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Research article

Global solvability of a chemotaxis-haptotaxis model in the whole 2-d space

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Abstract: This paper investigates a two-dimensional chemotaxis-haptotaxis model

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w), & x\in\mathbb{R}^2,\ t>0,\\ v_t = \Delta v - v + u, & x\in\mathbb{R}^2,\ t>0,\\ w_t = -vw, & x\in\mathbb{R}^2,\ t>0,\end{cases}$$

where χ and ξ are positive parameters. It is proved that, for any suitable smooth initial data (u_0, v_0, w_0), this model admits a unique global strong solution if $\|u_0\|_{L^1} < \frac{8\pi}{\chi}$. Compared to the result by Calvez and Corrias (Calvez and Corrias, 2008 [1]), we can see that the haptotaxis effect is almost negligible in terms of global existence, which is consistent with the result of bounded domain (Jin and Xiang, 2021 [2]). Moreover, to the best of our knowledge, this is the first analytical work for the well-posedness of chemotaxis-haptotaxis system in the whole space.

Keywords: chemotaxis; haptotaxis; global existence; whole space; Cauchy problem

1. Introduction

In the present work, we shall consider a chemotaxis-haptotaxis model

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w),\\ \tau v_t = \Delta v - v + u,\\ w_t = -vw + \eta u(1-u-w),\end{cases} \tag{1.1}$$

where χ and ξ are positive parameters. In the model (1.1), u represents the density of cancer cell, v and w denote the density of matrix degrading enzymes (MDEs) and the extracellular matrix (ECM) with the positive sensitivity χ, ξ, respectively. Such an important extension of chemotaxis to a more complex cell migration mechanism has been proposed by Chaplain and Lolas [3] to describe the cancer cell invasion of tissue. In that process, cancer invasion is associated with the degradation of ECM, which is degraded by MDEs secreted by cancer cells. Besides random motion, the migration of invasive cells is oriented both by a chemotaxis mechanism and by a haptotaxis mechanism.

In the past ten more years, the global solvability, boundedness and asymptotic behavior for the corresponding no-flux or homogeneous Neumann boundary-initial value problem in bounded domain and its numerous variants have been widely investigated for certain smooth initial data. For the full parabolic system of (1.1), Pang and Wang [4] studied the global boundedness of classical solution in the case $\tau = 1$ in 2D domains, and the global solvability also was established for three dimension. When $\eta = 0$ and $\tau = 1$, Tao and Wang [5] proved the existence and uniqueness of global classical solution for any $\chi > 0$ in 1D intervals and for small $\frac{\chi}{\mu} > 0$ in 2D domains, and Tao [6] improved the results for any $\mu > 0$ in two dimension; Cao [7] proved for small $\frac{\chi}{\mu} > 0$, the model (1.1) processes a global and bounded classical solution in 3D domains.

When $\tau = 0$, the second equation of (1.1) becomes an elliptic function. In the case of $\eta > 0$, Tao and Winkler [8] proved the global existence of classical solutions in 2D domains for any $\mu > 0$. In the case of $\eta = 0$, the global existence and boundedness for this simplified model under the condition of $\mu > \frac{(N-2)_+}{N}\chi$ in any N-D domains in [9]. Moreover, the stabilization of solutions with on-flux boundary conditions was discussed in [10]. For the explosion phenomenon, Xiang [11] proved that (1.1) possess a striking feature of finite-time blow-up for $N \geq 3$ with $\mu = \eta = \tau = 0$; the blow-up results for two dimension was discussed in [2] with $w_t = -vw + \eta w(1-w)$ and $\mu = 0$.

When $\chi = 0$, the system (1.1) becomes a haptotaxis-only system. The local existence and uniqueness of classical solutions was proved in [12]. In [13–15], the authors respectively established the global existence, the uniform-in-time boundedness of classical solutions and the asymptotic behavior. Very recently, Xiang [11] showed that the pure haptotaxis term cannot induce blow-up and pattern for $N \leq 3$ or $\tau = 0$ in the case of $\mu = \eta = 0$.

Without considering the effect of the haptotaxis term in (1.1), we may have the extensively-studied Keller-Segel system, which was proposed in [16] to describe the collective behavior of cells under the influence of chemotaxis

$$\begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \tau \partial_t v = \Delta u - \lambda v + u \end{cases} \tag{1.2}$$

with u and v denoting the cell density and chemosignal concentration, respectively. There have been a lot of results in the past years (see [17–21], for instance). Here we only mention some global existence and blow-up results in two dimensional space. For the parabolic-elliptic case of (1.2) with $\lambda = 0$, $\frac{8\pi}{\chi}$ was proved to be the mass threshold in two dimension in [22–24] (see also [25, 26] for related results in the bounded domain); namely, the chemotactic collapse (blowup) should occur if and only if $\|u_0\|_{L^1}$ is greater than $\frac{8\pi}{\chi}$. If $\|u_0\|_{L^1} < \frac{8\pi}{\chi}$, the existence of free-energy solutions were improved in [22]. Furthermore, the asymptotic behavior was given by a unique self-similar profile of the system (see also [27] for radially symmetric results concerning self-similar behavior). For the results in the threshold $\frac{8\pi}{\chi}$, we refer readers for [28–30] for more details. For the parabolic-elliptic model in higher dimensions ($N \geq 3$) in (1.2), the solvability results were discussed in [31–34] with small data in critical spaces like $L^{\frac{N}{2}}\left(\mathbb{R}^N\right), L_{\mathrm{w}}^{\frac{N}{2}}\left(\mathbb{R}^N\right), M^{\frac{N}{2}}\left(\mathbb{R}^N\right)$, i.e., those which are scale-invariant under the natural scaling. Blowing up solutions to the parabolic-elliptic model of (1.2) in dimension $N \geq 3$ have been studied in [35–38].

In the case $\tau = 1$, Calvez and Corrias [1] showed that under hypotheses $u_0 \ln\left(1 + |x|^2\right) \in L^1\left(\mathbb{R}^2\right)$

and $u_0 \ln u_0 \in L^1\left(\mathbb{R}^2\right)$, any solution exists globally in time if $\|u_0\|_{L^1} < \frac{8\pi}{\chi}$. In [39], the extra assumptions on u_0 were removed, while the condition on mass was restricted to $\|u_0\|_{L^1} < \frac{4\pi}{\chi}$. The value $\frac{4\pi}{\chi}$ appeared since a Brezis-Merle type inequality played an essential role there. These results were improved in [40, 41] to global existence of all solutions with $\|u_0\|_{L^1} < \frac{8\pi}{\chi}$ by two different method. Furthermore the global existence of solutions was also obtained under some condition on u_0 in the critical case $\|u_0\|_{L^1} = \frac{8\pi}{\chi}$ in [40]. The blow-up results of the parabolic-parabolic case in the whole space were discussed in [42, 43] with the second equation was replaced by $\partial_t v = \Delta u + u$.

However, the global solvability and explosion phenomenon of chemotaxis-haptotaxis model in the whole space have never been touched. Here we consider the global solvability of a simplified model of (1.1)

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w), & x \in \mathbb{R}^2,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \mathbb{R}^2,\ t > 0, \\ w_t = -vw, & x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{1.3}$$

Main results. We assume that the initial data satisfies the following assumptions:

$$(u_0, v_0, w_0) \in H^2(\mathbb{R}^2) \times H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2) \text{ and } u_0, v_0, w_0 \text{ are nonnegative,} \tag{1.4}$$

$$u_0 \in L^1\left(\mathbb{R}^2, \ln\left(1 + |x|^2\right) dx\right) \text{ and } u_0 \ln u_0 \in L^1\left(\mathbb{R}^2\right) \tag{1.5}$$

and

$$\Delta w_0 \in L^\infty\left(\mathbb{R}^2\right) \text{ and } \nabla \sqrt{w_0} \in L^\infty\left(\mathbb{R}^2\right). \tag{1.6}$$

Theorem 1.1. *Let $\chi > 0$, $\xi > 0$ and the initial data (u_0, v_0, w_0) satisfy (1.4)–(1.6). If $m := \|u_0\|_{L^1} < \frac{8\pi}{\chi}$, then the corresponding chemotaxis-haptotaxis system (1.3) possesses a unique global-in-time, nonnegative and strong solution (u, v, w) fulfilling that for any $T < \infty$*

$$(u, v, w) \in C\left(0, T; H^2\left(\mathbb{R}^2\right) \times H^3\left(\mathbb{R}^2\right) \times H^3\left(\mathbb{R}^2\right)\right).$$

Remark 1.1. Our theorem extends the previous results in two aspects. First, our result agrees with that in [1] by setting $w = 0$, which proved that if $\|u_0\|_{L^1} < \frac{8\pi}{\chi}$, then the Cauchy problem of the system (1.2) admits a global solution. Secondly, our theorem extends Theorem 1.1 in [2], where the authors proved that $\frac{4\pi}{\chi}$ is the critical mass of the system (1.3) in bounded domains, implying the negligibility of haptotaxis on global existence.

We obtain the critical mass value using the energy method in [1, 22]. The energy functional:

$$F(u, v, w)(t) = \int_{\mathbb{R}^2} u \ln u - \chi \int_{\mathbb{R}^2} uv - \xi \int_{\mathbb{R}^2} uw + \frac{\chi}{2} \int_{\mathbb{R}^2} \left(v^2 + |\nabla v|^2\right), \quad \forall t \in (0, T_{\max}) \tag{1.7}$$

as shown in [2] comes out to be the key ingredient leading to the global existence of solutions under the smallness condition for the mass. Under the assumption

$$\|u_0\|_{L^1} < \frac{8\pi}{\chi} \tag{1.8}$$

and (1.5), we can derive an integral-type Gronwall inequality for $F(t)$. As a result, we can get a priori estimate for the $\int_{\mathbb{R}^2} u \ln u$, which is the key step to establish the global existence of solutions to the system (1.3).

The rest of this paper is organized as follows. In Section 2, we prove local-in-time existence of the solution, and obtain the blow-up criteria for the solution. In Section 3, we give the proof of the Theorem 1.1.

In the following, $(u)_+$ and $(u)_-$ will denote the positive and negative part of u as usual, while $L^p := L^p\left(\mathbb{R}^2\right)$.

2. local existence

We now establish the local existence and uniqueness of strong solutions to system (1.3). Our strategy is first to construct an iteration scheme for (1.3) to obtain the approximate solutions and then to derive uniform bounds for the approximate solutions to pass the limit.

Lemma 2.1. *Let $\chi > 0$, $\xi > 0$ and $u_0 \geq 0$. Then, there exists a maximal existence time $T_{\max} > 0$, such that, if the initial data (u_0, v_0, w_0) satisfy (1.4), then there exists a unique solution (u, v, w) of (1.3) satisfying for any $T < T_{\max}$, and*

$$(u, v, w) \in C\left(0, T; H^2(\mathbb{R}^2) \times H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2)\right). \tag{2.1}$$

Furthermore, u, v and w are all nonnegative.

Proof. To obtain the local solution, we follow similar procedures of an iterative scheme developed in [45, 46]. We construct the solution sequence $\left(u^j, v^j, w^j\right)_{j\geq 0}$ by iteratively solving the Cauchy problems of the following system

$$\begin{cases} \partial_t u^{j+1} = \Delta u^{j+1} - \chi\nabla\cdot\left(u^{j+1}\nabla v^j\right) - \xi\nabla\cdot\left(u^{j+1}\nabla w^j\right), & x\in\mathbb{R}^2,\ t>0,\\ \partial_t v^{j+1} = \Delta v^{j+1} - v^{j+1} + u^j, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t w^{j+1} = -v^{j+1}w^{j+1}, & x\in\mathbb{R}^2,\ t>0,\\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x),\ \ w(x,0)=w_0(x), & x\in\mathbb{R}^2. \end{cases} \tag{2.2}$$

We first set $\left(u^0(x,t), v^0(x,t), w^0(x,t)\right) = (u_0(x), v_0(x), w_0(x))$. We point out that the system is decouple, then by the linear parabolic equations theory in [44, Theorem III.5.2], we can obtain the unique solution $u^1, v^1 \in V_2^{1,\frac{1}{2}}\left([0,T]\times\mathbb{R}^2\right)$, then we get $w^1 \in C^1\left([0,T], H^1(\mathbb{R}^2)\right)$ by directly solving the ordinary equation. Similarly, we define $\left(u^j, v^j, w^j\right)$ iteratively.

In the following, we shall prove the convergence of the iterative sequences $\left\{u^j, v^j, w^j\right\}_{j\geq 1}$ in $C(0,T;X)$ with $X := H^2\times H^3\times H^3$ for some small $T>0$. To obtain the uniform estimates, we may use the standard mollifying procedure. However, since the procedure is lengthy, we omit the details, like in the proofs of Theorem 1.1 in [45] and Theorem 2.1 in [46].

Uniform estimates: We will use the induction argument to show that the iterative sequences $\left\{u^j, v^j, w^j\right\}_{j\geq 1}$ are in $C(0,T;X)$ with $X := H^2\times H^3\times H^3$ for some small $T>0$, which means that there exists a constant $R>0$ such that, for any j, the following inequality holds for a small time interval

$$\sup_{0\leq t\leq T}\left(\left\|u^j\right\|_{H^2} + \left\|v^j\right\|_{H^3} + \left\|w^j\right\|_{H^3}\right) \leq R, \tag{2.3}$$

where $R = 2\{\|u_0\|_{H^2} + \|v_0\|_{H^3} + \|w_0\|_{H^3}\} + 8$. Due to the definition of R, the case $j = 0$ is obvious. Then, we need to show that (2.3) is also true for $j + 1$. This will be done by establishing the energy estimate for $\left(u^{j+1}, v^{j+1}, w^{j+1}\right)$. First, we begin with the estimate of v^{j+1}.

(i) *Estimates of* v^{j+1}. Taking the L^2 inner product of the second equation of (2.2) with v^{j+1}, integrating by parts and using Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|v^{j+1}(t)\right\|_{L^2}^2 + \left\|\nabla v^{j+1}\right\|_{L^2}^2 &= -\int_{\mathbb{R}^2}(v^{j+1})^2 + \int_{\mathbb{R}^2} v^{j+1}u^j \\ &\le -\frac{1}{2}\left\|v^{j+1}\right\|_{L^2}^2 + \frac{1}{2}\left\|u^j\right\|_{L^2}^2 .\end{aligned} \tag{2.4}$$

To show the H^1 estimate of v^{j+1}, we will multiply the second equation of (2.2) by $\partial_t v^{j+1}$, integrating by parts and then obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\nabla v^{j+1}(t)\right\|_{L^2}^2 + \left\|\partial_t v^{j+1}\right\|_{L^2}^2 &= -\int_{\mathbb{R}^2} v^{j+1}\partial_t v^{j+1} + \int_{\mathbb{R}^2} u^j\partial_t v^{j+1} \\ &\le \frac{1}{2}\left\|\partial_t v^{j+1}\right\|_{L^2}^2 + \left\|v^{j+1}\right\|_{L^2}^2 + \left\|u^j\right\|_{L^2}^2 .\end{aligned} \tag{2.5}$$

For the H^2 estimate of v^{j+1}, by Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\nabla^2 v^{j+1}(t)\right\|_{L^2}^2 + \left\|\nabla\Delta v^{j+1}\right\|_{L^2}^2 &= -\int_{\mathbb{R}^2}(\Delta v^{j+1})^2 + \int_{\mathbb{R}^2}\Delta v^{j+1}\Delta u^j \\ &\le -\frac{1}{2}\left\|\Delta v^{j+1}\right\|_{L^2}^2 + \frac{1}{2}\left\|\Delta u^j\right\|_{L^2}^2 .\end{aligned} \tag{2.6}$$

Similarly, integrating by parts, it is clear that for all $t \in (0, T)$

$$\begin{aligned}\frac{d}{dt}\left\|\nabla^3 v^{j+1}(t)\right\|_{L^2}^2 &= 2\int_{\mathbb{R}^2}\nabla^3 v^{j+1}\nabla^3(\Delta v^{j+1} - v^{j+1} + u^j) \\ &= -2\int_{\mathbb{R}^2}\left|\nabla^4 v^{j+1}\right|^2 - 2\int_{\mathbb{R}^2}\left|\nabla^3 v^{j+1}\right|^2 - 2\int_{\mathbb{R}^2}\nabla^4 v^{j+1}\nabla^2 u^j \\ &\le -\left\|\nabla^4 v^{j+1}\right\|_{L^2}^2 - 2\left\|\nabla^3 v^{j+1}\right\|_{L^2}^2 + \left\|\nabla^2 u^j\right\|_{L^2}^2 ,\end{aligned}$$

togethering with (2.3)–(2.6) and adjusting the coefficients carefully, we can find a positive constant α such that

$$\begin{aligned}&\frac{d}{dt}\left\|v^{j+1}(t)\right\|_{H^3}^2 + \alpha\left(\left\|v^{j+1}\right\|_{H^4}^2 + \left\|\partial_t v^{j+1}\right\|_{L^2}^2\right) \\ \le\ & c_1\left(\left\|v^{j+1}\right\|_{H^3}^2 + \left\|u^j\right\|_{H^2}^2\right)\end{aligned} \tag{2.7}$$

with $c_1 > 0$. Here after $c_i(i = 2, 3...)$ denotes the constant independent of R. Integrating on $(0, t)$, we can obtain for all $t \in (0, T)$

$$\left\|v^{j+1}(t)\right\|_{H^3}^2 + \alpha\int_0^t\left(\left\|v^{j+1}(s)\right\|_{H^4}^2 + \left\|\partial_t v^{j+1}(s)\right\|_{L^2}^2\right)$$

$$
\begin{aligned}
&\leq e^{c_1 T}\|v_0\|_{H^3}^2 + e^{c_1 T} T \sup_{t\in(0,T)} \left\|u^j(t)\right\|_{H^2}^2 \\
&\leq e^{c_1 T}\|v_0\|_{H^3}^2 + e^{c_1 T} T R^2 \\
&\leq 2\|v_0\|_{H^3}^2 + 2, \qquad (2.8)
\end{aligned}
$$

by choosing $T > 0$ small enough to satisfy $e^{c_1 T} < 2$ and $TR^2 < 1$.

(ii) *The estimate of* w^{j+1}. In fact, the third component of the above solution of (2.2) can be expressed explicitly in terms of v^{j+1}. This leads to the representation formulae

$$
w^{j+1}(x,t) = w_0(x) e^{-\int_0^t v^{j+1}(x,s)ds}, \qquad (2.9)
$$

$$
\nabla w^{j+1}(x,t) = \nabla w_0(x) \mathrm{e}^{-\int_0^t v^{j+1}(x,s)\mathrm{d}s} - w_0(x)\mathrm{e}^{-\int_0^t v^{j+1}(x,s)\mathrm{d}s} \int_0^t \nabla v^{j+1}(x,s)\mathrm{d}s \qquad (2.10)
$$

as well as

$$
\begin{aligned}
&\Delta w^{j+1}(x,t) \\
=&\Delta w_0(x)\mathrm{e}^{-\int_0^t v^{j+1}(x,s)\mathrm{d}s} - 2\mathrm{e}^{-\int_0^t v^{j+1}(x,s)\mathrm{d}s} \nabla w_0(x)\cdot \int_0^t \nabla v^{j+1}(x,s)\mathrm{d}s \\
&+ w_0(x)\mathrm{e}^{-\int_0^t v^{j+1}(x,s)\mathrm{d}s} \left|\int_0^t \nabla v^{j+1}(x,s)\mathrm{d}s\right|^2 - w_0(x)\mathrm{e}^{-\int_0^t v^{j+1}(x,s)\mathrm{d}s} \int_0^t \Delta v^{j+1}(x,s)\mathrm{d}s. \qquad (2.11)
\end{aligned}
$$

From (2.9), we can easily get for $t \in (0,T)$

$$
\|w^{j+1}\|_{L^p} \leq \|w_0\|_{L^p}, \forall p \in (1,\infty]. \qquad (2.12)
$$

From (2.10), by (2.8), the definition of R and the following inequality

$$
\begin{aligned}
&\left\|\int_0^t f(x,s)ds\right\|_{L^p} = \left\{\int_{\mathbb{R}^2} |\int_0^t f(x,s)ds|^p dx\right\}^{\frac{1}{p}} \\
\leq& \left\{t^{p-1}\int_{\mathbb{R}^2}\int_0^t |f|^p ds dx\right\}^{\frac{1}{p}} \leq t \sup_{s\in(0,t)} \|f(s)\|_{L^p}, \quad \text{for all } p\in(1,\infty), \qquad (2.13)
\end{aligned}
$$

we can obtain

$$
\begin{aligned}
\left\|\nabla w^{j+1}\right\|_{L^2} &\leq \|\nabla w_0\|_{L^2} + \left\|w_0 \mathrm{e}^{-\int_0^t v^{j+1}ds}\int_0^t \nabla v^{j+1}\mathrm{d}s\right\|_{L^2} \\
&\leq \|\nabla w_0\|_{L^2} + \|w_0\|_{L^\infty}\left\|\int_0^t \nabla v^{j+1} ds\right\|_{L^2} \\
&\leq \|\nabla w_0\|_{L^2} + \|w_0\|_{L^\infty} T \sup_{t\in(0,T)} \left\|\nabla v^{j+1}\right\|_{L^2} \\
&\leq \|\nabla w_0\|_{L^2} + c_2 R^2 T \\
&\leq \|\nabla w_0\|_{L^2} + 1 \qquad (2.14)
\end{aligned}
$$

by setting T small enough to satisfy $c_2R^2T < 1$.

Similarly, by the embedding $H^2 \hookrightarrow W^{1,4}$ and (2.13), we can obtain from (2.11) for all $t \in (0, T)$

$$
\begin{aligned}
\left\|\Delta w^{j+1}\right\|_{L^2} \leq\;& \|\Delta w_0\|_{L^2} + 2\left\|\nabla w_0 \int_0^t \nabla v^{j+1} ds\right\|_{L^2} + \left\|w_0 \left|\int_0^t \nabla v^{j+1} ds\right|^2\right\|_{L^2} \\
&+ \left\|w_0 \int_0^t \Delta v^{j+1} ds\right\|_{L^2} \\
\leq\;& \|\Delta w_0\|_{L^2} + 2\|\nabla w_0\|_{L^4} \left\|\int_0^t \nabla v^{j+1} ds\right\|_{L^4} + \|w_0\|_{L^\infty} \left\|\left|\int_0^t \nabla v^{j+1} ds\right|^2\right\|_{L^2} \\
&+ \|w_0\|_{L^\infty} \left\|\int_0^t \Delta v^{j+1} ds\right\|_{L^2} \\
\leq\;& \|\Delta w_0\|_{L^2} + 2\|\nabla w_0\|_{L^4} T \sup_{t\in(0,T)} \left\|\nabla v^{j+1}\right\|_{L^4} + \|w_0\|_{L^\infty} T^2 \sup_{t\in(0,T)} \left\|\nabla v^{j+1}\right\|_{L^2}^2 \\
&+ \|w_0\|_{L^\infty} T \sup_{t\in(0,T)} \left\|\Delta v^{j+1}\right\|_{L^2} \\
\leq\;& \|\Delta w_0\|_{L^2} + c_3R^2T + c_3R^3T^2 \\
\leq\;& \|\Delta w_0\|_{L^2} + 2
\end{aligned} \tag{2.15}
$$

by setting T small enough to satisfy $c_3R^2T < 1$ and $c_3R^3T^2 < 1$.

Now we deduce the L^2 norm of $\nabla^3 w^{j+1}$. According to the equation of w and Hölder inequality, we can easily get for all $t \in (0, T)$

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nabla^3 w^{j+1}(t)\right\|_{L^2}^2 \\
=&2\int_{\mathbb{R}^2} \nabla^3 w^{j+1} \nabla^3 w_t^{j+1} = -2\int_{\mathbb{R}^2} \nabla^3 w^{j+1} \nabla^3 (v^{j+1} w^{j+1}) \\
\leq& 2\left\|\nabla^3 w^{j+1}\right\|_{L^2} \left\|\nabla^3 (v^{j+1} w^{j+1})\right\|_{L^2} \\
\leq& c_4 \left\|\nabla^3 w^{j+1}\right\|_{L^2} \left\|\nabla^3 w^{j+1} v^{j+1} + 3\nabla^2 w^{j+1} \nabla v^{j+1} + 3\nabla^2 v^{j+1} \nabla w^{j+1} + \nabla^3 v^{j+1} w^{j+1}\right\|_{L^2} \\
\leq& c_4 \Big\{\left\|\nabla^3 w^{j+1}\right\|_{L^2}^2 \left\|v^{j+1}\right\|_{L^\infty} + \left\|\nabla^3 w^{j+1}\right\|_{L^2} \left\|\nabla^2 w^{j+1}\right\|_{L^4} \left\|\nabla v^{j+1}\right\|_{L^4} \\
&+ \left\|\nabla^3 w^{j+1}\right\|_{L^2} \left\|\nabla^2 v^{j+1}\right\|_{L^2} \left\|\nabla w^{j+1}\right\|_{L^\infty} + \left\|\nabla^3 w^{j+1}\right\|_{L^2} \left\|\nabla^3 v^{j+1}\right\|_{L^2} \left\|w^{j+1}\right\|_{L^\infty}\Big\}.
\end{aligned} \tag{2.16}
$$

By Galiardo-Nirenberg inequality, we have

$$
\begin{aligned}
\left\|\nabla^2 w^{j+1}\right\|_{L^4} &\leq c_5 \left\|\nabla^3 w^{j+1}\right\|_{L^2}^{\frac{3}{4}} \left\|w^{j+1}\right\|_{L^\infty}^{\frac{1}{4}}, \\
\left\|\nabla w^{j+1}\right\|_{L^\infty} &\leq c_5 \left\|\nabla^3 w^{j+1}\right\|_{L^2}^{\frac{1}{2}} \left\|w^{j+1}\right\|_{L^\infty}^{\frac{1}{2}}.
\end{aligned}
$$

Together with Young's inequality, (2.8), (2.12) and (2.16), we can get

$$
\begin{aligned}
\frac{d}{dt}\left\|\nabla^3 w^{j+1}(t)\right\|_{L^2}^2 &\leq c_6\left(\left\|v^{j+1}\right\|_{H^3}^2 + 1\right)\left\|\nabla^3 w^{j+1}\right\|_{L^2}^2 + c_6\left\|\nabla w^{j+1}\right\|_{L^\infty}^2 \\
&\leq c_7R^2\left\|\nabla^3 w^{j+1}\right\|_{L^2}^2 + c_6\|\nabla w_0\|_{L^\infty}^2
\end{aligned}
$$

$$\leq c_7 R^2 \left\|\nabla^3 w^{j+1}\right\|_{L^2}^2 + c_7 R^2. \tag{2.17}$$

Then, we can deduce from Gronwall's inequality that for all $t \in (0, T)$

$$\begin{aligned}\left\|\nabla^3 w^{j+1}\right\|_{L^2}^2 &\leq e^{c_7 R^2 T^3} \left\|\nabla^3 w_0\right\|_{L^2}^2 + e^{c_7 R^2 T^3} c_7 R^2 T \\ &\leq 2\left\|\nabla^3 w_0\right\|_{L^2}^2 + 2\end{aligned} \tag{2.18}$$

by setting T small enough to satisfy $e^{c_7 R^2 T^3} < 2$ and $c_7 R^2 T < 1$.

Combining with (2.12)–(2.15) and (2.18), we can see that

$$\left\|w^{j+1}\right\|_{H^3} \leq 2\left\|w_0\right\|_{H^3} + 5. \tag{2.19}$$

(iii) *Estimates of* u^{j+1}. Taking the L^2 inner product of the equation of u^{j+1} in (2.2), integrating by part we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|u^{j+1}(t)\right\|_{L^2}^2 + \left\|\nabla u^{j+1}\right\|_{L^2}^2 \\ &= \chi \int_{\mathbb{R}^2} u^{j+1}\nabla v^j \cdot \nabla u^{j+1} + \xi \int_{\mathbb{R}^2} u^{j+1}\nabla w^j \cdot \nabla u^{j+1} \\ &\leq \left\|u^{j+1}\right\|_{L^2}\left\|\nabla v^j\right\|_{L^\infty}\left\|\nabla u^{j+1}\right\|_{L^2} + \left\|u^{j+1}\right\|_{L^2}\left\|\nabla w^j\right\|_{L^\infty}\left\|\nabla u^{j+1}\right\|_{L^2}.\end{aligned} \tag{2.20}$$

By (2.8), (2.19) and (2.20) and the embedding $H^3 \hookrightarrow W^{1,\infty}$, we can see for all $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}\left\|u^{j+1}(t)\right\|_{L^2}^2 + \left\|\nabla u^{j+1}\right\|_{L^2}^2 \leq c_8 R^2\left\|u^{j+1}\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla u^{j+1}\right\|_{L^2}^2. \tag{2.21}$$

Now we turn to show the L^2-estimate of ∇u^{j+1}. Multiplying $-\Delta u^{j+1}$ to both sides of the first equation of (2.3) and integrating by parts, we obtain for all $t \in (0, T)$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla u^{j+1}(t)\right\|_{L^2}^2 + \left\|\Delta u^{j+1}\right\|_{L^2}^2 \\ &= \chi \int_{\mathbb{R}^2} \Delta u^{j+1}\nabla \cdot \left(u^{j+1}\nabla v^j\right) + \xi \int_{\mathbb{R}^2} \Delta u^{i+1}\nabla \cdot \left(u^{i+1}\nabla w^j\right) \\ &= I_1 + I_2.\end{aligned} \tag{2.22}$$

By Hölder inequality and Young's inequality, it yields that

$$\begin{aligned}I_1 &\leq c_9\left\|\Delta u^{j+1}\right\|_{L^2}\left\|\nabla\left(u^{j+1}\nabla v^j\right)\right\|_{L^2} \\ &\leq c_{10}\left\|\Delta u^{j+1}\right\|_{L^2}\left\{\left\|u^{j+1}\right\|_{H^1}\left\|v^j\right\|_{H^2}\right\} \\ &\leq \frac{1}{4}\left\|\Delta u^{j+1}\right\|_{L^2}^2 + c_{10}^2\left\|v^j\right\|_{H^2}^2\left\|u^{j+1}\right\|_{H^1}^2.\end{aligned}$$

Applying the similar procedure to I_2, we can obtain

$$I_2 \leq \frac{1}{4}\left\|\Delta u^{j+1}\right\|_{L^2}^2 + c_{10}^2\left\|w^j\right\|_{H^2}^2\left\|u^{j+1}\right\|_{H^1}^2,$$

which entails that for all $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla u^{j+1}(t)\right\|_{L^2}^2 + \frac{1}{2}\left\|\Delta u^{j+1}\right\|_{L^2}^2 \le c_{11}R^2\left\|u^{j+1}\right\|_{H^1}^2. \tag{2.23}$$

Similar as (2.16), we can get

$$\begin{aligned}
&\frac{d}{dt}\left\|\nabla^2 u^{j+1}(t)\right\|_{L^2}^2\\
&= 2\int_{\mathbb{R}^2}\nabla^2 u^{j+1}\nabla^2 u_t^{j+1}\\
&= 2\int_{\mathbb{R}^2}\nabla^2 u^{j+1}\left\{\nabla^2\left(\Delta u^{j+1} - \chi\nabla\cdot(u^{j+1}\nabla v^j) - \xi\nabla\cdot(u^{j+1}\nabla w^j)\right)\right\}\\
&= 2\int_{\mathbb{R}^2}\nabla^2 u^{j+1}\nabla^2\Delta u^{j+1} - 2\chi\int_{\mathbb{R}^2}\nabla^2 u^{j+1}\nabla^3\left(u^{j+1}\nabla v^j\right) - 2\xi\int_{\mathbb{R}^2}\nabla^2 u^{i+1}\nabla^3\left(u^{j+1}\nabla w^j\right)\\
&= -2\left\|\nabla^3 u^{j+1}\right\|_{L^2}^2 + 2\chi\int_{\mathbb{R}^2}\nabla^3 u^{j+1}\nabla^2\left(u^{j+1}\nabla v^j\right) + 2\xi\int_{\mathbb{R}^2}\nabla^3 u^{i+1}\nabla^2\left(u^{i+1}\nabla w^j\right)\\
&\le -\left\|\nabla^3 u^{j+1}\right\|_{L^2}^2 + c_{12}\left\{\left\|\nabla^2\left(u^{j+1}\nabla v^j\right)\right\|_{L^2}^2 + \left\|\nabla^2\left(u^{i+1}\nabla w^j\right)\right\|_{L^2}^2\right\}\\
&\le -\left\|\nabla^3 u^{j+1}\right\|_{L^2}^2 + c_{13}R^2\left\|u^{j+1}\right\|_{H^2}^2.
\end{aligned} \tag{2.24}$$

Together with (2.21), (2.23) and (2.24), and adjusting the coefficients carefully, we can find a positive constant β such that

$$\frac{d}{dt}\left\|u^{j+1}(t)\right\|_{H^2}^2 + \beta\left\|u^{j+1}\right\|_{H^3}^2 \le c_{14}R^2\left\|u^{j+1}\right\|_{H^2}^2, \tag{2.25}$$

which implies from Gronwall's inequality that

$$\left\|u^{j+1}\right\|_{H^2}^2 + \beta\int_0^t\left\|u^{j+1}\right\|_{H^3}^2 \leqslant e^{c_{14}R^2T}\left\|u_0\right\|_{H^2}^2 \le 2\left\|u_0\right\|_{H^2}^2, \quad \text{for all } t \in (0, T), \tag{2.26}$$

by choosing T small enough to satisfy $e^{c_{14}R^2T} < 2$.

Combining (2.8), (2.19) and (2.26), we can get for all $t \in (0, T)$

$$\left\|u^{j+1}\right\|_{H^2} + \left\|v^{j+1}\right\|_{H^2} + \left\|w^{j+1}\right\|_{H^3} \le 2\left\{\|u_0\|_{H^1} + \|v_0\|_{H^2} + \|w_0\|_{H^3}\right\} + 7 \le R, \tag{2.27}$$

by the definition of R.

Convergence: The derivation of the relevant estimates of $u^{j+1} - u^j$, $v^{j+1} - v^j$ and $w^{j+1} - w^j$ are similar to the ones of u^{j+1}, v^{j+1} and w^{j+1}, so we omit the details. For simplicity, we denote $\delta f^{j+1} := f^{j+1} - f^j$. Subtracting the j-th equations from the $(j+1)$-th equations, we have the following equations for $\delta u^{j+1}, \delta v^{j+1}$ and δw^{j+1} :

$$\begin{cases}
\partial_t\delta u^{j+1} = \Delta\delta u^{j+1} - \chi\nabla\cdot\left(\delta u^{j+1}\nabla v^j\right) - \chi\nabla\cdot\left(u^j\nabla\delta v^j\right) - \xi\nabla\cdot\left(\delta u^{j+1}\nabla w^j\right) - \xi\nabla\cdot\left(u^j\nabla\delta w^j\right),\\
\partial_t\delta v^{j+1} = \Delta\delta v^{j+1} - \delta v^{j+1} + \delta u^j,\\
\partial_t\delta w^{j+1} = -v^{j+1}\delta w^{j+1} - \delta v^{j+1}w^{j+1}.
\end{cases} \tag{2.28}$$

(i) *Estimates of* δv^{j+1}. Using the same procedure as proving (2.8), we can obtain that for all $t \in (0, T)$

$$\left\|\delta v^{j+1}(t)\right\|_{H^3}^2 + \alpha \int_0^t \left(\left\|\delta v^{j+1}(s)\right\|_{H^4}^2 + \left\|\partial_t \delta v^{j+1}(s)\right\|_{L^2}^2\right) ds \leq e^{c_{15}T} c_{15} T \sup_{0\leq t\leq T} \left\|\delta u^j(t)\right\|_{H^2}^2. \tag{2.29}$$

(ii) *Estimates of* δw^{j+1}. According to the third equation of (2.28), we have for all $t \in (0, T)$

$$\delta w^{j+1}(t) = -\int_0^t e^{-\int_s^t v^{j+1}(\tau)d\tau} w^{j+1}(s)\delta v^{j+1}(s) ds. \tag{2.30}$$

Using the same procedure as proving (2.19) entails that for all $t \in (0, T)$

$$\left\|\delta w^{j+1}\right\|_{L^2} \leqslant \sup_{t\in(0,T)} \left\|w^{j+1}\right\|_{L^\infty} \sup_{t\in(0,T)} \left\|\delta v^{j+1}\right\|_{L^2} T \leqslant c_{16} RT \sup_{t\in(0,T)} \left\|\delta v^{j+1}\right\|_{L^2},$$

$$\begin{aligned}
\left\|\nabla \delta w^{j+1}\right\|_{L^2} \leqslant & \left\|\int_0^t \nabla w^{j+1}(s)\delta v^{j+1}(s) ds\right\|_{L^2} + \left\|\int_0^t w^{j+1}(s)\nabla \delta v^{j+1}(s) ds\right\|_{L^2} \\
& + \left\|\int_0^t w^{j+1}\delta v^{j+1} \int_s^t \nabla v^{j+1} d\tau ds\right\|_{L^2} \\
\leqslant & c_{17} RT \sup_{t\in(0,T)} \left\|\delta v^{j+1}\right\|_{L^2} + c_{17} RT \sup_{t\in(0,T)} \left\|\nabla \delta v^{j+1}\right\|_{L^2} + c_{17} R^2 T^2 \sup_{t\in(0,T)} \left\|\delta v^{j+1}\right\|_{L^2} \\
\leqslant & c_{18}(R^2 + R) T \sup_{t\in(0,T)} \left\|\delta v^{j+1}\right\|_{H^1}, \\
\left\|\Delta \delta w^{j+1}\right\|_{L^2} = & \left\|-\int_0^t \Delta \left\{e^{-\int_s^t v^{j+1} d\tau} w^{j+1}\delta v^{j+1}\right\} ds\right\|_{L^2} \\
\leqslant & T \sup_{t\in(0,T)} \left\|\Delta \left\{e^{-\int_s^t v^{j+1} d\tau} w^{j+1}\delta v^{j+1}\right\}\right\|_{L^2} \\
\leqslant & T \sup_{t\in(0,T)} \left\{\left(\left\|v^{j+1}\right\|_{H^2}^2 T^2 + \left\|v^{j+1}\right\|_{H^2} T\right) \left\|w^{j+1}\right\|_{H^2} \left\|\delta v^{j+1}\right\|_{H^2}\right\} \\
\leqslant & c_{19}(R^3 + R^2) T \sup_{t\in(0,T)} \left\|\delta v^{j+1}\right\|_{H^2}
\end{aligned}$$

and

$$\left\|\nabla^3 \delta w^{j+1}\right\|_{L^2}^2 \leqslant e^{c_{20}R^2T} T \sup_{t\in(0,T)} \left\|\delta v^{j+1}\right\|_{H^3},$$

which imply that for all $t \in (0, T)$

$$\left\|\delta w^{j+1}\right\|_{H^3} \leq c_{21}(R^3 + R^2 + R) T \sup_{0\leq t\leq T} \left\|\delta u^j(t)\right\|_{H^2}. \tag{2.31}$$

by setting $T < 1$.

(iii) *Estimates of* δu^{j+1}. Using the same procedure as proving (2.26) entails that for all $t \in (0, T)$

$$\sup_{0\leq t\leq T} \left\|\delta u^{j+1}\right\|_{H^2}^2 + \beta \int_0^t \left\|\delta u^{j+1}\right\|_{H^3}^2 \leqslant e^{c_{22}RT} c_{22} RT \sup_{0\leq t\leq T} \left(\left\|\delta v^j\right\|_{H^3}^2 + \left\|\delta w^j\right\|_{H^2}^2\right). \tag{2.32}$$

Combining with (2.29), (2.31) and (2.32), we can obtain for all $t \in (0, T)$

$$\begin{aligned}&\sup_{0\le t\le T}\left(\left\|\delta u^{j+1}\right\|_{H^2}+\left\|\delta v^{j+1}\right\|_{H^3}+\left\|\delta w^{j+1}\right\|_{H^3}\right)\\ \le& e^{c_{23}(R^3+R^2+R)T}c_{23}(R^3+R^2+R)T\sup_{0\le t\le T}\left(\left\|\delta u^{j}\right\|_{H^2}+\left\|\delta v^{j}\right\|_{H^3}+\left\|\delta w^{j}\right\|_{H^3}\right).\end{aligned} \tag{2.33}$$

Taking $T > 0$ small enough, we can find a constant $r \in (0, 1)$ such that

$$\sup_{0\le t\le T}\left(\left\|\delta u^{j+1}\right\|_{H^2}+\left\|\delta v^{j+1}\right\|_{H^3}+\left\|\delta w^{j+1}\right\|_{H^3}\right)\le r\sup_{0\le t\le T}\left(\left\|\delta u^{j}\right\|_{H^2}+\left\|\delta v^{j}\right\|_{H^3}+\left\|\delta w^{j}\right\|_{H^3}\right) \tag{2.34}$$

for any $j \ge 1$ and $t \in (0, T)$. From the above inequality, we find that $\left(u^j, v^j, w^j\right)$ is a Cauchy sequence in the Banach space $C(0, T; X)$ for some small $T > 0$, and thus its corresponding limit denoted by (u, v, w) definitely exists in the same space.

Uniqueness: If $(\bar{u}, \bar{v}, \bar{w})$ is another local-in-time solution of system (1.3), $(\tilde{u}, \tilde{v}, \tilde{w}) := (u - \bar{u}, v - \bar{v}, w - \bar{w})$ solves

$$\begin{cases}\partial_t\tilde{u}=\Delta\tilde{u}-\chi\nabla\cdot(\tilde{u}\nabla\bar{v})-\chi\nabla\cdot(u\nabla\tilde{v})-\xi\nabla\cdot(\tilde{u}\nabla\bar{w})-\xi\nabla\cdot(u\nabla\tilde{w}), & x\in\mathbb{R}^2,\ 0<t\le T,\\ \partial_t\tilde{v}=\Delta\tilde{v}-\tilde{v}+\tilde{u}, & x\in\mathbb{R}^2,\ 0<t\le T,\\ \partial_t\tilde{w}=-v\tilde{w}-\tilde{v}\bar{w}, & x\in\mathbb{R}^2,\ 0<t\le T,\\ \tilde{u}(x,0)=\tilde{v}(x,0)=\tilde{w}(x,0)=0, & x\in\mathbb{R}^2,\end{cases}$$

where T is any time before the maximal time of existence. Following a same procedure as (2.34), we can deduce that $\tilde{u} = \tilde{v} = \tilde{w} = 0$, which implies the uniqueness of the local solution.

Nonnegativity: The nonnegativity of w^j can be easily obtained by (2.9) and the nonnegativity of w_0. We will use the induction argument to show that u^j and v^j are nonnegative for all $j > 0$. We assume that u^j and v^j are nonnegative. If we apply the maximum principle to the second equation of (2.2), we find v^{j+1} is nonnegative (u^j is nonnegative). Then we turn to deal with u^{j+1}. Let us decompose $u^{j+1} = u_+^{j+1} - u_-^{j+1}$, where $u_+^{j+1} = \begin{cases} u^{j+1} & u^{j+1} \ge 0 \\ 0 & u^{j+1} < 0 \end{cases}$ and $u_-^{j+1} = \begin{cases} -u^{j+1} & u^{j+1} \le 0 \\ 0 & u^{j+1} > 0 \end{cases}$. Now multiplying the negative part u_-^{j+1} on both sides of the first equation of (2.2) and integrating over $[0, t] \times \mathbb{R}^2$, we can get

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\partial_t u^{j+1}u_-^{j+1}dxds\\ =&-\int_0^t\left\|\nabla\left(u_-^{j+1}\right)\right\|_{L^2}^2ds+\chi\int_0^t\int_{\mathbb{R}^2}u^{j+1}\nabla v^j\cdot\nabla u_-^{j+1}+\xi\int_{\mathbb{R}^2}u^{j+1}\nabla w^j\cdot\nabla u_-^{j+1}\\ \le& C\int_0^t\left\|\left(u^{j+1}\right)_-\right\|_{L^2}^2\left(\left\|\nabla v^j\right\|_{L^\infty}^2+\left\|\nabla w^j\right\|_{L^\infty}^2\right)+\frac{1}{2}\left\|\nabla\left(u_-^{j+1}\right)\right\|_{L^2}^2ds\end{aligned}$$

by Young's inequality and the fact the weak derivative of u_-^{j+1} is $-\nabla u^{j+1}$ if $u_-^{j+1} < 0$, otherwise zero. Since $u_-^{j+1},\ \partial_t u^{j+1} \in L^2\left(0, T; L^2\left(\mathbb{R}^2\right)\right)$, we can have

$$\int_0^t\int_{\mathbb{R}^2}\partial_t u^{j+1}\left(u^{j+1}\right)_-dxds=\frac{1}{2}\left(\left\|\left(u_-^{j+1}\right)(0)\right\|_{L^2}-\left\|\left(u_-^{j+1}\right)(0)\right\|_{L^2}\right),$$

together with the above inequality, it holds that

$$\left\|\left(u_{-}^{j+1}\right)(t)\right\|_{L^2}^2 \leq \left\|\left(u_{-}^{j+1}\right)(0)\right\|_{L^2}^2 \exp\left(C\int_0^t \left(\left\|\nabla v^j\right\|_{L^\infty}^2 + \left\|\nabla w^j\right\|_{L^\infty}^2\right) ds\right).$$

Due to the fact $u_{-}^{j+1}(0)$ is nonnegative, we can deduce that u^{j+1} is nonnegative. This completes the proof of Lemma 2.1. □

Remark 2.1. Since the above choice of T depends only on $\|u_0\|_{H^2(\mathbb{R}^2)}$, $\|v_0\|_{H^3(\mathbb{R}^2)}$ and $\|w_0\|_{H^3(\mathbb{R}^2)}$, it is clear by a standard argument that (u, v, w) can be extended up to some $T_{\max} \leq \infty$. If $T_{\max} < \infty$ in Lemma 2.1, then

$$\limsup_{t\to T_{\max}}\left(\|u(t)\|_{H^2(\mathbb{R}^2)} + \|v(t)\|_{H^3(\mathbb{R}^2)} + \|w(t)\|_{H^3(\mathbb{R}^2)}\right) = \infty. \tag{2.35}$$

In order to show the $H^2\times H^3\times H^3$-boundedness of (u, v, w), it suffices to estimate a suitable L^p-norm of u, with some large, but finite p.

Lemma 2.2. *Suppose that $\chi, \xi > 0$ and the initial data (u_0, v_0, w_0) satisfy all the assumptions presented in Lemma 2.1. Then for every $K > 0$ there is $C > 0$ such that whenever $(u, v, w) \in C\left(0, T; H^2(\mathbb{R}^2)\times H^3(\mathbb{R}^2)\times H^3(\mathbb{R}^2)\right)$ solves (1.3) for some $T > 0$ and $q_0 > 2$ satisfies*

$$\|u(\cdot, t)\|_{L^{q_0}} \leq K, \text{ for all } t \in (0, T), \tag{2.36}$$

then

$$\|u(t)\|_{H^2(\mathbb{R}^2)} + \|v(t)\|_{H^3(\mathbb{R}^2)} + \|w(t)\|_{H^3(\mathbb{R}^2)} \leq C \text{ for all } t \in (0, T). \tag{2.37}$$

Proof. Firstly, we suppose that for some $q_0 > 2$ and $K > 0$

$$\|u(t)\|_{L^{q_0}(\mathbb{R}^2)} \leq K, \quad \text{for all } t \in (0, T). \tag{2.38}$$

By the Duhamel principle, we represent u and v of the following integral equations

$$u(t) = e^{t\Delta}u_0 - \chi\int_0^t e^{(t-\tau)\Delta}\nabla\cdot(u\nabla v)(\tau)d\tau - \xi\int_0^t e^{(t-\tau)\Delta}\nabla\cdot(u\nabla w)(\tau)d\tau \tag{2.39}$$

and

$$v(t) = e^{-t(-\Delta+1)}v_0 + \int_0^t e^{-(t-\tau)(-\Delta+1)}u(\tau)d\tau. \tag{2.40}$$

where $e^{t\Delta}f(x) = \int_{\mathbb{R}^2} G(x-y, t)f(y)dy$ and

$$G(x, t) = G_t(x) := \frac{1}{(4\pi t)}\exp\left(-\frac{|x|^2}{4t}\right)$$

is the Gaussian heat kernel. The following well-known $L^p - L^q$ estimates of the heat semigroup play an important role in the proofs [47,48]. For $1 \leqslant p \leqslant q \leqslant \infty$ and $f \in L^q\left(\mathbb{R}^2\right)$, we have

$$\left\|e^{t\Delta}f\right\|_{L^p} \leqslant (4\pi t)^{-\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q},$$
$$\left\|\nabla e^{t\Delta}f\right\|_{L^p} \leqslant C_3 t^{-\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q},$$

where C_3 is a constant depending on p and q. Then, according to (2.39), we can see that for $q_0 > 2$ and all $t \in (0, T)$

$$\begin{aligned}\|u(t)\|_{L^\infty} &\le \|u_0\|_{L^\infty} + \chi \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{q_0}} \|u \cdot \nabla v\|_{L^{q_0}} d\tau + \xi \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{q_0}} \|u \cdot \nabla w\|_{L^{q_0}} d\tau \\ &\le \|u_0\|_{L^\infty} + K\chi \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{q_0}} \|\nabla v\|_{L^\infty} d\tau + K\xi \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{q_0}} \|\nabla w\|_{L^\infty} d\tau. \end{aligned} \tag{2.41}$$

From (2.40) and the above $L^p - L^q$ estimates of the heat semigroup, we have

$$\|v(t)\|_{L^q} \le \|v_0\|_{L^q} + \int_0^t e^{-(t-\tau)} (t-\tau)^{-(\frac{1}{q_0}-\frac{1}{q})} \|u(t)\|_{L^{q_0}} d\tau \le C_4, \forall q \in (1, \infty] \tag{2.42}$$

and by the embedding $H^3(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$

$$\|\nabla v(t)\|_{L^\infty} \le \|\nabla v_0\|_{L^\infty} + \int_0^t e^{-(t-\tau)} (t-\tau)^{-\frac{1}{2}-\frac{1}{q_0}} \|u(t)\|_{L^{q_0}} d\tau \le C_5, \tag{2.43}$$

where C_4 and C_5 depend on $\|v_0\|_{H^3}$ and K in (2.38).

According to the equation of w, we can see that for some $C_6 = C_6(\|w_0\|_{H^3}, \|v_0\|_{H^2}, K, T) > 0$ and all $t \in (0, T)$

$$\begin{aligned}\|\nabla w(t)\|_{L^\infty} &\le \left\| \nabla w_0 e^{-\int_0^t v(s)ds} \right\|_{L^\infty} + \left\| w_0 e^{-\int_0^t v(s)ds} \int_0^t \nabla v(s) ds \right\|_{L^\infty} \\ &\le \|w_0\|_{H^3} + \|w_0\|_{L^\infty} \sup_{t \in (0,T)} \|\nabla v(t)\|_{L^\infty} T \le C_6 \end{aligned} \tag{2.44}$$

by the embedding $H^3(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$. Inserting (2.43) and (2.44) into (2.41), this yields for all $t \in (0, T)$

$$\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty} + KC_5\chi \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{q_0}} d\tau + KC_6\xi \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{1}{q_0}} d\tau \le C_7, \tag{2.45}$$

where C_7 depends on $\|u_0\|_{H^2}, \|v_0\|_{H^2}, \|w_0\|_{H^3}$, K and T.

Integrating by parts and by Young's inequality, we can obtain from the second equation of (1.3) that for all $t \in (0, T)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \|\partial_t v(t)\|_{L^2}^2 + \|\nabla \partial_t v\|_{L^2}^2 &= -\int_{\mathbb{R}^2} (\partial_t v)^2 + \int_{\mathbb{R}^2} \partial_t v \partial_t u \\ &\le -\frac{1}{2} \|\partial_t v\|_{L^2}^2 + \frac{1}{2} \|\partial_t u\|_{L^2}^2 \end{aligned} \tag{2.46}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \|\nabla v(t)\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 &= \int_{\mathbb{R}^2} v \Delta v + \int_{\mathbb{R}^2} u \Delta v \\ &\le -\|\nabla v\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2}^2 + \frac{1}{2} \|\Delta v\|_{L^2}^2 . \end{aligned} \tag{2.47}$$

Similarly, according to the first equation of (1.3), (2.43) and (2.44), we have for all $t \in (0, T)$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &= \chi \int_{\mathbb{R}^2} u\nabla v \cdot \nabla u + \xi \int_{\mathbb{R}^2} u\nabla w \cdot \nabla u \\
&\leq \frac{\chi^2}{2}\|u\nabla v\|_{L^2}^2 + \frac{\xi^2}{2}\|u\nabla w\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 \\
&\leq \frac{\chi^2 C_5^2}{2}\|u\|_{L^2}^2 + \frac{\xi^2 C_6^2}{2}\|u\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2
\end{aligned}
\tag{2.48}
$$

and by (2.27) for some $\theta > 0$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_t u(t)\|_{L^2}^2 + \|\nabla \partial_t u\|_{L^2}^2 =& \chi \int_{\mathbb{R}^2} \partial_t u\nabla v \cdot \nabla \partial_t u + \chi \int_{\mathbb{R}^2} u\partial_t \nabla v \cdot \nabla \partial_t u \\
&+ \xi \int_{\mathbb{R}^2} \partial_t u\nabla w \cdot \nabla \partial_t u + \xi \int_{\mathbb{R}^2} u\nabla \partial_t w \cdot \nabla \partial_t u \\
\leq& C_8 \left(\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty}\right)\left(\theta \|\partial_t u\|_{L^2}^2 + \frac{1}{\theta}\|\nabla \partial_t u\|_{L^2}^2\right) \\
&+ C_8\|u\|_{L^\infty}\left(\theta \|\nabla \partial_t v\|_{L^2}^2 + \theta \|\nabla \partial_t w\|_{L^2}^2 + \frac{2}{\theta}\|\nabla \partial_t u\|_{L^2}^2\right) \\
\leq& C_8(C_5 + C_6)\theta \|\partial_t u\|_{L^2}^2 + \frac{C_8(C_5 + C_6 + 2C_7)}{\theta}\|\nabla \partial_t u\|_{L^2}^2 \\
&+ \theta C_8 C_7 \left(\|\nabla \partial_t v\|_{L^2}^2 + \|\nabla \partial_t w\|_{L^2}^2\right).
\end{aligned}
\tag{2.49}
$$

Now we turn to estimate the last term of the right side of (2.49). According to the third equation of (1.3), (2.42) and (2.43), we obtain for some $C_9 > 0$

$$
\begin{aligned}
\|\nabla \partial_t w\|_{L^2}^2 &\leq \|\nabla v\|_{L^2}^2 \|w\|_{L^\infty}^2 + \|\nabla w\|_{L^2}^2 \|v\|_{L^\infty}^2 \\
&\leq \|w_0\|_{L^\infty}^2 \|\nabla v\|_{L^2}^2 + C_4^2 \left(\|\nabla w_0\|_{L^2}^2 + T^2 \sup_{t\in(0,T)} \|\nabla v\|_{L^\infty}^2 \|w_0\|_{L^2}^2\right) \\
&\leq C_9 \|\nabla v\|_{L^2}^2 + C_9.
\end{aligned}
\tag{2.50}
$$

Combining with (2.46)–(2.50) and setting $\theta > 0$ to satisfy $\frac{C_8(C_5+C_6+2C_7)}{\theta} < \frac{1}{2}$, we can obtain such Gronwall-type inequality

$$
\begin{aligned}
&\frac{d}{dt}\left\{\|\partial_t v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right\} \\
&+ C_{10}\left(\|\nabla \partial_t v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\nabla \partial_t u\|_{L^2}^2\right) \\
\leq& C_{10}\left(\|\partial_t v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2 + 1\right), \quad \text{for all } t \in (0, T),
\end{aligned}
\tag{2.51}
$$

then by direct integration, we can have for some $C_{11} = C_{11}(\|w_0\|_{H^3}, \|v_0\|_{H^2}, \|u_0\|_{H^2}, K, T) > 0$

$$
\begin{aligned}
&\int_0^t \left(\|\nabla \partial_t v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\nabla \partial_t u\|_{L^2}^2\right) \\
&+ \left\{\|\partial_t v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2\right\} \leq C_{11}, \quad \text{for all } t \in (0, T).
\end{aligned}
\tag{2.52}
$$

By (2.42) and (2.52) and the second equation of v in (1.3), we have

$$\|\Delta v\|_{L^2} \leq \|\partial_t v\|_{L^2} + \|v\|_{L^2} + \|u\|_{L^2} \leq C_{12}, \quad \text{for all } t \in (0, T), \tag{2.53}$$

where $C_{12} = C_{12}(\|w_0\|_{H^3}, \|v_0\|_{H^2}, \|u_0\|_{H^2}, K, T)$. Hence by the equation of w and Young's inequality, we obtain for some $C_{13} = C_{13}(\|w_0\|_{H^3}, \|v_0\|_{H^2}, \|u_0\|_{H^2}, K, T) > 0$

$$\begin{aligned}
\|\Delta w\|_{L^2} \leq& \|\Delta w_0\|_{L^2} + 2\|\nabla w_0\|_{L^4} \left\| \int_0^t \nabla v \right\|_{L^4} + \|w_0\|_{L^\infty} \left\| \left| \int_0^t \nabla v \right|^2 \right\|_{L^2} + \|w_0\|_{L^\infty} \left\| \int_0^t \Delta v \right\|_{L^2} \\
\leq& \|\Delta w_0\|_{L^2} + 2\|\nabla w_0\|_{L^4} \sup_{t\in(0,T)} \|\nabla v\|_{L^4} T \\
&+ \|w_0\|_{L^\infty} \sup_{t\in(0,T)} \|\nabla v\|_{L^4}^2 T + \|w_0\|_{L^\infty} \sup_{t\in(0,T)} \|\Delta v\|_{L^2} T \\
\leq& C_{13}, \quad \text{for all } t \in (0, T).
\end{aligned} \tag{2.54}$$

By (2.42) and (2.43) and the embedding $W^{2,2}(\mathbb{R}^2) \hookrightarrow W^{1,4}(\mathbb{R}^2)$, we can see

$$\begin{aligned}
\frac{d}{dt} \left\|\nabla^3 w(t)\right\|_{L^2}^2 \leqslant& C\Big\{ \left\|\nabla^3 w\right\|_{L^2}^2 \|v\|_{L^\infty} + \left\|\nabla^3 w\right\|_{L^2} \left\|\nabla^2 w\right\|_{L^4} \|\nabla v\|_{L^4} \\
&+ \left\|\nabla^3 w\right\|_{L^2} \left\|\nabla^2 v\right\|_{L^2} \|\nabla w\|_{L^\infty} + \left\|\nabla^3 w\right\|_{L^2} \|w\|_{L^\infty} \left\|\nabla^3 v\right\|_{L^2} \Big\} \\
\leq& C_{14} \left\|\nabla^3 w\right\|_{L^2}^2 + C_{14} \left\|\nabla^3 v\right\|_{L^2}^2 + C_{14}, \quad \text{for all } t \in (0, T).
\end{aligned} \tag{2.55}$$

Integrating on $(0, t)$, we have

$$\left\|\nabla^3 w\right\|_{L^2}^2 \leq C_{15} \left\|\nabla^3 w_0\right\|_{L^2}^2 + C_{15} \int_0^t \left\|\nabla^3 v\right\|_{L^2}^2 + C_{15}, \quad \text{for all } t \in (0, T). \tag{2.56}$$

Now we turn to estimate the second integral of the right side of (2.56). Applying ∇ to the second equation of (1.3) and rewriting the equation as $\nabla \Delta v = \nabla v_t + \nabla v - \nabla u$, then by (2.52) we have that

$$\begin{aligned}
\int_0^t \left\|\nabla^3 v\right\|_{L^2}^2 \leq& \int_0^t \|\nabla v_t\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 \\
\leq& C_{11}, \quad \text{for all } t \in (0, T).
\end{aligned} \tag{2.57}$$

Inserting (2.57) into (2.56), we can obtain that for some $C_{16} > 0$

$$\left\|\nabla^3 w\right\|_{L^2} \leq C_{16}, \quad \text{for all } t \in (0, T). \tag{2.58}$$

Now we deduce the L^2-norm of ∇u and $\nabla^2 u$. We multiply the first equation of (1.3) by $-\Delta u$, integrate by parts and then obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 =& \chi \int_{\mathbb{R}^2} \Delta u \nabla \cdot (u \nabla v) + \xi \int_{\mathbb{R}^2} \Delta u \cdot \nabla(u \nabla w) \\
=& \chi \int_{\mathbb{R}^2} \Delta u \nabla u \cdot \nabla v + \chi \int_{\mathbb{R}^2} u \Delta u \Delta v \\
&+ \xi \int_{\mathbb{R}^2} \Delta u \nabla u \cdot \nabla w + \xi \int_{\mathbb{R}^2} u \Delta u \Delta w
\end{aligned}$$

$$=I_1 + I_2 + I_3 + I_4. \tag{2.59}$$

Then by (2.42), (2.45) and (2.52), Hölder's inequality and Young's inequality, we have

$$\begin{aligned} I_1 + I_2 &\leqslant \chi\|\nabla v\|_{L^\infty}\|\Delta u\|_{L^2}\|\nabla u\|_{L^2} + \chi\|\Delta v\|_{L^2}\|\Delta u\|_{L^2}\|u\|_{L^\infty} \\ &\leqslant \frac{1}{4}\|\Delta u\|_{L^2}^2 + C_{17}\|\nabla u\|_{L^2}^2 + C_{17}. \end{aligned}$$

Similarly, according to (2.43), (2.45) and (2.56), we can obtain

$$\begin{aligned} I_3 + I_4 &\leqslant \chi\|\nabla v\|_{L^\infty}\|\Delta u\|_{L^2}\|\nabla u\|_{L^2} + \chi\|\Delta v\|_{L^2}\|\Delta u\|_{L^2}\|u\|_{L^\infty} \\ &\leqslant \frac{1}{4}\|\Delta u\|_{L^2}^2 + C_{17}\|\nabla u\|_{L^2}^2 + C_{18}. \end{aligned}$$

Then we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2 \leq C_{19}\|\nabla u\|_{L^2}^2 + C_{19}.$$

Integrating on $(0, t)$, we have for some $C_{20} > 0$

$$\|\nabla u\|_{L^2} \leq C_{20}, \quad \text{for all } t \in (0, T). \tag{2.60}$$

Rewriting the first equation of (1.3) as $\Delta u = u_t + \chi\nabla \cdot (u\nabla v) + \xi\nabla \cdot (u\nabla w)$, and by (2.42)–(2.44), (2.53) and (2.54), we have for some $C_{21} > 0$

$$\begin{aligned} \|\Delta u\|_{L^2} &\leq \|\partial_t u\|_{L^2} + \chi \|\nabla u \cdot \nabla v\|_{L^2} + \chi \|u\Delta v\|_{L^2} + \xi \|\nabla u \cdot \nabla w\|_{L^2} + \xi \|u\Delta w\|_{L^2} \\ &\leq \|\partial_t u\|_{L^2} + \chi \|\nabla u\|_{L^2} \|\nabla v\|_{L^\infty} + \chi \|\Delta v\|_{L^2} \|u\|_{L^\infty} \\ &\quad + \xi \|\nabla u\|_{L^2} \|\nabla w\|_{L^\infty} + \xi \|\Delta w\|_{L^2} \|u\|_{L^\infty} \\ &\leq C_{21}. \end{aligned} \tag{2.61}$$

For the L^2-norm of $\nabla^3 v$, integrating by parts, we deduce that

$$\begin{aligned} \frac{d}{dt}\left\|\nabla^3 v(t)\right\|_{L^2}^2 &= 2\int_{\mathbb{R}^2} \nabla^3 v \nabla^3(\Delta v - v + u) \\ &= -2\int_{\mathbb{R}^2} \left|\nabla^4 v\right|^2 - 2\int_{\mathbb{R}^2} \left|\nabla^3 v\right|^2 + 2\int_{\mathbb{R}^2} \nabla^3 v \nabla^3 u \\ &\leq -\left\|\nabla^4 v\right\|_{L^2}^2 - 2\left\|\nabla^3 v\right\|_{L^2}^2 + \left\|\nabla^2 u\right\|_{L^2}^2, \end{aligned}$$

then by (2.61) and Gronwall's inequality, we can see that for all $t \in (0, T)$

$$\left\|\nabla^3 v(t)\right\|_{L^2} \leq C_{22}. \tag{2.62}$$

Putting (2.52)–(2.54), (2.58) and (2.60)–(2.62) together, we conclude that for some $C > 0$

$$\|u(t)\|_{H^2(\mathbb{R}^2)} + \|v(t)\|_{H^3(\mathbb{R}^2)} + \|w(t)\|_{H^3(\mathbb{R}^2)} \leq C, \quad \forall t \in (0, T), \tag{2.63}$$

which completes the proof. □

3. proof of Theorem1.1

As a preparation, we first state some results concerning the system which will be used in the proof of Theorem 1.1.

Lemma 3.1. *The local-in-time classical solution* (u, v, w) *of system (1.3) satisfies*

$$\|u(t)\|_{L^1} = \|u_0\|_{L^1} := M, \quad \forall t \in (0, T_{\max}) \tag{3.1}$$

and

$$\|v(t)\|_{L^1} = \|u_0\|_{L^1} + (\|v_0\|_{L^1} - \|u_0\|_{L^1})\, e^{-t}, \quad \forall t \in (0, T_{\max})\,. \tag{3.2}$$

Proof. Integrating the first and second equation of (1.3) on $\mathbb{R}^2$, we can obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} u = \int_{\mathbb{R}^2} \Delta u - \chi \int_{\mathbb{R}^2} \nabla \cdot (u\nabla v) - \xi \int_{\mathbb{R}^2} \nabla \cdot (u\nabla w) = 0$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^2} v = \int_{\mathbb{R}^2} \Delta v - \int_{\mathbb{R}^2} v + \int_{\mathbb{R}^2} u = -\int_{\mathbb{R}^2} v + \int_{\mathbb{R}^2} u,$$

which can easily yield (3.1) and (3.2). □

The following energy

$$F(t) = \int_{\mathbb{R}^2} u \ln u - \chi \int_{\mathbb{R}^2} uv - \xi \int_{\mathbb{R}^2} uw + \frac{\chi}{2} \int_{\mathbb{R}^2} \left(v^2 + |\nabla v|^2\right)$$

plays a key role in the proof. The main idea of the proof is similar to the strategy introduced in [2].

Lemma 3.2. *Assume that (1.4) and (1.5) holds. Let* (u, v, w) *be the local-in-time classical solution of system (1.3). Then* $F(t)$ *satisfies*

$$F(t) + \chi \int_0^t \int_{\mathbb{R}^2} v_t^2 + \int_0^t \int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2 = F(0) + \xi \int_0^t \int_{\mathbb{R}^2} uvw, \quad \forall t \in (0, T_{\max})\,. \tag{3.3}$$

Proof. We use the same ideas as in the proofs of [45, Theorem 1.3], [46, Lemma 3.1] and [1, Theorem 3.2]. The equation of u can be written as $u_t = \nabla \cdot (u\nabla(\ln u - \chi v - \xi w))$. Multiplying by $\ln u - \chi v - \xi w$ and integrating by parts, we obtain

$$\begin{aligned} -\int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2 &= \int_{\mathbb{R}^2} u_t(\ln u - \chi v - \xi w) \\ &= \frac{d}{dt}\int_{\mathbb{R}^2} (u \ln u - \chi uv - \xi uw) + \chi \int_{\mathbb{R}^2} uv_t + \xi \int_{\mathbb{R}^2} uw_t. \end{aligned} \tag{3.4}$$

Substituting the second and third equation of (1.3) into (3.4) and integrating by parts, we have

$$\begin{aligned} &-\int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2 \\ =&\frac{d}{dt}\int_{\mathbb{R}^2} (u \ln u - \chi uv - \xi uw) + \chi \int_{\mathbb{R}^2} (v_t - \Delta v + v)\, v_t + \xi \int_{\mathbb{R}^2} uvw \\ =&\frac{d}{dt}\int_{\mathbb{R}^2} (u \ln u - \chi uv - \xi uw) - \chi \int_{\mathbb{R}^2} v_t^2 + \frac{\chi}{2}\frac{d}{dt} \int_{\mathbb{R}^2} \left(v^2 + |\nabla v|^2\right) + \xi \int_{\mathbb{R}^2} uvw, \end{aligned} \tag{3.5}$$

which, upon being integrated from 0 to t, yields simply that (3.3). □

We give some lemmas to deal with the term $\int_{\mathbb{R}^2} u \ln u$ in (1.7).

Lemma 3.3. ([1, Lemma 2.1]) *Let ψ be any function such that $e^{\psi} \in L^1\left(\mathbb{R}^2\right)$ and denote $\bar{u} = Me^{\psi}\left(\int_{\mathbb{R}^2} e^{\psi} dx\right)^{-1}$ with M a positive arbitrary constant. Let $E : L^1_+\left(\mathbb{R}^2\right) \to \mathbb{R} \cup \{\infty\}$ be the entropy functional*

$$E(u;\psi) = \int_{\mathbb{R}^2} (u \ln u - u\psi)\, dx$$

and let $RE : L^1_+\left(\mathbb{R}^2\right) \to \mathbb{R} \cup \{\infty\}$ defined by

$$RE(u \mid \bar{u}) = \int_{\mathbb{R}^2} u \ln(\frac{u}{\bar{u}}) dx$$

be the relative (to $\bar{u}$) entropy.

Then $E(u;\psi)$ and $RE(u \mid \bar{u})$ are finite or infinite in the same time and for all u in the set $\mathcal{U} = \left\{u \in L^1_+\left(\mathbb{R}^2\right), \int_{\mathbb{R}^2} u(x)dx = M\right\}$ and it holds true that

$$E(u;\psi) - E(\bar{u};\psi) = RE(u \mid \bar{u}) \geq 0.$$

Next, we give a Moser-Trudinger-Onofri inequality.

Lemma 3.4. ([1, Lemma 3.1]) *Let H be defined as $H(x) = \frac{1}{\pi}\frac{1}{\left(1+|x|^2\right)^2}$. Then*

$$\int_{\mathbb{R}^2} e^{\varphi(x)} H(x)dx \leq \exp\left\{\int_{\mathbb{R}^2} \varphi(x)H(x)dx + \frac{1}{16\pi}\int_{\mathbb{R}^2} |\nabla\varphi(x)|^2 dx\right\}, \tag{3.6}$$

for all functions $\varphi \in L^1\left(\mathbb{R}^2, H(x)dx\right)$ such that $|\nabla\varphi(x)| \in L^2\left(\mathbb{R}^2, dx\right)$.

Lemma 3.5. ([1, Lemma 2.4]) *Let ψ be any function such that $e^{\psi} \in L^1\left(\mathbb{R}^2\right)$, and let f be a non-negative function such that $\left(f\mathbf{1}_{\{f\leq 1\}}\right) \in L^1\left(\mathbb{R}^2\right) \cap L^1\left(\mathbb{R}^2, |\psi(x)|dx\right)$. Then there exists a constant C such that*

$$\int_{\mathbb{R}^2} f(x)(\ln f(x))_- dx \leq C - \int_{\{f\leq 1\}} f(x)\psi(x)dx.$$

With the help of Lemma 3.2–3.5, we now use the subcritical mass condition (1.8) to derive a Gronwall-type inequality and to get a time-dependent bound for $\|(u \ln u)(t)\|_{L^1}$.

Lemma 3.6. *Under the subcritical mass condition (1.8) and (1.5), there exists $C = C\left(u_0, v_0, w_0\right) > 0$ such that*

$$\|(u \ln u)(t)\|_{L^1} + \|v(t)\|^2_{H^1} \leq Ce^{\frac{\xi K}{\gamma}t}, \quad \forall t \in (0, T_{\max}), \tag{3.7}$$

where $K > 0$ and γ are defined by (3.8) and (3.10) below, respectively.

Proof. According to the third equation of (1.3), we have for all $t \in (0, T)$

$$\|w\|_{L^\infty} \leq \|w_0\|_{L^\infty} := K, \tag{3.8}$$

then we apply the estimate of (3.3) to find that

$$F(t) + \int_0^t \int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2 \le F(0) + \xi K \int_0^t \int_{\mathbb{R}^2} uv, \quad \forall t \in (0, T_{\max}). \tag{3.9}$$

For our later purpose, since $M < \frac{8\pi}{\chi}$, we first choose some positive constant $\gamma > 0$ small enough to satisfy

$$\chi - \frac{M(\chi+\gamma)^2}{8\pi} > 0, \tag{3.10}$$

then by the definition of $F(t)$ in (1.7), we use (3.1) and (3.8) to deduce that

$$\begin{aligned} F(t) &= \int_{\mathbb{R}^2} u \ln u - \chi \int_{\mathbb{R}^2} uv - \xi \int_{\mathbb{R}^2} uw + \frac{\chi}{2} \int_{\mathbb{R}^2} \left(v^2 + |\nabla v|^2\right) \\ &\ge \int_{\mathbb{R}^2} u \ln u - (\chi+\gamma) \int_{\mathbb{R}^2} uv - \xi K M + \frac{\chi}{2} \int_{\mathbb{R}^2} \left(v^2 + |\nabla v|^2\right) + \gamma \int_{\mathbb{R}^2} uv. \end{aligned} \tag{3.11}$$

Similar as the calculation shown in [1], we set $\bar{u}(x,t) = M e^{(\chi+\gamma)v(x,t)} H(x) \left(\int_{\mathbb{R}^2} e^{(\chi+\gamma)v(x,t)} H(x) dx\right)^{-1}$, where $H(x)$ is defined in Lemma 3.4. Then, we can apply the Entropy Lemma 3.3 with $\psi = (\chi+\gamma)v + \ln H$ to obtain

$$\begin{aligned} E(u; (\chi+\gamma)v + \ln H) &\ge E(\bar{u}; (\chi+\gamma)v + \ln H) \\ &= M \ln M - M \ln\left(\int_{\mathbb{R}^2} e^{(\chi+\gamma)v(x,t)} H(x) dx\right). \end{aligned} \tag{3.12}$$

Furthermore, applying Lemma 3.4 with $\varphi = (\chi+\gamma)v$ to the last term in the right hand side of (3.12), we have that

$$\begin{aligned} E(u; (\chi+\gamma)v + \ln H) &= \int_{\mathbb{R}^2} u \ln u - (\chi+\gamma) \int_{\mathbb{R}^2} uv - \int_{\mathbb{R}^2} u \ln H \\ &\ge M \ln M - M \ln\left(\int_{\mathbb{R}^2} e^{(\chi+\gamma)v(x,t)} H(x) dx\right) \\ &\ge M \ln M - M(\chi+\gamma) \int_{\mathbb{R}^2} vH - \frac{M(\chi+\gamma)^2}{16\pi} \int_{\mathbb{R}^2} |\nabla v|^2. \end{aligned} \tag{3.13}$$

Then by Young's inequality, we have $M(\chi+\gamma) \int_{\mathbb{R}^2} vH \le \frac{M(\chi+\gamma)^2}{16\pi} \int_{\mathbb{R}^2} v^2 + 4M\pi \int_{\mathbb{R}^2} H^2$. Together with (3.13) and the fact $\int_{\mathbb{R}^2} H^2(x) dx = \frac{1}{3\pi}$, we can easily obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} u \ln u - (\chi+\gamma) \int_{\mathbb{R}^2} uv - \int_{\mathbb{R}^2} u \ln H \\ \ge& M \ln M - M(\chi+\gamma) \int_{\mathbb{R}^2} vH - \frac{M(\chi+\gamma)^2}{16\pi} \int_{\mathbb{R}^2} |\nabla v|^2 \\ \ge& M \ln M - \frac{M(\chi+\gamma)^2}{16\pi} \int_{\mathbb{R}^2} v^2 - \frac{M(\chi+\gamma)^2}{16\pi} \int_{\mathbb{R}^2} |\nabla v|^2 - \frac{4}{3} M. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.11), we have

$$\begin{aligned} F(t) &\geq \int_{\mathbb{R}^2} u \ln u - (\chi + \gamma) \int_{\mathbb{R}^2} uv + \frac{\chi}{2} \int_{\mathbb{R}^2} \left(v^2 + |\nabla v|^2\right) + \gamma \int_{\mathbb{R}^2} uv - \xi K M \\ &\geq M \ln M + \int_{\mathbb{R}^2} u \ln H + \left(\frac{\chi}{2} - \frac{M(\chi+\gamma)^2}{16\pi}\right) \int_{\mathbb{R}^2} \left(v^2 + |\nabla v|^2\right) + \gamma \int_{\mathbb{R}^2} uv - (\xi K + \frac{4}{3})M \\ &\geq M \ln M + \int_{\mathbb{R}^2} u \ln H + \gamma \int_{\mathbb{R}^2} uv - (\xi K + \frac{4}{3})M \end{aligned} \tag{3.15}$$

by (3.10). Now we turn to estimate the second term on the right side of (3.15). We set $\phi(x) = \ln(1+|x|^2)$, then we can obtain by Young's inequality

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} u\phi &= \int_{\mathbb{R}^2} u_t \phi = \int_{\mathbb{R}^2} u \nabla\phi \cdot \nabla(\ln u - \chi v - \xi w) \\ &\leq \int_{\mathbb{R}^2} u|\nabla\phi|^2 + \frac{1}{4} \int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2, \quad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

By the fact $|\nabla\phi(x)| = \left|\frac{2x}{1+|x|^2}\right| \leq 1$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^2} u\phi \leq \int_{\mathbb{R}^2} u + \frac{1}{4} \int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2 \leq M + \frac{1}{4} \int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2,$$

upon being integrated from 0 to t, which yields simply that for all $t \in (0, T_{\max})$

$$\int_{\mathbb{R}^2} u \ln(1+|x|^2) \leq \int_{\mathbb{R}^2} u_0 \ln(1+|x|^2) + Mt + \frac{1}{4} \int_0^t \int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2. \tag{3.16}$$

By the definition of $H(x)$, we have for all $t \in (0, T_{\max})$

$$\begin{aligned} \int_{\mathbb{R}^2} u \ln H &= -2 \int_{\mathbb{R}^2} u \ln(1+|x|^2) - M \ln \pi \\ &\geq -2 \int_{\mathbb{R}^2} u_0 \ln(1+|x|^2) - 2Mt - \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} u|\nabla(\ln u - \chi v - \xi w)|^2 - M \ln \pi. \end{aligned} \tag{3.17}$$

Substituting (3.15) and (3.17) into (3.9), we have for all $t \in (0, T_{\max})$

$$\gamma \int_{\mathbb{R}^2} uv \leq \xi K \int_0^t \int_{\mathbb{R}^2} uv + 2Mt + F(0) + 2 \int_{\mathbb{R}^2} u_0 \ln(1+|x|^2) + (\ln \pi + \xi K + \frac{4}{3} - \ln M)M. \tag{3.18}$$

From (1.4), we have assumed for convenience that $u_0 \ln u_0$ and $u_0 \ln(1+|x|^2)$ belongs to $L^1(\mathbb{R}^2)$ for convenience. Then we conclude an integral-type Gronwall inequality as follows

$$\gamma \int_{\mathbb{R}^2} uv \leq \xi K \int_0^t \int_{\mathbb{R}^2} uv + 2Mt + C_1, \quad \forall t \in (0, T_{\max}), \tag{3.19}$$

where $C_1 = F(0) + 2 \int_{\mathbb{R}^2} u_0 \ln(1+|x|^2) + (\ln \pi + \xi K + \frac{4}{3} - \ln M)M$ is a finite number. Solving the integral-type Gronwall inequality (3.19) via integrating factor method, we infer that for some $C_2 > 0$

$$\int_{\mathbb{R}^2} uv + \int_0^t \int_{\mathbb{R}^2} uv \leq C_2 e^{\frac{\xi K}{\gamma} t}, \quad \forall t \in (0, T_{\max}).$$

Then by (3.9), one can simply deduce that $F(t)$ grows no great than exponentially as well:

$$F(t) \leq C_3 e^{\frac{\xi K}{\gamma} t}, \quad \forall t \in (0, T_{\max}) . \tag{3.20}$$

Similarly, this along with (1.7) shows that for some $C_4 > 0$

$$\int_{\mathbb{R}^2} u \ln u + \int_{\mathbb{R}^2} v^2 + \int_{\mathbb{R}^2} |\nabla v|^2 \leq C_4 e^{\frac{\xi K}{\gamma} t}, \quad \forall t \in (0, T_{\max}) . \tag{3.21}$$

According to Lemma 3.5 with $\psi = -(1+\delta)\ln\left(1+|x|^2\right)$, for arbitrary $\delta > 0$ in order to have $e^{-(1+\delta)\ln\left(1+|x|^2\right)} \in L^1\left(\mathbb{R}^2\right)$, we have for all $t \in (0, T_{\max})$

$$\begin{aligned}
&\int_{\mathbb{R}^2} u(\ln u)_- dx \\
\leq &(1+\delta) \int_{\mathbb{R}^2} u \ln\left(1+|x|^2\right) dx + C_5 \\
\leq &(1+\delta) \left\{ \int_{\mathbb{R}^2} u_0 \ln(1+|x|^2) + Mt + \frac{1}{4} \int_0^t \int_{\mathbb{R}^2} u |\nabla(\ln u - \chi v - \xi w)|^2 \right\} + C_5 \\
\leq &\frac{1+\delta}{4} \left\{ F(0) - F(t) + \xi K \int_0^t \int_{\mathbb{R}^2} uv \right\} + M(1+\delta)t + C_6 \\
\leq &C_7 e^{\frac{\xi K}{\gamma} t}
\end{aligned} \tag{3.22}$$

for some $C_i > 0$ $(i = 5, 6, 7)$. Finally, the identity

$$\int_{\mathbb{R}^2} |u \ln u| dx = \int_{\mathbb{R}^2} u \ln u dx + 2 \int_{\mathbb{R}^2} u(\ln u)_- dx \tag{3.23}$$

gives that $\|(u \ln u)(t)\|_{L^1} \leq C_8 e^{\frac{\xi K}{\gamma} t}$ for some $C_8 > 0$. Together with (3.21), this easily yield (3.7). □

Next, we wish to raise the regularity of u based on the local L^1-boundedness of $u \ln u$. In particular, for subcritical mass M, we have $\int_{\mathbb{R}^2} (u(x,t) - k)_+ dx \leq M$ for any $k > 0$, while for $k > 1$ we have for all $t \in (0, T_{\max})$

$$\begin{aligned}
\int_{\mathbb{R}^2} (u(x,t) - k)_+ dx &\leq \frac{1}{\ln k} \int_{\mathbb{R}^2} (u(x,t) - k)_+ \ln u(x,t) dx \\
&\leq \frac{1}{\ln k} \int_{\mathbb{R}^2} u(x,t) (\ln u(x,t))_+ dx \leq \frac{C e^{\frac{\xi K}{\gamma} t}}{\ln k}.
\end{aligned} \tag{3.24}$$

Lemma 3.7. *Under the condition (1.5) and (1.8), for any $T \in (0, T_{\max})$, there exists $C(T) > 0$ such that the local solution (u, v, w) of (1.1) verifies that for any $p \geq 2$*

$$\int_{\mathbb{R}^2} u^p(x,t) dx \leq C(T), \ \forall t \in (0, T] , \tag{3.25}$$

where $C(T) = 2^p \bar{C}(T) + (2k)^{p-1} M$ with k and $\bar{C}(T)$ respectively given by (3.37) and (3.40) below, which are finite for any $T > 0$.

Proof. Let $k > 0$, to be chosen later. We derive a non-linear differential inequality for the quantity $Y_p(t) := \int_{\mathbb{R}^2}(u(x,t)-k)_+^p dx$, which guarantees that the L^p-norm of u remains finite.

Multiplying the equation of u in (1.3) by $p(u-k)_+^{p-1}$ yields, using integration by parts,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(u-k)_+^p dx\\
=&-4\frac{(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla(u-k)_+^{\frac{p}{2}}\right|^2 dx-(p-1)\chi\int_{\mathbb{R}^2}(u-k)_+^p\Delta v dx-pk\chi\int_{\mathbb{R}^2}(u-k)_+^{p-1}\Delta v dx\\
&-(p-1)\xi\int_{\mathbb{R}^2}(u-k)_+^p\Delta w dx-pk\xi\int_{\mathbb{R}^2}(u-k)_+^{p-1}\Delta w dx\\
=&I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{3.26}$$

Now using the equation of v in (1.3) and the nonnegativity of v, one obtains

$$\begin{aligned}
I_2 &= -(p-1)\chi\int_{\mathbb{R}^2}(u-k)_+^p\Delta v dx\\
&= (p-1)\chi\int_{\mathbb{R}^2}(u-k)_+^p(-v_t-v+u)dx\\
&\le -(p-1)\chi\int_{\mathbb{R}^2}(u-k)_+^p v_t+(p-1)\chi\int_{\mathbb{R}^2}(u-k)_+^{p+1}dx+(p-1)k\chi\int_{\mathbb{R}^2}(u-k)_+^p dx
\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}
I_3 &= -pk\chi\int_{\mathbb{R}^2}(u-k)_+^{p-1}\Delta v dx\\
&= pk\chi\int_{\mathbb{R}^2}(u-k)_+^{p-1}(-v_t-v+u)dx\\
&\le -pk\chi\int_{\mathbb{R}^2}(u-k)_+^{p-1}v_t dx+pk\chi\int_{\mathbb{R}^2}(u-k)_+^p dx+pk^2\chi\int_{\mathbb{R}^2}(u-k)_+^{p-1}dx.
\end{aligned}\tag{3.28}$$

Using Gagliardo-Nirenberg inequality $\int_{\mathbb{R}^2} f^4(x)dx \le C\int_{\mathbb{R}^2} f^2(x)dx\int_{\mathbb{R}^2}|\nabla f(x)|^2 dx$ with $f=(u-k)_+^{\frac{p}{2}}$ and Hölder inequality, we obtain for $\varepsilon > 0$

$$\begin{aligned}
\left|\int_{\mathbb{R}^2}(u-k)_+^p v_t dx\right| &\le \left(\int_{\mathbb{R}^2}(u-k)_+^{2p}dx\right)^{\frac{1}{2}}\|v_t\|_{L^2}\\
&\le C\left(\int_{\mathbb{R}^2}(u-k)_+^p dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\left|\nabla(u-k)_+^{\frac{p}{2}}\right|^2 dx\right)^{1/2}\|v_t\|_{L^2}\\
&\le C(p)\varepsilon\|v_t\|_{L^2}^2\int_{\mathbb{R}^2}(u-k)_+^p dx+\frac{2}{\varepsilon p}\int_{\mathbb{R}^2}\left|\nabla(u-k)_+^{\frac{p}{2}}\right|^2 dx.
\end{aligned}\tag{3.29}$$

Similarly, we have, for $p \ge \frac{3}{2}$

$$\left|\int_{\mathbb{R}^2}(u-k)_+^{p-1}v_t dx\right| \le \left(\int_{\mathbb{R}^2}(u-k)_+^{2(p-1)}dx\right)^{\frac{1}{2}}\|v_t\|_{L^2}$$

$$
\begin{aligned}
&\le\left(C(M,p)+C(p)\int_{\mathbb{R}^2}(u-k)_+^{2p}dx\right)^{\frac{1}{2}}\|v_t\|_{L^2}\\
&\le C(M,p)\|v_t\|_{L^2}+C(p)\varepsilon\|v_t\|_{L^2}^2\int_{\mathbb{R}^2}(u-k)_+^{p}dx\\
&\quad+\frac{p-1}{\varepsilon p^2 k}\int_{\mathbb{R}^2}\left|\nabla(u-k)_+^{\frac{p}{2}}\right|^2dx. \qquad (3.30)
\end{aligned}
$$

Then we can see that

$$
\begin{aligned}
I_2+I_3 \le\ & (p-1)\int_{\mathbb{R}^2}(u-k)_+^{p+1}dx+\frac{(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla(u-k)_+^{\frac{p}{2}}\right|^2dx\\
&+C(p,\chi)(k+1)\|v_t\|_{L^2}^2\int_{\mathbb{R}^2}(u-k)_+^{p}dx+C(M,p,\chi)k\|v_t\|_{L^2}\\
&+(2p-1)k\chi\int_{\mathbb{R}^2}(u-k)_+^{p}dx+pk^2\chi\int_{\mathbb{R}^2}(u-k)_+^{p-1}dx \qquad (3.31)
\end{aligned}
$$

by setting $\varepsilon=4\chi$. According to the equation of w and v and (3.8), one obtains for all $t\in(0,T)$

$$
\begin{aligned}
-\Delta w(x,t)=&-\Delta w_0(x)\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}+2\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}\nabla w_0(x)\cdot\int_0^t\nabla v(x,s)\mathrm{d}s\\
&-w_0(x)\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}\left|\int_0^t\nabla v(x,s)\mathrm{d}s\right|^2+w_0(x)\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}\int_0^t\Delta v(x,s)\mathrm{d}s\\
\le&\|\Delta w_0\|_{L^\infty}-\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}\left(\sqrt{w_0}\int_0^t\nabla v(x,s)\mathrm{d}s-\frac{\nabla w_0}{\sqrt{w_0}}\right)^2+\mathrm{e}^{-\int_0^t v(x,s)ds}\frac{|\nabla w_0|^2}{w_0}\\
&+w_0(x)\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}\int_0^t(v_s(x,s)+v-u)\,\mathrm{d}s. \qquad (3.32)
\end{aligned}
$$

Here to estimate the last integral of the right side of (3.32) we first note (1.7) guarantees that

$$
\begin{aligned}
w_0(x)\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}\int_0^t(v_s(x,s)+v-u)\,\mathrm{d}s&\le\|w_0\|_{L^\infty}\mathrm{e}^{-\int_0^t v(x,s)\mathrm{d}s}\left[v(x,t)-v_0+\int_0^t v(x,s)\mathrm{d}s\right]\\
&\le\|w_0\|_{L^\infty}v+\frac{\|w_0\|_{L^\infty}}{\mathrm{e}},\quad\forall t\in(0,T)
\end{aligned}
$$

by the nonnegativity of w_0 and v_0 and the fact $\mathrm{e}^{-x}x\le\frac{1}{\mathrm{e}}$ for all $x>0$. Substituting (3.8) and (3.32) into (3.26), we have

$$
\begin{aligned}
I_4+I_5=&-(p-1)\xi\int_{\mathbb{R}^2}(u-k)_+^{p}\Delta w dx-pk\xi\int_{\mathbb{R}^2}(u-k)_+^{p-1}\Delta w dx\\
\le&(p-1)K\xi\int_{\mathbb{R}^2}(u-k)_+^{p}vdx+(p-1)K_1\xi\int_{\mathbb{R}^2}(u-k)_+^{p}dx\\
&+pkK\xi\int_{\mathbb{R}^2}(u-k)_+^{p-1}vdx+pkK_1\xi\int_{\mathbb{R}^2}(u-k)_+^{p-1}dx,
\end{aligned}
$$

where $K_1=\|\Delta w_0\|_{L^\infty}+4\|\nabla\sqrt{w_0}\|_{L^\infty}^2+\frac{K}{\mathrm{e}}$. Applying similar procedure as (3.29) and (3.30) to $\int_{\mathbb{R}^2}(u-k)_+^{p}vdx$ and $\int_{\mathbb{R}^2}(u-k)_+^{p-1}vdx$, this yields

$$
I_4+I_5\le\frac{(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla(u-k)_+^{\frac{p}{2}}\right|^2dx+C(p,K,\xi)k\|v\|_{L^2}^2\int_{\mathbb{R}^2}(u-k)_+^{p}dx+C(M,p,K,\xi)k\|v\|_{L^2}
$$

$$+ (p-1)K_1\xi \int_{\mathbb{R}^2} (u-k)_+^p dx + pkK_1\xi \int_{\mathbb{R}^2} (u-k)_+^{p-1} dx \tag{3.33}$$

by setting $\varepsilon = 2K\xi$. Combining (3.26), (3.31) and (3.33), we have for all $t \in (0, T)$

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} (u-k)_+^p dx \\
&\le -2\frac{(p-1)}{p}\int_{\mathbb{R}^2} \left|\nabla (u-k)_+^{\frac{p}{2}}\right|^2 dx + (p-1)\int_{\mathbb{R}^2} (u-k)_+^{p+1} dx \\
&\quad + [(2p-1)k\chi + (p-1)K_1\xi]\int_{\mathbb{R}^2} (u-k)_+^p dx + (pk^2\chi + pkK_1\xi)\int_{\mathbb{R}^2} (u-k)_+^{p-1} dx \\
&\quad + C(p, K, \chi, \xi)(k+1)\left(\|\partial_t v\|_{L^2}^2 + \|v\|_{L^2}^2\right)\int_{\mathbb{R}^2} (u-k)_+^p dx + C(M, p, K, \chi, \xi)k\left(\|\partial_t v\|_{L^2} + \|v\|_{L^2}\right).
\end{aligned} \tag{3.34}$$

Next, we estimate the nonlinear and negative contribution $-2\frac{(p-1)}{p}\int_{\mathbb{R}^2} |\nabla (u-k)_+^{\frac{p}{2}}|^2 dx$ in terms of $\int_{\mathbb{R}^2} (u-k)_+^{p+1} dx$, with the help of the Sobolev's inequality $\|f\|_{L^2}^2 \le c_1 \|\nabla f\|_{L^1}^2$. Indeed, by (3.24),

$$\begin{aligned}
\int_{\mathbb{R}^2} (u-k)_+^{p+1} dx &= \int_{\mathbb{R}^2} \left((u-k)_+^{\frac{(p+1)}{2}}\right)^2 dx \le c_1 \left(\int_{\mathbb{R}^2} \left|\nabla (u-k)_+^{\frac{(p+1)}{2}}\right| dx\right)^2 \\
&= C(p)\left(\int_{\mathbb{R}^2} (u-k)_+^{\frac{1}{2}} \left|\nabla (u-k)_+^{\frac{p}{2}}\right| dx\right)^2 \\
&\le C(p)\int_{\mathbb{R}^2} (u-k)_+ dx \int_{\mathbb{R}^2} \left|\nabla (u-k)_+^{p/2}\right|^2 dx \\
&\le C(p)\frac{Ce^{\frac{\xi K}{\gamma}T}}{\ln k}\int_{\mathbb{R}^2} \left|\nabla (u-k)_+^{p/2}\right|^2 dx, \quad \forall 0 < t \le T.
\end{aligned} \tag{3.35}$$

Moreover, since for $p \ge 2$ it holds true that

$$\int_{\mathbb{R}^2} (u-k)_+^{p-1} dx \le \int_{\mathbb{R}^2} (u-k)_+ dx + \int_{\mathbb{R}^2} (u-k)_+^p dx. \tag{3.36}$$

Inserting (3.35) and (3.36) into (3.34) gives for $p \ge 2$ and $0 < t \le T$ that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} (u-k)_+^p dx \\
&\le (p-1)\left(1 - \frac{2\ln k}{pC(p)Ce^{\frac{\xi K}{\gamma}T}}\right)\int_{\mathbb{R}^2} (u-k)_+^{p+1} dx \\
&\quad + C(p, K, \chi, \xi)k\left(1 + \|\partial_t v\|_{L^2}^2 + \|v\|_{L^2}^2\right)\int_{\mathbb{R}^2} (u-k)_+^p dx + C(M, p, K, \chi, \xi)k\left(\|\partial_t v\|_{L^2} + \|v\|_{L^2} + 1\right).
\end{aligned}$$

For any fixed p we can choose $k = k(p, T)$ sufficiently large such that

$$\delta := \frac{2\ln k}{pC(p)Ce^{\frac{\xi K}{\gamma}T}} - 1 > 0, \tag{3.37}$$

namely, $k = \exp\left(\frac{(1+\delta)pC(p)Ce^{\frac{\xi K}{\gamma}T}}{2}\right)$. For such a k, using the interpolation

$$\begin{aligned}\int_{\mathbb{R}^2}(u-k)_+^p dx &\le \left(\int_{\mathbb{R}^2}(u-k)_+ dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^2}(u-k)_+^{p+1}dx\right)^{\left(1-\frac{1}{p}\right)}\\ &\le M^{\frac{1}{p}}\left(\int_{\mathbb{R}^2}(u-k)_+^{p+1}dx\right)^{\left(1-\frac{1}{p}\right)},\end{aligned}$$

we end up with the following differential inequality for $Y_p(t)$, $p \ge 2$ fixed and $0 < t \le T$

$$\begin{aligned}\frac{d}{dt}Y_p(t) \le& -(p-1)M^{-\frac{1}{p-1}}\delta Y_p^{\beta}(t) + c_2(p,K,\chi,\xi)k\left(1+\|\partial_t v\|_{L^2}^2+\|v\|_{L^2}^2\right)Y_p(t)\\ &+ c_3(M,p,K,\chi,\xi)k\left(1+\|\partial_t v\|_{L^2}^2+\|v\|_{L^2}^2\right),\end{aligned} \tag{3.38}$$

where $\beta = \frac{p}{p-1} > 1$. Let us write the differential inequality (3.38) as follows for simplicity:

$$\frac{d}{dt}Y_p(t) \le -\tilde{C}Y_p^{\beta}(t) + g(t)Y_p(t) + g(t), \quad 0 < t \le T, \tag{3.39}$$

where $g(t) = \bar{C}(M,p,K,\chi,\xi)k\left(1+\|\partial_t v\|_{L^2}^2+\|v\|_{L^2}^2\right)$ and $\tilde{C} = (p-1)M^{-\frac{1}{p-1}}\delta > 0$. According to (3.7), (3.9) and (3.20), we can see that $g(t) \le \bar{C}(M,p,K,\chi,\xi)ke^{\frac{\xi Kt}{\gamma}}$. Then by comparison inequality, we show that there exists a constant $\bar{C}(T)$ such that for all $t \in (0,T)$

$$\begin{aligned}Y_p(t) &\le Y_p(0)\exp\left(\int_0^t g(s)ds\right) + \int_0^t g(\tau)\exp\left(\int_\tau^t g(s)ds\right)d\tau\\ &\le Y_p(0)\bar{C}(M,p,K,\chi,\xi)ke^{\frac{\xi KT}{\gamma}}T + \bar{C}(M,p,K,\chi,\xi)ke^{\frac{\xi KT}{\gamma}}e^{\bar{C}(M,p,K,\chi,\xi)ke^{\frac{\xi KT}{\gamma}}}T := \bar{C}(T).\end{aligned} \tag{3.40}$$

It is sufficient to observe that for any $k > 0$

$$\begin{aligned}\int_{\mathbb{R}^2}u^p(x,t)dx &= \int_{\{u\le 2k\}}u^p(x,t)dx + \int_{\{u>2k\}}u^p(x,t)dx\\ &\le (2k)^{p-1}M + 2^p\int_{\{u>2k\}}(u(x,t)-k)^p dx\\ &\le (2k)^{p-1}M + 2^p\int_{\mathbb{R}^2}(u(x,t)-k)_+^p dx,\end{aligned} \tag{3.41}$$

where the inequality $x^p \le 2^p(x-k)^p$, for $x \ge 2k$, has been used. Therefore, (3.25) follows for any $p \ge 2$ by (3.40) and (3.41) choosing $k = k(p,T)$ sufficiently large such that (3.37) holds true. □

Proof of Theorem 1.1. According to the local L^p−boundedness of Lemma 3.7 and Lemma 2.2 we must have the local $H^2 \times H^3 \times H^3$-boundedness of (u,v,w), which contracts the extensibility criteria in (2.35). Then we must obtain that $T_{\max} = \infty$, that is, the strong solution (u,v,w) of (1.3) exists globally in time and is locally bounded as in (2.2).

□

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Conflict of interest

The authors declare there is no conflict of interest.

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