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Research article

Numerical analysis of multi-dimensional time-fractional diffusion problems under the Atangana-Baleanu Caputo derivative

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Abstract: This paper presents the Elzaki homotopy perturbation transform scheme (**E**HPTS) to analyze the approximate solution of the multi-dimensional fractional diffusion equation. The Atangana-Baleanu derivative is considered in the Caputo sense. First, we apply Elzaki transform (**E**T) to obtain a recurrence relation without any assumption or restrictive variable. Then, this relation becomes very easy to handle for the implementation of the homotopy perturbation scheme (HPS). We observe that HPS produces the iterations in the form of convergence series that approaches the precise solution. We provide the graphical representation in 2D plot distribution and 3D surface solution. The error analysis shows that the solution derived by **E**HPTS is very close to the exact solution. The obtained series shows that **E**HPTS is a very simple, straightforward, and efficient tool for other problems of fractional derivatives.

Keywords: Elzaki transform; diffusion problems; homotopy perturbation scheme; approximate solution; Atangana-Baleanu derivative

1. Introduction

The study of fractional calculus (FC) yields the development of ordinary calculus with the history of more than 300 years earlier. In real-world, fractional-order derivatives are nonlocal, whereas integer-order derivatives are local. Many physical phenomena are designed by fractional partial differential equations arising in biology, sociology, medicine, hydrodynamics, computational modeling, chemical kinetics and among others [1–3]. One of the most exciting and challenging study to investigate the exact solution of some differential problem in physical science. Dong and Gao [4] derived an integral formulation of the nonlocal operator Ginzburg-Landau equation with the half Laplacian. To overcome this situation, numerous mathematical strategies have been put forth to configure the approximate solutions of these problems, such that Laplace iterative transform method [5], q-homotopy analysis Sumudu transform method [6], ρ-Laplace transform method [7, 8], Haar wavelet method [9], Chebyshev spectral collocation method [10], extended modified auxiliary [11] and many others. Recently, various type of concepts and formulas of fractional operators are studied such as Riemann and Liouville [12], Caputo and Fabrizio [13], Atangana and Baleanu [14], and Liouville and Caputo [15]. Later, Abro and Atangana [16] showed that Liouville-Caputo and Atangana-Baleanu operators have excellent fractional retrieves. Caputo and Fabrizio [17] proposed a new concept of fractional derivative with a stabilize kernel to represent the temporal and spatial variables in two different ways. Toufik and Atangana [18] established a novel notion of fractional differentiation with a non-local and non-singular kernel to expand the limitations of the traditional Riemann-Liouville and Caputo fractional derivatives to solve linear and non-linear fractional differential equations. Gao et al. [19, 20] presented a new method to achieve a smooth decay rates for the damped wave problems with nonlinear acoustic boundary conditions.

The diffusion equation with time fractional derivative presents the density dynamics in a material undergoing diffusion. Jaradat et al. [21] provided the extended fractional power series approach for the analytical solution of 2D diffusion, wave-like, telegraph, and Burgers models. They obtained the results and claimed that both schemes are in excellent agreement. Dehghan and Shakeri [22] provided variational iteration method for solving the Cauchy reaction–diffusion problem. Singh and Srivastava [23] obtained the approximate series solution of multi-dimensional with time-fractional derivative using reduced differential transform method. Shah et al. [24] used natural transform method for the analytical solution of fractional order diffusion equations. Kumar et al. [25] used Laplace transform for the analytical solution of fractional multi-dimensional diffusion equations.

He [26, 27] studied an idea of the HPS for the analytical results of ordinary and partial differential problems. HPS provided the excellent findings and show the rate of convergence toward the precise solution than other analytical approaches in literature. Odibat and Momani [28] have demonstrated the significance of HPS in large number of fields and showed that HPS has an excellent treatment in providing the exact solution of these problems. Tarig M. Elzaki [29] established a new approach named as Elzaki transform (**E**T) to evaluate the approximate solutions in a wide range of areas. The **E**T is a remarkable tool in order to show the physical nature of the differential problems compared to other schemes. Recently, many authors studied the Elzaki transform involving Atangana-Baleanu fractional derivative operator for various fields such as alcohol drinking model [30], Hirota-Satsuma KdV equations [31], nonlinear regularized long-wave models, but all these approaches have some limitations and restrictions.

In this paper, we eliminate these draw backs and study the Elzaki transform combined with the HPS involving Atangana-Baleanu fractional derivative operator in Caputo sense for the approximate solution multi-dimensional diffusion problems. The reason for using Atangana-Baleanu fractional derivatives is its nonlocal properties and its capability to deal the complex behavior more efficiently than other operators. The obtained series show the significant results and we see that the computational series approaches the precise results with few repetitions. This paper is designed as: In Section 2, we define a few basic definitions of Atangana-Baleanu fractional derivative operator in Caputo sense and Elazki transform. We formulate the strategy of **E**HPTS to achieve the numerical solution of the differential problems in Section 3. We provide a three-example approach for assessing the validity and dependability of **E**HPTS in Section 4 and we depict the conclusion in last Section 5.

2. Basic definitions

Definition 2.1. The Caputo fractional derivative (CFD) is given as [32]

$$D_\eta^\alpha \vartheta(\eta) = \frac{1}{(m-\alpha)} \int_0^\eta \frac{\vartheta^m(v)}{(\eta - v)^{\alpha+1+m}} dv, \quad m-1 < \alpha \leq m. \tag{2.1}$$

Definition 2.2. The Atangana-Baleanu Caputo (ABC) operator is defined as [33]

$$D_\eta^\alpha \vartheta(\eta) = \frac{N(\alpha)}{1-\alpha} \int_m^\eta \vartheta'(v) \mathbf{E}_\alpha \Big[-\frac{\alpha(\eta - v)^\alpha}{1-\alpha} \Big] dv, \tag{2.2}$$

where $\vartheta \in H^1(\alpha', \beta')$, $\beta' > \alpha', \alpha \in [0,1]$, E_α is Mittag Leffler function, $N(\alpha)$ is normalisation function and $N(0) = N(1) = 1$.

Definition 2.3. The fractional integral operator in ABC sense is given as [33]

$$I_\eta^\alpha(\vartheta(\eta)) = \frac{1-\alpha}{N(\alpha)} \vartheta(\eta) + \frac{\alpha}{\Gamma(\alpha) N(\alpha)} \int_m^\eta \vartheta(v)(\eta - v)^{\alpha-1} dv. \tag{2.3}$$

Definition 2.4. The Elzaki transform is given as [34]

$$\mathbf{E}[\vartheta(\eta)] = R(f) = f \int_0^\infty e^{-\frac{\eta}{f}} \vartheta(\eta) d\eta, \quad k_1 \leq f \leq k_2. \tag{2.4}$$

Propositions: The differential properties of **E**T are defined as [35]

$$\begin{aligned} \mathbf{E}[\eta^n] &= n! f^{n+2}, \\ \mathbf{E}[\vartheta'(\eta)] &= \frac{\mathbf{E}[\vartheta(\eta)]}{f} - f\vartheta(0), \\ \mathbf{E}[\vartheta''(\eta)] &= \frac{\mathbf{E}[\vartheta(\eta)]}{f^2} - \vartheta(0) - f\vartheta'(0), \end{aligned} \tag{2.5}$$

Definition 2.5. The Elzaki transform of $D_\eta^\alpha \vartheta(\eta)$ CFD operator is as

$$\mathbf{E}[D_\eta^\alpha \vartheta(\eta)] = f^{-\alpha} R(f) - \sum_{k=0}^{m-1} f^{2-\alpha+k} \vartheta^k(0), \qquad m-1 < \alpha < m. \tag{2.6}$$

Definition 2.6. The Elzaki transform of ${}^{ABC}D_\eta^\alpha \vartheta(\eta)$ under ABC operator is as

$$\mathbf{E}[{}^{ABC}D_\eta^\alpha \vartheta(\eta)] = \frac{N(\alpha)}{\alpha f^\alpha + 1 - \alpha} \left(\frac{R(f)}{f} - f\vartheta(0) \right), \tag{2.7}$$

where f is the transfer parameter of η such that $\mathbf{E}[\vartheta(\eta)] = R(f)$.

3. Formulation of EHTM

Consider a fractional partial differential equation in the following form,

$$ {}^{ABC}D_{\eta}^{\alpha}\vartheta(\theta_1,\eta) + L\vartheta(\theta_1,\eta) + M\vartheta(\theta_1,\eta) = g(\theta_1,\eta), \tag{3.1} $$

with the following initial condition

$$ \vartheta(\theta_1, 0) = a, \tag{3.2} $$

here ${}^{ABC}D_{\eta}^{\alpha}\vartheta$ represents ABC fractional derivative operator, a is constants. where L and M are linear and nonlinear operators, $g(\theta_1,\eta)$ in known term.
Employing **E**T on Eq (3.1), we obtain

$$ \mathbf{E}\Big[{}^{ABC}D_{\eta}^{\alpha}\vartheta(\theta_1,\eta) + L\vartheta(\theta_1,\eta) + M\vartheta(\theta_1,\eta)\Big] = \mathbf{E}[g(\theta_1,\eta)]. \tag{3.3} $$

By the property of the **E**T differentiation, we have

$$ \frac{N(\alpha)}{\alpha f^{\alpha} + 1 - \alpha}\Big[\mathbf{E}[\vartheta(\theta_1,\eta)] - f^2\vartheta(\theta_1,0)\Big] = \mathbf{E}[g(\theta_1,\eta)] - \mathbf{E}\Big[L\vartheta(\theta_1,\eta) + Mu(\theta_1,\eta)\Big], $$

which can be written as

$$ \mathbf{E}[\vartheta(\theta_1,\eta)] = f^2\vartheta(\theta_1,0) + \frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}[g(\theta_1,\eta)] - \frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big[L\vartheta(\theta_1,\eta) + M\vartheta(\theta_1,\eta)\Big]. $$

Employing the inverse **E**T, we get

$$ \vartheta(\theta_1,\eta) = \mathbf{E}^{-1}\Big[f^2\vartheta(\theta_1,0) + \frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}[g(\theta_1,\eta)]\Big] - \Big[\frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{L\vartheta(\theta_1,\eta) + M\vartheta(\theta_1,\eta)\Big\}\Big]. $$

In other words, we may also write it as

$$ \vartheta(\theta_1,\eta) = G(\theta_1,\eta) - \Big[\frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{L\vartheta(\theta_1,\eta) + M\vartheta(\theta_1,\eta)\Big\}\Big]. \tag{3.4} $$

where

$$ \vartheta(\theta_1,\eta) = G(\theta_1,\eta) - \mathbf{E}^{-1}\Big[f^2\vartheta(\theta_1,0) + \frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}[g(\theta_1,\eta)]\Big]. $$

Now, we apply HPS on Eq (3.4). Let

$$ \vartheta(\eta) = \sum_{i=0}^{\infty} p^i\vartheta_i(n) = \vartheta_0 + p^1\vartheta_1 + p^2\vartheta_2 + \cdots, \tag{3.5} $$

where p is homotopy parameter and $M\vartheta(\theta_1,\eta)$ can be calculated by using formula,

$$ M\vartheta(\theta_1,\eta) = \sum_{i=0}^{\infty} p^i H_i(\vartheta) = H_0 + p^1H_1 + p^2H_2 + \cdots, \tag{3.6} $$

where He's polynomial are calculated as

$$H_n(\vartheta_0 + \vartheta_1 + \cdots + \vartheta_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(M\Big(\sum_{i=0}^{\infty} p^i \vartheta_i\Big)\right)_{p=0}, \quad n = 0, 1, 2, \cdots \tag{3.7}$$

Put Eqs (3.5)–(3.7) in Eq (3.4), we get

$$\sum_{i=0}^{\infty} p^i \vartheta(\theta_1, \eta) = G(\theta_1, \eta) - \Big[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{L\sum_{i=0}^{\infty} p^i \vartheta_i(\theta_1, \eta) + \sum_{i=0}^{\infty} p^i H_i\Big\}\Big]. \tag{3.8}$$

and similar power of p produces the following iterations, we get

$$\begin{aligned}
p^0 &: \vartheta_0(\theta_1, \eta) = G(\theta_1, \eta),\\
p^1 &: \vartheta_1(\theta_1, \eta) = -\mathbf{E}^{-1}\left[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\vartheta_0(\theta_1, \eta) + H_0(\vartheta)\Big\}\right],\\
p^2 &: \vartheta_2(\theta_1, \eta) = -\mathbf{E}^{-1}\left[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\vartheta_1(\theta_1, \eta) + H_1(\vartheta)\Big\}\right],\\
p^3 &: \vartheta_3(\theta_1, \eta) = -\mathbf{E}^{-1}\left[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\vartheta_2(\theta_1, \eta) + H_2(\vartheta)\Big\}\right],\\
&\vdots
\end{aligned} \tag{3.9}$$

on continuing, these iterations can be written in the following series

$$\vartheta(\theta_1, \eta) = \vartheta_0(\theta_1, \eta) + \vartheta_1(\theta_1, \eta) + \vartheta_2(\theta_1, \eta) + \vartheta_3(\theta_1, \eta) + \cdots = \sum_{i=0}^{\infty} \vartheta_i. \tag{3.10}$$

which represents the approximate solution of the differential problem (3.1).

4. Numerical applications

Some numerical applications are provided to confirm the significance of **E**HPTS and the physical behavior through the graphical representation. It is noticed that only few iterations are enough to demonstrate the accuracy of **E**HPTS.

4.1. Example 1

Consider a one-dimensional fractional diffusion problem

$$\frac{\partial^\alpha \vartheta}{\partial \eta^\alpha} = \frac{\partial^2 \vartheta}{\partial \theta_1^2} + \sin\theta_1, \tag{4.1}$$

with the initial condition

$$\vartheta(\theta_1, 0) = \cos\theta_1, \tag{4.2}$$

and boundary condition

$$\vartheta(0,\eta) = \mathbf{E}^{-\eta}, \qquad \vartheta(\pi,\eta) = -\mathbf{E}^{-\eta}. \tag{4.3}$$

Taking **E**T on Eq (4.1), we get

$$\mathbf{E}\Big[\frac{\partial^\alpha \vartheta}{\partial \eta^\alpha}\Big] = \mathbf{E}\Big[\frac{\partial^2 \vartheta}{\partial \theta_1^2} + \sin\theta_1\Big].$$

Employing the differential properties of **E**T under ABC operator, we get

$$\frac{N(\alpha)}{\alpha f^\alpha + 1 - \alpha}\Big[\mathbf{E}[\vartheta(\theta_1,\eta)] - f^2\vartheta(\theta_1,0)\Big] = \mathbf{E}\Big[\frac{\partial^2 \vartheta}{\partial \theta_1^2}\Big] + f^2 \sin\theta_1,$$

it may also be written as

$$\mathbf{E}[\vartheta(\theta_1,\eta)] = f^2\vartheta(\theta_1,0)) + \frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)} f^2 \sin\theta_1 + \frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big[\frac{\partial^2 \vartheta}{\partial \theta_1^2}\Big]. \tag{4.4}$$

Taking inverse **E**T on Eq (4.4), we get, we get

$$\vartheta(\theta_1,\eta) = \cos\theta_1 + \sin\theta_1\Big[\frac{\alpha\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\Big] + \mathbf{E}^{-1}\Bigg[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2 \vartheta}{\partial \theta_1^2}\Big\}\Bigg]. \tag{4.5}$$

Applying HPS on Eq (4.5), we get

$$\sum_{i=0}^{\infty} p^i \vartheta(\theta_1,\eta) = \cos\theta_1 + \sin\theta_1\Big[\frac{\alpha\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\Big] + \mathbf{E}^{-1}\Bigg[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\sum_{i=0}^{\infty} p^i \frac{\partial^2 \vartheta}{\partial \theta_1^2}\Big\}\Bigg]. \tag{4.6}$$

Equating p on both sides, we have

$$\begin{aligned}
p^0 &: \vartheta_0(\theta_1,\eta) = \cos\theta_1 + \sin\theta_1\Big[\frac{\alpha\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\Big],\\
p^1 &: \vartheta_1(\theta_1,\eta) = \mathbf{E}^{-1}\Bigg[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2 \vartheta_0}{\partial \theta_1^2}\Big\}\Bigg],\\
p^2 &: \vartheta_2(\theta_1,\eta) = \mathbf{E}^{-1}\Bigg[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2 \vartheta_1}{\partial \theta_1^2}\Big\}\Bigg],\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\vartheta_0(\theta_1,\eta) &= \cos\theta_1 + \sin\theta_1\Big[\frac{\alpha\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\Big],\\
\vartheta_1(\theta_1,\eta) &= -\frac{\alpha\eta^\alpha}{\Gamma(\alpha+1)}\cos\theta_1 + (1-\alpha)\cos\theta_1 - \sin\theta_1\Big[\frac{2\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(1-\alpha^2)\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)^2\Big],\\
\vartheta_2(\theta_1,\eta) &= \frac{\alpha\eta^{2\alpha}}{\Gamma(2\alpha+1)}\cos\theta_1 + \frac{(1-\alpha)\eta^\alpha}{\Gamma(\alpha+1)}\cos\theta_1 + \frac{\alpha(1-\alpha)\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha^2)\cos\theta_1
\end{aligned}$$

$$+\sin\theta_1\Big[\frac{\alpha\eta^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{\alpha(1-\alpha)\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha(1-\alpha^2)\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(1-\alpha)(1-\alpha^2)\eta^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha(1-\alpha)^2\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^3\Big],$$

similarly proceeding this process, we can obtain this iteration series such as

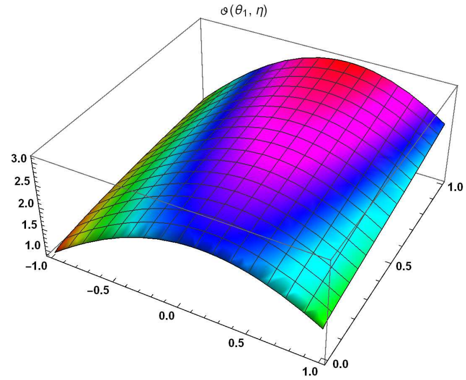
$$\begin{aligned}\vartheta(\theta_1,\eta)=&\cos\theta_1+\sin\theta_1\Big[\frac{\alpha\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\Big]\\&-\frac{\alpha\eta^{\alpha}}{\Gamma(\alpha+1)}\cos\theta_1+(1-\alpha)\cos\theta_1-\sin\theta_1\Big[\frac{\alpha\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(1-\alpha^2)\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^2\Big]\\&+\frac{\alpha\eta^{2\alpha}}{\Gamma(2\alpha+1)}\cos\theta_1+\frac{(1-\alpha)\eta^{\alpha}}{\Gamma(\alpha+1)}\cos\theta_1+\frac{\alpha(1-\alpha)\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha^2)\cos\theta_1\\&+\sin\theta_1\Big[\frac{\alpha\eta^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{\alpha(1-\alpha)\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha(1-\alpha^2)\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(1-\alpha)(1-\alpha^2)\eta^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha(1-\alpha)^2\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^3\Big],\end{aligned}\tag{4.7}$$

which provides the close contact at $\alpha = 1$ such that

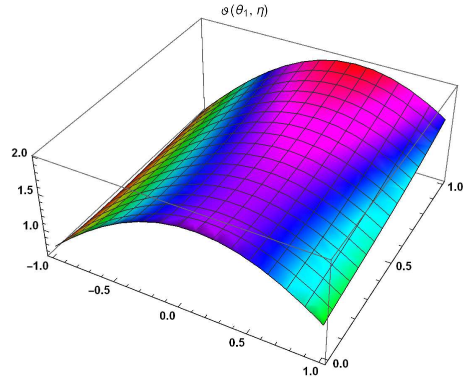
Table 1. The **E**HPTS, exact and absolute error of $\vartheta(\theta_1,\eta)$ for Problem 1 at various value of θ_1 with $\alpha = 1$ and $\eta = 0.01$.

θ_1	**E**HPTS values	Exact values	Absolute error
0.1	0.986099	0.986097	2×10^{-7}
0.2	0.972295	0.972292	3×10^{-6}
0.3	0.948776	0.948771	5×10^{-6}
0.4	0.915778	0.915771	7×10^{-6}
0.5	0.873629	0.873621	8×10^{-6}
0.6	0.822751	0.822742	9×10^{-6}
0.7	0.763653	0.763642	1.1×10^{-6}
0.8	0.696924	0.696912	1.2×10^{-5}
0.9	0.623232	0.623219	1.3×10^{-5}
1.0	0.543313	0.543299	8.6×10^{-4}

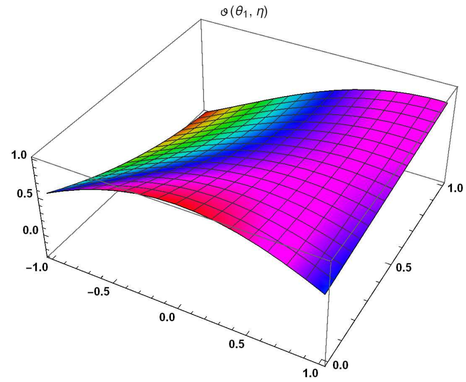
$$\vartheta(\theta_1,\eta)=(1-e^{-\eta})\sin\theta_1+e^{-\eta}\cos\theta_1.\tag{4.8}$$



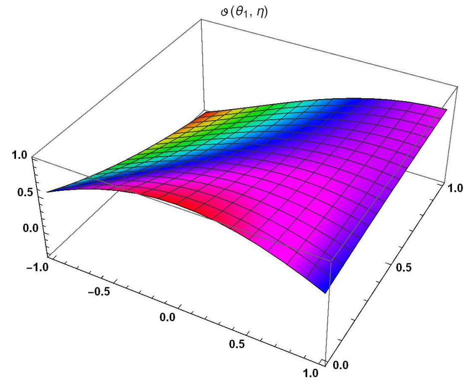
(a) The solution surface of Eq (4.7) at $\alpha = 0.50$



(b) The solution surface of Eq (4.7) at $\alpha = 0.75$

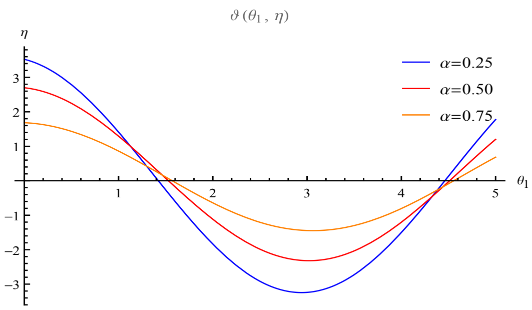


(c) The solution surface of Eq (4.7) at $\alpha = 1$

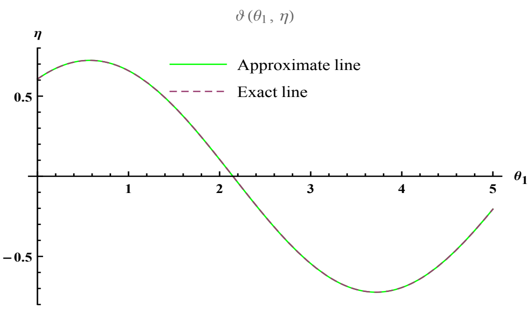


(d) The solution surface of Eq (4.8) at $\alpha = 1$

Figure 1. The approximate and exact solution surface for one-dimensional equation.



(a) 2D Plot distribution at various fractional order of α



(b) 2D Plot distribution at fractional order of $\alpha = 1$

Figure 2. Graphical error between the **EHPTS** and the exact results.

In Figure 1, we plot (a) surface solution for approximate results (b) surface solution for exact results. We indicate the performance of the **EHPTS** at $\alpha = 1$ with $-1 \leq \theta_1 \leq 1$ and $0 \leq \eta \leq 1$ respectively. Figure 2 represents the graphical error between the approximate solution obtained by the **EHPTS** for (4.7) under ABC fractional derivative operators and the exact solutions for (4.8) at $0 \leq \theta_1 \leq 5$ and

$\eta = 0.5$. We observe that both solutions are in close contact and present that **E**HPTS is extremely reliable and achieves the convenient findings.

4.2. Example 2

Next, consider a two-dimensional fractional diffusion problem

$$\frac{\partial^{\alpha}\vartheta}{\partial\eta^{\alpha}} = \frac{\partial^2\vartheta}{\partial\theta_1^2} + \frac{\partial^2\vartheta}{\partial\theta_2^2} - \vartheta, \tag{4.9}$$

with the initial condition

$$\vartheta(\theta_1, \theta_2, 0) = \sin\theta_1 \cos\theta_2, \tag{4.10}$$

and boundary condition

$$\begin{aligned} &\vartheta(\theta_1, 0, \eta) = -\vartheta(\theta_1, \pi, \eta) = e^{-3\eta}\sin\theta_1, \\ &\vartheta(0, \theta_2, \eta) = \vartheta(\pi, \theta_2, \eta) = 0, \end{aligned} \tag{4.11}$$

Taking **E**T on Eq (4.9), we get

$$\mathbf{E}\Big[\frac{\partial^{\alpha}\vartheta}{\partial\eta^{\alpha}}\Big] = \mathbf{E}\Big[\frac{\partial^2\vartheta}{\partial\theta_1^2} + \frac{\partial^2\vartheta}{\partial\theta_2^2} - \vartheta\Big],$$

Employing the differential properties of **E**T under ABC operator, we get

$$\frac{N(\alpha)}{\alpha f^{\alpha} + 1 - \alpha}\Big[\mathbf{E}[\vartheta(\theta_1, \eta)] - f^2\vartheta(\theta_1, 0)\Big] = \mathbf{E}\Big[\frac{\partial^2\vartheta}{\partial\theta_1^2} + \frac{\partial^2\vartheta}{\partial\theta_2^2} - \vartheta\Big],$$

it may also be written as

$$\mathbf{E}[\vartheta(\theta_1, \eta)] = f^2\vartheta(\theta_1, 0)) + \frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big[\frac{\partial^2\vartheta}{\partial\theta_1^2} + \frac{\partial^2\vartheta}{\partial\theta_2^2} - \vartheta\Big], \tag{4.12}$$

Taking inverse **E**T on Eq (4.12), we get

$$\vartheta(\theta_1, \eta) = \vartheta(\theta_1, 0) + \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2\vartheta}{\partial\theta_1^2} + \frac{\partial^2\vartheta}{\partial\theta_2^2} - \vartheta\Big\}\right]. \tag{4.13}$$

Applying HPS on Eq (4.13), we get

$$\sum_{i=0}^{\infty} p^i\vartheta(\theta_1, \eta) = \sin\theta_1\cos\theta_2 + \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\sum_{i=0}^{\infty} p^i\frac{\partial^2\vartheta}{\partial\theta_1^2} + \sum_{i=0}^{\infty} p^i\frac{\partial^2\vartheta}{\partial\theta_2^2} - \sum_{i=0}^{\infty} p^i\vartheta\Big\}\right].$$

Equating p on both sides, we have

$$\begin{aligned} &p^0 : \vartheta_0(\theta_1, \theta_2, \eta) = \vartheta(\theta_1, 0, \eta), \\ &p^1 : \vartheta_1(\theta_1, \theta_2, \eta) = \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha} + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2\vartheta_0}{\partial\theta_1^2} + \frac{\partial^2\vartheta_0}{\partial\theta_2^2} - \vartheta_0\Big\}\right], \end{aligned}$$

$$p^2 : \vartheta_2(\theta_1, \theta_2, \eta) = \mathbf{E}^{-1}\Big[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2 \vartheta_1}{\partial \theta_1^2} + \frac{\partial^2 \vartheta_1}{\partial \theta_2^2} - \vartheta_1\Big\}\Big],$$
$$p^3 : \vartheta_3(\theta_1, \theta_2, \eta) = \mathbf{E}^{-1}\Big[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2 \vartheta_2}{\partial \theta_1^2} + \frac{\partial^2 \vartheta_2}{\partial \theta_2^2} - \vartheta_2\Big\}\Big],$$
$$p^4 : \vartheta_4(\theta_1, \theta_2, \eta) = \mathbf{E}^{-1}\Big[\frac{\alpha f^\alpha + 1 - \alpha}{N(\alpha)}\mathbf{E}\Big\{\frac{\partial^2 \vartheta_3}{\partial \theta_1^2} + \frac{\partial^2 \vartheta_3}{\partial \theta_2^2} - \vartheta_3\Big\}\Big],$$
$$\vdots$$

$$\vartheta_0(\theta_1, \theta_2, \eta) = \sin\theta_1 \cos\theta_2,$$
$$\vartheta_1(\theta_1, \theta_2, \eta) = -3\sin\theta_1\cos\theta_2\Big[\frac{\alpha\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\Big],$$
$$\vartheta_2(\theta_1, \theta_2, \eta) = 9\sin\theta_1\cos\theta_2\Big[\frac{\alpha^2\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2\alpha(1-\alpha)\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)^2\Big],$$
$$\vartheta_3(\theta_1, \theta_2, \eta) = -27\sin\theta_1\cos\theta_2\Big[\frac{\alpha^3\eta^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{3\alpha^2(1-\alpha)\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{3\alpha(1-\alpha)^2\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)^3\Big],$$
$$\vartheta_4(\theta_1, \theta_2, \eta) = 81\sin\theta_1\cos\theta_2\Big[\frac{\alpha^4\eta^{3\alpha}}{\Gamma(4\alpha+1)} + \frac{4\alpha^3(1-\alpha)\eta^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{6\alpha^2(1-\alpha)^2\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2\alpha(1-\alpha)^3\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)^4\Big],$$

similarly proceeding this process, we can obtain this iteration series such as

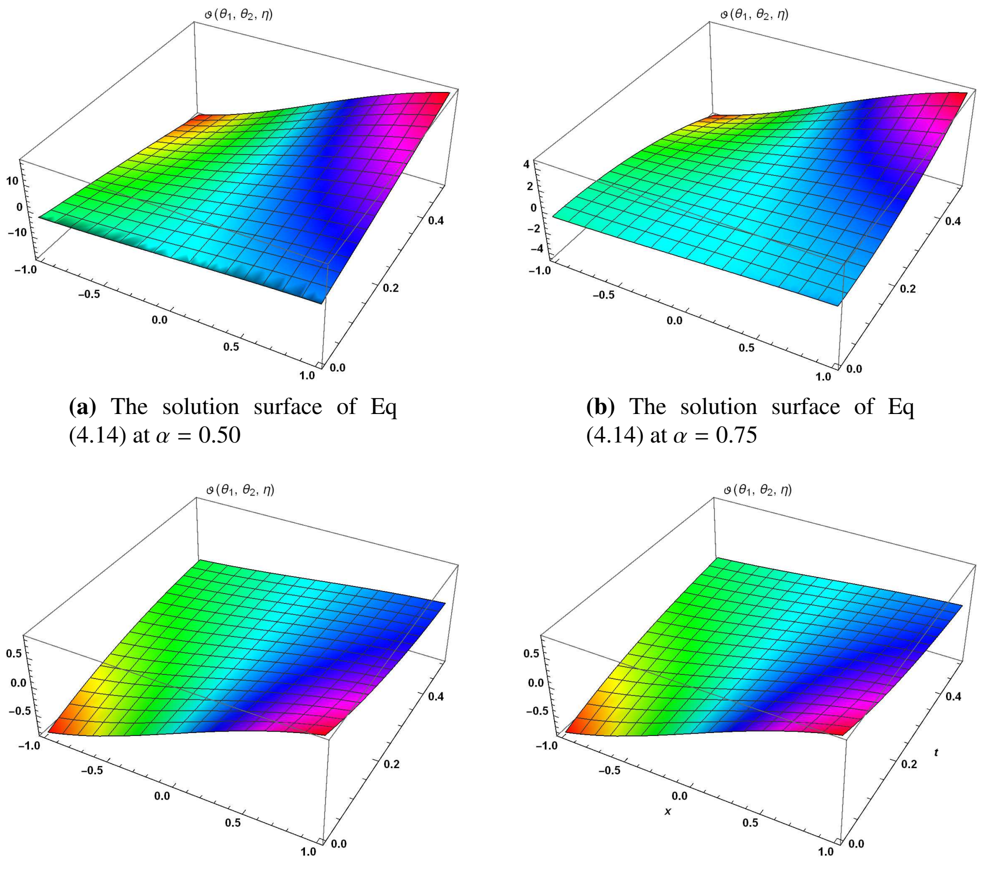
$$\begin{aligned}\vartheta(\theta_1, \theta_2, \eta) = {} & \sin(\theta_1)\cos(\theta_2) - 3\sin\theta_1\cos\theta_2\Big[\frac{\alpha\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)\Big] \\ & + 9\sin\theta_1\cos\theta_2\Big[\frac{\alpha^2\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2\alpha(1-\alpha)\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)^2\Big] \\ & - 27\sin\theta_1\cos\theta_2\Big[\frac{\alpha^3\eta^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{3\alpha^2(1-\alpha)\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{3\alpha(1-\alpha)^2\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)^3\Big] \\ & + 81\sin\theta_1\cos\theta_2\Big[\frac{\alpha^4\eta^{3\alpha}}{\Gamma(4\alpha+1)} + \frac{4\alpha^3(1-\alpha)\eta^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{6\alpha^2(1-\alpha)^2\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2\alpha(1-\alpha)^3\eta^\alpha}{\Gamma(\alpha+1)} + (1-\alpha)^4\Big],\end{aligned} \tag{4.14}$$

Table 2. The **E**HPTS, exact and absolute error of $\vartheta(\theta_1, \theta_2, \eta)$ for Problem 2 at various value of θ_1 with $\alpha = 1$ and $\theta_2 = 0.1, \eta = 0.1$.

θ_1	**E**HPTS values	Exact values	Absolute error
0.1	0.0735908	0.0735889	1.9×10^{-6}
0.2	0.146446	0.1046443	3×10^{-6}
0.3	0.217839	0.217833	6×10^{-6}
0.4	0.287054	0.287047	7×10^{-6}
0.5	0.353402	0.353393	9×10^{-6}
0.6	0.416219	0.416208	1.1×10^{-5}
0.7	0.474876	0.474864	1.2×10^{-5}
0.8	0.528789	0.528775	1.4×10^{-5}
0.9	0.577419	0.577404	1.5×10^{-5}
1.0	0.620279	0.620263	1.6×10^{-5}

which provides the close contact at $\alpha = 1$ such that

$$\vartheta(\theta_1, \theta_2, \eta) = e^{-3\eta} \sin\theta_1 \cos\theta_2. \tag{4.15}$$



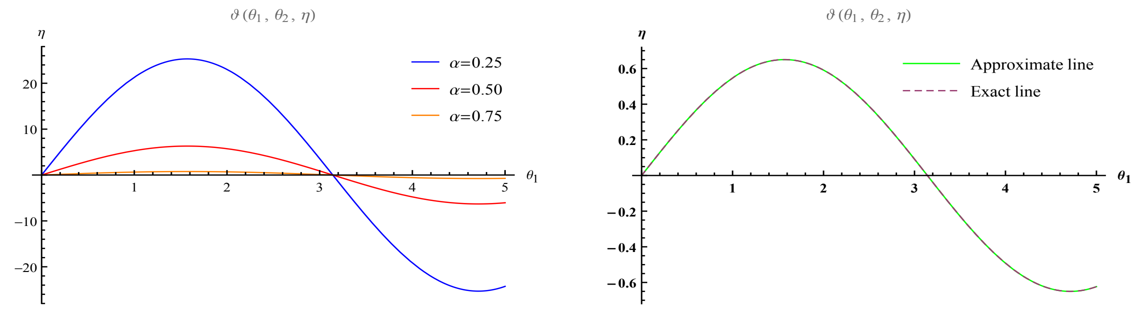
(a) The solution surface of Eq (4.14) at $\alpha = 0.50$

(b) The solution surface of Eq (4.14) at $\alpha = 0.75$

(c) The solution surface of Eq (4.14) at $\alpha = 1$

(d) The exact surface of Eq (4.15) at $\alpha = 1$

Figure 3. The approximate and exact solution surface for two-dimensional equation.



(a) 2D Plot distribution at various fractional order of α

(b) 2D Plot distribution at fractional order of $\alpha = 1$

Figure 4. Graphical error between the **EHPTS** and the exact results.

In Figure 3, we plot (a) surface solution for approximate results (b) surface solution for exact results. We indicate the performance of the **EHPTS** at $\alpha = 1$ with $-1 \leq \theta_1 \leq 1$, $\theta_2 = 0.1$ and $0 \leq \eta \leq 0.5$ respectively. Figure 4 represents the graphical error between the approximate solution obtained by the **EHPTS** for (4.14) under ABC fractional derivative operators and the exact solutions for (4.23) at $0 \leq \theta_1 \leq 5$, $\theta_2 = 0.5$ and $\eta = 0.25, 0.50, 0.75$ and 1. We observe that both solutions are in close contact and present that **EHPTS** is extremely reliable and achieves the convenient findings.

4.3. Example 3

Finally, consider a three-dimensional fractional diffusion problem

$$\frac{\partial^\alpha \vartheta}{\partial \eta^\alpha} = \frac{\partial^2 \vartheta}{\partial \theta_1^2} + \frac{\partial^2 \vartheta}{\partial \theta_2^2} + \frac{\partial^2 \vartheta}{\partial \theta_3^2} - 2\vartheta, \tag{4.16}$$

with the initial condition

$$\vartheta(\theta_1, \theta_2, \theta_3, 0) = \sin\theta_1 \sin\theta_2 \sin\theta_3, \tag{4.17}$$

and boundary condition

$$\begin{aligned} \vartheta(0, \theta_2, \theta_3, \eta) &= \vartheta(\pi, \theta_2, \theta_3, \eta) = 0, \\ \vartheta(\theta_1, 0, \theta_3, \eta) &= \vartheta(\theta_1, \pi, \theta_3, \eta) = 0, \\ \vartheta(\theta_1, \theta_2, 0, \eta) &= \vartheta(\theta_1, \theta_2, \pi, \eta) = 0, \end{aligned} \tag{4.18}$$

Taking **ET** on Eq (4.16), we get

$$\mathbf{E}\Big[\frac{\partial^\alpha \vartheta}{\partial \eta^\alpha}\Big] = \mathbf{E}\Big[\frac{\partial^2 \vartheta}{\partial \theta_1^2} + \frac{\partial^2 \vartheta}{\partial \theta_2^2} + \frac{\partial^2 \vartheta}{\partial \theta_3^2} - 2\vartheta\Big],$$

Employing the differential properties of **ET** under ABC operator, we get

$$\frac{N(\alpha)}{\alpha f^\alpha + 1 - \alpha}\Big[\mathbf{E}[\vartheta(\theta_1, \eta)] - f^2 \vartheta(\theta_1, 0)\Big] = \mathbf{E}\Big[\frac{\partial^2 \vartheta}{\partial \theta_1^2} + \frac{\partial^2 \vartheta}{\partial \theta_2^2} + \frac{\partial^2 \vartheta}{\partial \theta_3^2} - 2\vartheta\Big], \tag{4.19}$$

it may also be written as

$$\mathbf{E}[\vartheta(\theta_1,\eta)] = f^2\vartheta(\theta_1,0)) + \frac{\alpha f^{\alpha}+1-\alpha}{N(\alpha)}\mathbf{E}\Big[\frac{\partial^2\vartheta}{\partial\theta_1^2}+\frac{\partial^2\vartheta}{\partial\theta_2^2}+\frac{\partial^2\vartheta}{\partial\theta_3^2}-2\vartheta\Big], \tag{4.20}$$

Taking inverse **E**T on Eq (4.20), we get, we get

$$\vartheta(\theta_1,\eta) = \vartheta(\theta_1,0) + \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha}+1-\alpha}{N(\alpha)}\mathbf{E}\left\{\frac{\partial^2\vartheta}{\partial\theta_1^2}+\frac{\partial^2\vartheta}{\partial\theta_2^2}+\frac{\partial^2\vartheta}{\partial\theta_3^2}-2\vartheta\right\}\right]. \tag{4.21}$$

Applying HPS on Eq (4.21), we get, we get

$$\sum_{i=0}^{\infty} p^i\vartheta(\theta_1,\eta) = \sin\theta_1\sin\theta_2\sin\theta_3 + \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha}+1-\alpha}{N(\alpha)}\mathbf{E}\left\{\sum_{i=0}^{\infty}p^i\frac{\partial^2\vartheta}{\partial\theta_1^2}+\sum_{i=0}^{\infty}p^i\frac{\partial^2\vartheta}{\partial\theta_2^2}+\sum_{i=0}^{\infty}p^i\frac{\partial^2\vartheta}{\partial\theta_3^2}-2\sum_{i=0}^{\infty}p^i\vartheta\right\}\right].$$

Equating p on both sides, we have

$$\begin{aligned}
p^0 &: \vartheta_0(\theta_1,\theta_2,\theta_3,\eta) = \vartheta(\theta_1,\theta_2,\theta_3,0),\\
p^1 &: \vartheta_1(\theta_1,\theta_2,\theta_3,\eta) = \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha}+1-\alpha}{N(\alpha)}\mathbf{E}\left\{\frac{\partial^2\vartheta_0}{\partial\theta_1^2}+\frac{\partial^2\vartheta_0}{\partial\theta_2^2}+\frac{\partial^2\vartheta_0}{\partial\theta_3^2}-2\vartheta_0\right\}\right],\\
p^2 &: \vartheta_2(\theta_1,\theta_2,\theta_3,\eta) = \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha}+1-\alpha}{N(\alpha)}\mathbf{E}\left\{\frac{\partial^2\vartheta_1}{\partial\theta_1^2}+\frac{\partial^2\vartheta_1}{\partial\theta_2^2}+\frac{\partial^2\vartheta_1}{\partial\theta_3^2}-2\vartheta_1\right\}\right],\\
p^3 &: \vartheta_3(\theta_1,\theta_2,\theta_3,\eta) = \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha}+1-\alpha}{N(\alpha)}\mathbf{E}\left\{\frac{\partial^2\vartheta_2}{\partial\theta_1^2}+\frac{\partial^2\vartheta_2}{\partial\theta_2^2}+\frac{\partial^2\vartheta_2}{\partial\theta_3^2}-2\vartheta_2\right\}\right],\\
p^4 &: \vartheta_4(\theta_1,\theta_2,\theta_3,\eta) = \mathbf{E}^{-1}\left[\frac{\alpha f^{\alpha}+1-\alpha}{N(\alpha)}\mathbf{E}\left\{\frac{\partial^2\vartheta_3}{\partial\theta_1^2}+\frac{\partial^2\vartheta_3}{\partial\theta_2^2}+\frac{\partial^2\vartheta_3}{\partial\theta_3^2}-2\vartheta_3\right\}\right],\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\vartheta_0(\theta_1,\theta_2,\theta_3,\eta) &= \sin\theta_1\sin\theta_2\sin\theta_3,\\
\vartheta_1(\theta_1,\theta_2,\theta_3,\eta) &= -5\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\Big],\\
\vartheta_2(\theta_1,\theta_2,\theta_3,\eta) &= 25\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha^2\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2\alpha(1-\alpha)\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^2\Big],\\
\vartheta_3(\theta_1,\theta_2,\theta_3,\eta) &= -125\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha^3\eta^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{3\alpha^2(1-\alpha)\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{3\alpha(1-\alpha)^2\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^3\Big],\\
\vartheta_4(\theta_1,\theta_2,\theta_3,\eta) &= 625\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha^4\eta^{3\alpha}}{\Gamma(4\alpha+1)}+\frac{4\alpha^3(1-\alpha)\eta^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{6\alpha^2(1-\alpha)^2\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2\alpha(1-\alpha)^3\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^4\Big],
\end{aligned}$$

similarly proceeding this process, we can obtain this iteration series such as

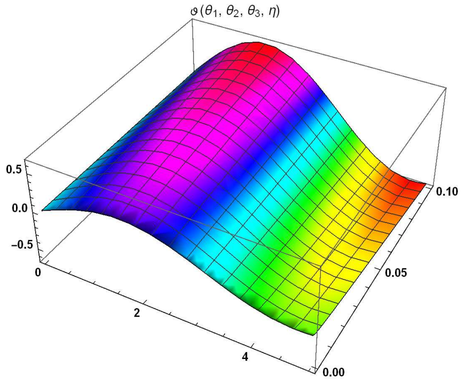
$$\begin{aligned}\vartheta(\theta_1,\theta_2,\theta_3,\eta) &= \sin\theta_1\sin\theta_2\sin\theta_3 - 5\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)\Big]\\
&+25\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha^2\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2\alpha(1-\alpha)\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^2\Big]\\
&-125\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha^3\eta^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{3\alpha^2(1-\alpha)\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{3\alpha(1-\alpha)^2\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^3\Big]\\
&+625\sin\theta_1\sin\theta_2\sin\theta_3\Big[\frac{\alpha^4\eta^{3\alpha}}{\Gamma(4\alpha+1)}+\frac{4\alpha^3(1-\alpha)\eta^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{6\alpha^2(1-\alpha)^2\eta^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2\alpha(1-\alpha)^3\eta^{\alpha}}{\Gamma(\alpha+1)}+(1-\alpha)^4\Big],\end{aligned} \tag{4.22}$$

Table 3. The **E**HPTS, exact and absolute error of $\vartheta(\theta_1,\theta_2,\theta_3,\eta)$ for Problem 3 at various value of θ_1 with $\alpha = 1$ and $\theta_2 = 0.1$, $\theta_3 = 0.1$ and $\eta = 0.0.5$.

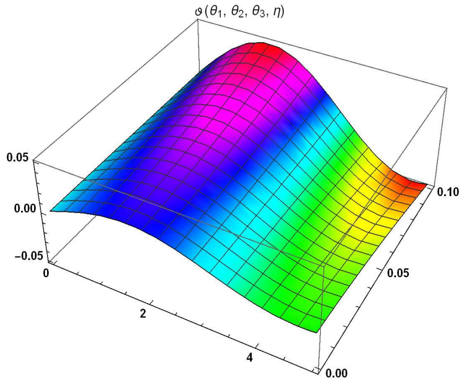
x	Approximate values	Exact values	Absolute error
0.1	0.00077427	0.000774393	2.7×10^{-7}
0.2	0.0015408	0.00154105	2.5×10^{-7}
0.3	0.00229194	0.00229231	3.7×10^{-7}
0.4	0.00302018	0.00302066	4.8×10^{-7}
0.5	0.00371824	0.00371883	5.9×10^{-7}
0.6	0.00437915	0.00437985	7×10^{-7}
0.7	0.00499631	0.0049971	7.9×10^{-7}
0.8	0.00556354	0.00556443	8.9×10^{-7}
0.9	0.00607518	0.00607615	9.7×10^{-7}
1.0	0.00652613	0.00652717	1.04×10^{-6}

which provides the close contact at $\alpha = 1$ such that

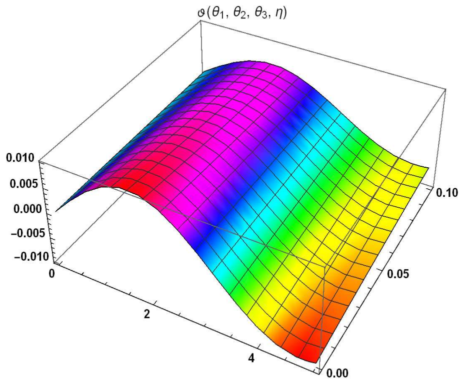
$$\vartheta(\theta_1,\theta_2,\theta_3,\eta) = e^{-5\eta}\sin\theta_1\sin\theta_2\sin\theta_3. \tag{4.23}$$



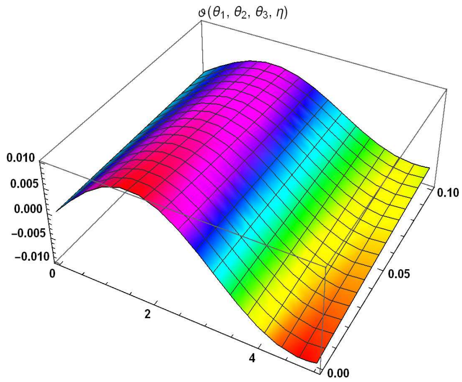
(a) The solution surface of Eq (4.22) at $\alpha = 0.50$



(b) The solution surface of Eq (4.22) at $\alpha = 0.75$

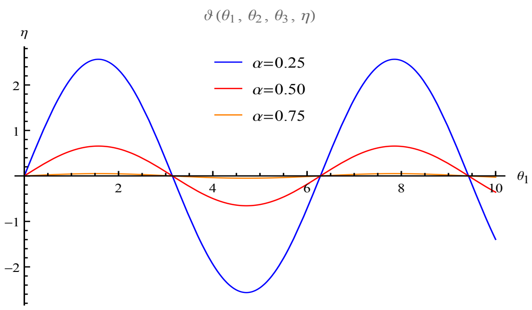


(c) The solution surface of Eq (4.22) at $\alpha = 1$

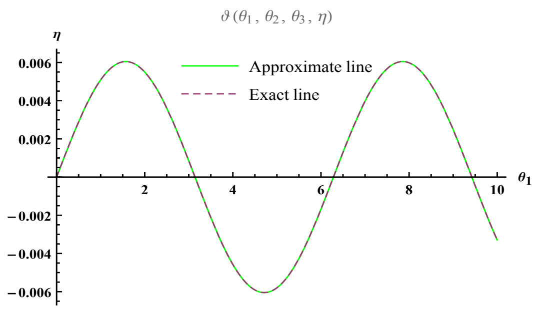


(d) The exact surface of Eq (4.23) at $\alpha = 1$

Figure 5. The approximate and exact solution surface for three-dimensional equation.



(a) 2D Plot distribution at various fractional order of α



(b) 2D Plot distribution at fractional order of $\alpha = 1$

Figure 6. Graphical error between the **EHPTS** and the exact results.

In Figure 5, we plot (a) surface solution for approximate results at $\alpha = 0.50, 0.75, 1$ (b) surface solution for exact results. We indicate the performance of the **EHPTS** at $\alpha = 1$ with $0 \leq \theta_1 \leq 10$, $\theta_2 = 0.1$, $\theta_3 = 0.1$ and $0 \leq \eta \leq 0.1$ respectively. Figure 6 represents the graphical error between the approximate solution obtained by the **EHPTS** for (4.22) under ABC fractional derivative operators and the exact solutions for (4.23) at $0 \leq \theta_1 \leq 10$, $\theta_2 = 0.1$, $\theta_3 = 0.1$ and $\eta = 0.25, 0.50, 0.75$ and 1. We observe that both solutions are in close contact and present that **E**HPTS is extremely reliable and achieves the convenient findings.

5. Conclusion and Future Interact

This paper presents the study of **E**HPTS for obtaining the approximate solution of multi-dimensional diffusion problems under ABC fractional order derivative. In addition, HPS produces successive iterations and shows the results in the form of a series. This strategy does not involve rectified constants, steady constraints, or massive integrals due to the noise-free results. Some examples are carried out to provide the efficiency of **E**HPTS and showed the results in better obligations towards the precise results. We compute the values of iterations and graphical results using the Mathematica software 11. The physical solutions behavior of the graphical representation and plot distribution yield that **E**HPTS is a very powerful and efficient method to produce the approximate solution of partial differential equations that arise in science and engineering. This method evaluates and controls the series of solutions that quickly arrive at the precise solution in a condensed acceptable domain. In future, we consider the strategy of **E**HPTS for other fractional differential problems and compete with other exceedingly fractional order systems of equations.

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Conflict of interest

The authors declare there is no conflict of interest.

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