Research article

Dynamics of a delayed reaction-diffusion predator-prey model with the effect of the toxins

Meiling Zhu and Huijun Xu[*]

Yangzhou Polytechnic Institute, Yangzhou 225127, China

* Correspondence: Email: xuhj@ypi.edu.cn.

Abstract: In this study, we investigate a delayed reaction-diffusion predator-prey system with the effect of toxins. We first investigate whether the internal equilibrium exists. We then provide certain requirements for the presence of Turing and Hopf bifurcations by examining the corresponding characteristic equation. We also study Turing-Hopf and Hopf bifurcations brought on by delays. Finally, numerical simulations that exemplify our theoretical findings are provided. The quantitatively obtained properties are in good agreement with the findings that the theory had predicted. The effects of toxins on the system are substantial, according to theoretical and numerical calculations.

Keywords: toxins; delay; diffusion; bifurcation; spatial pattern

1. Introduction

The release of unregulated toxins into the environment has been linked to the extinction of countless species in recent years, such as car exhaust pollution, industrial wastewater pollution, toxic metal pollution, water pollution, military and ship noise pollution in the ocean, etc. The impact of these pollutions on species is like "chronic poison", its degree of harm is very horrible. For example, the albino dolphin, which has survived on earth for more than 20 million years and is only found in China's Yangtze River, was officially declared extinct on August 8, 2007. Studies have shown that global warming and environmental pollution are the main causes of the extinction of the golden toad [1]. Studying how toxins affect ecosystems is essential from a conservation and environmental standpoint.

In order to investigate the effect of toxins on the population, a large number of researchers have developed mathematical models that are based on population models. First, Hallam et al. [2] put forth a toxicant-population model, which places an emphasis on the influence that toxins have on a population. They proposed that, despite the existence of limiting toxins, a colony's susceptibility to extinction may be influenced by the abundance of terminal organisms in the environment. Das et al. [3] explored the bioeconomics of harvesting a prey-predator fishery in which both species

are affected by certain toxins emitted by some other species. By considering a predator-prey model with simultaneous harvesting, Rani and Gakkhar [4] found that the presence of toxins alters the quality of food for both species, resulting in slower growth. Chakraborty and Das [5] investigated a two-zooplankton and one phytoplankton system in the presence of toxicity. Wu and Wei [6] and Wei and Chen [7] studied population models with psychological effects and partial tolerance in polluted environments. Wei et al. [8] proposed a single-species population model with partial pollution tolerance in a polluted environment and introduced the random perturbation of the birth rate of a single-species population into this model.

It is noted that most of the models mentioned above didn't consider the factor of delay. It is well known that delay differential equations have more complicated dynamics than traditional differential equations because of the instability of equilibrium, bifurcation, and chaotic behavior that results from delays [9–15]. So, when researching an ecosystem, we should consider the factor of delay. Pal et al. [16] investigated two competing fish species, one of which obeys the law of logistic growth and produces a chemical that is harmful to the other. The delay in reproduction caused by the gestation period was taken into account.

Furthermore, it is common knowledge that in order to improve its chances of survival, the species will always move to less populated places [17]. But few researchers have looked at the idea of population diffusion in hazardous situations [18]. Yan et al. [19] analyzed a diffusive predator-prey model with toxins. The authors of [20] also thought about the effect of diffusion. However, they did not investigate how the presence of poisonous compounds affects the spatiotemporal dynamics of the system [21, 22]. In addition, some researchers have studied the reaction-diffusion systems with delays. Zuo and Wei [23] investigated a delayed predator-prey system with a diffusion effect, and the stability of the positive equilibrium and the existence of spatially homogeneous and spatially inhomogeneous periodic solutions are investigated by analyzing the distribution of the eigenvalues. Chen et al. [24] proposed a new technique to study the stability and associated Hopf bifurcation of a constant equilibrium of a general reaction-diffusion system or a system of ordinary differential equations with delay effects. Chen et al. [25] considered a delayed diffusive Leslie-Gower predator-prey system and the stability/instability of the coexistence equilibrium and associated Hopf bifurcation are investigated by analyzing the characteristic equations.

Motivated by the preceding debates, we consider harmful compounds that will impact the quality of the food consumed by these two populations. The system then appears as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = ru\left(1 - \dfrac{u}{K}\right) - \dfrac{b_1 uv}{u + mv} - \alpha u^3 + d_1 \Delta u, & (x,t) \in (0, l\pi) \times (0, +\infty), \\ \dfrac{\partial v}{\partial t} = \dfrac{b_2 uv}{u + mv} - dv - \beta v^2(t - \tau) + d_2 \Delta v, & (x,t) \in (0, l\pi) \times (0, +\infty), \\ u_x(0,t) = v_x(0,t) = 0, u_x(l\pi, t) = v_x(l\pi, t) = 0, \end{cases} \quad (1.1)$$

where u and v represent the prey and predator population, respectively. The parameters r, K, b_1, α, b_2, d, β, d_1 and d_2 are all positive. r is the growth rate of prey, K is the environmental capacity, m is the consumption rate, α and β are the efficiency of toxicity. b_1 is the maximum value which per capita reduction rate of u can attain, b_2 is the conversion rate, and d is the death rate of the predator. The delay τ represents that the toxicants there is a delay in the damage of the toxin to the predator. $\Delta = \partial^2 / \partial x^2$ is the Laplace operator which indicates that the population is moving freely in space, and $u_x(0,t) = v_x(0,t) = 0$ represents the homogeneous Neumann border conditions which means that the

population is confined within the domain.

Our working hypothesis is that the rate at which hazardous chemicals are synthesized increases in tandem with the density of the prey. Prey species are more likely to die out than predator ones. Accordingly, we assume that αu^3 is the effect of toxins on the prey and βv^2 is the effect of toxicity to the predator with $0 < \alpha < \beta < 1$. Here, prey species are the direct targets of toxicants, but predators are also indirectly impacted because of the poisoned prey. It is clear that no population is entering or leaving the system thanks to the Neumann boundary constraints requiring all points to be homogeneous.

By letting

$$\bar{u} = \frac{u}{K}, \quad \bar{v} = \frac{b_1}{rK}v, \quad \bar{t} = rt, \quad \bar{\alpha} = \frac{\alpha k^2}{r}, \quad \bar{m} = \frac{mr}{b_1},$$
$$\bar{d}_1 = \frac{d_1}{r}, \quad \bar{d}_2 = \frac{d_2}{r}, \quad \bar{d} = \frac{db_1}{r\beta K}, \quad \bar{b} = \frac{b_2 b_1}{r\beta K}, \quad \bar{\beta} = \frac{\beta K}{b_1}.$$

(1.2)

By getting rid of the bars to make things easier, system (1.1) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + u(1-u) - \frac{uv}{u+mv} - \alpha u^3, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \beta \left(\frac{buv}{u+mv} - dv - v^2(t-\tau) \right), \\ u_x(0,t) = v_x(0,t) = 0, u_x(l\pi, t) = v_x(l\pi, t) = 0. \end{cases}$$

(1.3)

In this paper, we shall attempt to answer the following questions: In what ways do toxins's delay and diffusion alter the dynamics of a system? Can they lead to Hopf, Turing, or even Turing-Hopf bifurcations?

The rest of the paper is organized as follows. In section 2, we discuss the existence of the interior equilibrium, and the stability, Hopf bifurcations and Turing instability of system (1.3). In section 3, theoretical analysis is verified by numerical simulations.

2. Stability and bifurcation of the interior equilibrium

2.1. The existence of the equilibria

Obviously, model (1.3) has a predator-free equilibrium $E_1 = (\frac{-1+\sqrt{1+4\alpha}}{2\alpha}, 0)$ and the interior equilibrium satisfies the following two non-trivial prey and predator nullclines simultaneously in the interior of the first quadrant

$$\Phi(u,v) = 1 - u - \frac{v}{u+mv} - \alpha u^2 = 0,$$

(2.1)

$$\Psi(u,v) = \frac{bu}{u+mv} - d - v = 0.$$

(2.2)

From (2.2), we obtain that $u = \frac{(d+v)mv}{b-(d+v)}$, and substitute this into (2.1), we have

$$A_0 v^4 + 4A_1 v^3 + 6A_2 v^2 + 4A_3 v + A_4 = 0,$$

(2.3)

where

$$A_0 = \alpha b m^3,$$

$$A_1 = \frac{1}{4}(-1 - bm^2 + 2\alpha bdm^3),$$

$$A_2 = \frac{1}{6}(b^2m^2 - bm + 3b - 2bdm^2 + \alpha bd^2m^3 - 3d), \qquad (2.4)$$

$$A_3 = \frac{1}{4}(b^2dm^2 + 2b^2m - 3b^2 - bd^2m^2 - 2bdm + 6bd - 3d^2),$$

$$A_4 = 3bd^2 - 3b^2d - b^3m + b^3 - d^3 - bd^2m + 2b^2dm.$$

Clearly, if

$$3bd^2 - 3b^2d - b^3m + b^3 - d^3 - bd^2m + 2b^2dm < 0, \qquad (2.5)$$

then Eq.(2.3) has at least a positive root v^*. Moreover, if $b - (d + \beta v^*) > 0$, then system (1.3) has an interior $E^* = (u^*, v^*)$. According to [26], we define

$$S = A_0A_4 - 4A_1A_3 + 3A_2^2,$$

$$I = Det \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_2 & A_3 \\ A_2 & A_3 & A_4 \end{pmatrix},$$

$$D = S^3 - 27I^2.$$

Then D is the discriminant of Eq (2.3). When $D < 0$, the equation has two real roots (one of which is positive, and the other is negative), as well as two imaginary roots. As a consequence of this, the equation has one and only one positive real root in this particular example. On the other hand, when $D > 0$, every root can be either real or imaginary, and because this equation includes at least two real roots, we can deduce that every root is real in this particular instance. Therefore, there is a possibility that multiple positive roots will occur.

On the other hand, we can demonstrate the existence of unique or multiple coexistence steady states geometrically by analyzing non-trivial nullclines (2.1) and (2.2) (see Figure 1).
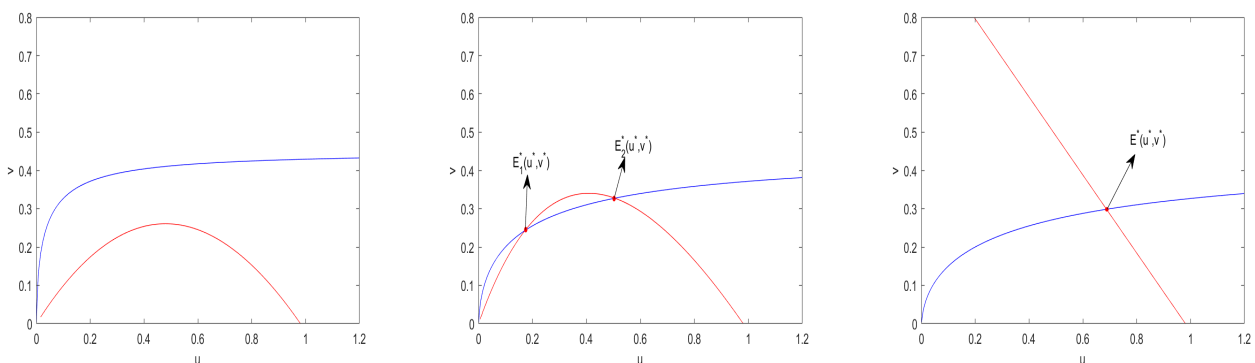

Figure 1. The number of coexistence equilibrium points can vary based on the possible crossing of non-trivial prey and predator nullclines. Both the blue and red curves represent the non-trivial nullclines for prey and predators, respectively. The blue curve represents the prey nullcline, while the red curve represents the predator nullcline.

In the following, we discuss the stability of the equilibria. First, we define the real-valued Sobolev space

$$X = \left\{ u, v \in H^2(0, l\pi) \times H^2(0, l\pi) : (u_x, v_x)\big|_{x=0, l\pi} = 0 \right\},$$

and the corresponding complexification space is given by

$$X_{\mathbb{C}} = X \oplus iX = \{a_1 + ib_1 : a_1, b_1 \in X\}$$

Let

$$U(t) = (u(\cdot, t), v(\cdot, t))^T.$$

Thus, we linearize system (1.3) around $E^* = (u^*, v^*)$ in the phase space $C_\tau = C([-\tau, 0], X_{\mathbb{C}})$, we have

$$\dot{U} = D\Delta U(t) + L(\beta, \tau)(U_t), \tag{2.6}$$

where $D = \text{diag}(d_1, d_2)$, $U_t = \mathbf{col}(u(x, t), v(x, t)) \in C_\tau$ and $L : C_\tau \to X_{\mathbb{C}}$ is given by

$$L(\beta, \tau)(\varphi) = \begin{pmatrix} a_{11} & -a_{12} \\ \beta a_{21} & \beta a_{22} \end{pmatrix} \varphi(0) + \begin{pmatrix} 0 & 0 \\ 0 & -\beta s \end{pmatrix} \varphi(-\tau) \tag{2.7}$$

with $\varphi = (\varphi_1, \varphi_2)^T$ and

$$a_{11} = \frac{u^* v^*}{(u^* + mv^*)^2} - u^* - 2\alpha u^{*2}, a_{12} = \frac{u^{*2}}{(u^* + mv^*)^2},$$

$$a_{21} = \frac{bmv^{*2}}{(u^* + mv^*)^2}, a_{22} = \frac{bu^{*2}}{(u^* + mv^*)^2} - d, s = 2v^*.$$

From [17], we obtain the characteristic equation of system (2.6) is

$$\lambda y - D\Delta y - L(e^{\lambda \cdot} y) = 0, \tag{2.8}$$

where $y \in dom(\Delta)$, and $y \neq 0$, $dom(\Delta) \subset X$, and $e^{\lambda \cdot}$ stands for $(x, t) \longmapsto e^{\lambda t}$.

From the properties of the Laplacian operator defined on the bounded domain, the operator Δ on X has the eigenvalues $\frac{n^2}{l^2}(n = 0, 1, 2, \cdots,)$ with the relative eigenfunctions

$$\psi_n^1 = \begin{pmatrix} \cos nx \\ 0 \end{pmatrix}, \psi_n^2 = \begin{pmatrix} 0 \\ \cos nx \end{pmatrix}, n = 0, 1, 2, \cdots.$$

Clearly, $(\psi_n^1, \psi_n^2))_{n=0}^\infty$ construct a basis of the phase space X and therefore any element y in X can be expanded as Fourier series in the following form:

$$y = \sum_{n=0}^\infty Y_n^T \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}, Y_n^T = (\langle y, \psi_n^1 \rangle, \langle y, \psi_n^2 \rangle) \tag{2.9}$$

From (2.8), (2.9) is equivalent to

$$\sum_{n=0}^\infty Y_n^T \left[\lambda I_2 + D\frac{n^2}{l^2} - \begin{pmatrix} a_{11} - d_1 \frac{n^2}{l^2} & -a_{12} \\ \beta a_{21} & \beta a_{22} - d_2 \frac{n^2}{l^2} - \beta s e^{-\lambda \tau} \end{pmatrix} \right] \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix} = 0 \tag{2.10}$$

Therefore, that characteristic Eq (2.8) admits a non-zero solution y is equivalent to equation

$$W_n(\lambda, \beta, \tau) = \lambda^2 + A_n\lambda + B_n + \beta s(\lambda + C_n)e^{-\lambda\tau} = 0, \tag{2.11}$$

being satisfied for at least one $n \in \{0, 1, 2, \dots\}$, where

$$
\begin{aligned}
A_n &= (d_1 + d_2)\frac{n^2}{l^2} - a_{11} - \beta a_{22}, \\
B_n &= d_1 d_2 \frac{n^4}{l^4} - (d_1\beta a_{22} + d_2 a_{11})\frac{n^2}{l^2} + \beta(a_{22}a_{11} + a_{12}a_{21}), \\
C_n &= d_1\frac{n^2}{l^2} - a_{11}.
\end{aligned}
\tag{2.12}
$$

In the following, we will analyze stability and bifurcation of the interior equilibrium of system (1.3) by analysis the characteristic Eq (2.11).

2.2. Dynamics analyze of system (1.3) without delay

For $\tau = 0$, Eq (2.11) is shown as

$$R_n(\lambda) \triangleq W_n(\lambda, \beta, 0) = \lambda^2 - T_n\lambda + D_n = 0, \tag{2.13}$$

with

$$
\begin{aligned}
T_n &= -(d_1 + d_2)\frac{n^2}{l^2} + a_{11} - \beta q, \\
D_n &= d_1 d_2 \frac{n^4}{l^4} - (d_2 a_{11} - d_1\beta q)\frac{n^2}{l^2} + \beta(a_{12}a_{21} - a_{11}q), \\
q &= s - a_{22} = v^* + \frac{bmu^*v^*}{(u + mv^*)^2} > 0.
\end{aligned}
\tag{2.14}
$$

Denote

$$
\begin{aligned}
\beta_n^H &= \frac{1}{q}\left(a_{11} - (d_1 + d_2)\frac{n^2}{l^2}\right), \quad \beta_n^T = \frac{d_2 a_{11}\frac{n^2}{l^2} - d_1 d_2 \frac{n^4}{l^4}}{d_1 q \frac{n^2}{l^2} + a_{12}a_{21} - a_{11}q}, \\
\beta^* &= \max_{n\geq 0}\beta_n^T.
\end{aligned}
$$

We make the following assumption

(H_1) $a_{12}a_{21} - a_{11}q > 0, \beta > \dfrac{a_{11}}{q}.$

Theorem 2.1. *Suppose that (H_1) and $\beta > \beta^*$ hold, then the interior equilibrium E^* of system (1.3) is locally asymptotically stable with $\tau = 0$.*

Proof. Differential β_n^T with $\frac{n^2}{l^2}$, we have

$$\frac{d\beta_n^T}{d\frac{n^2}{l^2}} = \frac{-d_2\left(qd_1^2\frac{n^4}{l^4} + 2d_1(a_{12}a_{21} - qa_{11})\frac{n^2}{l^2} - a_{11}(a_{12}a_{21} - qa_{11})\right)}{\left(a_{12}a_{21} - qa_{11} + qd_1\frac{n^2}{l^2}\right)^2}, \tag{2.15}$$

which means that β_n^T is monotonically increasing and then monotonically decreasing with respect to $\frac{n^2}{l^2}$. So, there must exists a $n_c \in \mathbb{N}_0$ such that

$$\beta^* \triangleq \max_{n \geq 0} \beta_n^T = \beta_{n_c}^T. \tag{2.16}$$

When (H_1) holds and $\beta > \beta^*$, one has

$$T_n < 0, \quad D_n > 0, \, n = 0, 1, 2, \cdots,$$

which implies that the interior equilibrium E^* is locally asymptotically stable. $\qquad\square$

Denote

$$
\begin{aligned}
n_H &= \left\lfloor \max\left\{ n \,\middle|\, a_{11} - (d_1 + d_2)\frac{n^2}{l^2} > 0 \right\} \right\rfloor, \\
n_T &= \left\lfloor \max\left\{ n \,\middle|\, -d_1^2 \frac{n^4}{l^4} + (2a_{11}d_1 - \frac{a_{12}a_{21}(d_1 + d_2)}{q})\frac{n^2}{l^2} + \frac{a_{11}}{q}(a_{12}a_{21} - a_{11}q) > 0 \right\} \right\rfloor, \\
n^* &= \min\{n_H + 1, n_T + 1\}.
\end{aligned}
\tag{2.17}
$$

Theorem 2.2. *Suppose that $a_{12}a_{21} - a_{11}q > 0$ and $d_2q < d_1$. Thus, Hopf bifurcation occurs at $\beta = \beta_0^H = \frac{a_{11}}{q}$ with $\tau = 0$.*

Proof. When $\beta = \beta_0^H = \frac{a_{11}}{q}$, $T_0 = 0$ an $T_n < 0$ for $n \neq 0$. Then Substitute β_0^H into $D_n(\beta)$ yields

$$D_n(\beta_0^H) = d_1 d_2 \frac{n^4}{l^4} + (\frac{a_{11}}{q}d_1 - d_2 a_{11})\frac{n^2}{l^2} + \frac{a_{11}}{q}(a_{12}a_{21} - a_{11}q). \tag{2.18}$$

$a_{12}a_{21} - a_{11}q > 0$ and $d_2q < d_1$, so $D_n(\beta_0^H) > 0$ for any $n \geq 0$. Consequently, Eq (2.13) has roots

$$\lambda(\beta) = \xi(\beta) \pm i\omega(\beta),$$

where

$$\xi(\beta_0^H) = 0, \omega(\beta_0^H) = \sqrt{D_0(\beta_0^H)}.$$

Thus, when β is near β_0^H

$$\xi(\beta) = \frac{T_0(\beta)}{2}, \omega(\beta) = \sqrt{D_0(\beta) - \xi^2(\beta)}.$$

So, we have $\xi'(\beta_0^H) = -\frac{q}{2} < 0$, which meets the transversal condition. $\qquad\square$

Theorem 2.3. *If $\beta^* > \frac{a_{11}}{q}$ and (H_1) hold, a Turing bifurcation occurs for $\beta = \beta^*$ with $\tau = 0$.*

Proof. When $\beta = \beta^*$, we have

$$T_{n_c}(\beta) < 0, D_{n_c}(\beta) = 0 \text{ and } T_n(\beta) < 0, D_n(\beta) > 0, \text{ for } n \neq n_c.$$

It follows that the characteristic Eq (2.13) just has a unique zero root and the remaining roots with negative real parts.

Obviously, we have $W_{n_c}(\lambda,\beta,0)|_{\beta=\beta^*,\lambda=0} = 0$, and $\frac{\partial W_{n_c}(\lambda,\beta,0)}{\partial\lambda} = 2\lambda - T_n$ and $\frac{\partial W_{n_c}(\lambda,\beta,0)}{\partial\beta} = -q\lambda + d_1 q\frac{n_c^2}{l^2} + a_{12}a_{21} - a_{11}q \neq 0$. By the existence theorem of the implicit function, we know that $W_n(\lambda,\beta,0) = 0$ determines an implicit function $\lambda = \lambda(\beta)$, which satisfies $\lambda(\beta^*) = 0$ and $R_{n_c}(\lambda(\beta)) = 0$ when β near β^*. So,

$$\frac{\partial}{\partial\beta}\lambda(\beta)\bigg|_{\beta=\beta^*} = \frac{d_1 q\frac{n_c^2}{l^2} + a_{12}a_{21} - a_{11}q}{-(d_1 + d_2)\frac{n_c^2}{l^2} + a_{11} - \beta^* q} \neq 0.$$

□

2.3. Dynamics analysis of system (1.3) with delay

Now, we discuss the effect of delay τ. Assume that $\lambda = \pm i\omega$ ($\omega > 0$) are a pair of pure imaginary roots of Eq (2.11). So, ω satisfies

$$-\omega^2 + iA_n\omega + B_n + \beta s(i\omega + C_n)e^{-i\omega\tau} = 0. \tag{2.19}$$

It follows that

$$\begin{cases} \omega^2 - B_n = \omega\beta s\sin(\omega\tau) + \beta s C_n\cos(\omega\tau), \\ -A_n\omega = -\beta s C_n\sin(\omega\tau) + \omega\beta s\cos(\omega\tau). \end{cases} \tag{2.20}$$

Taking the modulus of Eq (2.20), we have

$$\omega^4 + P_n\omega^2 + Q_n = 0, \tag{2.21}$$

where

$$P_n = A_n^2 - 2B_n - \beta^2 s^2, \quad Q_n = B_n^2 - \beta^2 s^2 C_n^2.$$

Denote

$$D^* = P_n^2 - 4Q_n. \tag{2.22}$$

Then, it is easily obtained the following conclusion.

Lemma 2.1. *(i) Eq.(2.21) has only one positive root* $\omega_1^+(n) = \sqrt{\frac{-P_n + \sqrt{D^*}}{2}}$ *if any of the following condition holds*

    $(H_2) : Q_n < 0$;
    $(H_3) : P_n = -2\sqrt{Q_n}, Q_n > 0$;
    $(H_4) : Q_n = 0, P_n < 0$;
    (ii) Eq.(2.21) has two positive roots $\omega_2^\pm(n) = \sqrt{\frac{-P_n \pm \sqrt{D^*}}{2}}$ *if* (H_5) *holds:*
    $(H_5) : P_n < 0, Q_n > 0, D^* > 0$;
    (iii) Eq.(2.21) has no positive root if one of the assumptions holds:
    $(H_6) : D^* < 0$;
    $(H_7) : D^* > 0, Q_n \geq 0, P_n \geq 0$;
    $(H_8) : D^* = 0, P_n \geq 0$.

From (2.20) and 2.1, we obtained that when Eq (2.21) has only one positive root $\omega_1^+(n) = \sqrt{\frac{-P_n + \sqrt{D^*}}{2}}$, and by Eq (2.20), we obtain

$$\tau_{1,j}^+(n) = \frac{1}{\omega_1^+(n)} \left\{ \arccos\left[\frac{C_n(\omega_1^{+^2}(n) - B_n) - \omega_1^{+^2}(n)A_n}{\alpha\omega_1^{+^2}(n) + \alpha C_n^2} \right] + 2j\pi \right\}, \, j \in \mathbb{N}_0. \tag{2.23}$$

When Eq (2.21) has two positive real roots, and again by Eq (2.20), we have

$$\tau_{2,j}^\pm(n) = \frac{1}{\omega_2^+(n)} \left\{ \arccos\left[\frac{C_n(\omega_2^{\pm^2}(n) - B_n) - \omega_2^{\pm^2}(n)A_n}{\alpha\omega_2^{\pm^2}(n) + \alpha C_n^2} \right] + 2j\pi \right\}, \, \, j \in \mathbb{N}_0. \tag{2.24}$$

Lemma 2.2. *Assume that $\lambda(\tau) = \eta(\tau) \pm i\omega(\tau)$ are the roots of Eq (2.11), and it satisfies*

$$\eta(\tau_{1,j}^+(n)) = 0 \, (\eta(\tau_{2,j}^\pm(n)) = 0),$$
$$\omega(\tau_{1,j}^+(n)) = \omega_1^+(n) \, (\omega(\tau_{2,j}^\pm(n)) = \omega_2^\pm(n)).$$

So, we have

$$\frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\bigg|_{\tau=\tau_{1,j}^+(n)} > 0, \, \frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\bigg|_{\tau=\tau_{2,j}^+(n)} > 0, \, \frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\bigg|_{\tau=\tau_{2,j}^-(n)} < 0. \tag{2.25}$$

Proof. Taking the derivative of Eq (2.11) with respect to λ, we have

$$2\lambda + \alpha e^{-\lambda\tau} + A_n - \beta s(\lambda + C_n)(\tau + \lambda\frac{d\tau}{d\lambda})e^{-\lambda\tau} = 0.$$

Therefore,

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(2\lambda + A_n)e^{\lambda\tau}}{\lambda(\beta s\lambda + \beta s C_n)} + \frac{1}{\lambda(\lambda + C_n)} - \frac{\tau}{\lambda}.$$

Combining with Eq (2.20), we have

$$\mathrm{Re}\left(\left[\frac{d\lambda}{d\tau}\right]^{-1}\right)_{\lambda=\pm i\omega_2^\pm(n)} = \left[\frac{2\omega^2 + A_n^2 - 2B_n - \beta^2 s^2}{\beta^2 s^2 C_n^2 + \beta^2 s^2 \omega^2} \right]_{\lambda=\pm i\omega_2^\pm(n)}$$
$$= \left[\frac{2\omega^2 + Q_n}{\beta^2 s^2 C_n^2 + \beta^2 s^2 \omega^2} \right]_{\lambda=\pm i\omega_2^\pm(n)} = \pm\frac{\sqrt{\Delta_0}}{\beta^2 s^2 C_n^2 + \beta^2 s^2 \omega_2^{\pm^2}(n)}.$$

Consequently, one has

$$\frac{d\mathrm{Re}(\lambda)}{d\tau} = \frac{\sqrt{\Delta_0}}{\beta^2 s^2 C_n^2 + \beta^2 s^2 \omega_1^{+^2}(n)} > 0, \tau = \tau_{1,j}^+(n), \omega = \omega_1^+(n),$$

$$\frac{d\mathrm{Re}(\lambda)}{d\tau} = \frac{\sqrt{\Delta_0}}{\beta^2 s^2 C_n^2 + \beta^2 s^2 \omega_2^{+^2}(n)} > 0, \tau = \tau_{2,j}^+(n), \omega = \omega_2^+(n),$$

$$\frac{d\mathrm{Re}(\lambda)}{d\tau} = -\frac{\sqrt{\Delta_0}}{\beta^2 s^2 C_n^2 + \beta^2 s^2 \omega_2^{-^2}(n)} < 0, \tau = \tau_{2,j}^-(n), \omega = \omega_2^-(n).$$

$\square$

Denote

$$\Gamma = \{n \in \mathbb{N}_0 | (H_2) \text{ or } (H_3) \text{ or } (H_4) \text{ holds}\}.$$

Obviously, $\tau_{1,j}^+(n)$ increases monotonically with respect to j, and we define

$$\tau^* = \min_{n \in \Gamma, j \in \mathbb{N}_0} \{\tau_{1,j}^+(n)\} = \min_{n \in \Gamma} \{\tau_{1,0}^+(n)\}. \tag{2.26}$$

From Theorem 2.1, 2.1 and 2.2, as with the literature [23–25], we can conclude we have the following result.

Theorem 2.4. *Suppose that (H_1) and $\beta > \beta^*$ hold, then we have the following statements:*
(i) If (H_6) or (H_7) or (H_8) holds, then the interior equilibrium E^* is spectrally stable for all $\tau \geq 0$.
(ii) If (H_2) or (H_3) or (H_4) holds, then the interior equilibrium E^* is spectrally stable for $\tau \in [0, \tau^*)$, and unstable for $\tau \in (\tau^*, +\infty)$. Hopf bifurcation occurs at E^* when $\tau = \tau^*$;

In the following, we discuss the Turing bifurcation of system (1.3) with $\tau > 0$. First, we have the following conclusion.

Lemma 2.3. *Assume that (H_1) holds, and $\tau \neq \tau_c$, where*

$$\tau_c = \frac{(d_1 + d_2)\frac{n^2}{l^2} - a_{11} + \beta q}{\beta s(d_1 \frac{n^2}{l^2} - a_{11})}.$$

Then, when $\beta = \beta^*$ the interior equilibrium is Turing unstable.

Proof. When $\beta = \beta^*$, Theorem 2.3 implies that (2.11) always has a zero root. Furthermore, by (2.11), when $(\lambda, \beta, \tau) = (0, \beta^*, \tau_c)$, we have

$$\frac{\partial}{\partial \lambda} W_{n_c}(\lambda, \beta, \tau) = (d_1 + d_2)\frac{n^2}{l^2} - a_{11} - \beta a_{22} + \beta s - \tau_c \beta s(d_1 \frac{n^2}{l^2} - a_{11}) = 0$$

$$\frac{\partial^2}{\partial \lambda^2} W_{n_c}(\lambda, \beta, \tau) = 2 - 2\beta s \tau + \tau^2 \beta s(d_1 \frac{n^2}{l^2} - a_{11}) \neq 0. \tag{2.27}$$

$$= 2 - \frac{2(d_1 + d_2)\frac{n^2}{l^2} - 2a_{11} + 2\beta^* q}{d_1 \frac{n^2}{l^2} - a_{11}} + \frac{((d_1 + d_2)\frac{n^2}{l^2} - a_{11} + \beta^* q)^2}{\beta^* s(d_1 \frac{n^2}{l^2} - a_{11})} \neq 0.$$

It follows that (2.11) has a zero double eigenvalue. Now, let λ be an eigenvalue of Eq (2.11), and that it satisfies the equation $\lambda(\beta^*) = 0$ and $W_{n_c} = 0$ when β is near β^*. Thus

$$\frac{\partial}{\partial \beta} \lambda(\beta)\Big|_{\beta=\beta^*} = \frac{-q d_1 \frac{n^2}{l^2} + q a_{11} - a_{12} a_{21}}{\beta s - \beta s \tau_c(d_1 \frac{n^2}{l^2} - a_{11})} \neq 0.$$

This completes the proof. □

The Turing-Hopf bifurcation theorem follows from 2.3 and Theorem 2.4.

Theorem 2.5. *Assume that (H_1) holds, and (H_2) or (H_3) or (H_4) holds, then the Turing-Hopf bifurcation occurs when $(\beta, \tau) = (\beta^*, \tau_n^j)$ for $n \in \Gamma$, $j = 0, 1, 2. \cdots$.*

3. Numerical simulations

This section will involve running some numerical simulations to verify the theoretical analysis and demonstrate the many spatiotemporal patterns that might arise in system (1.3).

3.1. Hopf bifurcation caused by β

Choose the parameters of system (1.3) as $m = 0.68$, $\alpha = 0.02$, $b = 0.9$, $d = 0.06$, and the diffusion coefficients $d_1 = 0.2$ and $d_2 = 2$, and $l = 1$. A direct calculation shown that system (1.3) has two interior equilibria $E_1^* = (0.2469, 0.3798)$ and $E_2^* = (0.2053, 0.3541)$. It is easily obtained that E_2^* is always unstable. For the interior equilibrium E_1^*, we obtain the Hopf bifurcation curve is

$$\mathcal{H}_0 : \beta = \frac{a_{11}}{q} = \frac{\frac{u^*v^*}{(u^*+mv^*)^2} - u^* - 2\alpha u^{*2}}{2v^* - \frac{bu^{*2}}{(u^*+mv^*)^2} + d} = 0.1954.$$

Therefore, when $\beta > 0.1954$ the interior equilibrium E_1^* is locally asymptotically stable (see Figure 2). While $\beta < 0.1954$, E_1^* loses its stability and becomes unstable, and Hopf bifurcation occurs, see Figure 3 where we use the spatial average density.

.
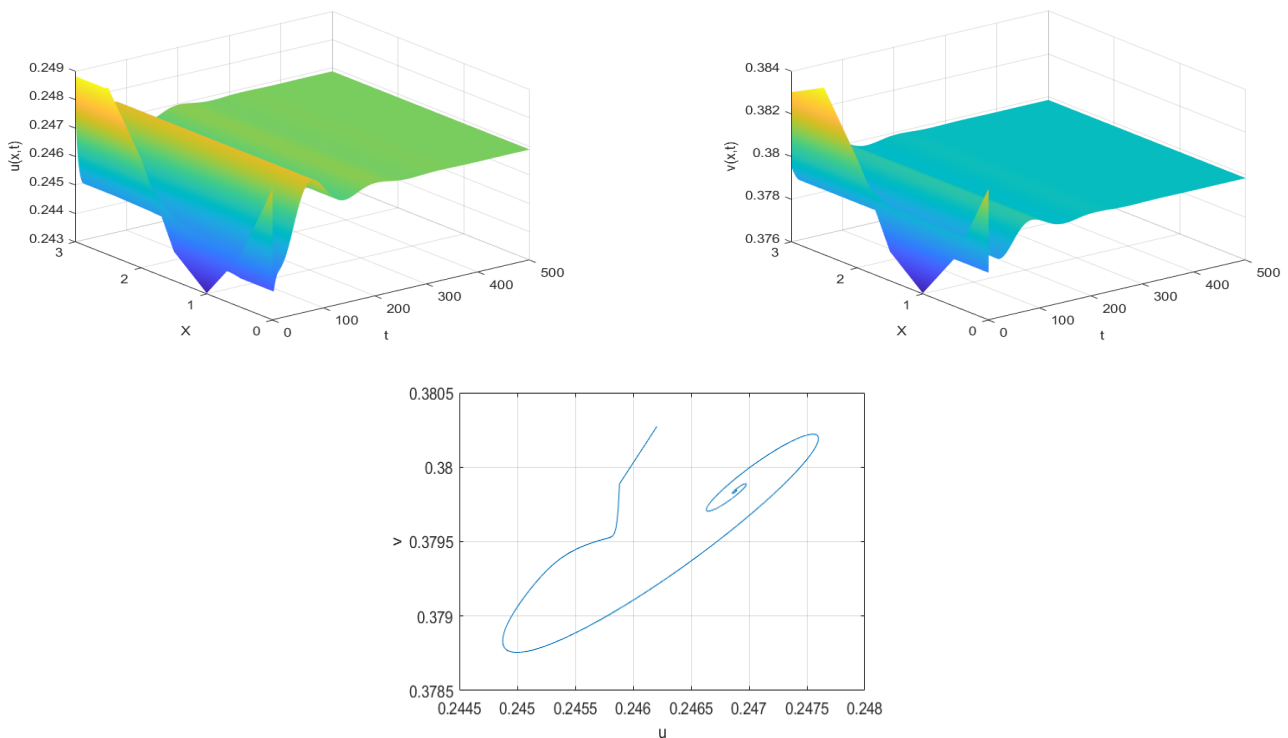
Figure 2. The interior equilibrium E_1^* is locally asymptotically stable with $\beta = 0.25$.
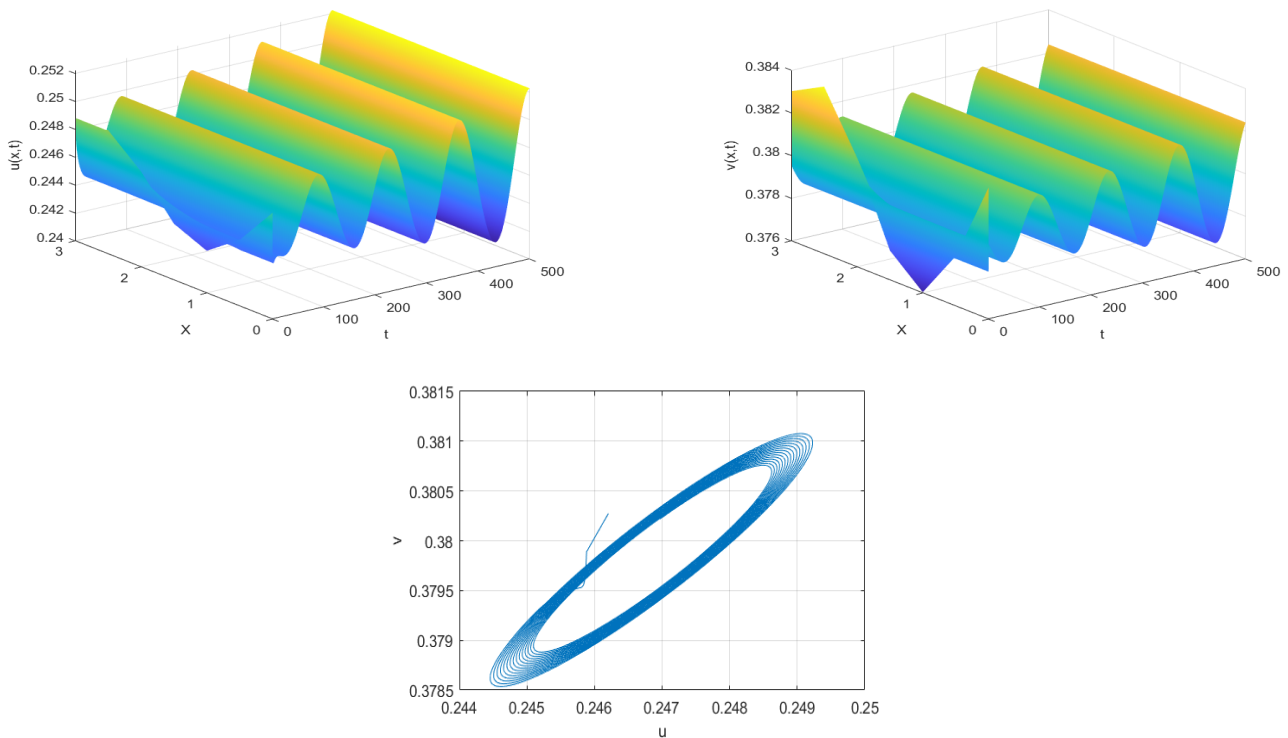
Figure 3. Hopf bifurcation occurs when $\beta = 0.19$.

3.2. Spatial patterns of system (1.3) without delay

Choose the parameters of system (1.3) as $m = 0.7$, $\alpha = 0.02$, $b = 0.9$, $d = 0.06$, $\beta = 0.3$, and the diffusion coefficients $d_1 = 0.002$ and $d_2 = 4$, and $l = 10$. A direct calculation show that system (1.3) has two interior equilibria $E_1^* = (0.32, 0.4129)$ and $E_2^* = (0.1344, 0.2949)$. It is easily obtained that E_2^* is always unstable. For the interior equilibrium E_1^*, the instability region is depicted in Figure 4, and the critical value of β is $\beta^* \approx 7.4428$ when $n = 28$. Thus, according to Theorem 2.1 and Theorem 2.3, when $\beta > 7.4428$, E_1^* is locally asymptotically stable (see Figure 5). While $\beta < 7.4428$, Turing bifurcation occurs and E_1^* becomes unstable (see Figure 6).
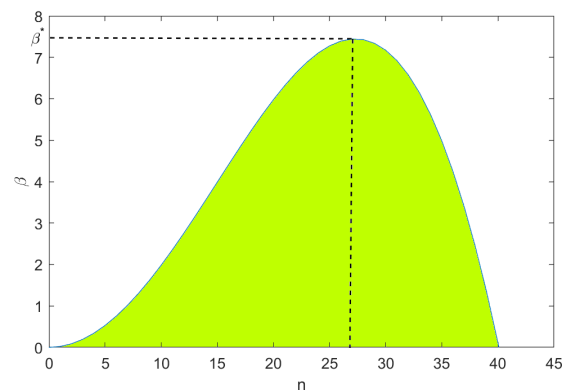
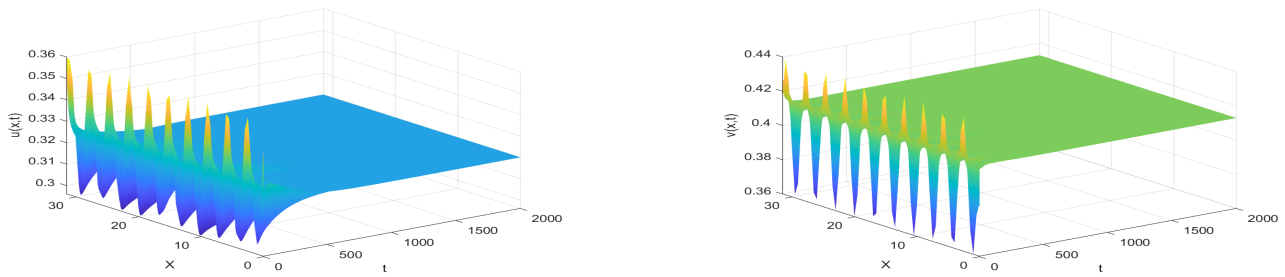**Figure 4.** A graph of $\beta = \beta_n^T$ in $n - \beta$ plane. The green region is unstable region of E^*.

Figure 5. The interior equilibrium E_1^* is locally asymptotically stable with $\beta = 10$.
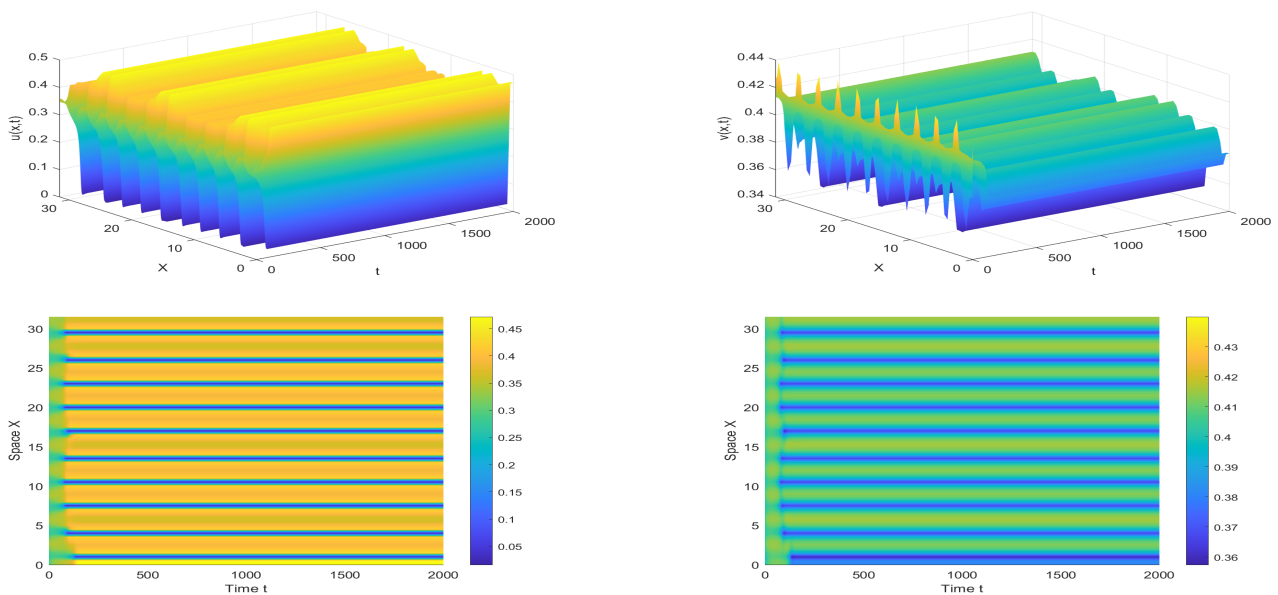

Figure 6. Turing bifurcation occurs and E_1^* becomes unstable when $\beta = 7$.

3.3. *System* (1.3) *with delay*

Taking the parameters as: $m = 0.63$, $\alpha = 0.02$, $b = 0.7$, $d = 0.02$, and the diffusion coefficients $d_1 = 0.2$ and $d_2 = 2$, and $l = 10$. Calculations show that system (1.3) has an interior equilibrium $E_1^* = (0.3098, 0.3765)$.

It is easily obtained that $\beta^* = 0.3301$. Therefore, when $\beta > \beta^* = 0.3301$, the interior equilibrium E_1^* is locally asymptotically stable with $\tau = 0$. While $\tau > 0$, let $\beta = 0.4$, we obtain that the critical value $\tau^* = 3.4807$. According to Theorem 2.4, when $\tau = 3 < \tau^* = 3.4807$, E_1^* is locally asymptotically stable (see Figure 7), and when $\tau = 4$, Hopf bifurcation occurs (see Figure 8).

According to Theorem 2.5, when (β, τ) near $(0.3301, 4.0431)$, from Figure 9, we can see that system (1.3) oscillates periodically in the time direction and is spatially distributed in an inhomogeneous manner, which means that Turing-Hopf bifurcation occurs.

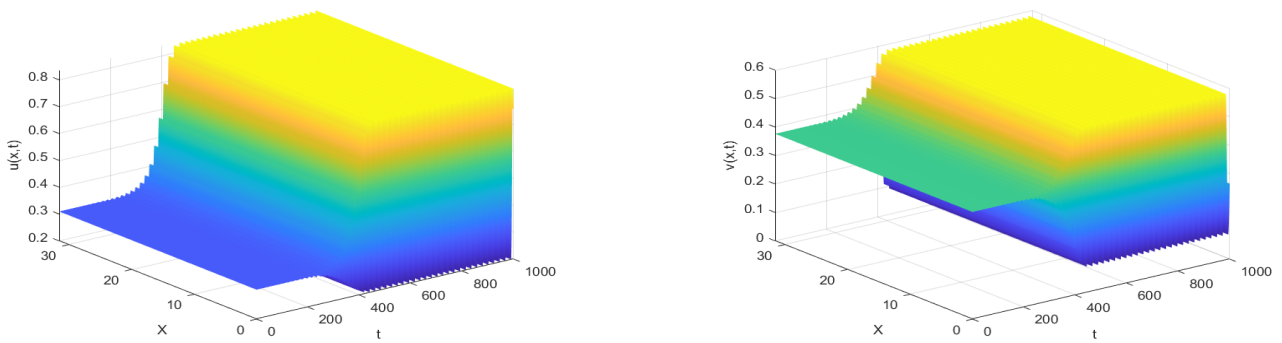**Figure 7.** The interior equilibrium E_1^* is locally asymptotically stable with $\tau = 3$.

Figure 8. E_1^* is unstable and Hopf bifurcation occurs with $\tau = 3.8$.
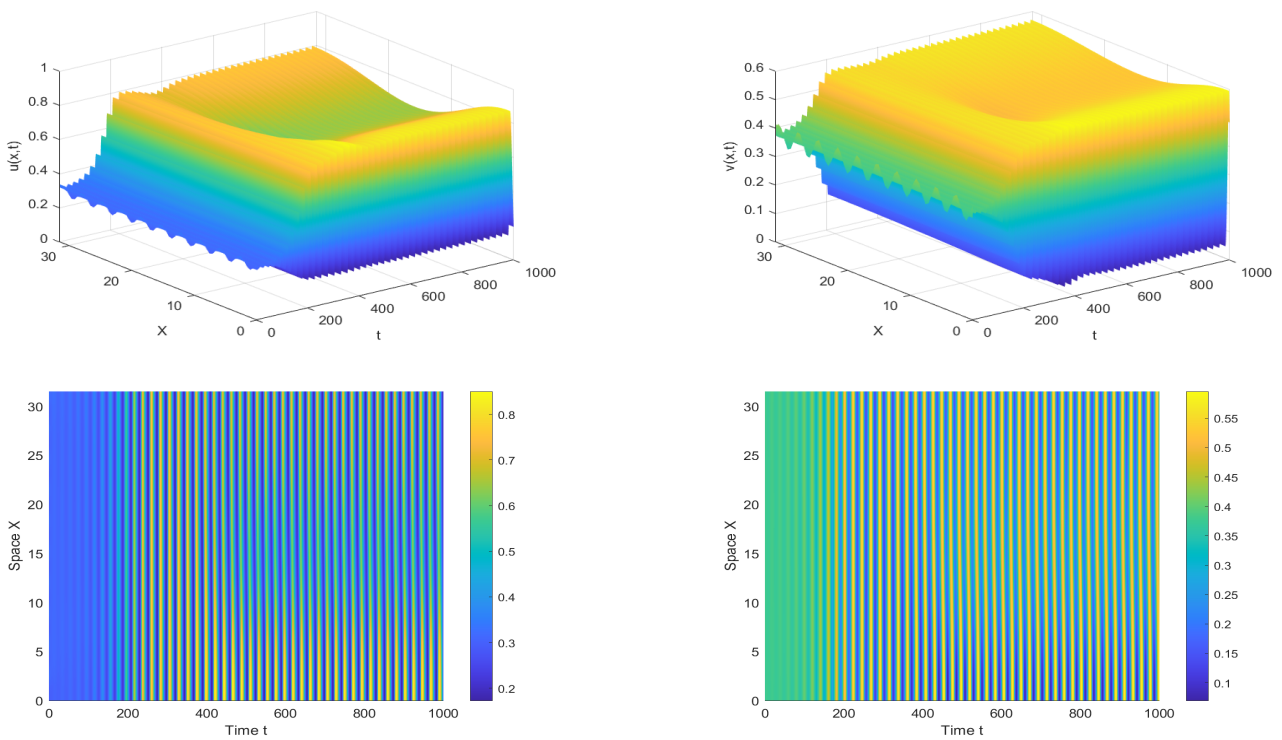

Figure 9. When $(\beta, \tau) = (0.35, 4.2)$, Turing-Hopf bifurcation occurs.

4. Results and discussions

In this study, the spectral stability of a delayed reaction-diffusion predator-prey model under the influence of poison was examined. We began by analyzing the presence of the internal equilibrium of the system (1.3). Afterward, we evaluated the system's dynamics. Using τ and β as bifurcation parameters, we studied the bifurcation. In addition, numerical simulations of the system's dynamics are shown. Figure 8 and Figure 9 illustrate the spatial patterns that are induced by τ and β.

In the following, we further discuss the effect of toxins. We choose the parameters $m = 0.63$, $\alpha = 0.02$, $b = 0.7$, $d = 0.02$, and the diffusion coefficients $d_1 = 0.2$ and $d_2 = 2$, $l = 10$, which are the same as those in subsection 3.3. If we let $\beta = 0.4$, then from the discussions in subsection 3.3 the interior equilibrium E^* is stable when $\tau = 0$ (see Figure 10). However, if we change the value of the parameter α from 0.02 to 0.05, by a direct calculation, we obtained that $\beta^* = 0.5980$. $\beta = 0.4 < \beta^* = 0.5980$, so according to Theorem 2.3, Turing bifurcation occurs. Therefore, the interior equilibrium E^* becomes unstable, and the system has a non-constant steady-state solution, which means that the Turing bifurcation occurs (Figure 11). Therefore, the prey and predator are distributed unevenly in space If the value of parameter α is further changed to 0.1, we find that the prey and the predator will both extinct (Figure 12), which means that the toxins can cause the extinction of the population, which is in line with reality.

The model without the effect of toxins has been studied in [27]. According to Theorem 2.5 in [27], the condition of the stability of the predator-free equilibrium is related to the consumption rate m. However, in our model, the condition of the predator-free equilibrium is $\beta b < d$ which is not related to the consumption rate m. In addition, due to the introduction of toxins and the delay, system (1.3) exhibits more dynamic behaviors, such as Hopf bifurcation, Turing-Hopf bifurcation, etc.

All of these results show that toxins play an essential role in the formation of complex spatial patterns.
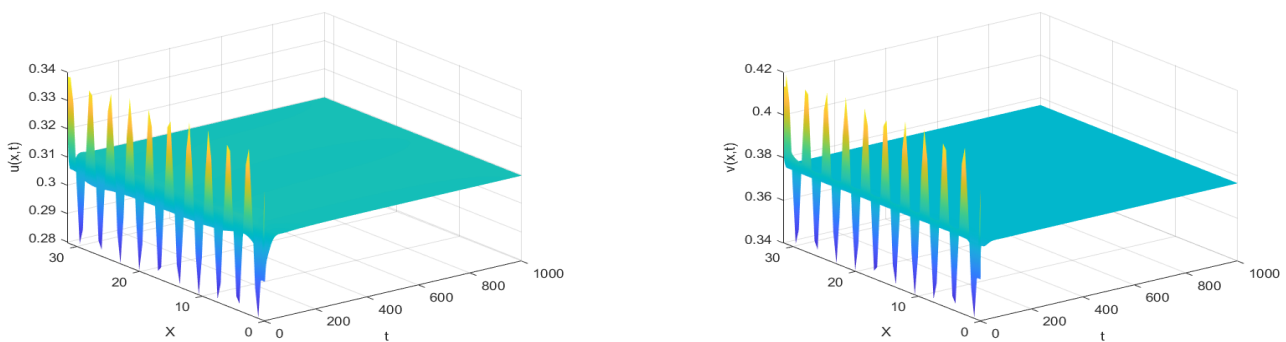

Figure 10. When $\alpha = 0.02$, the interior equilibrium E^* is stable.
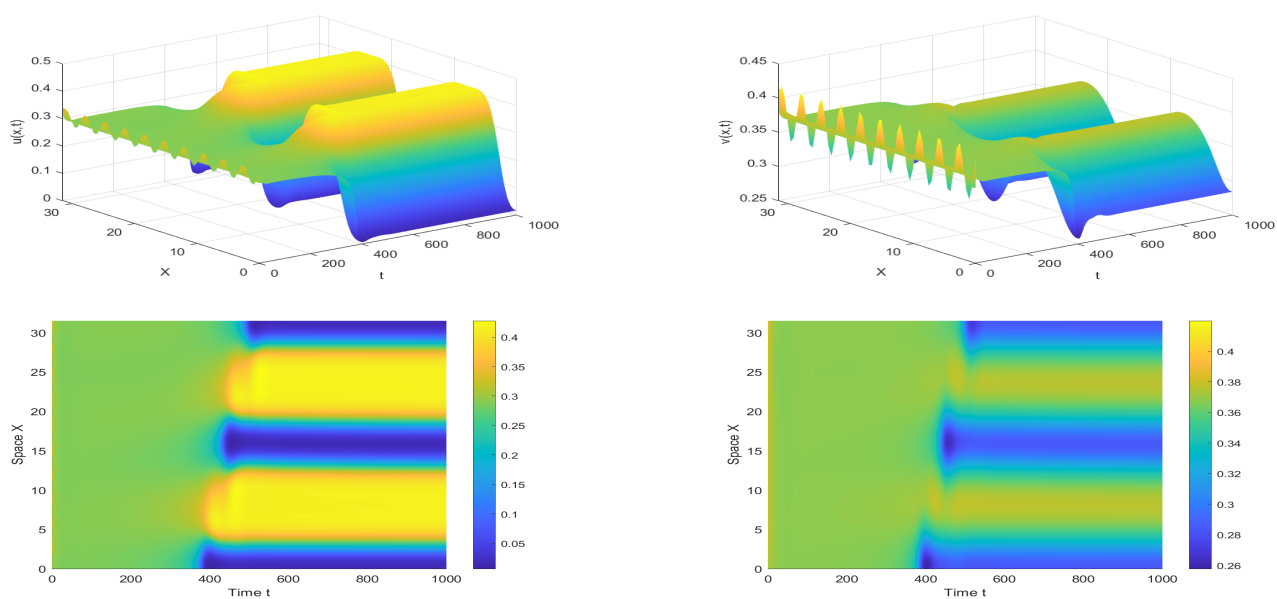
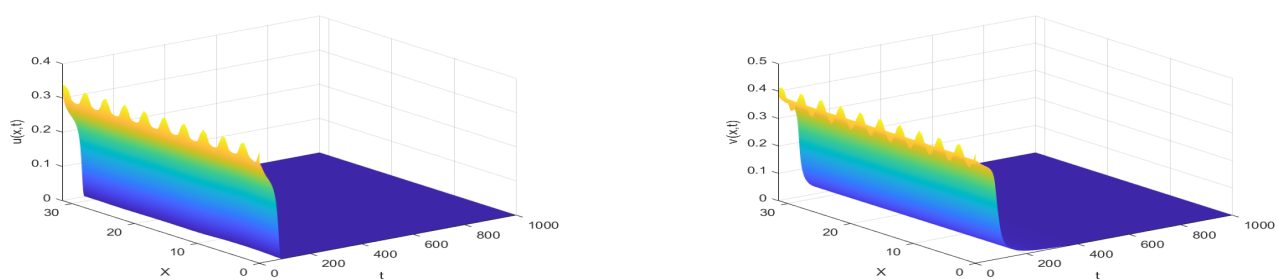**Figure 11.** When $\alpha = 0.05$, the interior equilibrium E^* is unstable and Turing bifurcation occurs.

Figure 12. When $\alpha = 0.1$, the prey and the predator both extinct.

Acknowledgments

Conflict of interest

The authors declare there is no conflict of interest.

References

1. M. L. Crump, F. R. Hensley, K. L. Clark, Apparent decline of the golden toad: Underground or extinct?, *Copeia*, **2** (1992), 413–420. https://doi.org/10.2307/1446201

2. T. G. Hallam, C. E. Clark, R. R. Lassiter, Effects of toxicants on populations: A qualitative approach I. Equilibrium environmental exposure, *Ecol. Modell.*, **18** (1983), 291–304. https://doi.org/10.1016/0304-3800(83)90019-4

3. T. Das, R. N. Mukherjee, K. S. Chaudhuri, Harvesting of a prey–predator fishery in the presence of toxicity, *Appl. Math. Modell.*, **33** (2009), 2282–2292. https://doi.org/10.1016/j.apm.2008.06.008

4. R. Rani, S. Gakkhar, The impact of provision of additional food to predator in predator–prey model with combined harvesting in the presence of toxicity, *J. Appl. Math. Comput.*, **60** (2019), 673–701. https://doi.org/10.1007/s12190-018-01232-z

5. K. Chakraborty, K. Das, Modeling and analysis of a two-zooplankton one-phytoplankton system in the presence of toxicity, *Appl. Math. Modell.*, **39** (2015), 1241–1265. https://doi.org/10.1016/j.apm.2014.08.004

6. X. Wu, F. Wei, Single-species population models with stage structure and partial tolerance in polluted environments, *Math. Biosci. Eng.*, **19** (2022), 9590–9611. https://doi.org/10.3934/mbe.2022446

7. F. Wei, L. Chen, Psychological effect on single-species population models in a polluted environment, *Math. Biosci.*, **290** (2017), 22–30. https://doi.org/10.1016/j.mbs.2017.05.011

8. F. Wei, S. A. H. Geritz, J. Cai, A stochastic single-species population model with partial pollution tolerance in a polluted environment, *Appl. Math. Lett.*, **63** (2017), 130–136. https://doi.org/10.1016/j.aml.2016.07.026

9. X. Zhang, H. Zhao, Bifurcation and optimal harvesting of a diffusive predator–prey system with delays and interval biological parameters, *J. Theor. Biol.*, **363** (2014), 390–403. https://doi.org/10.1016/j.jtbi.2014.08.031

10. X. Zhang, H. Zhao, Dynamics analysis of a delayed reaction-diffusion predator-prey system with non-continuous threshold harvesting, *Math. Biosci.*, **289** (2017), 130–141. https://doi.org/10.1016/j.mbs.2017.05.007

11. B. K. Das, D. Sahoo, G. Samanta, Impact of fear in a delay-induced predator–prey system with intraspecific competition within predator species, *Math. Comput. Simul.*, **191** (2022), 134–156. https://doi.org/10.1016/j.matcom.2021.08.005

12. Y. Wang, X. Zou, On a predator–prey system with digestion delay and anti-predation strategy, *J. Nonlinear Sci.*, **30** (2020), 1579–1605. https://doi.org/10.1007/s00332-020-09618-9

13. X. Jiang, X. Chen, T. Huang, H. Yan, Bifurcation and control for a predator-prey system with two delays, *IEEE Trans. Circuits and Syst. II: Express Briefs*, **68** (2021), 376–380. https://doi.org/10.1109/TCSII.2020.2987392

14. D. Duan, B. Niu, J. Wei, Hopf-hopf bifurcation and chaotic attractors in a delayed diffusive predator-prey model with fear effect, *Chaos Solitons Fractals*, **123** (2019), 206–216. https://doi.org/10.1016/j.chaos.2019.04.012

15. J. Liu, Y. Cai, J. Tan, Y. Chen, Dynamical behaviours of a delayed diffusive eco-epidemiological model with fear effect, *Chaos, Solitons Fractals*, **161** (2022), 112349. https://doi.org/10.1016/j.chaos.2022.112349

16. D. Pal, G. P. Samanta, G. S. Mahapatra, Selective harvesting of two competing fish species in the presence of toxicity with time delay, *Appl. Math. Comput.*, **313** (2017), 74–93. https://doi.org/10.1016/j.amc.2017.05.069

17. J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences, Springer, New York,1996.

18. H. Zhu, X. Zhang, G. Wang, L. Wang, Effect of toxicant on the dynamics of a delayed diffusive predator-prey model, *J. Appl. Math. Comput.*, **69** (2023), 355–379. https://doi.org/10.1007/s12190-022-01744-9

19. X. Yan, Y. Li, G. Guo, Qualitative analysis on a diffusive predator-prey model with toxins, *J. Math. Anal. Appl.*, **486** (2020), 123868. https://doi.org/10.1016/j.jmaa.2020.123868

20. G. P. Samanta, A two-species competitive system under the influence of toxic substances, *Appl. Math. Comput.*, **216** (2010), 291–299. https://doi.org/10.1016/j.amc.2010.01.061

21. X. Zhang, Q. An, L. Wang, Spatiotemporal dynamics of a delayed diffusive ratio-dependent predator–prey model with fear effect, *Nonlinear Dyn.*, **105** (2021), 3775–3790. https://doi.org/10.1007/s11071-021-06780-x

22. W. Ni, M. Wang, Dynamics and patterns of a diffusive Leslie–Gower prey–predator model with strong Allee effect in prey, *J. Differ. Equations*, **261** (2016), 4244–4274. https://doi.org/10.1016/j.jde.2016.06.022

23. W. Zuo, J. Wei, Stability and hopf bifurcation in a diffusive predator–prey system with delay effect, *Nonlinear Anal.: Real World Appl.*, **12** (2011), 1998–2011. https://doi.org/10.1016/j.nonrwa.2010.12.016

24. S. Chen, J. Shi, J. Wei, Time delay-induced instabilities and hopf bifurcations in general reaction-diffusion systems, *J. Nonlinear Sci.*, **23** (2013), 1–38. https://doi.org/10.1007/s00332-012-9138-1

25. S. Chen, J. Shi, J. Wei, Global stability and hopf bifurcation in a delayed diffusive Leslie–Gower predator-prey system, *Int. J. Bifurcation Chaos*, **22** (2012), 1250061. https://doi.org/10.1142/S0218127412500617

26. L. A. Howland, A solution of the biquadratic equation, *Am. Math. Mon.*, **18** (1911), 102–108. https://doi.org/10.1080/00029890.1911.11997617

27. P. Y. Pang, M. Wang, Qualitative analysis of a ratio-dependent predator–prey system with diffusion, *Proc. R. Soc. Edinburgh Sect. A: Math.*, **133** (2003), 919–942. https://doi.org/10.1017/S0308210500002742

AIMS Press