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Theory article

Stability analysis of multi-point boundary conditions for fractional differential equation with non-instantaneous integral impulse

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Abstract: This paper considers the stability of a fractional differential equation with multi-point boundary conditions and non-instantaneous integral impulse. Some sufficient conditions for the existence, uniqueness and at least one solution of the aforementioned equation are studied by using the Diaz-Margolis fixed point theorem. Secondly, the Ulam stability of the equation is also discussed. Lastly, we give one example to support our main results. It is worth pointing out that these two non-instantaneous integral impulse and multi-point boundary conditions factors are simultaneously considered in the fractional differential equations studied for the first time.

Keywords: Caputo fractional derivative; non-instantaneous integral impulse; multi-point boundary conditions; existence; stability

1. Introduction

In the past decades, a lot of complex dynamic phenomena have been produced by multi-point boundary conditions for fractional differential equations, which are more general than classical integer differential equations, so more and more researchers are attracted to studying the stability analysis of multi-point boundary conditions for fractional differential equations. In the modeling of many physical phenomena, fractional differential equations have been used as strong tools. Thus, some scholars [1–7] provided the most theoretical method for qualitative analysis in this research fieldsuch as, medicine, mechanical engineering, ecology, biology and astronomy.

Because impulse fractional differential equation calculus can describe the dynamic properties of system, it has attracted extensive attention of many researchers. For example, Zhao [8] considered multiple positive solutions of integral boundary value problems (BVPs) for high-order nonlinear fractional differential equations with impulses and distributed delays. Zhao [9] studied an impulsive integral boundary value problems of the higher-order fractional differential equation with eigenvalue arguments. Tian and Bai [10] studied impulsive boundary value problem for differential equations with fractional order. In addition, some papers also studied the dynamic properties of impulsive fractional differential equations. For example, solutions of impulsive fractional Langevin equations and existence results were studied by [11]. A new class of impulsive fractional differential equations was considered in [12].

Many researchers developed some interesting results about the existence of solutions for different boundary value problems, using different fixed point theorems [13–17]. Often, it is a challenging task for researchers to find the exact solutions of nonlinear differential equations. Thus, in this situation different approximation techniques were introduced [18, 19]. The difference between approximate and exact solutions can be treated with the help of Hyers-Ulam (HU) stability, which was first introduced in 1940 by Ulam [20–22]. Based on this method, many scholars have conducted further research on the stability of the solutions of fractional equations. For example, Zada et al. [23] presented the existence and uniqueness of solutions and different types of Ulam-Hyers stability for a class of nonlinear implicit fractional differential equations with non-instantaneous integral impulses and nonlinear integral boundary conditions. Subsequently, Zada and Ali [24] studied existence, uniqueness, and generalized different type of Ulam stability of fractional differential equations with non-instantaneous impulses. There are many interesting results to see [25–28]. As far as we know, a fractional differential equation simultaneous consideration of the multi-point boundary conditions and non-instantaneous integral impulse is not found in the existing literature.

Motivated by the existing works [29–36], in this manuscript, we deal with a multi-point boundary conditions for fractional differential equation with non-instantaneous integral impulse

$$\begin{cases} {}^cD^{\alpha}(D+\lambda)x(t) = f(t, x(t), {}^cD^{\beta}x(t)), & t \in (t_k, s_k] \subset J, k = 0, 1, \ldots, m, \\ x(t) = I^{\alpha}_{s_{k-1},t_k}(g_k(t, x(t))) & t \in (s_{k-1}, t_k] \subset J, k = 1, 2, \ldots, m, \\ ax(t_k) + bx(s_k) = c, & x(0) = 0, \end{cases} \tag{1.1}$$

where $0 < \beta < \alpha \leq 1$. λ, a, b and c are constants, and $\lambda > 0$, $b \neq 0$, $J = [0, T]$. ${}^*D^{\alpha}$ stands for the Caputo fractional derivatives of order $*$, and D stands for the ordinary derivative. I^{α} is Caputo fractional integral of order α. As we have $0 = t_0 < s_0 < t_1 < s_1 < \cdots < t_m < s_m = T$, T is a fixed number. $f : C(J \times \mathbb{R}^2) \to \mathbb{R}$ is continuous, and $g_k \in C((s_{k-1}, t_k] \times \mathbb{R} \to \mathbb{R}$ is also continuous for all $k = 1, 2, \ldots, m$.

Based on the method of [37–44], in this paper, we study the multi-point boundary conditions for a general fractional differential equation with non-instantaneous integral impulse. We consider the existence and stability analysis of multi-point boundary conditions for the general fractional differential equation with non-instantaneous integral impulse. By using the Diaz-Margolis fixed point theorem, we discuss some sufficient conditions for the existence, uniquenes, and at least one solution of the aforementioned equation. Secondly, the Ulam stability of Eq (1.1) is also given. The method of proving stability is only one of the results. The major innovations are Theorems 1, 3 and 4 of this paper. We only refer to references [23, 24] to prove the stability method of the solution of our system.

The novelty and difficulty are the following.

For the main results: 1) Although both this paper and [23, 24] discuss existence and generalized different type of Ulam stability for fractional differential equation, this paper first gives different the function value at the boundary of each pulse interval $(t_k, s_k]$ has a certain relationship value, that is, the value of the function at the boundary point t_k is related to the value of the function at s_k. 2) In [24], a stability analysis of a multi-point boundary value problem for sequential fractional differential equations with non-instantaneous impulses is considered. The general fractional differential equations with (1.1) considered in our paper have non-instantaneous impulses and multi-point boundary conditions, and [24] have considered sequential fractional differential equations with non-instantaneous impulses. Using the Diaz-Margolis fixed point theorem, some general sufficient conditions for the existence, uniqueness and at least one solution of the aforementioned equation are given in our article. 3) It should be noticed that [23] considered the existence and different type of Ulam stability for a fractional differential equation. Different from [23], in this paper, we improve it more generally; for example the second impulse equation is introduced into our equation. We point out that the non-instantaneous integral impulse and multi-point boundary conditions of two factors are simultaneously considered in the general fractional differential equations studied for the first time.

For the difficulty in analysis method of this article: 1) The traditional continuity theory cannot be applied due to the multi-point boundary conditions for the general fractional differential equation in our paper. For example, when proving the existence, uniqueness and at least one solution of the systems with non-instantaneous integral impulse, the traditional stability theorem cannot be similarly constructed. 2) The fixed point theorem of continuous systems cannot be used to prove the existence of stability of general Eq (1.1). In this paper, by using the Diaz-Margolis fixed point theorem, we obtain the Ulam stability of the Eq (1.1).

This paper is organized as follows: in Section 2, we give some basic Definitions, Lemmas and the existence of solution. Section 3 gives the Ulam stabilities analysis. Section 4 gives one example to illustrate the main results. Finally, we summarize the main results of this paper in Section 5.

Notations: Let $J = [0, T]$ and $C(J, \mathbb{R})$ be the space of all continuous functions from J to $\mathbb{R}$. Let $\mathbb{B} = PC^2(J, \mathbb{R})$ represent the space of piecewise continuous and two times differentiable functions. For a function $u : J \to \mathbb{R}$, the Caputo fractional derivative of order α is defined as

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^n(s) ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α. For a function $u : J \to \mathbb{R}$, the sequential fractional derivative is defined as

$$D^{\alpha}u(t) = D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_k} u(t),$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is any multi-index. In general, the operator D^{α} can either be Riemann-Liouville or Caputo or any other kind of integro-differential operator.

Let $y \in \mathbb{B}$, $\varepsilon > 0$, $\nu > 0$, $\lambda \in \mathbb{R}^+$ and $\theta \in C(J, \mathbb{R}^+)$ be a non-decreasing function. Let us consider the following set:

$$\begin{cases} |^{c}D^{\alpha}(D+\lambda)y(t) - f(t, y(t), {}^{c}D^{\beta}y(t))| \leqslant \varepsilon, & t \in (t_k, s_k] \subset J, k = 0, 1, \dots, m, \\ |y(t) - I^{\alpha}_{s_{k-1}, t_k} g_k(t, y(t))| \leqslant \varepsilon, & t \in (s_{k-1}, t_k] \subset J, k = 1, 2, \dots, m, \end{cases} \tag{1.2}$$

$$\begin{cases} |{}^cD^{\alpha}(D+\lambda)y(t) - f(t, y(t), {}^cD^{\beta}y(t))| \leqslant \theta(t), & t \in (t_k, s_k] \subset J, k = 0, 1, \ldots, m, \\ |y(t) - I^{\alpha}_{s_{k-1},t_k} g_k(t, y(t))| \leqslant v, & t \in (s_{k-1}, t_k] \subset J, k = 1, 2, \ldots, m, \end{cases} \quad (1.3)$$

and

$$\begin{cases} |{}^cD^{\alpha}(D+\lambda)y(t) - f(t, y(t), {}^cD^{\beta}y(t))| \leqslant \varepsilon\theta(t), & t \in (t_k, s_k] \subset J, k = 0, 1, \ldots, m, \\ |y(t) - I^{\alpha}_{s_{k-1},t_k} g_k(t, y(t))| \leqslant \varepsilon v, & t \in (s_{k-1}, t_k] \subset J, k = 1, 2, \ldots, m. \end{cases} \quad (1.4)$$

2. Preliminaries and the existence of solution

In this part, we give some basic definitions, lemmas, theorems and the existence conditions of solution in Eq (1.1).

2.1. Preliminaries

Definition 1. *[7] For a function $u : J \to \mathbb{R}$, the Caputo fractional integral of order α is defined as*

$${}^cI^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t > 0, \quad \alpha > 0,$$

where Euler gamma function Γ is defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$.

As in [19], Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability of Eq (1.1) are given as follows.

Definition 2. *[19] For a given $\varepsilon > 0$, the following conditions hold.*

(1)) For each solution $y \in \mathbb{B}$ of Eq (1.2), there are a positive constant $c_{m,\alpha,\beta} > 0$ and a solution $x \in \mathbb{B}$ of Eq (1.1) satisfying $|y(t) - x(t)| \leq c_{m,\alpha,\beta}\varepsilon, t \in J$. Then, the solution of Eq (1.1) is Ulam-Hyers stable.

(2) For each solution $y \in \mathbb{B}$ of Eq (1.3), there are a constant $\phi_{m,\alpha,\beta} \in C(\mathbb{R}^+, \mathbb{R}^+), \phi_{m,\alpha,\beta}(0) = 0$ and solution $x \in \mathbb{B}$ of Eq (1.1) satisfying $|y(t) - x(t)| \leq \phi_{m,\alpha,\beta}(\varepsilon), t \in J$. Then, the solution of Eq (1.1) is generalized Ulam-Hyers stable.

(3) For each solution $y \in \mathbb{B}$ of Eq (1.4), there are a positive constant $c_{m,\alpha,\beta} > 0$ and a solution $x \in \mathbb{B}$ of Eq (1.1) satisfying $|y(t) - x(t)| \leq c_{m,\alpha,\beta}\varepsilon(\theta(t) + v), t \in J$. Then, the solution of Eq (1.1) is Ulam-Hyers-Rassias stable with respect to (θ, v).

(4) For each solution $y \in \mathbb{B}$ of Eq (1.3), there are a positive constant $c_{m,\alpha,\beta} > 0$ and a solution $x \in \mathbb{B}$ of Eq (1.1) satisfying $|y(t) - x(t)| \leq c_{m,\alpha,\beta}(\theta(t) + v), t \in J$. Then, the solution of Eq (1.1) is generalized Ulam-Hyers-Rassias stable.

Definition 3. *[25] Let X be a non-empty set, and a function $d : X \times X \to [0, \infty]$, for $a, b, c \in X$ satisfying $d(a, b) \geq 0$; $d(a, b) = 0$ if and only if $a = b$; $d(a, b) = d(b, a)$; $d(a, b) \leq d(a, c) + d(c, b)$. Then, X is a generalized metric space.*

Definition 4. *[25] Let X be a generalized metric space. If every d Cauchy sequence in X is d-convergent, i.e., if* $\{a_n\}$ *is a sequence in X satisfying* $\lim_{m,n\to\infty} d(a_n, a_m) = 0$, *and further, there is* $u \in X$ *that satisfies* $\lim_{n\to\infty} d(a_n, u) = 0$, *then, X is generalized complete metric space.*

Lemma 1. *[26] Suppose* (X, d) *is a generalized complete metric space, and an operator* $\bigwedge : X \to X$ *is strictly contractive with Lipschitz constant* $L < 1$. *If there is an integer* $n \geq 0$ *such that* $d(\bigwedge^{n+1} x, \bigwedge^{n} x) < \infty$ *for some* $x \in X$, *then the following conditions hold. (i) The sequence* $\{\bigwedge^{n} x\}$ *converges to a fixed point* θ^* *of* $\bigwedge$. *(ii)* θ^* *is the unique fixed point of* $\bigwedge$ *in* $X^* = \{y \in X | d(\bigwedge^{n} \theta, y) < \infty\}$. *(iii) If* $y \in X^*$, *then* $d(y, x^*) \leq \frac{1}{1-L} d(\bigwedge y, y)$.

Lemma 2. *[7] For any* $\alpha > 0$ *and* $u \in \mathbb{B}$, *the following conditions hold.*

1). The Caputo fractional differential equation ${}^{c}D^{\alpha}u(t) = 0$ *has a solution of the following form:* $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, *where* $c_i \in \mathbb{R}, i = 0, 1, \cdots n-1$, *and* $n = [\alpha] + 1$.

2). $I_0 +^{\alpha} (D^{\alpha}_{0+} u(t)) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1}$, *where* $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$, *and* $n = [\alpha] + 1$.

To give main results of Eq (1.1), the following assumptions are necessary

(H1) $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function.

(H2) There exist two numbers $L_f > 0$ and $0 < \widetilde{L_f} < 1$ such that $|f(t, w_1, \overline{w}_1) - f(t, w_2, \overline{w}_2)| \leq L_f |w_1 - w_2| + \widetilde{L_f} |\overline{w}_1 - \overline{w}_2|$, where $t \in J$ and $w_1, \overline{w}_1, w_2, \overline{w}_2 \in \mathbb{R}^+$.

(H3) For $g_k \in C([s_{k-1}, t_k], \mathbb{R}^+, \mathbb{R}^+)$ and there are $L_{g_k} > 0, k = 1, 2, \cdots, m$ such that $|g_k(t, w_1) - g_k(t, w_2)| \leq L_{g_k} |w_1 - w_2|$, where $t \in (s_{k-1}, t_k]$ and $w_1, w_2 \in \mathbb{R}^+$.

(H4) Letting $\theta(t) \in C(J, \mathbb{R}^+)$ be a non-decreasing function, for each $t \in J$, there are $c^{\alpha}_{\theta}, c^{\alpha-\beta}_{\theta}$ such that $\int_0^t I^{\alpha}\theta(s) ds \leq c^{\alpha}_{\theta}\theta(t)$ $\int_0^t I^{\alpha-\beta}\theta(s) ds \leq c^{\alpha-\beta}_{\theta}\theta(t)$.

For convenience, we give the following notations

$$\Lambda = \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda s_0}}, B_k = \frac{1 - e^{-\lambda(t - t_k)}}{1 - e^{-\lambda(s_k - t_k)}}.$$

2.2. Existence of solution

In this part, by using Definition 1 and Lemma 2, we address the existence of solution in Eq (1.1) as follows.

Theorem 1. *Let* $0 < \alpha \leq 1$ *and* $f : J \to \mathbb{R}$ *be a given continuous function. A solution* $x \in \mathbb{B}$ *satisfies the linear impulsive problem*

$$\begin{cases} {}^{c}D^{\alpha}(D + \lambda)x(t) = f(t), & t \in (t_k, s_k] \subset J, k = 0, 1, \ldots, m, 0 < \beta < \alpha \leq 1, \\ x(t) = I^{\alpha}_{s_{k-1}, t_k}(g_k(t)), & t \in (s_{k-1}, t_k] \subset J, k = 1, 2, \ldots, m, \\ ax(t_k) + bx(s_k) = c, & x(0) = 0, \end{cases} \tag{2.1}$$

if and only if the solution $x \in \mathbb{B}$ *satisfies*

$$x(t)=\begin{cases}\int_0^t e^{-\lambda(t-s)}I^\alpha f(s)ds+\Lambda[\frac{c}{b}-\int_0^{s_0}e^{-\lambda(s_0-s)}I^\alpha f(s)ds], & t\in[t_0,s_0],\\ I^\alpha_{s_{k-1},t_k}(g_k(t)), k=1,2,\cdots,m, & t\in(s_{k-1},t_k],\\ \int_{t_k}^t e^{-\lambda(t-s)}I^\alpha f(s)ds+B_k[\frac{c}{b}-\frac{a}{b}I^\alpha_{s_{k-1},t_k}(g_k(t_k))-\int_{t_k}^{s_k}e^{-\lambda(s_k-s)}I^\alpha f(s)ds & \\ -e^{-\lambda(s_k-t_k)}I^\alpha_{s_{k-1},t_k}(g_k(t_k))]+e^{-\lambda(t-t_k)}I^\alpha_{s_{k-1},t_k}(g_k(t_k)),\ k=1,2,\cdots,m, & t\in(t_k,s_k].\end{cases} \tag{2.2}$$

Proof. To prove the sufficiency, let $x\in\mathbb{B}$ be the solution of (2.1). For $t\in[0,s_0]$, we first consider

$$^cD^\alpha_{0,t}(D+\lambda)x(t)=f(t). \tag{2.3}$$

By using Lemma 2, we have

$$x(t)=\int_0^t e^{-\lambda(t-s)}I^\alpha f(s)ds+c_0\frac{1-e^{-\lambda t}}{\lambda}+d_0e^{-\lambda t}, \tag{2.4}$$

where c_0 and d_0 are arbitrary constants.

From (2.4) and $x(0)=0$, we get $d_0=0$. In view of $ax(0)+bx(s_0)=c$, we obtain $x(s_0)=\frac{c}{b}$. From (2.4), $x(s_0)=\int_0^{s_0}e^{-\lambda(s_0-s)}I^\alpha f(s)ds+c_0\frac{1-e^{-\lambda s_0}}{\lambda}+d_0e^{-\lambda s_0}=\frac{c}{b}$, and then

$$c_0=\frac{\lambda}{1-e^{-\lambda s_0}}[\frac{c}{b}-\int_0^{s_0}e^{-\lambda(s_0-s)}I^\alpha f(s)ds].$$

So,

$$x(t)=\int_0^t e^{-\lambda(t-s)}I^\alpha f(s)ds+\frac{1-e^{-\lambda t}}{1-e^{-\lambda s_0}}[\frac{c}{b}-\int_0^{s_0}e^{-\lambda(s_0-s)}I^\alpha f(s)ds]. \tag{2.5}$$

Moreover, we assume that $t\in(s_0,t_1]$, and from the second equation of (2.1), we then have $x(t)=I^\alpha_{s_{k-1},t_k}(g_k(t))$.

If $t\in(t_1,s_1]$, then

$$x(t)=\int_{t_1}^t e^{-\lambda(t-s)}I^\alpha f(s)ds+c_1\frac{1-e^{-\lambda(t-t_1)}}{\lambda}+d_1e^{-\lambda t}. \tag{2.6}$$

Based on $x(t_1)=I^\alpha_{s_0,t_1}(g_1(t_1))$, we get $x(t_1)=d_1e^{-\lambda t_1}$, and then $d_1=e^{\lambda t_1}I^\alpha_{s_0,t_1}(g_1(t_1))$. In view of the condition $ax(t_1)+bx(s_1)=c$, with

$$x(s_1)=\int_{t_1}^{s_1}e^{-\lambda(s_1-s)}I^\alpha f(s)ds+c_1\frac{1-e^{-\lambda(s_1-t_1)}}{\lambda}+d_1e^{-\lambda s_1},$$

we obtain

$$c_1=\frac{\lambda}{1-e^{-\lambda(s_1-t_1)}}[\frac{c}{b}-\frac{a}{b}I^\alpha_{s_0,t_1}(g_1(t_1))-\int_{t_1}^{s_1}e^{-\lambda(s_1-s)}I^\alpha f(s)ds-e^{-\lambda(s_1-t_1)}I^\alpha_{s_0,t_1}(g_1(t_1))].$$

So,

$$\begin{aligned}x(t)=&\int_{t_1}^t e^{-\lambda(t-s)}I^\alpha f(s)ds+\frac{1-e^{-\lambda(t-t_1)}}{1-e^{-\lambda(s_1-t_1)}}[\frac{c}{b}-\frac{a}{b}I^\alpha_{s_0,t_1}(g_1(t_1))-\int_{t_1}^{s_1}e^{-\lambda(s_1-s)}I^\alpha f(s)ds\\&-e^{-\lambda(s_1-t_1)}I^\alpha_{s_0,t_1}(g_1(t_1))]+e^{-\lambda(t-t_1)}I^\alpha_{s_0,t_1}(g_1(t_1)).\end{aligned} \tag{2.7}$$

If $t \in (t_k, s_k]$, the solution of (2.1) with $x(t_k) = I^{\alpha}_{s_{k-1},t_k}(g_k(t_k)), ax(t_k) + bx(s_k) = c$, is given as

$$x(t) = \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha} f(s)ds + \frac{1 - e^{-\lambda(t-t_k)}}{1 - e^{-\lambda(s_k-t_k)}}[\frac{c}{b} - \frac{a}{b} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k)) - \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha} f(s)ds - e^{-\lambda(s_k-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k))] + e^{-\lambda(t-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k)). \tag{2.8}$$

Conversely, if $x \in \mathbb{B}$ is a solution of fractional integral (2.2), by the fact that ${}^cD^{\alpha}_{0,t}$ is the left inverse of ${}^cI^{\alpha}_{0,t}$, we can easily verify our result.

Theorem 2. *Letting (H1)–(H4) hold and a solution y satisfy* (1.4)*, for all time $t \in J$, there is a unique solution y_0 of* (1.1) *that satisfies*

$$y_0(t) = \begin{cases} \int_0^t e^{-\lambda(t-s)} I^{\alpha} f(s, y_0(s), {}^cD^{\beta} y_0(s))ds \\ \quad +\Lambda[\frac{c}{b} - \int_0^{s_0} e^{-\lambda(s_0-s)} I^{\alpha} f(s, y_0(s), {}^cD^{\beta} y_0(s)ds], & t \in [t_0, s_0], \\ I^{\alpha}_{s_{k-1},t_k}(g_k(t, (y_0(t))), k = 1, 2, \cdots, m, & t \in (s_{k-1}, t_k], \\ \int_{t_k}^t e^{-\lambda(t-s)} I^{\alpha} f(s, y_0(s), {}^cD^{\beta} y_0(s)ds + B_k[\frac{c}{b} - \frac{a}{b} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k, (y_0(t_k))) \\ \quad - \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha} f(s, y_0(s), {}^cD^{\beta} y_0(s))ds - e^{-\lambda(s_k-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k, (y_0(t_k)))] \\ \quad +e^{-\lambda(t-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k, (y_0(t_k))), k = 1, 2, \cdots, m, & t \in (t_k, s_k], \end{cases} \tag{2.9}$$

where (X, d) is a generalized complete metric space, $y, y_0 \in X$.

Proof. Let X be the space of piecewise continuous functions, i.e., $X = \{p : J \to \mathbb{R} | p \in \mathbb{B}\}$ with generalized metric on Y, which is defined as

$$d(p, q) = \inf\{C_1 + C_2 \in [0, +\infty] \mid |p(t) - q(t)| \leq \varepsilon(C_1 + C_2)(\theta(t) + \nu)\}, \tag{2.10}$$

for $t \in J$, where

$$C_1 \in \{C \in [0, +\infty] \mid |p(t) - q(t)| \leq C\varepsilon\theta(t)\}, t \in (t_k, s_k], k = 0, 1, \ldots, m\},$$

and

$$C_2 \in \{C \in [0, +\infty] \mid |p(t) - q(t)| \leq C\varepsilon\nu\}, t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m\}.$$

Now, we prove that there exists at least one positive solution.

Based on Lemma 1, we prove that (X, d) is a complete generalized metric space.

For all $x \in X$ and $t \in J$, define an operator $T : X \times X \to \mathbb{R}^+$ by

$$(Tx)(t) = \begin{cases} \int_0^t e^{-\lambda(t-s)} I^{\alpha} f(s, x(s), {}^cD^{\beta} x(s))ds \\ \quad +\Lambda[\frac{c}{b} - \int_0^{s_0} e^{-\lambda(s_0-s)} I^{\alpha} f(s, x(s), {}^cD^{\beta} x(s))ds], & t \in [t_0, s_0], \\ I^{\alpha}_{s_{k-1},t_k}(g_k(t, (x(t))), k = 1, 2, \cdots, m, & t \in (s_{k-1}, t_k], \\ \int_{t_k}^t e^{-\lambda(t-s)} I^{\alpha} f(s, x(s), {}^cD^{\beta} x(s))ds + B_k[\frac{c}{b} - \frac{a}{b} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k, (x(t_k))) \\ \quad - \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha} f(s, x(s), {}^cD^{\beta} x(s))ds - e^{-\lambda(s_k-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k, (x(t_k)))] \\ \quad +e^{-\lambda(t-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k, (x(t_k))),\ k = 1, 2, \cdots, m, & t \in (t_k, s_k], \end{cases} \tag{2.11}$$

where T is a defined operator according to assumption (H1). To verify (X,d) is a complete generalized metric space, we can prove that T of (2.11) is strictly contractive on X.

From (2.10), for any number $p,q\in X$, we can find $C_1,C_2\in[0,+\infty]$ such that

$$|p(t)-q(t)|\leq\begin{cases}C_1\varepsilon\theta(t), & t\in(t_k,s_k],k=0,1,\ldots,m,\\ C_2\varepsilon\nu, & t\in(s_{k-1},t_k],k=1,2,\ldots,m.\end{cases}\tag{2.12}$$

Let $L=\max\{L_1,L_2\}<1, L_1=\max\{(L_f+\widetilde{L_f})c_\theta[\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)]+[B_k|\frac{a}{b}|+B_ke^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]L_{g_k}\}, L_2=\max\{(L_f+\widetilde{L_f})[\frac{(t-t_k)^\alpha}{\Gamma(\alpha+1)}+\frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}]\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)+[B_k|\frac{a}{b}|+B_ke^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]L_{g_k}\}$. From (H1)–(H4), (2.10) and (2.12), four Cases are considered as follows.

Case 1 For $t\in[0,s_0]$, we have

$$\begin{aligned}\Gamma_1&=|(Tp)(t)-(Tq)(t)|\\
&=|\int_0^t e^{-\lambda(t-s)}I_{0,t}^\alpha f(s,p(s),{}^cD_{0,t}^\beta p(s))ds+\Lambda[\tfrac{c}{b}-\int_0^{s_0}e^{-\lambda(s_0-s)}I_{0,t}^\alpha f(s,p(s),{}^cD_{0,t}^\beta p(s))ds]\\
&\quad-\int_0^t e^{-\lambda(t-s)}I_{0,t}^\alpha f(s,q(s),{}^cD_{0,t}^\beta q(s))ds-\Lambda[\tfrac{c}{b}-\int_0^{s_0}e^{-\lambda(s_0-s)}I_{0,t}^\alpha f(s,q(s),{}^cD_{0,t}^\beta q(s))ds]|\\
&\leq\int_0^t e^{-\lambda(t-s)}I_{0,t}^\alpha|f(s,p(s),{}^cD_{0,t}^\beta p(s))-f(s,q(s),{}^cD_{0,t}^\beta q(s))|ds\\
&\quad+\Lambda\int_0^{s_0}e^{-\lambda(s_0-s)}I_{0,t}^\alpha|f(s,p(s),{}^cD_{0,t}^\beta p(s))-f(s,q(s),{}^cD_{0,t}^\beta q(s))|ds\\
&\leq\int_0^t e^{-\lambda(t-s)}I_{0,t}^\alpha(L_f|p(s)-q(s)|+\widetilde{L_f}|{}^cD_{0,t}^\beta p(s)-{}^cD_{0,t}^\beta q(s)|)ds\\
&\quad+\Lambda\int_0^{s_0}e^{-\lambda(s_0-s)}I_{0,t}^\alpha(L_f|p(s)-q(s)|+\widetilde{L_f}|{}^cD_{0,t}^\beta p(s)-{}^cD_{0,t}^\beta q(s)|)ds\\
&\leq L_fC_1\varepsilon\int_0^t e^{-\lambda(t-s)}I_{0,t}^\alpha\theta(s)ds+\widetilde{L_f}C_1\varepsilon\int_0^t e^{-\lambda(t-s)}I_{0,t}^{\alpha}{}^cD_{0,t}^\beta\theta(s)ds\\
&\quad+\Lambda L_fC_1\varepsilon\int_0^{s_0}e^{-\lambda(s_0-s)}I_{0,t}^\alpha\theta(s)ds+A\widetilde{L_f}C_1\varepsilon\int_0^{s_0}e^{-\lambda(s_0-s)}I_{0,t}^{\alpha}{}^cD_{0,t}^\beta\theta(s)ds\\
&\leq L_fC_1\varepsilon(\int_0^t e^{-\lambda(t-s)}ds)(\int_0^t I_{0,t}^\alpha\theta(s)ds)+\widetilde{L_f}C_1\varepsilon(\int_0^t e^{-\lambda(t-s)}ds)(\int_0^t I_{0,t}^{\alpha}{}^cD_{0,t}^\beta\theta(s)ds)\\
&\quad+\Lambda L_fC_1\varepsilon(\int_0^{s_0}e^{-\lambda(s_0-s)}ds)(\int_0^{s_0}I_{0,t}^\alpha\theta(s)ds)+A\widetilde{L_f}C_1\varepsilon(\int_0^{s_0}e^{-\lambda(s_0-s)}ds)(\int_0^{s_0}I_{0,t}^{\alpha}{}^cD_{0,t}^\beta\theta(s)ds)\\
&\leq L_fC_1\varepsilon\tfrac{1-e^{-\lambda t}}{\lambda}c_\theta^\alpha\theta(t)+\widetilde{L_f}C_1\varepsilon\tfrac{1-e^{-\lambda t}}{\lambda}c_\theta^{\alpha-\beta}\theta(t)+\Lambda L_fC_1\varepsilon\tfrac{1-e^{-\lambda s_0}}{\lambda}c_\theta^\alpha\theta(s_0)\\
&\quad+\Lambda\widetilde{L_f}C_1\varepsilon\tfrac{1-e^{-\lambda s_0}}{\lambda}c_\theta^{\alpha-\beta}\theta(s_0)\\
&\leq C_1\varepsilon\theta(t)\tfrac{1-e^{-\lambda t}}{\lambda}(L_fc_\theta^\alpha+\widetilde{L_f}c_\theta^{\alpha-\beta})+\Lambda C_1\varepsilon\theta(s_0)\tfrac{1-e^{-\lambda s_0}}{\lambda}(L_fc_\theta^\alpha+\widetilde{L_f}c_\theta^{\alpha-\beta})\\
&\leq(L_f+\widetilde{L_f})c_\theta[\tfrac{1-e^{-\lambda s_0}}{\lambda}(1+\Lambda)]C_1\varepsilon\theta(s_0),\end{aligned}$$

where $c_\theta=\max\{c_\theta^\alpha,c_\theta^{\alpha-\beta}\}$.

Case 2 For $t\in(s_{k-1},t_k],k=1,2,\ldots,m$, we have

$$\begin{aligned}|(Tp)(t)-(Tq)(t)|&=|I_{s_{k-1},t_k}^\alpha g_k(t,p(t))-I_{s_{k-1},t_k}^\alpha g_k(t,q(t))|\leq L_{g_k}|p(t)-q(t)|\\
&\leq L_{g_k}C_2\varepsilon\nu.\end{aligned}$$

Case 3 For $t\in(t_k,s_k]$ and $s\in(t_k,s_k],k=1,2,\ldots,m$, we have

$$\begin{aligned}\Gamma_2&=|(Tp)(t)-(Tq)(t)|\\
&=|\int_{t_k}^t e^{-\lambda(t-s)}I^\alpha f(s,p(s),{}^cD^\beta p(s))ds+B_k[\tfrac{c}{b}-\tfrac{a}{b}I_{s_{k-1},t_k}^\alpha(g_k(t_k,p(t_k)))\\
&\quad-\int_{t_k}^{s_k}e^{-\lambda(s_k-s)}I^\alpha f(s,p(s),{}^cD^\beta p(s))ds-e^{-\lambda(s_k-t_k)}I_{s_{k-1},t_k}^\alpha(g_k(t_k,p(t_k)))]\\
&\quad+e^{-\lambda(t-t_k)}I_{s_{k-1},t_k}^\alpha(g_k(t_k,p(t_k)))-\int_{t_k}^t e^{-\lambda(t-s)}I^\alpha f(s,q(s),{}^cD^\beta q(s))ds\\
&\quad-B_k[\tfrac{c}{b}-\tfrac{a}{b}I_{s_{k-1},t_k}^\alpha(g_k(t_k,(q(t_k)))-\int_{t_k}^{s_k}e^{-\lambda(s_k-s)}I^\alpha f(s,q(s),{}^cD^\beta q(s))ds\\
&\quad-e^{-\lambda(s_k-t_k)}I_{s_{k-1},t_k}^\alpha(g_k(t_k,q(t_k)))]-e^{-\lambda(t-t_k)}I_{s_{k-1},t_k}^\alpha(g_k(t_k,q(t_k)))|\end{aligned}$$

$$
\begin{aligned}
&\le \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha}|f(s,p(s),{}^{c}D^{\beta}p(s)) - f(s,q(s),{}^{c}D^{\beta}q(s))|ds \\
&\quad + B_k|\tfrac{a}{b}||I^{\alpha}_{s_{k-1},t_k} g_k(t_k,p(t_k)) - I^{\alpha}_{s_{k-1},t_k} g_k(t_k,q(t_k))| \\
&\quad + B_k \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha}|f(s,p(s),{}^{c}D^{\beta}p(s)) - f(s,q(s),{}^{c}D^{\beta}q(s))|ds \\
&\quad + B_k e^{-\lambda(s_k-t_k)}|I^{\alpha}_{s_{k-1},t_k} g_k(t_k,p(t_k)) - I^{\alpha}_{s_{k-1},t_k} g_k(t_k,q(t_k))| \\
&\quad + e^{-\lambda(t-t_k)}|I^{\alpha}_{s_{k-1},t_k} g_k(t_k,p(t_k)) - I^{\alpha}_{s_{k-1},t_k} g_k(t_k,q(t_k))| \\
&\le \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha}(L_f|p(s)-q(s)| + \widetilde{L_f}|{}^{c}D^{\beta}p(s) - {}^{c}D^{\beta}q(s)|)ds + B_k|\tfrac{a}{b}|L_{g_k}|p(t_k)-q(t_k)| \\
&\quad + B_k \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha}(L_f|p(s)-q(s)| + \widetilde{L_f}|{}^{c}D^{\beta}p(s) - {}^{c}D^{\beta}q(s)|)ds \\
&\quad + B_k e^{-\lambda(s_k-t_k)} L_{g_k}|p(t_k)-q(t_k)| + e^{-\lambda(t-t_k)} L_{g_k}|p(t_k)-q(t_k)|
\end{aligned}
$$

$$
\begin{aligned}
&\le L_f C_1\varepsilon \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha}\theta(s)ds + \widetilde{L_f} C_1\varepsilon \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha c}D^{\beta}\theta(s)ds + B_k|\tfrac{a}{b}|L_{g_k}C_2\varepsilon v \\
&\quad + B_k L_f C_1\varepsilon \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha}\theta(s)ds + B_k\widetilde{L_f} C_1\varepsilon \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha c}D^{\beta}\theta(s)ds \\
&\quad + B_k e^{-\lambda(s_k-t_k)} L_{g_k} C_2\varepsilon v + e^{-\lambda(t-t_k)} L_{g_k} C_2\varepsilon v
\end{aligned}
$$

$$
\begin{aligned}
&\le L_f C_1\varepsilon\Big(\int_{t_k}^{t} e^{-\lambda(t-s)}ds\Big)\Big(\int_{t_k}^{t} I^{\alpha}\theta(s)ds\Big) + \widetilde{L_f} C_1\varepsilon\Big(\int_{t_k}^{t} e^{-\lambda(t-s)}ds\Big)\Big(\int_{t_k}^{t} I^{\alpha c}D^{\beta}\theta(s)ds\Big) \\
&\quad + B_k|\tfrac{a}{b}|L_{g_k}C_2\varepsilon v + B_k L_f C_1\varepsilon\Big(\int_{t_k}^{s_k} e^{-\lambda(s_k-s)}\Big)\Big(\int_{t_k}^{s_k} I^{\alpha}\theta(s)ds\Big) \\
&\quad + B_k\widetilde{L_f} C_1\varepsilon\Big(\int_{t_k}^{s_k} e^{-\lambda(s_k-s)}\Big)\Big(\int_{t_k}^{s_k} I^{\alpha c}D^{\beta}_{0,t}\theta(s)ds\Big) + B_k e^{-\lambda(s_k-t_k)} L_{g_k} C_2\varepsilon v \\
&\quad + e^{-\lambda(t-t_k)} L_{g_k} C_2\varepsilon v
\end{aligned}
$$

$$
\begin{aligned}
&\le L_f C_1\varepsilon \tfrac{1-e^{-\lambda(t-t_k)}}{\lambda} c^{\alpha}_{\theta}\theta(t) + \widetilde{L_f} C_1\varepsilon \tfrac{1-e^{-\lambda(t-t_k)}}{\lambda} c^{\alpha-\beta}_{\theta}\theta(t) + B_k|\tfrac{a}{b}|L_{g_k}C_2\varepsilon v \\
&\quad + B_k L_f C_1\varepsilon \tfrac{1-e^{-\lambda(s_k-t_k)}}{\lambda} c^{\alpha}_{\theta}\theta(t) + B_k\widetilde{L_f} C_1\varepsilon \tfrac{1-e^{-\lambda(s_k-t_k)}}{\lambda} c^{\alpha-\beta}_{\theta}\theta(t) + B_k e^{-\lambda(s_k-t_k)} L_{g_k} C_2\varepsilon v \\
&\quad + e^{-\lambda(t-t_k)} L_{g_k} C_2\varepsilon v
\end{aligned}
$$

$$
\begin{aligned}
&\le \{(L_f + \widetilde{L_f})c_{\theta}[\tfrac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)] + [B_k|\tfrac{a}{b}| + B_k e^{-\lambda(s_k-t_k)} + e^{-\lambda(t-t_k)}]L_{g_k}\} \\
&\quad (C_1+C_2)\varepsilon(\theta(t)+v).
\end{aligned}
$$

Case 4 For $t \in (t_k, s_k]$ and $s \in (s_{k-1}, t_k]$, $\quad k = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
\Gamma_3 &= |(Tp)(t) - (Tq)(t)| \\
&= |\int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha} f(s,p(s),{}^{c}D^{\beta}p(s))ds + B_k[\tfrac{c}{b} - \tfrac{a}{b} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,p(t_k))) \\
&\quad - \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha} f(s,p(s),{}^{c}D^{\beta}p(s))ds - e^{-\lambda(s_k-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,p(t_k)))] \\
&\quad + e^{-\lambda(t-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,p(t_k))) - \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha} f(s,q(s),{}^{c}D^{\beta}q(s))ds \\
&\quad - B_k[\tfrac{c}{b} - \tfrac{a}{b} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,q(t_k))) - \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha} f(s,q(s),{}^{c}D^{\beta}q(s))ds \\
&\quad - e^{-\lambda(s_k-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,q(t_k)))] - e^{-\lambda(t-t_k)} I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,q(t_k)))| \\
&\le \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha}|f(s,p(s),{}^{c}D^{\beta}p(s)) - f(s,q(s),{}^{c}D^{\beta}q(s))|ds \\
&\quad + B_k|\tfrac{a}{b}||I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,p(t_k))) - I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,q(t_k)))| \\
&\quad + B_k \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha}|f(s,p(s),{}^{c}D^{\beta}p(s)) - f(s,q(s),{}^{c}D^{\beta}q(s))|ds \\
&\quad + B_k e^{-\lambda(s_k-t_k)}|I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,p(t_k))) - I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,q(t_k)))| \\
&\quad + e^{-\lambda(t-t_k)}|I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,p(t_k))) - I^{\alpha}_{s_{k-1},t_k}(g_k(t_k,q(t_k))) \\
&\le \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha}(L_f|p(s)-q(s)| + \widetilde{L_f}|{}^{c}D^{\beta}p(s) - {}^{c}D^{\beta}q(s)|)ds + B_k|\tfrac{a}{b}|L_{g_k}|p(t_k)-q(t_k)| \\
&\quad + B_k \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha}(L_f|p(s)-q(s)| + \widetilde{L_f}|{}^{c}D^{\beta}p(s) - {}^{c}D^{\beta}q(s)|)ds \\
&\quad + B_k e^{-\lambda(s_k-t_k)} L_{g_k}|p(t_k)-q(t_k)| + e^{-\lambda(t-t_k)} L_{g_k}|p(t_k)-q(t_k)| \\
&\le \int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha}(L_f C_2\varepsilon v + \widetilde{L_f}^{c} D^{\beta} C_2\varepsilon v)ds + B_k|\tfrac{a}{b}|L_{g_k}C_2\varepsilon v
\end{aligned}
$$

$$
\begin{aligned}
&+B_k\int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha}(L_f C_2\varepsilon v+\widetilde{L_f}^{c} D^{\beta} C_2\varepsilon v)ds\\
&+B_k e^{-\lambda(s_k-t_k)} L_{g_k} C_2\varepsilon v+e^{-\lambda(t-t_k)} L_{g_k} C_2\varepsilon v\\
\leq{}& [L_f\int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha} ds+\widetilde{L_f}\int_{t_k}^{t} e^{-\lambda(t-s)} I^{\alpha c} D^{\beta} ds+B_k|\tfrac{a}{b}|L_{g_k}\\
&+B_k L_f\int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha} ds+B_k\widetilde{L_f}\int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^{\alpha c} D^{\beta} ds\\
&+B_k e^{-\lambda(s_k-t_k)} L_{g_k}+e^{-\lambda(t-t_k)} L_{g_k}]C_2\varepsilon v\\
\leq{}& [\tfrac{L_f(t-t_k)^{\alpha}(1-e^{-\lambda(t-t_k)})}{\lambda\Gamma(\alpha+1)}+\tfrac{\widetilde{L_f}(1-e^{-\lambda(t-t_k)})(t-t_k)^{\alpha-\beta}}{\lambda\Gamma(\alpha-\beta+1)}+B_k|\tfrac{a}{b}|L_{g_k}+B_k\tfrac{L_f(t-t_k)^{\alpha}(1-e^{-\lambda(s_k-t_k)})}{\lambda\Gamma(\alpha+1)}\\
&+B_k\tfrac{\widetilde{L_f}(1-e^{-\lambda(s_k-t_k)})(t-t_k)^{\alpha-\beta}}{\lambda\Gamma(\alpha-\beta+1)}+B_k e^{-\lambda(s_k-t_k)} L_{g_k}+e^{-\lambda(t-t_k)} L_{g_k}]C_2\varepsilon v\\
\leq{}& \{(L_f+\widetilde{L_f})[\tfrac{(t-t_k)^{\alpha}}{\Gamma(\alpha+1)}+\tfrac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}]\tfrac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)\\
&+[B_k|\tfrac{a}{b}|+B_k e^{-\lambda(s_k-t_k)]}+e^{-\lambda(t-t_k)}]L_{g_k}\}(C_1+C_2)\varepsilon(\theta(t)+v).
\end{aligned}
$$

From the above four Cases, for any number $p, q \in X$, one obtains

$$|(Tp)(t)-(Tq)(t)| \leq L(C_1+C_2)\varepsilon(\theta(t)+v), t \in J.$$

Thus,

$$\mathsf{d}(Tp, Tq) \leq L(C_1+C_2)\varepsilon(\theta(t)+v), t \in J,$$

which implies that T is strictly contractive on X. Based on Definitions 3 and 4, we know that (X, d) is a complete generalized metric space.

3. Ulam stability analysis

In this section, based on [19] and Definition 2, Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability of (1.1) are given as follows.

Theorem 3. *Letting (H1)–(H4) hold and a solution y satisfy* (1.4)*, for all $t \in J$, there is an unique solution y_0 of* (1.1) *that satisfies* (2.9) *and*

$$|y(t)-y_0(t)| \leq \frac{D_k\varepsilon(\theta(t)+v)}{1-L}. \tag{3.1}$$

Then, the solution of (1.1) *is Ulam-Hyers-Rassias stable with respect to* (θ, v)*, where* (X, d) *is a generalized complete metric space,* $y, y_0 \in X$. $D_k = c_\theta\{\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)+B_k|\frac{a}{b}|+B_k e^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}\}, L = \max\{L_1, L_2\} < 1, L_1 = \max\{(L_f+\widetilde{L_f})c_\theta[\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)]+[B_k|\frac{a}{b}|+B_k e^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]L_{g_k}\}, L_2 = \max\{(L_f+\widetilde{L_f})[\frac{(t-t_k)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}]\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)+[B_k|\frac{a}{b}|+B_k e^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]L_{g_k}\}$.

Proof. From Theorem 2, we know that (X, d) is a complete generalized metric space. Next, based on the third case of Definition 2, we prove that the solution of (1.1) is Ulam-Hyers-Rassias stable with respect to (θ, v). Two steps are given as follows.

Step 1 We verify that $\{p \in X | d(p_0, p) < \infty\} = X$.

From Eqs (3.2) and (3.3), for arbitrary number $p_0 \in X$, we know that there is a constant $M_1 > 0$ that satisfies

$$
\begin{aligned}
|(Tp_0)(t) - p_0(t)| = &| \int_0^t e^{-\lambda(t-s)} I^\alpha f(s, y_p(s), {}^c D^\beta p_0(s)) ds \\
&+ \Lambda[\frac{c}{b} - \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha f(s, p_0(s), {}^c D^\beta p_0(s) ds] - p_0(t)| \\
\leq &M_1 \varepsilon \theta(t) \\
\leq &M_1 \varepsilon(\theta(t) + v), t \in [0, s_0].
\end{aligned}
$$

For $t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m$, it shows that there is an $M_2 > 0$ such that

$$
\begin{aligned}
|(Tp_0)(t) - p_0(t)| = |I^\alpha_{s_{k-1}, t_k}(g_k(t, (p_0(t))) - p_0(t)| \quad &\leq M_2 \varepsilon v \\
&\leq M_2 \varepsilon(\theta(t) + v).
\end{aligned}
$$

Then, for $t \in (t_k, s_k],\ k = 1, 2, \ldots, m$, we can find a number $M_3 > 0$ such that

$$
\begin{aligned}
|(Tp_0)(t) - p_0(t)| = &| \int_{t_k}^t e^{-\lambda(t-s)} I^\alpha f(s, p_0(s), {}^c D^\beta p_0(s) ds + B_k[\frac{c}{b} - \frac{a}{b} I^\alpha_{s_{k-1}, t_k}(g_k(t_k, (p_0(t_k))) \\
&- \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^\alpha f(s, p_0(s), {}^c D^\beta p_0(s)) ds - e^{-\lambda(s_k-t_k)} I^\alpha_{s_{k-1}, t_k}(g_k(t_k, (p_0(t_k)))] \\
&+ e^{-\lambda(t-t_k)} I^\alpha_{s_{k-1}, t_k}(g_k(t_k, (p_0(t_k))) - p_0(t)| \\
\leq &M_3 \varepsilon(\theta(t) + v).
\end{aligned}
$$

In view of number p, g_k and p_0 being bounded on J and $\theta(\cdot) + v > 0$, (3.2) implies that

$$d(Tp_0, p_0) < \infty.$$

By using Lemma 1(i), there is a continuous function $y_0 : J \to \mathbb{R}$ that satisfies $T^n p_0 \to y_0$ in (X, d) as $n \to \infty$ and $Ty_0 = y_0$, for all $t \in J$.

For any $p \in X$, in view of p and p_0 being bounded on J and $\min_{t \in J} \varepsilon(\theta(t) + v) > 0$, we know that there exists a constant $0 < C_p < \infty$ such that

$$|p_0(t) - p_(t)| \leq C_p \varepsilon(\theta(t) + v), \quad t \in J.$$

Therefore, we get $d(Tp_0, p_0) < \infty$ for all $p \in X$, that is,

$$\{p \in X | d(p_0, p) < \infty\} = X.$$

Hence, in view of Lemma 1(ii), we conclude that p_0 is the unique continuous function with (2.9).
Step 2 We verify that $|y(t) - y_0(t)| \leq \frac{cD_k \varepsilon(\theta(t)+v)}{1-L}$.

From Lemma 4 in the Appendix and hypotheses (H1)–(H4), for $t \in [0, s_0]$, we have

$$\begin{aligned}
\Upsilon_1 &= |y(t) - \int_0^t e^{-\lambda(t-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - \Lambda[\frac{c}{b} - \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds]| \\
&\leq \varepsilon \int_0^t e^{-\lambda(t-s)} I^\alpha \theta(s)ds + \Lambda\varepsilon \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha \theta(s)ds \\
&\leq \varepsilon(\int_0^t e^{-\lambda(t-s)}ds)(\int_0^t I^\alpha \theta(s)ds) + \Lambda\varepsilon(\int_0^{s_0} e^{-\lambda(s_0-s)}ds)(\int_0^{s_0} I^\alpha \theta(s)ds) \\
&\leq \varepsilon \frac{1-e^{-\lambda t}}{\lambda} c_\theta^\alpha \theta(t) + \Lambda\varepsilon \frac{1-e^{-\lambda s_0}}{\lambda} c_\theta^\alpha \theta(t) \\
&\leq c \frac{(1+\Lambda)(1-e^{-\lambda s_0})}{\lambda} \varepsilon\theta(t) \\
&\leq c \frac{(1+\Lambda)(1-e^{-\lambda s_0})}{\lambda} \varepsilon(\theta(t) + v).
\end{aligned}$$

For $t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m$, we have

$$|y(t) - I^\alpha_{s_{k-1,t_k}}(g_k(t, y(t)))| \leq \varepsilon v \leq \varepsilon(\theta(t) + v).$$

For $t \in (t_k, s_k], k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\Upsilon_2 &= |y(t) - \int_{t_k}^t e^{-\lambda(t-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - B_k[\frac{c}{b} - \frac{a}{b} I^\alpha_{s_{k-1,t_k}}(g_k(t_k), y(t_k)) \\
&\quad - \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - e^{-\lambda(s_k-t_k)} I^\alpha_{s_{k-1,t_k}}(g_k(t_k), y(t_k))] \\
&\quad - e^{-\lambda(t-t_k)} I^\alpha_{s_{k-1,t_k}}(g_k(t_k), y(t_k))| \\
&\leq \varepsilon \int_{t_k}^t e^{-\lambda(t-s)} I^\alpha \theta(s)ds + B_k\varepsilon \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^\alpha \theta(s)ds + B_k(|\frac{a}{b}| + e^{-\lambda(s_k-t_k)})\varepsilon v + e^{-\lambda(t-t_k)}\varepsilon v \\
&\leq \varepsilon(\int_{t_k}^t e^{-\lambda(t-s)}ds)(\int_{t_k}^t I^\alpha \theta(s)ds) + B_k\varepsilon(\int_{t_k}^{s_k} e^{-\lambda(s_k-s)}ds)(\int_{t_k}^{s_k} I^\alpha \theta(s)ds) \\
&\quad + B_k(|\frac{a}{b}| + e^{-\lambda(s_k-t_k)})\varepsilon v + e^{-\lambda(t-t_k)}\varepsilon v \\
&\leq \varepsilon \frac{1-e^{-\lambda(t-t_k)}}{\lambda} c_\theta^\alpha \theta(t) + B_k\varepsilon \frac{1-e^{-\lambda(s_k-t_k)}}{\lambda} c_\theta^\alpha \theta(t) + B_k(|\frac{a}{b}| + e^{-\lambda(s_k-t_k)})\varepsilon v + e^{-\lambda(t-t_k)}\varepsilon v \\
&\leq c_\theta^\alpha[\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)]\varepsilon\theta(t) + [B_k(|\frac{a}{b}| + e^{-\lambda(s_k-t_k)}) + e^{-\lambda(t-t_k)}]\varepsilon v \\
&\leq c_\theta\{[\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)] + B_k(|\frac{a}{b}| + e^{-\lambda(s_k-t_k)}) + e^{-\lambda(t-t_k)}\}\varepsilon(\theta(t) + v).
\end{aligned}$$

From the above four cases, we get

$$d(y, Ty) \leq D_k,$$

where, $D_k = c_\theta\{\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k) + B_k|\frac{a}{b}| + B_k e^{-\lambda(s_k-t_k)} + e^{-\lambda(t-t_k)}\}$.

Moreover, we have

$$d(y, y_0) \leq \frac{d(y, Ty)}{1 - L} \leq \frac{D_k}{1 - L},$$

where $L = \max\{L_1, L_2\} < 1, L_1 = \max\{(L_f + \widetilde{L_f})c_\theta[\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1 + B_k)] + [B_k|\frac{a}{b}| + B_k e^{-\lambda(s_k-t_k)} + e^{-\lambda(t-t_k)}]L_{g_k}\}, L_2 = \max\{(L_f + \widetilde{L_f})[\frac{(t-t_k)^\alpha}{\Gamma(\alpha+1)} + \frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}]\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1 + B_k) + [B_k|\frac{a}{b}| + B_k e^{-\lambda(s_k-t_k)} + e^{-\lambda(t-t_k)}]L_{g_k}\}$. this implies that

$$|y(t) - y_0(t)| \leq \frac{D_k\varepsilon(\theta(t) + v)}{1 - L},$$

and then (3.1) is true for all $t \in J$. Lemma 1(i) holds. Based on the third case of Definition 2 and Lemma 1, we know that (1.1) is Ulam-Hyers-Rassias stable with respect to (θ, v).

Theorem 4. *Letting (H1)–(H4) hold and a solution y satisfy* (1.3)*, for all $t \in J$, there is a unique solution y_0 of* (1.1) *that satisfies* (2.9) *and*

$$|y(t) - y_0(t)| \leq \frac{cD_k(\theta(t) + v)}{1 - L}. \tag{3.2}$$

Then, the solution of (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to (θ, v), where (X, d) is a generalized complete metric space, $y, y_0 \in X$.*

Proof. By Definition 2(4), choosing $\varepsilon = 1$, similar to the proof of Theorem 3, we know that the solution of (1.1) is generalized Ulam-Hyers-Rassias stable with respect to (θ, v). Here we omit it.

Theorem 5. *Letting (H1)–(H4) hold and a solution $y \in \mathbb{B}$ satisfy* (1.2)*.Then, there is a unique solution $y_0 \in X$ of* (1.1) *that satisfies* (2.9) *and*

$$|y(t) - y_0(t)| \leq c_{m,\alpha,\beta}\varepsilon$$

with $c_{m,\alpha,\beta} = \frac{E_k}{1-L}$. Then, (1.1) is Ulam-Hyers stable, where $E_k = [\frac{(t-t_k)^\alpha}{\Gamma(\alpha+1)} + \frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}][\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1 + B_k)] + [B_k|\frac{a}{b}| + B_k e^{-\lambda(s_k-t_k)} + e^{-\lambda(t-t_k)}]$, $L = max\{(L_f + \widetilde{L_f})[\frac{(t-t_k)^\alpha}{\Gamma(\alpha+1)} + \frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}][\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1 + B_k)] + [B_k|\frac{a}{b}| + B_k e^{-\lambda(s_k-t_k)} + e^{-\lambda(t-t_k)}]L_{g_k}\} < 1$, $k = 1, 2, \cdots, m$. $c_{m,\alpha,\beta}$ is a positive number, and (X, d) is a generalized complete metric space that satisfies $y, y_0 \in X$.

Proof. Let X be the space of piecewise continuous functions, i.e., $X = \{p : J \to \mathbb{R} | p \in \mathbb{B}\}$ with generalized metric on Y, which is defined as

$$d(p, q) = \inf\{C_1 + C_2 \in [0, +\infty] \mid |p(t) - q(t)| \leq \varepsilon(C_1 + C_2)\}, \tag{3.3}$$

for all $t \in J$, where $C_1 \in \{C \in [0, +\infty] \mid |p(t) - q(t)| \leq C\varepsilon\}, t \in (t_k, s_k], k = 0, 1, \ldots, m\}$, and $C_2 \in \{C \in [0, +\infty] \mid |p(t) - q(t)| \leq C\varepsilon\}, t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m\}$.

Next, we prove that the solution of (1.1) is Ulam-Hyers stable. Two steps are given as follows.

Step 1 We verify the condition that $\{p \in X | d(p_0, p) < \infty\} = X$.

From (3.3), for arbitrary $p_0 \in X$, we know that there exists an $M_1 > 0$ such that

$$\begin{aligned}|(Tp_0)(t) - p_0(t)| = |&\int_0^t e^{-\lambda(t-s)} I^\alpha f(s, y_p(s), {}^c D^\beta p_0(s))ds \\ &+ \Lambda[\frac{c}{b} - \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha f(s, p_0(s), {}^c D^\beta p_0(s)ds] - p_0(t)| \\ \le M_1\varepsilon,&\quad t \in [0, s_0].\end{aligned}$$

For $t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m$, we know that there also exists a positive number M_2 such that

$$|(Tp_0)(t) - p_0(t)| \quad = |I^\alpha_{s_{k-1},t_k}(g_k(t, (p_0(t))) - p_0(t)| \le M_2\varepsilon,$$

and then, for $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we can find an $M_3 > 0$ such that

$$\begin{aligned}|(Tp_0)(t) - p_0(t)| = |&\int_{t_k}^t e^{-\lambda(t-s)} I^\alpha f(s, p_0(s), {}^c D^\beta p_0(s)ds + B_k[\frac{c}{b} - \frac{a}{b} I^\alpha_{s_{k-1},t_k}(g_k(t_k, (p_0(t_k))) \\ &- \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^\alpha f(s, p_0(s), {}^c D^\beta p_0(s))ds - e^{-\lambda(s_k-t_k)} I^\alpha_{s_{k-1},t_k}(g_k(t_k, (p_0(t_k)))] \\ &+ e^{-\lambda(t-t_k)} I^\alpha_{s_{k-1},t_k}(g_k(t_k, (p_0(t_k))) - p_0(t)| \\ \le M_3\varepsilon.&\end{aligned}$$

In view of p, g_k and p_0 being bounded on J, (3.2) implies that

$$d(Tp_0, p_0) < \infty.$$

By using Lemma 1(i), there is a continuous function $y_0 : J \to R$ that satisfies $T^n p_0 \to y_0$ in (X, d) as $n \to \infty$ and $Ty_0 = y_0$, for all $t \in J$.

For any $p \in X$, note that p and p_0 being bounded on J and $\min_{t\in J} \varepsilon > 0$, and we know that there is a constant $0 < C_p < \infty$ such that

$$|p_0(t) - p_(t)| \le C_p\varepsilon, \quad t \in J.$$

Therefore, we get $d(Tp_0, p_0) < \infty$ for all $p \in X$, that is,

$$\{p \in X | d(p_0, p) < \infty\} = X.$$

Thus, in light of Lemma 1(ii), we conclude that p_0 is a unique continuous function with (2.9).

Step 2 We verify the condition that $|y(t) - y_0(t)| \le \frac{E_k\varepsilon}{1-L}$.

From Lemma 3 in the Appendix and hypotheses (H1)–(H4), for $t \in [0, s_0]$, one gets

$$\begin{aligned}\Lambda_1 &= |y(t) - \int_0^t e^{-\lambda(t-s)} I^\alpha f(s, y(s), {}^c D^\beta y(s))ds - \Lambda[\frac{c}{b} - \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha f(s, y(s), {}^c D^\beta y(s))ds]| \\ &\le \varepsilon \int_0^t e^{-\lambda(t-s)} I^\alpha ds + \Lambda\varepsilon \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha ds \\ &\le \varepsilon(\int_0^t e^{-\lambda(t-s)} ds)(\int_0^t I^\alpha ds) + \Lambda\varepsilon(\int_0^{s_0} e^{-\lambda(s_0-s)} ds)(\int_0^{s_0} I^\alpha ds)\end{aligned}$$

$$\leq \varepsilon\frac{t^{\alpha}(1-e^{-\lambda t})}{\lambda\Gamma(\alpha+1)}+\Lambda\varepsilon\frac{t^{\alpha}(1-e^{-\lambda s_0})}{\lambda\Gamma(\alpha+1)}$$
$$\leq \frac{(1+\Lambda)t^{\alpha}(1-e^{-\lambda s_0})}{\lambda\Gamma(\alpha+1)}\varepsilon.$$

For $t\in(s_{k-1},t_k], k=1,2,\dots,m$, one has

$$|y(t)-I^{\alpha}_{s_{k-1},t_k}(g_k(t,y(t)))|\leq\varepsilon.$$

For $t\in(t_k,s_k], k=1,2,\dots,m$, one gets

$$\begin{aligned}\Lambda_2 =& |y(t)-\int_{t_k}^{t}e^{-\lambda(t-s)}I^{\alpha}f(s,y(s),{}^{c}D^{\beta}y(s))ds-B_k[\frac{c}{b}-\frac{a}{b}I^{\alpha}_{s_{k-1},t_k}(g_k(t_k),y(t_k))-\\
&\int_{t_k}^{s_k}e^{-\lambda(s_k-s)}I^{\alpha}f(s,y(s),{}^{c}D^{\beta}y(s))ds-e^{-\lambda(s_k-t_k)}I^{\alpha}_{s_{k-1},t_k}(g_k(t_k),y(t_k))]\\
&-e^{-\lambda(t-t_k)}I^{\alpha}_{s_{k-1},t_k}(g_k(t_k),y(t_k))|\\
\leq& \frac{\varepsilon(t-t_k)^{\alpha}}{\lambda\Gamma(\alpha+1)}[(1-e^{-\lambda(t-t_k)})+B_k(1-e^{-\lambda(s_k-t_k)})]+B_k(|\frac{a}{b}|+e^{-\lambda(s_k-t_k)})\varepsilon+e^{-\lambda(t-t_k)}\varepsilon\\
\leq& \varepsilon\frac{(t-t_k)^{\alpha}(1-e^{-\lambda(t-t_k)})}{\lambda\Gamma(\alpha+1)}+B_k\varepsilon\frac{(t-t_k)^{\alpha}(1-e^{-\lambda(s_k-t_k)})}{\lambda\Gamma(\alpha+1)}+B_k(|\frac{a}{b}|+e^{-\lambda(s_k-t_k)})\varepsilon+e^{-\lambda(t-t_k)}\varepsilon\\
\leq& [\frac{(t-t_k)^{\alpha}(1-e^{-\lambda(s_k-t_k)})}{\lambda\Gamma(\alpha+1)}(1+B_k)]\varepsilon+[B_k(|\frac{a}{b}|+e^{-\lambda(s_k-t_k)})+e^{-\lambda(t-t_k)}]\varepsilon\\
\leq& \{[\frac{(t-t_k)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}][\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)]+[B_k|\frac{a}{b}|+B_ke^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]\}\varepsilon.\end{aligned}$$

Similar to the proof of Theorem 3, we get $d(y,Ty)\leq E_k$. That is,

$$d(y,y_0)\leq\frac{d(y,Ty)}{1-L}\leq\frac{E_k}{1-L},$$

where $E_k=[\frac{(t-t_k)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}][\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)]+[B_k|\frac{a}{b}|+B_ke^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]$, $L=\max\{(L_f+\widetilde{L_f})[\frac{(t-t_k)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}][\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)]+[B_k|\frac{a}{b}|+B_ke^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]L_{g_k}\}<1,\ k=1,2,\cdots,m$, which implies that

$$|y(t)-y_0(t)|\leq\frac{E_k\varepsilon}{1-L}.$$

Hence, based on the first case of Definition 2, the solution of (1.1) is Ulam-Hyers stable.

Theorem 6. *Let $\phi_{m,\alpha,\beta}(\varepsilon)$ be a positive number, and (X,d) is a generalized complete metric space that satisfies $y,y_0\in X$. If (H1)–(H4) hold, and a solution $y\in\mathbb{B}$ satisfies* (1.3)*, then there is a unique solution $y_0\in X$ of* (1.1) *that satisfies* (2.9) *and*

$$|y(t)-y_0(t)|\leq\phi_{m,\alpha,\beta}(\varepsilon).$$

Proof. When $c_{m,\alpha,\beta}\varepsilon=\phi_{m,\alpha,\beta}(\varepsilon)$ with $\phi_{m,\alpha,\beta}(0)=0$, we know that the solution of (1.1) is generalized Ulam-Hyers stable. Here we omit it.

4. Example

In order to better check the correctness of this results, we give the following example to verify the above theorem. Choose $J=[0,6], \alpha=\frac{1}{2}, \beta=\frac{1}{3}, \lambda=6$ and $0=t_0<s_0=2<t_1=4<s_1=6$. Then, the following equation is given.

$$\begin{cases} {}^cD_{0,t}^{\frac{1}{2}}(D+6)x(t)=\frac{{}^cD_{0,t}^{\frac{1}{3}}x(t)+|x(t)|}{(t+\sqrt{10})^2(1+x(t))}, & t\in(0,2]\bigcup(4,6], \\ x(t)=I_{2,4}^{\frac{1}{2}}(g_1(t,x(t)))=\frac{1}{\Gamma(\frac{1}{2})}\int_2^4(t-s)^{-\frac{1}{2}}sin|x(s)|ds, & t\in(2,4], \\ 2x(t_k)+7x(s_k)=10, x(0)=0. \end{cases} \tag{4.1}$$

Represent $f(t,x(t),{}^cD_{0,t}^{\beta}x(t))=\frac{{}^cD^{\frac{1}{3}}x(t)+|x(t)|}{(t+\sqrt{10})^2(1+x(t))}$ with $L_f=\widetilde{L_f}=\frac{1}{10}$ for $t\in(0,2]\bigcup(4,6]$ and $I_{2,4}^{\frac{1}{2}}(g_1(t,x(t)))$ with $L_{g_1}=\frac{1}{2}$ for $t\in(2,4]$.

By Theorem 2, we easily know that there exists an unique solution $y_0:[0,6]\to\mathbb{R}$ such that

$$y_0(t)=\begin{cases} \int_0^t e^{-6(t-s)}I^{\frac{1}{2}}\frac{{}^cD^{\frac{1}{3}}y_0(s)+|y_0(t)|}{(s+\sqrt{10})^2(1+y_0(s))}ds+A[\frac{10}{7}-\int_0^2 e^{-6(2-s)}I^{\frac{1}{2}}\frac{{}^cD^{\frac{1}{3}}y_0(s)+|y_0(t)|}{(s+\sqrt{10})^2(1+y_0(s))}ds], & t\in[0,2], \\ I_{2,4}^{\frac{1}{2}}(g_1(t,x(t)))=\frac{1}{\Gamma(\frac{1}{2})}\int_2^4(t-s)^{-\frac{1}{2}}sin|x(s)|ds, & t\in(2,4], \\ \int_{t_1}^t e^{-6(t-s)}I^{\frac{1}{2}}\frac{{}^cD^{\frac{1}{3}}y_0(s)+|y_0(t)|}{(s+\sqrt{10})^2(1+y_0(s))}ds+B_k[\frac{10}{7}-\frac{2}{7}I_{s_0,t_1}^{\frac{1}{2}}(g_1(t_1,(y_0(t_1))) & \\ \quad -\int_{t_k}^{s_k}e^{-6(s_k-s)}I^{\frac{1}{2}}\frac{{}^cD^{\frac{1}{3}}y_0(s)+|y_0(t)|}{(s+\sqrt{10})^2(1+y_0(s))}ds-e^{-6(s_k-t_k)}I_{s_{s_0,t_1}}^{\frac{1}{2}}(g_1(t_1,(y_0(t_1)))] & \\ \quad +e^{-6(t-t_k)}I_{s_0,t_1}^{\frac{1}{2}}(g_1(t_1,(y_0(t_1))), & t\in[4,6], \end{cases} \tag{4.2}$$

where

$$\Lambda=\frac{1-e^{-6t}}{1-e^{-6\times2}}, B_k=\frac{1-e^{-6(t-t_k)}}{1-e^{-6(s_k-t_k)}}=\frac{1-e^{-6(t-t_1)}}{1-e^{-6(s_1-t_1)}}=\frac{1-e^{-6(t-4)}}{1-e^{-6(6-4)}}.$$

Next, we check the conditions of Theorems 3–6.

1) We first check the conditions of Theorem 3:

$$\begin{cases} |{}^cD_{0,t}^{\frac{1}{2}}(D+6)y(t)-\frac{{}^cD_{0,t}^{\frac{1}{3}}y(t)+|y(t)|}{(t+\sqrt{10})^2(1+y(t))}|\le\varepsilon\theta(t), & t\in(0,2]\bigcup(4,6], \\ y(t)-\frac{1}{\Gamma(\frac{1}{2})}\int_2^4(t-s)^{-\frac{1}{2}}sin|y(s)|ds|\le\varepsilon v, & t\in(2,4]. \end{cases} \tag{4.3}$$

Choosing again $\varepsilon=\frac{1}{4}, \theta(t)=e^t, v=\frac{1}{5}$ and $c_\theta^\alpha=c_\theta^{\alpha-\beta}=6$, we have

$$\int_0^t I^{\frac{1}{2}}e^s ds\le 6e^t, \int_0^t I^{\alpha-\beta}e^s ds\le 6e^t,$$

$$L=\max\{L_1,L_2\},$$

$$
\begin{aligned}
L_1 &= \max\{(L_f+\widetilde{L_f})c[\tfrac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)]+[B_k|\tfrac{a}{b}|+B_ke^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]L_{g_k}\mid k=1\}\\
&= \max\{(\tfrac{1}{10}+\tfrac{1}{10})\times 10\times[\tfrac{1-e^{-6\times(6-4)}}{6}(1+\tfrac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}})]+[\tfrac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}}|\tfrac{2}{7}|\\
&\quad+\tfrac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}}e^{-6\times(6-4)}+e^{-6(t-4)}]\tfrac{1}{2}\}\\
&\le 0.5429, t\in(4,6], L_2=\max\{(L_f+\widetilde{L_f})[\tfrac{(t-t_k)^{\alpha}}{\Gamma(\alpha+1)}+\tfrac{(t-t_k)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}]\tfrac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)\\
&\quad+[B_k|\tfrac{a}{b}|+B_ke^{-6(s_k-t_k)}+e^{-\lambda(t-t_k)}]L_{g_k}\mid k=1\}\\
&= \max\{(\tfrac{1}{10}+\tfrac{1}{10})[\tfrac{(t-4)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}+\tfrac{(t-4)^{\frac{1}{2}-\frac{1}{3}}}{\Gamma(\frac{1}{2}-\frac{1}{3}+1)}]\tfrac{1-e^{-6\times(6-4)}}{6}(1+\tfrac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}})\\
&\quad+[\tfrac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}}|\tfrac{2}{7}|+\tfrac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}}e^{-6\times(6-4)}+e^{-6(t-4)}]\times\tfrac{1}{2}\}\\
&\le 0.3226 \text{ for},\quad t\in(4,6].
\end{aligned}
$$

Thus, $L=0.5429$, and then, one has

$$
\begin{aligned}
|y(t)-y_0(t)| &\le \frac{c[\frac{1-e^{-\lambda(s_k-t_k)}}{\lambda}(1+B_k)+B_k|\frac{a}{b}|+B_ke^{-\lambda(s_k-t_k)}+e^{-\lambda(t-t_k)}]\varepsilon(\theta(t)+v)}{1-L}\\
&= \frac{10\times[\frac{1-e^{-6\times(6-4)}}{6}(1+\frac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}})+\frac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}}|\frac{2}{7}|+k_2]\times\frac{1}{4}(e^t+\frac{1}{5})}{1-0.5429}\\
&\le 8.1259\times\frac{1}{4}(e^t+\frac{1}{5}),\quad t\in(4,6],
\end{aligned}
$$

where $k_2 = 1-e^{-6(t-4)}1-e^{-6\times(6-4)}e^{-6\times(6-4)}+e^{-6(t-4)}$. Thus, the solution of (4.1) is Ulam-Hyers-Rassias stable.

2) We verify Theorem 4, and choose $\varepsilon = 1$. Other parameters are the same as (4.1), and then (4.1) is generalized Ulam-Hyers-Rassias stable.

3) We verify the conditions of Theorem 5. Consider (4.1) and

$$
\begin{cases}
|{}^cD_{0,t}^{\frac{1}{2}}(D+6)y(t)-\frac{{}^cD_{0,t}^{\frac{1}{3}}y(t)+|y(t)|}{(t+\sqrt{10})^2(1+y(t))}|\le\varepsilon, & t\in(0,2]\bigcup(4,6],\\
y(t)-\frac{1}{\Gamma(\frac{1}{2})}\int_2^4(t-s)^{-\frac{1}{2}}sin|y(s)|ds|\le\varepsilon, & t\in(2,4].
\end{cases}
\tag{4.4}
$$

As

$$
\begin{aligned}
L = max\{&(L_f+\widetilde{L_f})[\tfrac{(t-t_1)^{\alpha}}{\Gamma(\alpha+1)}+\tfrac{(t-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}][\tfrac{1-e^{-\lambda(s_1-t_1)}}{\lambda}(1+B_1)]+[B_1|\tfrac{a}{b}|+B_1e^{-\lambda(s_1-t_1)}\\
&+e^{-\lambda(t-t_1)}]L_{g_1}\}\le 0.5429<1,
\end{aligned}
$$

via calculations, we know that there exists a unique solution $y_0 : [0,6]\to\mathbb{R}$ that satisfies (2.9) and

$$|y(t)-y_0(t)|\le c_{m,\alpha,\beta}\varepsilon$$

with

$$
\begin{aligned}
c_{m,\alpha,\beta} &= \frac{[\frac{(t-t_1)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(t-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}][\frac{1-e^{-\lambda(s_1-t_1)}}{\lambda}(1+B_1)]+[B_1|\frac{a}{b}|+B_1e^{-\lambda(s_1-t_1)}+e^{-\lambda(t-t_1)}]}{1-L}\\
&= \frac{[\frac{(t-4)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}+\frac{(t-4)^{\frac{1}{2}-\frac{1}{3}}}{\Gamma(\frac{1}{2}-\frac{1}{3}+1)}][\frac{1-e^{-6\times(6-4)}}{\lambda}(1+\frac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}})]+[\frac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}}|\frac{2}{7}|+k_1]}{1-0.5429}\\
&\le 4.7788>0,
\end{aligned}
$$

where $k_1 = \frac{1-e^{-6(t-4)}}{1-e^{-6\times(6-4)}} e^{-6\times(6-4)} + e^{-6(t-4)}$. That is, $|y(t) - y_0(t)| \le 4.7788\varepsilon$. Thus, the solution of (4.1) is Ulam-Hyers stable.

4) We verify the conditions of Theorem 6. By choosing $\phi_{m,\alpha,\beta}(\varepsilon) = 4.7788\varepsilon$ with $\phi_{m,\alpha,\beta}(0) = 0$, the solution of (4.1) is generalized Ulam-Hyers stable.

5. Conclusions

In this manuscript, the existence and Ulam stability for a fractional differential equation is considered with multi-point boundary conditions and non-instantaneous integral impulse. First, some sufficient conditions for the existence, uniquenes, and at least one solution of the aforementioned equation are discussed by using the generalized Diaz-Margolis fixed point theorem. Then, we obtain the Ulam stability of the equation. Lastly, we give one example to support our main results. In addition, in this paper, we only consider the stability analysis of multi-point boundary conditions for a fractional differential equation. However, the reaction-diffusion multi-point boundary conditions for fractional differential equation, the dynamical behaviors of system (1.1) and the situation of the method of proving global stability are not yet fully clear, which would be our further topic.

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Conflict of interest

The authors have no conflict of interest to declare in carrying out this research work.

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Appendix

First, a function $y \in \mathbb{B}$ is the solution of (1.2), if and only if there exists a function $\tau \in \mathbb{B}$ and y dependent $\tau_k, k = 1, 2, \cdots, m$, such that

◇ $|\tau(t)| \leq \varepsilon, \quad t \in J.$

◇ $|\tau_k| \leq \varepsilon, \quad t = 1, 2, \cdots, m.$

◇ ${}^cD^\alpha(D + \lambda)y(t) = f(t, y(t), {}^cD^\beta y(t)) + \tau(t), \quad t \in (t_k, s_k] \subset J, k = 0, 1, \ldots, m.$

◇ $y(t) = I^\alpha_{s_{k-1},t_k} g_k(t, y(t)) + \tau_k, t \in (s_{k-1}, t_k] \subset J, k = 1, 2, \ldots, m.$

Thus, we have the following Lemma 3:

Lemma 3. *Let $y \in \mathbb{B}$ be a solution of inequality (Eq 1.2), and then y is a solution of the following*

$$\begin{cases} |y(t) - \int_0^t e^{-\lambda(t-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - \Lambda[\frac{c}{b} \\ - \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds]| \leq \frac{\varepsilon t^\alpha}{\lambda\Gamma(\alpha+1)}[(1-e^{-\lambda t}) + \Lambda(1-e^{-\lambda s_0})], & t \in [t_0, s_0], \\ |y(t) - I^\alpha_{s_{k-1},t_k}(g_k(t, y(t)))| \leq \varepsilon, k = 1, 2, \cdots, m, & t \in (s_{k-1}, t_k], \\ |y(t) - \int_{t_k}^t e^{-\lambda(t-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - B_k[\frac{c}{b} - \frac{a}{b} I^\alpha_{s_{k-1},t_k}(g_k(t_k), y(t_k)) - \\ \quad \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - e^{-\lambda(s_k-t_k)} I^\alpha_{s_{k-1},t_k}(g_k(t_k), y(t_k))] \\ \quad -e^{-\lambda(t-t_k)} I^\alpha_{s_{k-1},t_k}(g_k(t_k), y(t_k))| \\ \leq \frac{\varepsilon(t-t_k)^\alpha}{\lambda\Gamma(\alpha+1)}[(1-e^{-\lambda(t-t_k)}) + B_k(1-e^{-\lambda(s_k-t_k)})] + B_k(|\frac{a}{b}| \\ +e^{-\lambda(s_k-t_k)})\varepsilon + e^{-\lambda(t-t_k)}\varepsilon, k = 1, 2, \cdots, m, & t \in (t_k, s_k]. \end{cases}$$

In addition, a function $y \in \mathbb{B}$ is the solution of (1.4), if and only if there exists a function $\tau \in \mathbb{B}$ and y dependent sequenze $\tau_k, k = 1, 2, \cdots, m$, such that

- ⋄ $|\tau(t)| \leq \varepsilon\theta(t), \quad t \in J.$
- ⋄ $|\tau_k| \leq \varepsilon\nu, \quad t = 1, 2, \cdots, m.$
- ⋄ ${}^cD^\alpha(D + \lambda)y(t) = f(t, y(t), {}^cD^\beta y(t)) + \tau(t), \quad t \in (t_k, s_k] \subset J, \quad k = 0, 1, \ldots, m.$
- ⋄ $y(t) = I^\alpha_{s_{k-1},t_k} g_k(t, y(t)) + \tau_k, \quad t \in (s_{k-1}, t_k] \subset J, \quad k = 1, 2, \ldots, m.$

Thus, we have the following Lemma 4:

Lemma 4. *Let $y \in \mathbb{B}$ be a solution of inequality (Eq 1.4), and then y is a solution of the following.*

$$\begin{cases} |y(t) - \int_0^t e^{-\lambda(t-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - \Lambda[\frac{c}{b} - \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds]| \\ \leq \varepsilon \int_0^t e^{-\lambda(t-s)} I^\alpha \theta(s)ds + \Lambda\varepsilon \int_0^{s_0} e^{-\lambda(s_0-s)} I^\alpha \theta(s)ds, & t \in [t_0, s_0], \\ |y(t) - I^\alpha_{s_{k-1},t_k}(g_k(t, y(t)))| \leq \varepsilon\nu, k = 1, 2, \cdots, m, & t \in (s_{k-1}, t_k], \\ |y(t) - \int_{t_k}^t e^{-\lambda(t-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - B_k[\frac{c}{b} - \frac{a}{b} I^\alpha_{s_{k-1},t_k}(g_k(t_k), y(t_k)) - \\ \quad \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^\alpha f(s, y(s), {}^cD^\beta y(s))ds - e^{-\lambda(s_k-t_k)} I^\alpha_{s_{k-1},t_k}(g_k(t_k), y(t_k))] \\ \quad -e^{-\lambda(t-t_k)} I^\alpha_{s_{k-1},t_k}(g_k(t_k), y(t_k))| \\ \leq \varepsilon \int_{t_k}^t e^{-\lambda(t-s)} I^\alpha \theta(s)ds + B_k\varepsilon \int_{t_k}^{s_k} e^{-\lambda(s_k-s)} I^\alpha \theta(s)ds + B_k(|\frac{a}{b}| \\ +e^{-\lambda(s_k-t_k)})\varepsilon\nu + e^{-\lambda(t-t_k)}\varepsilon\nu, k = 1, 2, \cdots, m, & t \in (t_k, s_k]. \end{cases}$$