(t, 0) = 0$, and $S \subset R^n$ containing the origin.

Definition 1 [4]: The equilibrium $\xi = 0$ of (2.1) is globally finite-time stable with $S = R^n$ if it is globally asymptotically stable and moreover any trajectory $\xi(t, \xi_0)$ of (2.1) arrives at the equilibrium at a finite time instant $T(\xi_0)$, i.e., $\xi(t, \xi_0) = 0$, $\forall\, t \geq T(\xi_0)$, where $T(\xi(0)) \geq 0$ is the settling-time function.

Definition 2 [23]: The equilibrium $\xi = 0$ of (2.1) is globally fixed-time stable if it is globally finite-time stable and the settling-time function $T(\xi_0)$ is bounded by a fixed number, i.e., there is a positive constant $T_{\max}$ independent of $\xi(0)$ such that $T(\xi_0) \leq T_{\max}$, $\forall\, \xi_0 \in R^n$.

Definition 3: The adaptive fixed-time stabilization problem is to find a continuously differential control law

$$\begin{cases} u(t) &= \mu(\xi(t), \chi(t)), \ \mu(0, \chi(t)) = 0, \\ \dot{\chi}(t) &= \nu(\xi(t), \chi(t)), \ \nu(0, \chi(t)) = 0, \end{cases} \tag{2.2}$$

where $\mu(\cdot)$ and $\nu(\cdot)$ are continuous functions, and $\chi(t)\in R$ is an adaptive variable to estimate the uncertainties, such that the solutions $(\xi(t)^{\mathrm{T}},\chi(t))^{\mathrm{T}}$ of system (1.1) with controller (2.2) is globally uniformly bounded. Additionally, for any $(\xi(0)^{\mathrm{T}},\chi(0))^{\mathrm{T}}\in R^{n+1}$, $\xi(t)$ converges to its equilibrium in fixed time. That is, for any $(\xi(0)^{\mathrm{T}},\chi(0))^{\mathrm{T}}$, there exists a time instant $T > 0$ such that $\xi(t) = 0$ for any $t > T$, where T is the settling time and does not dependent on $(\xi(0)^{\mathrm{T}},\chi(0))^{\mathrm{T}}$.

We provide some useful lemmas for the following control design and analysis.

Lemma 1 [28]: For the system: $\dot{z} = -c_1 z^{\frac{h}{k}} - c_2 z^{\frac{m}{n}}$ with $z \in R$ and $z(0) = z_0$, where h, k, m, n are odd positive integers with $h < k$ and $m > n$, $c_1 > 0$, and $c_2 > 0$. Then the origin of system is globally fixed-time stable and its settling time T is bounded by $T^* \leq \frac{1}{c_1}\frac{k}{k-h} + \frac{1}{c_2}\frac{m}{m-n}$.

Lemma 2 [40]: Suppose that the continuous function $W(\xi) : R^n \to R$ is positive definite and radially unbounded satisfying

$$\dot{W}(\xi) \leq -c_1 W^{r_1}(\xi) - c_2 W^{r_2}(\xi) + \eta, \tag{2.3}$$

where $c_1 > 0$, $c_2 > 0$, $0 < r_1 < 1$ and $r_2 > 1$ and $0 < \eta < \infty$. Then, the system (2.1) is practically fixed-time stable. Additionally, the trajectory converges into the residual set given by

$$\Omega = \left\{ \lim_{t\to T} \xi \,\Big|\, W(\xi) \leq \min\left\{ \Big(\frac{\eta}{c_1(1-\sigma)}\Big)^{\frac{1}{r_1}}, \Big(\frac{\eta}{c_2(1-\sigma)}\Big)^{\frac{1}{r_2}} \right\}\right\}, \tag{2.4}$$

where $0 < \sigma < 1$, and the settling time has the following upper bound

$$T \leq \frac{1}{c_1\sigma(1-r_1)} + \frac{1}{c_2\sigma(r_2-1)}. \tag{2.5}$$

Lemma 3 [8]: Given $0 < \nu = \frac{m}{n} \leq 1$ with $m, n > 0$ odd integers, the following holds

$$|a^{\nu} - b^{\nu}| \leq 2^{1-\nu}|a-b|^{\nu},\ a, b \in R. \tag{2.6}$$

Lemma 4 [8]: Assume $p > 0, q > 0$, $p \in R$, $q \in R$, and the function $\varepsilon(a,b) > 0$, $a, b \in R$. Then,

$$a^p b^q \leq \frac{p\varepsilon(a,b)|x|^{p+q}}{p+q} + \frac{q\varepsilon^{-\frac{p}{q}}(a,b)|y|^{p+q}}{p+q}. \tag{2.7}$$

Lemma 5 [28]: Assume $a_1 > 0, \cdots, a_m > 0, b > 0$, and $a_1 \in R, \cdots, a_m \in R, b \in R$, then there holds

$$(a_1 + \cdots + a_m)^b \leq \max\{m^{b-1}, 1\}(a_1^b + \cdots + a_m^b). \tag{2.8}$$

Throughout this article, we need the following hypothesis.

Assumption 1: For each $g_i(t,\xi,\theta)$, there exists a smooth function $\varphi_i(\xi_1,\cdots,\xi_i) \geq 0$, satisfying

$$\left| g_i(t,\xi,\theta) \right| \leq (|\xi_1| + \cdots + |\xi_i|)\,\varphi_i(\xi_1,\cdots,\xi_i)\,\sigma, \tag{2.9}$$

where $\sigma \geq 1$ is a constant dependent on θ.

Assumption 2: The uncertain control coefficients $\lambda_i(t,\xi)$ satisfy

$$\lambda_{i1} \leq \lambda_i(t,\xi) \leq \lambda_{i2}, \tag{2.10}$$

where $\lambda_{i1}, \lambda_{i2} > 0$ are known positive real numbers.

Remark 1: It is noted that the Assumptions 1–2 are commonly used in literature. Assumption 1 shows that $x = 0$ is the equilibrium of (1.1). It is actually a requirement if the stabilizing control without residual sets. This assumption can also be seen in existing works [45,46]. Assumption 2 shows that the uncertain control coefficients $\lambda_i(t,\xi)$, $i = 1, \cdots, n$, have known lower and upper bounds, which avoids the possible singularity [47].

3. Design of adaptive fixed-time stabilizer

Before the control design, we choose the candidate Lyapunov functions in the form of

$$U_i = \int_{z_i^*}^{z_i} (\tau^{\frac{1}{r_i}} - z_i^{*\frac{1}{r_i}})^{2-r_i} \mathrm{d}\tau, \ 1 \le \mathrm{i} \le \mathrm{n}, \tag{3.1}$$

with the following positive real numbers:

$$r = \frac{4n}{2n+1}, \ r_1 = 1, \ r_k + \frac{2}{2n+1} = r_{k-1}, \ k = 2, \cdots, n+1. \tag{3.2}$$

It can be shown that U_i's are positive definite functions (see [8]). In the following control design, we construct the continuously differentiable function

$$W_i^* = \sum_{k=1}^{i} U_k = \sum_{k=1}^{i} \int_{z_k^*}^{z_k} (\tau^{\frac{1}{r_k}} - \xi_k^{*\frac{1}{r_k}})^{2-r_k} \mathrm{d}\tau. \tag{3.3}$$

Choose the following virtual control laws together as well as the errors

$$\begin{aligned}
\xi_1^* &= 0, & z_1 &= \xi_1^{\frac{1}{r_1}} - \xi_1^{*\frac{1}{r_1}}, \\
\xi_2^* &= -\frac{1}{\lambda_{11}} z_1^{r_2} \zeta_1(\xi_1, \widehat{\Psi}), & z_2 &= \xi_2^{\frac{1}{r_2}} - \xi_2^{*\frac{1}{r_2}}, \\
&\vdots & &\vdots \\
\xi_{n+1}^* &= -\frac{1}{\lambda_{n1}} z_n^{r_{n+1}} \zeta_n(\xi, \widehat{\Psi}), & z_n &= \xi_n^{\frac{1}{r_n}} - \xi_n^{*\frac{1}{r_n}},
\end{aligned} \tag{3.4}$$

and the parameter updating law

$$\dot{\widehat{\Psi}} = \Gamma_n(\xi, \widehat{\Psi}) \tag{3.5}$$

with $\Gamma_1 = z_1^r L_1(\xi_1)$, $\Gamma_i = \Gamma_{i-1} + z_i^r L_i(\bar{\xi}_i, \widehat{\Psi})$, $2 \le i \le n$. Particularly, $u = \xi_{n+1}^*$, $\zeta_i(\cdot) \ge 0$ and $L_i(\cdot) \ge 0$ are some smooth functions determined later.

The Propositions 1–6 are used in control design. We provide their proofs in Appendices A–E.

Proposition 1: Consider

$$W_n^* = \sum_{k=1}^{n} \int_{\xi_k^*}^{\xi_k} (\tau^{\frac{1}{r_k}} - \xi_k^{*\frac{1}{r_k}})^{2-r_k} \mathrm{d}\tau, \tag{3.6}$$

and one have

$$W_n^* \le 2(z_1^2 + \cdots + z_n^2). \tag{3.7}$$

Proposition 2: One can find a positive number H_{i1} satisfying

$$\lambda_{i-1}(t, \xi) z_{i-1}^{2-r_{i-1}} (\xi_i - \xi_i^*) \le \frac{1}{4} z_{i-1}^r + z_i^r H_{i1}, \ i = 2, \cdots, n. \tag{3.8}$$

Proposition 3: There is a continuously differentiable function $L_{i1}(\bar{\xi}_i, \widehat{\Psi}) \geq 0$ such that

$$z_i^{2-r_i} g_i(t, z, \theta) \leq \sum_{j=1}^{i-1} \frac{1}{4} z_j^r + z_i^r L_{i1}(\bar{\xi}_i, \widehat{\Psi})\Psi, \ i = 2, \cdots, n. \tag{3.9}$$

Proposition 4: There are positive continuously differentiable functions $H_{i2}(\bar{\xi}_{i-1}, \widehat{\Psi}) \geq 0$ and $L_{i2}(\bar{\xi}_i, \widehat{\Psi})$ such that

$$\sum_{j=1}^{i-1} \frac{\partial U_i}{\partial \xi_j} \dot{\xi}_j \leq \sum_{j=1}^{i-1} \frac{1}{4} z_j^r + z_i^r H_{i2}(\bar{\xi}_{i-1}, \widehat{\Psi}) + z_i^r L_{i2}(\bar{\xi}_i, \widehat{\Psi})\Psi, \ i = 2, \cdots, n. \tag{3.10}$$

Proposition 5: There exist smooth functions $\omega_i(\bar{\xi}_i, \widehat{\Psi}) \geq 0$ and $\varpi_i(\bar{\xi}_i, \widehat{\Psi}) \geq 0$ satisfying

$$\Gamma_i(\bar{\xi}_i, \widehat{\Psi}) \leq \Big(\sum_{j=1}^{i} z_i^r\Big) \omega_i(\bar{\xi}_i, \widehat{\Psi}) \leq \Big(\sum_{j=1}^{i} \xi_i^r\Big) \varpi_i(\bar{\xi}_i, \widehat{\Psi}), \ i = 1, \cdots, n. \tag{3.11}$$

Then, the AOPI method is invoked to show the controller design procedure.

Step 1: Let

$$\Psi = \max\Big\{\sigma, \sigma^r, \sigma^{\frac{r}{r_i}}, \sigma^{1+r_i}, \sigma^{1+\frac{1}{r_i}}, \sigma^{\frac{r}{2-r_i}}, \sigma^{\frac{rr_2}{r_i}} \Big| \, i = 1, \cdots, n\Big\}. \tag{3.12}$$

In what follows, we denote $\widehat{\Psi}$ as the estimate of Ψ with the error $\widetilde{\Psi} = \Psi - \widehat{\Psi}$.

We choose the candidate Lyapunov function

$$W_1 = U_1 + \frac{1}{2}\widetilde{\Psi}^2. \tag{3.13}$$

Then, from (3.1), the time-derivative of W_1 along the ξ_1-subsystem in (1.1) is

$$\dot{W}_1 = \lambda_1(t, \xi)\xi_1(\xi_2 - \xi_2^*) + \lambda_1(t, \xi)\xi_1\xi_2^* + \xi_1 g_1(t, z, \theta) - \widetilde{\Psi}\dot{\widehat{\Psi}}. \tag{3.14}$$

For notational convenience, let $L_1(\xi_1) = z_1^{2-d}\phi_1(\xi_1) \geq 0$, then, we know from Assumption 1 that

$$\begin{aligned} \xi_1 g_1(t, z, \theta) &\leq z_1^r L_1(\xi_1)\widehat{\Psi} + z_1^r L_1(\xi_1)\widetilde{\Psi} \\ &\leq z_1^r L_1(\xi_1)\sqrt{1 + \widehat{\Psi}^2} + z_1^r L_1(\xi_1)\widetilde{\Psi}. \end{aligned} \tag{3.15}$$

Substitute (3.15) into (3.14), and denote $\zeta_1(\xi_1, \widehat{\Psi}) = n + l_1 + \bar{l}_1 z_1^{\frac{r_0+1}{r_0}} + L_1(\xi_1)\sqrt{1 + \widehat{\Psi}^2}$, with $l_1 > 0$ a design constant, $\bar{r} = r + \frac{r_0+1}{r_0}$, and $r_0 > 0$ an odd positive integer, and then one get

$$\dot{W}_1 \leq z_1\Big(\lambda_1(t, \xi)\xi_2^* + z_1^{r-1}\zeta_1(\xi_1, \widehat{\Psi})\Big) + \widetilde{\Psi}(z_1^r L_1(\xi_1) - \dot{\widehat{\Psi}}) + \lambda_1(t, \xi)z_1(\xi_2 - \xi_2^*) - nz_1^r - l_1 z_1^r - \bar{l}_1 z_1^{\bar{r}}. \tag{3.16}$$

As a result, we choose the following virtual controller

$$\xi_2^* = -\frac{1}{\lambda_{11}} z_1^{r_2}\zeta_1(\xi_1, \widehat{\Psi}), \tag{3.17}$$

and from (3.16), we get

$$\dot{W}_1 \leq -nz_1^r - l_1 z_1^r - \bar{l}_1 z_1^{\bar{r}} + \lambda_1(t,\xi) z_1(\xi_2 - \xi_2^*) + \widetilde{\Psi}(\Gamma_1 - \dot{\widehat{\Psi}}). \tag{3.18}$$

Step $i(2 \leq i \leq n)$: Suppose that in **Step** $i-1$, the following holds

$$\begin{aligned} \dot{W}_{i-1} \leq & -\Big(n-(i-2)\Big)\sum_{j=1}^{i-1} z_j^r - \sum_{j=1}^{i-1} l_j z_j^r - \sum_{j=1}^{i-1} \bar{l}_j z_j^{\bar{r}} + (\widetilde{\Psi} + \Omega_{i-1})(\Gamma_{i-1} - \dot{\widehat{\Psi}}) \\ & + \lambda_{i-1}(t,\xi) z_{i-1}^{2-r_{i-1}}(\xi_i - \xi_i^*), \end{aligned} \tag{3.19}$$

where the notation of Ω_{i-1}, $i = 2, \cdots, n+1$, is defined by

$$\Omega_{i-1} = -\sum_{j=1}^{i-1} \frac{\partial U_j}{\partial \widehat{\Psi}}. \tag{3.20}$$

Then, we choose the function

$$W_i = \sum_{j=1}^{i} \int_{\xi_j^*}^{\xi_j} (\tau^{\frac{1}{r_j}} - \xi_j^{*\frac{1}{r_j}})^{2-r_j} \mathrm{d}\tau + \frac{1}{2}\widetilde{\Psi}^2, \tag{3.21}$$

and in terms of (3.19), its derivative satisfies

$$\begin{aligned} \dot{W}_i \leq & -\Big(n-(i-2)\Big)\sum_{j=1}^{i-1} z_j^r - \sum_{j=1}^{i-1} l_j z_j^r - \sum_{j=1}^{i-1} \bar{l}_j z_j^{\bar{r}} + (\widetilde{\Psi} + \Omega_{i-1})(\Gamma_{i-1} - \dot{\widehat{\Psi}}) + \frac{\partial U_i}{\partial \widehat{\Psi}} \dot{\widehat{\Psi}} \\ & + \lambda_{i-1}(t,\xi) z_{i-1}^{2-r_{i-1}}(\xi_i - \xi_i^*) + \sum_{j=1}^{i-1} \frac{\partial U_i}{\partial \xi_j} \dot{\xi}_j + z_i^{2-r_i} g_i(t,z,\theta) \\ & + \lambda_i(t,\xi) z_i^{2-r_i} \xi_{i+1}^* + \lambda_i(t,\xi) z_i^{2-r_i}(\xi_{i+1} - \xi_{i+1}^*). \end{aligned} \tag{3.22}$$

The following Proposition 6 is used to derive the virtual controller ξ_{i+1}^*, whose proof can be seen in Appendix F.

Proposition 6: There exists a nonnegative continuous function $H_{i3}(\bar{\xi}_i, \widehat{\Psi})$ satisfying

$$\begin{aligned} & (\widetilde{\Psi} + \Omega_{i-1})(\Gamma_{i-1} - \dot{\widehat{\Psi}}) + z_i^r L_i(\bar{\xi}_i, \widehat{\Psi})\widetilde{\Psi} + \frac{\partial U_i}{\partial \widehat{\Psi}} \dot{\widehat{\Psi}} \\ & \leq (\widetilde{\Psi} + \Omega_i)(\Gamma_i - \dot{\widehat{\Psi}}) + \sum_{j=1}^{i-1} \frac{1}{4} z_j^r + z_i^r H_{i3}(\bar{\xi}_i, \widehat{\Psi}), \ i = 3, \cdots, n. \end{aligned} \tag{3.23}$$

Define $\zeta_i(\bar{\xi}_i, \widehat{\Psi}) = n - i + 1 + l_i + H_i(\bar{\xi}_i, \widehat{\Psi}) + L_i(\bar{\xi}_i, \widehat{\Psi})\sqrt{1 + \widehat{\Psi}^2} + \bar{l}_i z_i^{\frac{r_0+1}{r_0}}$, with $H_i(\bar{\xi}_i, \widehat{\Psi}) = H_{i1} + H_{i2}(\bar{\xi}_{i-1}, \widehat{\Psi}) + H_{i3}(\bar{\xi}_i, \widehat{\Psi})$ and $L_i(\bar{\xi}_i, \widehat{\Psi}) = L_{i1}(\bar{\xi}_i, \widehat{\Psi}) + L_{i2}(\bar{\xi}_i, \widehat{\Psi})$, $l_i, \bar{l}_i > 0$. Then, the virtual control law is taken as follows

$$\xi_{i+1}^* = -\frac{1}{\lambda_{i1}} z_i^{r_{i+1}} \zeta_i(\bar{\xi}_i, \widehat{\Psi}), \ i = 1, \cdots, n. \tag{3.24}$$

Substitute (3.24) into (3.22), and one get

$$\dot{W}_i \leq -(n-i+1)\sum_{j=1}^{i} z_j^r - \sum_{j=1}^{i} l_j z_j^r - \sum_{j=1}^{i} \bar{l}_j z_j^{\bar{r}} + \lambda_i(t,\xi)\cdot z_i^{2-r_2}(\xi_{i+1} - \xi_{i+1}^*) + (\widetilde{\Psi} + \Omega_i)(\Gamma_i - \dot{\widehat{\Psi}}). \quad (3.25)$$

In particular, we design the actual controller when $i = n$ in (3.24) as follows

$$u = \xi_{n+1}^* = -\frac{1}{\lambda_{n1}} z_n^{r_{n+1}} \zeta_n(\xi, \widehat{\Psi}), \quad (3.26)$$

as well as the parameter updating law

$$\dot{\widehat{\Psi}} = \Gamma_n = \sum_{i=1}^{n} z_i^r L_i(\bar{\xi}_i, \widehat{\Psi}). \quad (3.27)$$

Then, based on the above calculations, we know that the Lyapunov function

$$W_n = \sum_{k=1}^{n} \int_{\xi_k^*}^{\xi_k} (\tau^{\frac{1}{r_k}} - \xi_k^{*\frac{1}{r_k}})^{2-r_k} \mathrm{d}\tau + \frac{1}{2}\widetilde{\Psi}^2 \quad (3.28)$$

satisfy

$$\dot{W}_n \leq -\sum_{k=1}^{n} z_k^r - \sum_{k=1}^{n} l_k z_k^r - \sum_{k=1}^{n} \bar{l}_k z_k^{\bar{r}}. \quad (3.29)$$

So far, we complete the adaptive fixed-time stabilizer design.

4. Main results

Now we summarize the main results contributed in this article in the following Theorem 1.

Theorem 1: The considered system (1.1) with the adaptive controller (3.26), is globally fixed-time stable in the context of Definition 3.

Proof: According to (3.28) and (3.29), we can conclude that all solutions $(\xi(t), \widetilde{\Psi}(t))$ are bounded. In view of Ψ a constant, it can be concluded that the adaptive estimate $\widehat{\Psi}(t)$ is also bounded. Specially, $\widehat{\Psi}(t) \geq 0$ if $\widehat{\Psi}(0) \geq 0$ according to $\dot{\widehat{\Psi}} = \Gamma_n(\xi, \widehat{\Psi}) \geq 0$. Then, one can find a positive constant Λ such that

$$\widehat{\Psi}(t) \in [0, \Lambda]. \quad (4.1)$$

In accordance of

$$W_n^* = \sum_{i=1}^{n} \int_{\xi_i^*}^{\xi_i} (\tau^{\frac{1}{r_i}} - \xi_i^{*\frac{1}{r_i}})^{2-r_i} \mathrm{d}\tau, \quad (4.2)$$

we know from (3.29) that

$$\dot{W}_n^* \leq -\sum_{i=1}^{n} z_i^r - \sum_{i=1}^{n} l_i z_i^r - \sum_{i=1}^{n} \bar{l}_i z_i^{\bar{r}} + \widetilde{\Psi}\Gamma_n. \quad (4.3)$$

Considering $d < 2$ and $2^{\frac{r}{2}-1} < 1$, we know from Lemma 5 that

$$\sum_{i=1}^{n} z_i^r \geq \frac{1}{\max\{2^{\frac{r}{2}-1}, 1\}}\Big(\frac{W_n^*}{2}\Big)^{\frac{r}{2}} = 2^{-\frac{r}{2}} W_n^{*\frac{r}{2}}. \tag{4.4}$$

Similarly, from $\bar{r} > 2$ and $2^{\frac{\bar{r}}{2}-1} > 1$, we have

$$\sum_{i=1}^{n} z_i^{\bar{r}} \geq \frac{1}{\max\{2^{\frac{\bar{r}}{2}-1}, 1\}}\Big(\sum_{i=1}^{n} z_i^2\Big)^{\frac{\bar{r}}{2}} = 2^{1-\bar{r}} W_n^{*\frac{\bar{r}}{2}}. \tag{4.5}$$

For notational convenience, let

$$c_1 = 2^{-\frac{r}{2}} \min\{l_1, \cdots, l_n\},\ c_2 = 2^{1-\bar{r}} \min\{\bar{l}_1, \cdots, \bar{l}_n\}, \tag{4.6}$$

then, (4.3) turns into

$$\dot{W}_n^* \leq -c_1 W_n^{*\frac{r}{2}} - c_2 W_n^{*\frac{\bar{r}}{2}} - \sum_{i=1}^{n} z_i^r + \widetilde{\Psi}\Gamma_n. \tag{4.7}$$

Define the function

$$\widetilde{W}(\xi, \widehat{\Psi}) = (\Psi + \Lambda)\,\omega_n(\xi, \widehat{\Psi}). \tag{4.8}$$

According to Proposition 5 with $i = n$ and (4.1), the following calculations hold

$$\begin{aligned} \dot{W}_n^* &\leq -c_1 W_n^{*\frac{r}{2}} - c_2 W_n^{*\frac{\bar{r}}{2}} - \sum_{i=1}^{n} z_i^r + (\Psi + \Lambda)\Big(\sum_{i=1}^{n} z_i^r\Big)\omega_n(\xi, \widehat{\Psi}) \\ &= -c_1 W_n^{*\frac{r}{2}} - c_2 W_n^{*\frac{\bar{r}}{2}} - \sum_{i=1}^{n} z_i^r(1 - \widetilde{W}(\xi, \widehat{\Psi})). \end{aligned} \tag{4.9}$$

In view of

$$\widetilde{W}(0, \widehat{\Psi}) = 0,\ \forall\, \widehat{\Psi} \in [0, \Lambda], \tag{4.10}$$

and the continuous property of $\widetilde{W}(\xi, \widehat{\Psi})$, it is known that there is a real number $\rho > 0$ satisfying for each $\xi \in N_1$ with

$$N_1 = \{(\xi, \widehat{\Psi}) : W_n^*(\xi, \widehat{\Psi}) \leq \rho\}, \tag{4.11}$$

there holds

$$\widetilde{W}(\xi, \widehat{\Psi}) \leq 1. \tag{4.12}$$

As a result, if $(\xi, \widehat{\Psi}) \in N_1$, that is $W_n^*(\xi, \widehat{\Psi}) \leq \rho$, in view of the definition of N_1, which further implies that

$$\dot{W}_n^* \leq -c_1 W_n^{*\frac{r}{2}} - c_2 W_n^{*\frac{\bar{r}}{2}}. \tag{4.13}$$

This shows that once $(\xi, \widehat{\Psi}) \in N_1$, it will be always in N_1.

In what follows, the fixed-time convergence analysis is separated into the following Case I and II.

Case I: If $(\xi(0), \widehat{\Psi}(0)) \in N_1$, it can be seen that, if $(\xi, \widehat{\Psi}) \in N_1$, there holds

$$\dot{W}_n^* \leq -c_1 W_n^{*\frac{r}{2}} - c_2 W_n^{*\frac{\bar{r}}{2}}. \tag{4.14}$$

Then, W_n^* is fixed-time convergent with local property. Since $W_n^* = 0$ if and only if $x = 0$, according to **Lemma 1**, one can conclude that x turns to be 0 within T_1:

$$T_1 \leq \frac{1}{c_1(1 - \frac{r}{2})} + \frac{1}{c_2(\frac{\bar{r}}{2} - 1)}. \tag{4.15}$$

Clearly, the real constants $c_1,\ c_2,\ r,\ \bar{r}$ are independent on the initial values $(\xi(0), \widehat{\Psi}(0))$.

Case II: If $(\xi(0), \widehat{\Psi}(0)) \notin N_1$, one can calculate its maximum arriving time moment T_2 into N_1.

Since ξ, $\widehat{\Psi}$ and $\widetilde{\Psi}$ are bounded, then there is a real number $C > 0$ satisfying

$$\widetilde{\Psi}\Gamma_n(\xi, \widehat{\Psi}) \quad \leq \quad C, \tag{4.16}$$

which leads to

$$\dot{W}_n^* \leq -c_1 W_n^{*\frac{r}{2}} - c_2 W_n^{*\frac{\bar{r}}{2}} + C. \tag{4.17}$$

Motivated by the recent work [40, 42], we define the following set:

$$N_2 = \left\{ (\xi, \widehat{\Psi}) : W_n^* \leq \min \left\{ \Big(\frac{C}{(1-\epsilon)c_1} \Big)^{\frac{2}{r}}, \Big(\frac{C}{(1-\epsilon)c_2} \Big)^{\frac{2}{\bar{r}}} \right\} \right\} \tag{4.18}$$

with any positive constant $\epsilon \in (0, 1)$.

Choose the constants l_i's and $\bar{l}_i$'s large enough, $i = 1, \cdots, n$, which in terms of (4.6) renders c_1 and c_2 large, and further implies $\left(\frac{C}{(1-\epsilon)c_1}\right)^{\frac{2}{r}}$ and $\left(\frac{C}{(1-\epsilon)c_2}\right)^{\frac{2}{\bar{r}}}$ sufficiently small, such that

$$\min \left\{ \Big(\frac{C}{(1-\epsilon)c_1} \Big)^{\frac{2}{r}}, \Big(\frac{C}{(1-\epsilon)c_2} \Big)^{\frac{2}{\bar{r}}} \right\} \leq \rho, \tag{4.19}$$

and then, we can get

$$N_2 \subseteq N_1. \tag{4.20}$$

As a result, in view of the Lemma 2, after the fixed-time T_1:

$$T_1 = \frac{1}{c_1\epsilon\Big(1 - \frac{r}{2}\Big)} + \frac{1}{c_2\epsilon\Big(\frac{\bar{r}}{2} - 1\Big)}, \tag{4.21}$$

$(\xi, \widehat{\Psi})$ goes into N_2, and hence enters N_1. According to the analysis in Case I, when $(\xi, \widehat{\Psi}) \in N_1$, after the fixed-time T_2:

$$T_2 = \frac{1}{c_1\Big(1 - \frac{r}{2}\Big)} + \frac{1}{c_2\Big(\frac{\bar{r}}{2} - 1\Big)}, \tag{4.22}$$

$(\xi, \widehat{\Psi})$ arrives at zero. Thus, in this situation, ξ approaches the origin within $T = T_1 + T_2$:

$$T = \frac{1}{c_1\epsilon\left(1 - \frac{r}{2}\right)} + \frac{1}{c_2\epsilon\left(\frac{\bar{r}}{2} - 1\right)} + \frac{1}{c_1\left(1 - \frac{r}{2}\right)} + \frac{1}{c_2\left(\frac{\bar{r}}{2} - 1\right)}. \tag{4.23}$$

Noting that fact of the positive design parameters c_1, c_2, r, $\bar{r}$ irrespective of the initial values, we can see that the closed-loop system states ξ globally converge to zero in fixed-time. Consequently, the problem of global adaptive fixed-time stabilization stated in Definition 3 is well addressed.

5. Simulation results

In this subsection, a practical example of a pendulum system is used to illustrate the proposed fixed-time control strategy. It is known that the simple pendulum motion can be described by [48]

$$ML\ddot{\upsilon} = -Mg\sin(\upsilon) - kL\dot{\upsilon} + \frac{1}{L}u, \tag{5.1}$$

where the torque $u \in R$ is viewed as the control variable, the angular displacement $\upsilon \in R$ is the state. The constants M, L, k, and g denote the mass, length, friction coefficient of the rod, and the gravity acceleration, respectively. It does not require that the parameters M, L, and k are known a priori.

The control task is to construct a control input torque using the presented control methodology developed here, so that the angular displacement of pendulum is regulated at the angle $\upsilon = \pi$ in a finite time irrespective of system initial conditions.

Towards this end, we need the following additional Assumption 3 to characterize the unknown parameters M, L, and k.

Assumption 3: The parameters m and l are assumed to satisfy

$$\underline{M} \le M \le \overline{M},\ \underline{L} \le L \le \bar{L}. \tag{5.2}$$

According to Assumption 3, we have $\frac{1}{\overline{M}\bar{L}^2} \le \frac{1}{ML^2} \le \frac{1}{\underline{M}\underline{L}^2}$.

To be first, we perform the following coordinates changes

$$\xi_1 = \upsilon - \pi,\ \xi_2 = \dot{\upsilon}, \tag{5.3}$$

and we get

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \frac{1}{ML^2}u + \frac{g}{L}\sin(\xi_1) - \frac{k}{M}\xi_2. \end{cases} \tag{5.4}$$

It follows that the new system (5.4) has the same form with the considered system (1.1) with $g_1(t, \xi, \theta) = 0$, $g_2(t, \xi, \theta) = \frac{g}{L}\sin(\xi_1) - \frac{k}{M}\xi_2$, and $\theta = \max\{\frac{g}{L}, \frac{k}{M}\}$.

According to the developed controller design algorithm presented in Section 3, we construct the following adaptive fixed-time controller

$$u = -\overline{M}\,\bar{L}^2\xi_2^{r_3}\zeta_2(\bar{\xi}_2, \widehat{\Psi}), \tag{5.5}$$

$$\dot{\widehat{\Psi}} = \xi_2^{r}L_2(\bar{\xi}_2, \widehat{\Psi}), \tag{5.6}$$

with $\zeta_1(\xi_1) = 2 + l_1 + \bar{l}_1\xi_1^{\frac{r_0+1}{r_0}}$, $\widehat{\mu}_1(\xi_1, \widehat{\Psi}) = \zeta_1^{\frac{1}{r_2}}(\xi_1) + \frac{1}{r_2}\frac{r_0+1}{r_0}\bar{l}_1\zeta_1^{\frac{1}{r_2}-1}(\xi_1)\xi_1^{\frac{r_0+1}{r_0}}$, $\zeta_2(\bar{\xi}_2, \widehat{\Psi}) = 1 + l_2 + \bar{l}_2\xi_2^{\frac{r_0+1}{r_0}} + H_2(\bar{\xi}_2, \widehat{\Psi}) + L_2(\bar{\xi}_2, \widehat{\Psi})\sqrt{1 + \widehat{\Psi}^2}$.

In simulation, we take the design constants: $r_1 = 1$, $r_2 = \frac{3}{5}$, $r_3 = \frac{1}{5}$, $r = \frac{8}{5}$, $r_0 = 5$, $\bar{r} = \frac{14}{5}$, and the gain functions: $H_2(\bar{\xi}_2, \widehat{\Psi}) = \frac{r_2}{r}(\frac{2}{5})^{-\frac{1}{r_2}}2^{\frac{(1-r_2)(1+r_2)}{r_2}} + (2-r_2)2^{2(1-r_2)}\widehat{\mu}_1(\xi_1, \widehat{\Psi}) + \frac{1}{r}(\frac{2}{3})^{-r_2}\Big((2-r_2)2^{1-r_2}\widehat{\mu}_1(\xi_1, \widehat{\Psi})\Big)^r$, $L_2(\bar{\xi}_2, \widehat{\Psi}) = \left(\frac{2}{5}\right)^{-\frac{5}{8}(1+\frac{1}{r_2})}\frac{r_2}{r}\xi_2^{\frac{2-2r_2}{r_2}r} + 2^{1-r_2}\xi_2^{2-r} + \frac{1}{r}(\frac{2}{3})^{-r_2}\Big(\xi_2^{1-r_2}\zeta_1(\xi_1, \widehat{\Psi})\Big)^{1+r_2}$.

For simulation use, the parameters M, L, and k are chosen as $ML^2 = 1$, $L = g$, $k = M$, and $l_1 = 1$, $l_2 = 0.1$, $\bar{l}_1 = 1$, $\bar{l}_2 = 0.1$, and the initial values $\xi_1(0) = 0.1$, $\xi_2(0) = 0.5$, $\widehat{\Psi}(0) = 0.5$. The simulation results are shown in Figures 1–2. Particularly, Figure 1 depicts the profiles of the angular displacement υ, its desired angular displacement π, the velocity $\dot{\upsilon}$, parameter estimate $\widehat{\Psi}$, and input torque u in (5.1). Figure 2 verifies fixed-time convergence property of the system states in closed-loop system (5.4)–(5.6). From the simulation results, we can see that the designed fixed-time stabilizer could achieve the fixed-time stabilization with zero error for the pendulum.

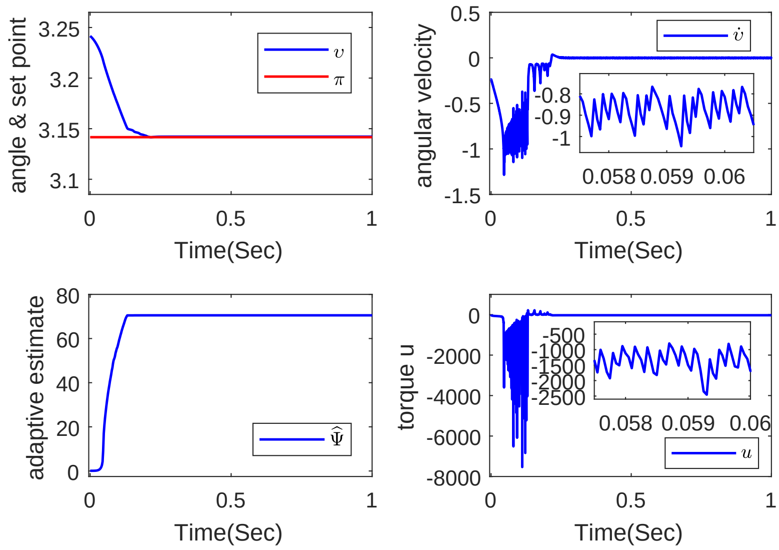

Figure 1. The closed-loop responses of the pendulum system (5.1).

6. Conclusions

The paper presents an adaptive fixed-time stabilization strategy for a kind of nonlinear systems perturbed by nonlinear parametric uncertainty and unknown control coefficients. We combine the adding one power integrator tool and backstepping method to present a systematic fixed-time controller design scheme. The proposed adaptive stabilizer guarantees that the states can converge to its equilibrium in fixed-time, and all closed-loop solutions are bounded. It provides a basic fixed-time stable approach to realize the adaptive stabilizing control for the class of nonlinear uncertain systems with parametric uncertainty and uncertain control coefficients. The obtained result in this article is an improvement of the existing results in the adaptive fixed-time control direction. The simulation results demonstrate the efficacy of the proposed control scheme by means of a pendulum system.

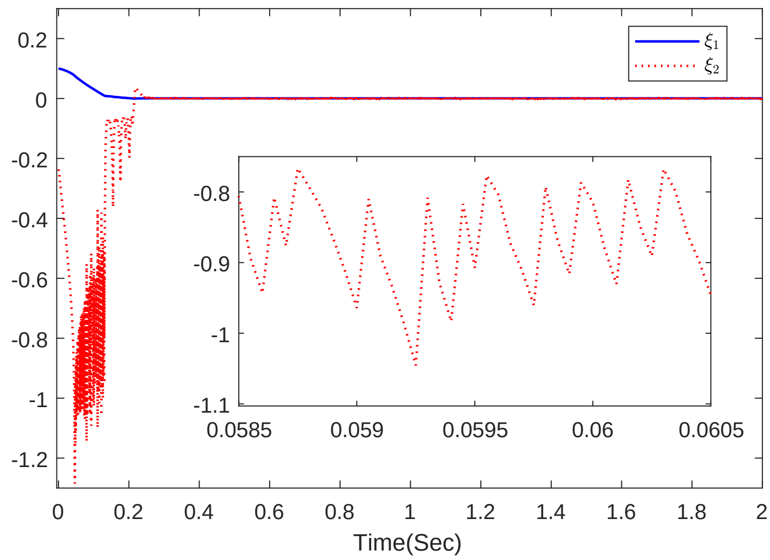

Figure 2. The system states of the closed-loop system (5.4)–(5.6).

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Conflict of interest

The authors declare that there is no conflict of interest.

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Appendix

A. Proof of Proposition 1:

This proposition can be referred to Proposition 2 together with its proof in [8].

B. Proof of Proposition 2:

Using Lemma 3, one can verify that

$$\begin{aligned}\lambda_{i-1}(t,\xi)\xi_{i-1}^{2-r_{i-1}}(\xi_i-\xi_i^*) &\leq 2^{1-r_i}\lambda_{i-1,2}\left|\xi_{i-1}^{2-r_{i-1}}\right|\left|\xi_i\right|^{r_i} \\ &\leq \frac{1}{4}\xi_{i-1}^r+\xi_i^r H_{i1},\end{aligned} \tag{A1}$$

with $H_{i1}=\frac{r_i}{r}\left(\frac{r}{4(2-r_{i-1})}\right)^{-\frac{2-r_{i-1}}{r_i}}2^{\frac{(1-r_2)(2-r_{i-1}+r_i)}{r_i}}\lambda_{i-1,2}^{\frac{2-r_{i-1}+r_i}{r_i}}$, and then, the proof is completed.

C. Proof of Proposition 3:

In terms of Assumption 1 and $\xi_j^*=-\frac{1}{\lambda_{j-1,1}}\xi_{j-1}^{r_i}\zeta_{j-1}(\bar{\xi}_{j-1},\widehat{\Psi})$, $j=2,\cdots,i$, one obtain

$$\left|\xi_j-\xi_j^*\right|\leq 2^{1-r_j}\left|\xi_j^{\frac{1}{r_j}}-\xi_j^{*\frac{1}{r_j}}\right|^{r_j}=2^{1-r_j}\left|\xi_j\right|^{r_j}. \tag{A2}$$

Then, there exists a continuous function $\bar{\phi}_i(\bar{\xi}_i,\widehat{\Psi})\geq 0$ such that

$$\left|g_i(t,z,\theta)\right|\leq\sum_{j=1}^{i}|\xi_j|^{r_{i+1}}\bar{\phi}_i(\bar{\xi}_i,\widehat{\Psi})\sigma. \tag{A3}$$

Using Lemma 4, we have the following calculations

$$|\xi_i^{2-r_i}||\xi_j|^{r_{i+1}}\bar{\phi}_i(\bar{\xi}_i,\widehat{\Psi})\sigma\leq\frac{1}{4}\xi_j^r+\xi_i^r L_{i1}(\bar{\xi}_i,\widehat{\Psi})\Psi, \tag{A4}$$

with $L_{i1}(\bar{\xi}_i,\widehat{\Psi})=\sum_{j=1}^{i}\frac{2-r_i}{d}\left(\frac{d}{4r_{i+1}}\right)^{-\frac{r_{i+1}}{2-r_i}}\left(\bar{\phi}_i(\bar{\xi}_i,\widehat{\Psi})\right)^{\frac{d}{2-r_i}}$, and then we complete the proof of proposition.

D. Proof of Proposition 4:

Firstly, it can be verified that

$$\frac{\partial U_i}{\partial\xi_k}=(2-r_i)\frac{\partial(-\xi_i^{*\frac{1}{r_i}})}{\partial\xi_k}\int_{\xi_i^*}^{\xi_i}(\tau^{\frac{1}{r_i}}-\xi_i^{*\frac{1}{r_i}})^{1-r_i}\mathrm{d}\tau,\ k=1,\cdots,i-1. \tag{A5}$$

Secondly, we know from $\xi_i^*=-\frac{1}{\lambda_{i-1,1}}\xi_{i-1}^{r_i}\zeta_{i-1}(\bar{\xi}_{i-1},\widehat{\Psi})$ that

$$\xi_i^{*\frac{1}{r_i}}=-\left(\frac{1}{\lambda_{i-1,1}}\right)^{\frac{1}{r_i}}\left(\xi_{i-1}^{\frac{1}{r_{i-1}}}-\xi_{i-1}^{*\frac{1}{r_{i-1}}}\right)\zeta_{i-1}^{\frac{1}{r_i}}(\bar{\xi}_{i-1},\widehat{\Psi}). \tag{A6}$$

Thanks to the inductive method, one can find a smooth function $\widehat{\mu}_{ij}(\bar{\xi}_{i-1}, \widehat{\Psi}) \geq 0$ satisfying

$$\left|\frac{\partial(-\xi_i^{*\frac{1}{r_i}})}{\partial \xi_j}\right| \leq \Big(\sum_{k=1}^{i-1} \xi_k^{1-r_j}\Big)\widehat{\mu}_{ij}(\bar{\xi}_{i-1}, \widehat{\Psi}). \tag{A7}$$

Additionally, according to (A7), one get

$$\begin{aligned}\frac{\partial(-\xi_i^{*\frac{1}{r_i}})}{\partial \xi_j}\dot{\xi}_j &\leq \Big(\sum_{k=1}^{i-1} \xi_k^{1-r_j}\Big)\widehat{\mu}_{ij}(\bar{\xi}_{i-1}, \widehat{\Psi})\Big(\lambda_{j2}2^{1-r_{j+1}}|\xi_{j+1}|^{r_{j+1}} + \frac{\lambda_{j2}}{\lambda_{j1}}|\xi_j|^{r_{j+1}}\zeta_j(\bar{\xi}_j, \widehat{\Psi}) + \Big(\sum_{k=1}^{j}|\xi_k|^{r_{j+1}}\Big)\bar{\phi}_j(\bar{\xi}_j, \widehat{\Psi})\sigma\Big) \\ &\leq \Big(\sum_{k=1}^{i-1} \xi_k^{1-r_j+r_{j+1}}\Big)\widehat{H}_{ij}(\bar{\xi}_j, \widehat{\Psi}) + \Big(\sum_{k=1}^{i-1} \xi_k^{1-r_j+r_{j+1}}\Big)\widehat{L}_{ij}(\bar{\xi}_j, \widehat{\Psi})\,\sigma^{\frac{1-r_j+r_{j+1}}{r_{j+1}}},\end{aligned} \tag{A8}$$

where $\widehat{H}_{ij}(\bar{\xi}_j\widehat{\Psi})$ and $\widehat{L}_{ij}(\bar{\xi}_j, \widehat{\Psi})$ are some nonnegative functions.

From

$$\begin{aligned}\int_{\xi_i^*}^{\xi_i} (\tau^{\frac{1}{r_i}} - \xi_i^{*\frac{1}{r_i}})^{1-r_i}\mathrm{d}\tau &\leq |\xi_i - \xi_i^*|\left|\xi_i^{\frac{1}{r_i}} - \xi_i^{*\frac{1}{r_i}}\right|^{1-r_i} \\ &\leq 2^{1-r_i}|\xi_i|,\end{aligned} \tag{A9}$$

one can find two nonnegative C^1 functions $H_{i2}(\bar{\xi}_{i-1}, \widehat{\Psi})$, $L_{i2}(\bar{\xi}_i, \widehat{\Psi})$, such that

$$\begin{aligned}\sum_{j=1}^{i-1}\frac{\partial U_i}{\partial \xi_j}\dot{\xi}_j &\leq \sum_{j=1}^{i-1}(2-r_i)\left|\int_{\xi_i^*}^{\xi_i} (\tau^{\frac{1}{r_i}} - \xi_i^{*\frac{1}{r_i}})^{1-r_i}\mathrm{d}\tau\right| \cdot \left|\frac{\partial(-\xi_\mathrm{i}^{*\frac{1}{\mathrm{r_i}}})}{\partial \xi_\mathrm{j}}\dot{\xi}_\mathrm{j}\right| \\ &\leq \sum_{j=1}^{i-1}(2-r_i)2^{1-r_i}|\xi_i|\Big(\Big(\sum_{k=1}^{i-1}\xi_k^{1-r_j+r_{j+1}}\Big)\widehat{H}_{ij}(\bar{\xi}_j, \widehat{\Psi}) + \Big(\sum_{k=1}^{i-1}\xi_k^{1-r_j+r_{j+1}}\Big)\widehat{L}_{ij}(\bar{\xi}_j, \widehat{\Psi})\,\sigma^{\frac{1-r_j+r_{j+1}}{r_{j+1}}}\Big) \\ &\leq \sum_{j=1}^{i-1}\frac{1}{4}\xi_j^r + \xi_i^r H_{i2}(\bar{\xi}_{i-1}, \widehat{\Psi}) + \xi_i^r L_{i2}(\bar{\xi}_i, \widehat{\Psi})\Psi.\end{aligned} \tag{A10}$$

Then, we complete the proof.

E. Proof of Proposition 5:

Firstly, one can choose

$$\omega_1(\xi_1, \widehat{\Psi}) = \varpi_1(\xi_1, \widehat{\Psi}) = L_1(\xi_1). \tag{A11}$$

Then, the function $\omega_i(\cdot)$ can be chosen as $\omega_i(\bar{\xi}_i, \widehat{\Psi}) = \max\{L_1(\xi_1), L_2(\bar{\xi}_2, \widehat{\Psi}), \cdots, L_i(\bar{\xi}_i, \widehat{\Psi})\}$, implying

$$\Gamma_i(\bar{\xi}_i, \widehat{\Psi}) \leq (\xi_1^r + \cdots + \xi_i^r)\omega_i(\bar{\xi}_i, \widehat{\Psi}),\ \ i = 2, \cdots, n. \tag{A12}$$

Considering $\xi_j^* = -\frac{1}{\lambda_{j-1,2}}\xi_{j-1}^{r_j}\zeta_{j-1}(\bar{\xi}_{j-1}, \widehat{\Psi})$, and $\xi_j = \xi_j^{\frac{1}{r_j}} - \xi_j^{*\frac{1}{r_j}}$, $j = 2, \cdots, i$, and then, we have

$$\xi_j^r = (\xi_j^{\frac{1}{r_j}} - \xi_j^{*\frac{1}{r_j}})^r$$

$$
\begin{aligned}
&\leq \max\{2^{d-1},1\}(\xi_j^{\frac{d}{r_j}} + \xi_j^{*\frac{d}{r_j}}) \\
&= \xi_j^r \cdot 2^{d-1}\xi_j^{\frac{1-r_j}{r_j}d} + 2^{d-1}\Big(\frac{1}{\lambda_{j-1,2}}\Big)^{\frac{d}{r_j}}\xi_{j-1}^r\Big(\zeta_{j-1}(\bar{\xi}_{j-1},\widehat{\Psi})\Big)^{\frac{d}{r_j}}. \qquad \text{(A13)}
\end{aligned}
$$

Then, in view of (A13), there exists a nonnegative function $\varpi_i(\bar{\xi}_i,\widehat{\Psi})$ satisfying

$$\Gamma_i(\bar{\xi}_i,\widehat{\Psi}) \leq (\xi_1^r + \cdots + \xi_i^r)\varpi_i(\bar{\xi}_i,\widehat{\Psi}),\ i = 1,\cdots,n. \qquad \text{(A14)}$$

Then, we complete the proof of Proposition 5.

F. The proof of Proposition 6:

Firstly, the following holds

$$\frac{\partial U_i}{\partial\widehat{\Psi}} = (2-r_i)\frac{\partial(-\xi_i^{*\frac{1}{r_i}})}{\partial\widehat{\Psi}}\int_{\xi_i^*}^{\xi_i}(\tau^{\frac{1}{r_i}} - \xi_i^{*\frac{1}{r_i}})^{1-r_i}\mathrm{d}\tau,\ \mathrm{i} = 1,\cdots,\mathrm{n}. \qquad \text{(A15)}$$

Then, considering $\xi_i^{*\frac{1}{r_i}} = -\Big(\frac{1}{\lambda_{i-1,1}}\Big)^{\frac{1}{r_i}}\xi_{i-1}\zeta_{i-1}^{\frac{1}{r_i}}(\bar{\xi}_{i-1},\widehat{\Psi})$ and $\frac{1}{r_i} > 1$, one can find a smooth function $\widehat{\nu}_i(\bar{\xi}_i,\widehat{\Psi}) \geq 0$ satisfying

$$\Big|\frac{\partial(-\xi_i^{*\frac{1}{r_i}})}{\partial\widehat{\Psi}}\Big| \leq \widehat{\nu}_i(\bar{\xi}_i,\widehat{\Psi}). \qquad \text{(A16)}$$

Furthermore, it is known from (A9) that

$$\Big|\frac{\partial U_i}{\partial\widehat{\Psi}}\Big| \leq |\xi_i|(2-r_i)2^{1-r_i}\widehat{\nu}_i(\bar{\xi}_i,\widehat{\Psi}). \qquad \text{(A17)}$$

According to Lemma 4, we have

$$
\begin{aligned}
&\xi_j^r|\xi_i|(2-r_i)2^{1-r_i}\widehat{\nu}_i(\bar{\xi}_i,\widehat{\Psi})\omega_i(\bar{\xi}_i,\widehat{\Psi}) \\
&\leq \frac{1}{4}\xi_j^r + \xi_i^r\cdot\xi_j^r\Big(\frac{r}{4(r-1)}\Big)^{-(r-1)}\frac{1}{r}\cdot\Big((2-r_i)2^{1-r_i}\widehat{\nu}_i(\bar{\xi}_i,\widehat{\Psi})\omega_i(\bar{\xi}_i,\widehat{\Psi})\Big)^r,\ j = 1,\cdots,i. \qquad \text{(A18)}
\end{aligned}
$$

Define $\bar{H}_{i3}(\bar{\xi}_i,\widehat{\Psi}) = \Big(\sum_{j=1}^{i}\xi_j^r\Big)\Big(\frac{r}{4(r-1)}\Big)^{-(r-1)}\frac{1}{r}\Big((2-r_i)2^{1-r_i}\cdot\widehat{\nu}_i(\bar{\xi}_i,\widehat{\Psi})\omega_i(\bar{\xi}_i,\widehat{\Psi})\Big)^r$, and then, we know from **Proposition 5** that

$$
\begin{aligned}
\Big|\frac{\partial U_i}{\partial\widehat{\Psi}}\Gamma_i\Big| &\leq |\xi_i|(2-r_i)2^{1-r_i}\widehat{\nu}_i(\bar{\xi}_i,\widehat{\Psi})\cdot(\sum_{j=1}^{i}\xi_j^r)\omega_i(\bar{\xi}_i,\widehat{\Psi}) \\
&\leq \sum_{j=1}^{i}\frac{1}{4}\xi_j^r + \xi_i^r\bar{H}_{i3}(\bar{\xi}_i,\widehat{\Psi}). \qquad \text{(A19)}
\end{aligned}
$$

Let $H_{i3}(\bar{\xi}_i,\widehat{\Psi}) = \bar{H}_{i3}(\bar{\xi}_i,\widehat{\Psi}) + L_i(\bar{\xi}_i,\widehat{\Psi})\cdot(2-r_{i-1})2^{1-r_{i-1}}\sqrt{1+\xi_{i-1}^2}\,\widehat{\nu}_{i-1}(\bar{\xi}_{i-1},\widehat{\Psi})$, and one can verify that

$$\Big(\widetilde{\Psi} + \Omega_{i-1}\Big)\Big(\Gamma_{i-1} - \dot{\widehat{\Psi}}\Big) + \xi_i^r L_i(\cdot)\cdot\widetilde{\Psi} + \frac{\partial U_i}{\partial\widehat{\Psi}}\dot{\widehat{\Psi}}$$

$$
\begin{aligned}
&= \left(\widetilde{\Psi} + \Omega_i\right)\left(\Gamma_i - \dot{\widehat{\Psi}}\right) + \xi_i^r L_i(\cdot) \cdot \frac{\partial U_{i-1}}{\partial \widehat{\Psi}} + \frac{\partial U_i}{\partial \widehat{\Psi}} \Gamma_i \\
&\leq \left(\widetilde{\Psi} + \Omega_i\right)\left(\Gamma_i - \dot{\widehat{\Psi}}\right) + \xi_i^r L_i(\cdot) \cdot (2 - r_{i-1}) 2^{1-r_{i-1}} \\
&\cdot \sqrt{1 + \xi_{i-1}^2}\, \widehat{\nu}_{i-1}(\bar{\xi}_{i-1}, \widehat{\Psi}) + \xi_i^r \bar{H}_{i3}(\bar{\xi}_i, \widehat{\Psi}) + \sum_{j=1}^{i-1} \frac{1}{4} \xi_i^r \\
&= \left(\widetilde{\Psi} + \Omega_i\right)\left(\Gamma_i - \dot{\widehat{\Psi}}\right) + \xi_i^r H_{i3}(\bar{\xi}_i, \widehat{\Psi}) + \sum_{j=1}^{i-1} \frac{1}{4} \xi_i^r. \qquad \text{(A20)}
\end{aligned}
$$

Then, the proof is completed.