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Research article

Sliding dynamics and bifurcations of a human influenza system under logistic source and broken line control strategy

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Abstract: This paper proposes a non-smooth human influenza model with logistic source to describe the impact on media coverage and quarantine of susceptible populations of the human influenza transmission process. First, we choose two thresholds I_T and S_T as a broken line control strategy: Once the number of infected people exceeds I_T, the media influence comes into play, and when the number of susceptible individuals is greater than S_T, the control by quarantine of susceptible individuals is open. Furthermore, by choosing different thresholds I_T and S_T and using Filippov theory, we study the dynamic behavior of the Filippov model with respect to all possible equilibria. It is shown that the Filippov system tends to the pseudo-equilibrium on sliding mode domain or one endemic equilibrium or bistability endemic equilibria under some conditions. The regular/virtual equilibrium bifurcations are also given. Lastly, numerical simulation results show that choosing appropriate threshold values can prevent the outbreak of influenza, which implies media coverage and quarantine of susceptible individuals can effectively restrain the transmission of influenza. The non-smooth system with logistic source can provide some new insights for the prevention and control of human influenza.

Keywords: Filippov human influenza model; logistic source; two-thresholds policy; bistability; bifurcations

1. Introduction

With the development of society, the number of deaths caused by influenza infection has had gradually increased. For example, the global pandemic in 1918 caused more deaths in the world than in the First World War [1, 2]. According to statistical analysis, the global influenza pandemic is caused by H2N2 virus, which is called "Asian influenza" because it first occurred in Asia. The incidence rate was about 15–30% [3,4]. Recently, COVID-19 also has a great impact on human life. According to the latest real-time statistics of the World Health Organization, as of October 10, 2022, the cumulative number of confirmed cases worldwide exceeded 600 million, and the cumulative number of deaths was about 6.5 million [5]. Therefore, in order to better control and reduce the number of deaths, how to effectively and quickly control the spread of disease is worthy of further study.

In the face of the global pandemic, many countries have responded to and mitigated the impact of influenza on human life through media reports, vaccination, disinfection, the use of protective equipment (such as masks), quarantine and other measures. In addition, some biological mathematicians have also established many mathematical models in order to better understand influenza infection and quantify the effectiveness of various control measures [6–13]. For example, considering social factors such as vaccination, media reports and protective measures, [14–21] established some different forms of disease models to analyze the impact of disease transmission. Practice shows that frequent hand washing, disinfection or wearing protective masks, as well as reducing or avoiding close contact with infected persons, can effectively reduce transmission [22–29]. However, media reports on influenza cases and deaths may have a great impact on the public, because the public will take some protective measures after media reports. In addition, quarantine and other measures will greatly reduce the effective contact rate between vulnerable people and infected people, thus reducing the spread of disease.

Recently, many different control strategies have been proposed to apply to the control of some disease models. For example, by using multiple optimal control, Ndii and Adi [12] have studied the effects of individual awareness and vector controls on Malaria transmission dynamics. By using a non-smooth control strategy, Li et al. [13] have considered the bifurcations and dynamics of a plant disease system. By using the two thresholds control strategy, Li et al. [30] have considered the global dynamics of a Filippov predator-prey model. Chen et al. [31] have considered a two-thresholds policy for a Filippov model in combating influenza. Zhou et al. [32] have discussed a two-thresholds policy to interrupt transmission of West Nile Virus to birds. Meanwhile, based on the above mentioned transmission factors, some scholars have also done a lot of meaningful work. For example, Dong et al. [33] studied a kind of nonlinear incidence Filippov epidemic model to describe the impact of media in the process of epidemic transmission. They found that choosing an appropriate threshold value and control intensity can prevent the outbreak of infectious diseases, and media coverage can reduce the burden of disease outbreaks and shorten the duration of disease outbreaks. Can et al. [25] have studied a Filippov model describing the effects of media coverage and quarantine on the spread of human influenza. Xiao et al. [28] have discussed a media impact switching surface during an infectious disease outbreak. In real life, when the influenza started to spread, that is, when the epidemic was not serious, the public paid little attention to the influenza. Generally, only when the number of people who have already felt it reaches and exceeds a certain threshold level will the mass media begin to coverage in large numbers and the public begin take some countermeasures [11,17,19,20]. Currently, many control managements have been developed, including threshold [30], harvesting control management [34–36], impulsive control management [37,38] and other controls [39–45]. However, the implementation of this measure may cause some social impacts, such as socioeconomic decline and psychological impact on the quarantined people. It is important to consider when to take these control measures. Therefore, an appropriate threshold policy is needed to deal with the influenza outbreak, or at least reduce the number of infected people to an acceptable level. As far as we know, under logistic source and broken line control strategy, the sliding dynamics and bifurcations of a human influenza system have been seldom reported in existing work.

Motivation and inspiration come from the discussion above. In this paper, we propose a human influenza system under logistic source and broken line control strategy. The main contributions of this paper include three points: First, choose two thresholds I_T and S_T as a broken line control strategy: Once the number of infected people exceeds I_T, the media influence comes into play, and when the number of susceptible individuals is greater than S_T, the control by quarantine of susceptible individuals control is open. Furthermore, by choosing different thresholds I_T and S_T, the existence and stability of all possible equilibria considered, and then the Filippov system tends to the pseudo-equilibrium on sliding mode domain or one endemic equilibrium or bistability endemic equilibria under some conditions. It is worth pointing out that in our paper there exists a new phenomenon of two real equilibrium points co-existing.

In fact, from the perspective of control strategy, our paper puts forward a broken line control strategy. Different from the previous single-stage control strategy [30,34,35,37,38], our control strategy is divided into two stages and can control and simulate the real life situation well.

This paper is structured as follows. In Section 2, we propose a non-smooth model under logistic source and broken line control strategy. By changing the infection threshold values and susceptibility threshold values, in Section 3, we consider the global dynamics of the system under Case 1: $S_T < S_1^* < S_2^* = S_3^*$. In Section 4, we study the global dynamics of the system under Case 2: $S_1^* < S_T < S_2^* = S_3^*$. Section 5 considers the global dynamics of the system under Case 3: $S_2^* = S_3^* < S_T$.The regular/virtual equilibrium bifurcations are given in Section 6. Finally, we summarize the main results of this paper and discuss some biological conclusions in Section 7.

2. Model description and basic knowledge

2.1. Model description

In [25], a Filippov human influenza model with effects of media coverage and quarantine was considered. Because a logistic source can better depict the real situation, motivated by the above discussion and the existing multiple threshold conditions [25, 30–32, 45], we consider the logistic source factor with the susceptible population, and then the new human influenza model can be described by

$$\begin{cases} S_t = \gamma S\left(1 - \dfrac{S}{K}\right) - \beta S I, \\ I_t = \beta S I - (d + \delta + r)I, \\ R_t = rI - dI, \end{cases} \tag{2.1}$$

where S is susceptible individuals, I is infected individuals, and R is recovered individuals. γ is an intrinsic growth rate of susceptible individuals, and K is carrying capacity. β is the transmission rate. d is the natural death rate. δ is the death rate caused by disease, and r is the recovered rate.

To better control the spread of the disease, we give the following broken line control strategy. If the infected populations is lower than I_T, no control is required. When the infected populations is greater than I_T, the media coverage control is open. That is, the mass media being coverage of information about the disease, including the route of transmission, the number of infected cases and the number of deaths.

Therefore, the public realizes the harm of the disease, and they changed their behavior, leading to a decline in the contact rate β. In this paper, we consider that the positive number $v_1(0 \le v_1 \le 1)$ is the reduction amount of contact rate. In addition, we consider the quarantine control as follows. If the number of susceptible individuals is less than S_T, we do not quarantine susceptible individuals. If $S > S_T$ holds, the quarantine control is open, and this time we assume that the positive number $v_2(0 \le v_2 \le 1)$ is the quarantine rate of susceptible individuals. Figure 1 shows the broken line control strategy.

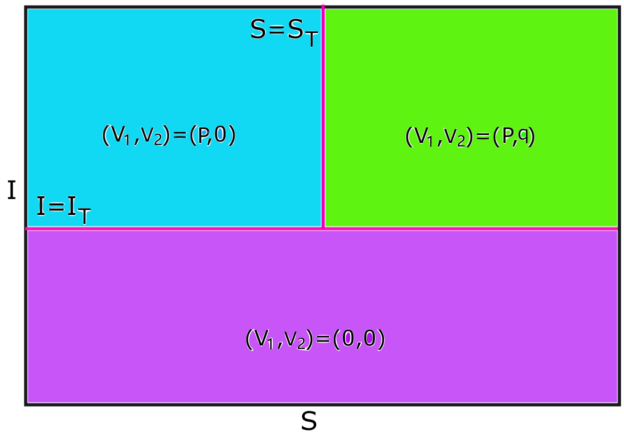


Figure 1. Schematic diagram of the broken line control strategy.

Based on system (2.1) and the broken line control strategy, in this paper, we propose a Filippov influenza system with media control and quarantine of susceptible populations as follows:

$$\begin{cases} S_t = \gamma S\left(1 - \dfrac{S}{K}\right) - \beta(1 - v_1)SI - v_2 S, \\ I_t = \beta(1 - v_1)SI - (d + \delta + r)I, \\ R_t = rI - dI \end{cases} \tag{2.2}$$

with

$$(v_1, v_2) = \begin{cases} (0, 0), & \text{for } I < I_T, \\ (p, 0), & \text{for } I > I_T \text{ and } S < S_T, \\ (p, q), & \text{for } I > I_T \text{ and } S > S_T. \end{cases} \tag{2.3}$$

Notice that the third equation R does not contain the variables S and I of (2.2), so we can not consider the third equation R of the system. Then, in this paper, we investigate a Filippov system

$$\begin{cases} S_t = \gamma S\left(1 - \dfrac{S}{K}\right) - \beta(1 - v_1)SI - v_2 S, \\ I_t = \beta(1 - v_1)SI - (d + \delta + r)I. \end{cases} \tag{2.4}$$

Then, the first quadrant is divided by the following five regions:

$$\begin{aligned}\Gamma_1 &= \{(S, I) \in \mathbb{R}_+^2 : I < I_T\},\\ \Gamma_2 &= \{(S, I) \in \mathbb{R}_+^2 : I > I_T \text{ and } S < S_T\},\\ \Gamma_3 &= \{(S, I) \in \mathbb{R}_+^2 : I > I_T \text{ and } S > S_T\},\\ \Pi_1 &= \{(S, I) \in \mathbb{R}_+^2 : I = I_T\},\\ \Pi_2 &= \{(S, I) \in \mathbb{R}_+^2 : I > I_T \text{ and } S = S_T\}.\end{aligned}$$

The non-smooth system in region Γ_i for $i = 1, 2, 3$ is described by

$$\begin{aligned}\begin{pmatrix} S_t \\ I_t \end{pmatrix} &= \begin{pmatrix} \gamma S(1-\frac{S}{K}) - \beta S I \\ \beta S I - (d+\delta+r)I \end{pmatrix} = F_1(S, I), && (S, I) \in \Gamma_1;\\ \begin{pmatrix} S_t \\ I_t \end{pmatrix} &= \begin{pmatrix} \gamma S(1-\frac{S}{K}) - \beta(1-p) S I \\ \beta(1-p) S I - (d+\delta+r)I \end{pmatrix} = F_2(S, I), && (S, I) \in \Gamma_2;\\ \begin{pmatrix} S_t \\ I_t \end{pmatrix} &= \begin{pmatrix} \gamma S(1-\frac{S}{K}) - \beta(1-p) S I - qS \\ \beta(1-p) S I - (d+\delta+r)I \end{pmatrix} = F_3(S, I), && (S, I) \in \Gamma_3.\end{aligned} \tag{2.5}$$

2.2. *Basic knowledge*

The normal vector of the system is defined as $n_1 = (0, 1)^T$ in Π_1, while the normal vector of the system is $n_2 = (1, 0)^T$ in Π_2 . Denote the right-hand side of system (2.4) by f. The following definitions are necessary in this paper [14, 46, 47].

Definition 1. *[14] A point E is called a real equilibrium of system (2.4) if there exists* $i \in \{1, 2, 3\}$ *such that* $F_i(E) = 0$ *and* $E \in \Gamma_i$*, denoted by* E^R.

Definition 2. *[14] A point E is called a virtual equilibrium of system (2.4) if there exists* $i \in \{1, 2, 3\}$ *such that* $F_i(E) = 0$ *and* $E \notin \overline{\Gamma_i}$*, where* $\overline{\Gamma_i}$ *is the closure of* Γ_i*, denoted by* E^V.

Denote the equations that describe the sliding mode dynamics on the sliding mode domain $\ell_i \subset \Pi_i$ by $F_{si}(S, I), i \in \{1, 2\}$.

Definition 3. *[14] A point E is called a pseudo-equilibrium of system (2.4) if point E is an equilibrium of* $F_{si}(S, I)$ *on sliding mode domain* ℓ*. That is,* $F_{si}(E) = 0$*, and point* $E \in \ell \subset \Pi_i, i \in \{1, 2\}$.

2.3. *Positiveness, boundedness and dynamics analysis of (2.4) in* Γ_i*(i=1,2,3)*

Lemma 1. *The solution* $(S(t), I(t))$ *of system (2.4) with the positive initial value* $S(0)$ *and* $I(0)$ *satisfy* $S(t) > 0$ *and* $I(t) > 0$ *for* $t \in [0, +\infty)$.

Proof. First, we prove that $S(t) > 0$. If not, we assume that there is a time t_1 satisfying the solution $S(t_1) \le 0$. Then we know that there exists the other time $t^* > 0$ such that the solution $S(t^*) = 0$ and $S(t) > 0$ for $t \in [0, t^*)$. From the first equation of system (2.4), one has

$$\frac{dS}{dt} = \gamma S\left(1-\frac{S}{K}\right) - \beta(1-v_1)SI - v_2 S = S\left[r\left(1-\frac{S}{K}\right) - \beta(1-v_1)I - v_2\right].$$

Further, we obtain

$$S(t^*) = S(0)e^{\int_0^{t^*}\left[r\left(1-\frac{S}{K}\right)-\beta(1-v_1)I-v_2\right]ds} > 0,$$

which is a contradiction with $S(t^*) = 0$. Thus, $S(t) > 0$, for all $t > 0$.

Next, we prove that $I(t) > 0$. If not, we assume there is a time t_2 satisfying the solution $I(t_2) \leq 0$. Then, there exists the other time $\tilde{t} > 0$ such that the solution $I(\tilde{t}) = 0$, and $I(t) > 0$ for time $t \in [0, \tilde{t})$. From the second equation of system (2.4), one gets

$$\frac{dI}{dt} = I\left[(\beta(1 - v_1)S - (d + \delta + r)\right] \geq -(d + \delta + r)I, \quad t \in [0, \tilde{t}].$$

Then, if $t = \tilde{t}$, we have

$$I(\tilde{t}) \geq I_0 e^{-(d+\delta+r)\tilde{t}} > 0,$$

which is a contradiction with $I(\tilde{t}) = 0$. Thus, $I(t) > 0$, for all $t > 0$. In a word, we have that $(S(t), I(t))$ of (2.4) with the positive initial value $S(0)$ and $I(0)$ satisfies $S(t) > 0$ and $I(t) > 0$ for $t \in [0, +\infty)$. The proof is finished.

Lemma 2. *The solution (S, I) of system (2.4) is bounded.*

Proof. Notice the first equation of system (2.4). Then,

$$\frac{dS}{dt}\Big|_{x=K} = -\beta(1 - v_1)KI - v_2 S < 0, \quad \text{and} \quad \frac{dS}{dt}\Big|_{x>K} < 0.$$

Hence, there is a positive number T such that the solution $S(t) < K$ for $t \geq T$.

Let $N = S + I$, and then for $t \geq T$, we obtain

$$\begin{aligned}\frac{dN}{dt} &= \gamma S(1 - \frac{S}{K}) - v_2 S - (d + \delta + r)I \\ &\leq \gamma K(1 - \frac{S}{K}) - v_2 S - (d + \delta + r)I \\ &\leq \gamma K - \min\{\gamma + v_2, d + \delta + r\}N.\end{aligned}$$

Moreover,

$$\limsup_{t\to\infty} N \leq \frac{\gamma K}{\min\{\gamma + v_2, d + \delta + r\}}.$$

Therefore, under Lemma 1, we know that the solution of (2.4) is bounded. The proof is finished.

Next, we give the basic reproduction number $R_{0i}, i = 1, 2, 3$, of the system. For example, in region Γ_1, the basic reproduction number is defined as $R_{01} = \frac{K\beta}{d+\delta+r}$. In region Γ_2, the basic reproduction number is defined as $R_{02} = \frac{K\beta(1-p)}{d+\delta+r}$. In region Γ_3, the basic reproduction number is defined as $R_{03} = \frac{(\gamma-p)K\beta(1-p)}{\gamma(d+\delta+r)}$.

For system (2.4), clearly, in region Γ_1, (2.4) has three equilibria, i.e., $E_{10} = (0, 0)$, $E_{11} = (K, 0)$ and $E_1 = (S_1^*, I_1^*) = (\frac{d+\delta+r}{\beta}, \frac{\gamma}{\beta}(1 - \frac{d+\delta+r}{K\beta}))$, which is a stable node if $R_{01} > 1$.

In region Γ_2, system (2.4) has three equilibria, i.e., $E_{20} = (0, 0)$, $E_{21} = (K, 0)$ and $E_2 = (S_2^*, I_2^*) = (\frac{d+\delta+r}{\beta(1-p)}, \frac{\gamma}{\beta(1-p)}(1 - \frac{d+\delta+r}{K\beta(1-p)}))$, which is a stable node (focus) if $R_{02} > 1$.

In region Γ_3, system (2.4) has three equilibria, i.e., $E_{30} = (0, 0)$, $E_{31} = (K\frac{\gamma-q}{\gamma}, 0)$ and $E_3 = (S_3^*, I_3^*) = (\frac{d+\delta+r}{\beta(1-p)}, \frac{\gamma}{\beta(1-p)}(1 - \frac{d+\delta+r}{K\beta(1-p)} - \frac{q}{r}))$, which is a stable node (focus) if $R_{03} > 1$.

Theorem 1. *Suppose that $R_{0i} < 1$, and the disease free equilibrium $E_{i1} (i = 1, 2, 3)$ of (2.4) is globally asymptotically stable. In addition, the endemic equilibrium $E_i (i = 1, 2, 3)$ of (2.4) is globally asymptotically stable if $R_{0i} > 1$.*

Proof. Since $R_{0i} < 1, i = 1, 2, 3$, a Lyapunov function is considered as

$$L(t) = S - S_{i1} - S_{i1} \ln \frac{S}{S_{i1}} + I.$$

Applying LaSalle's invariance principle [13, 14], we know that $E_{i1}(i = 1, 2, 3)$ of system (2.4) is globally asymptotically stable.

When $R_{0i} > 1, i = 1, 2, 3$, the following Lyapunov function is considered:

$$L(t) = S - S_i^* - S_i^* \ln \frac{S}{S_i^*} + I - I_i^* - I_i^* \ln \frac{I}{I_i^*}.$$

Using LaSalle's invariance principle [13, 14], we know that points $E_i(i = 1, 2, 3)$ of system (2.4) are-globally asymptotically stable. The proof is finished.

This paper only considers the global dynamics of (2.4) under case $R_{0i} > 1$ $(i = 1, 2, 3)$. Next, we aim to address the richness of the possible equilibria and sliding modes on Π_1 and Π_2 that the system with can exhibit.

From (2.4), when $S_1^* < S_2^* = S_3^*$ and $I_3^* < I_2^*$, we consider the following three cases: $S_T < S_1^*, S_1^* < S_T < S_2^* = S_3^*$, and $S_2^* = S_3^* < S_T$ with varied I_T. Further, according to the dynamics in each case, the biological phenomena of (2.4) are described in this section.

Throughout the paper, the S-nullclines and I-nullclines of (2.4) are represented by the dashed curves and dash-dot lines, respectively. Thus, $S = S_1^*, S_2^*$ and S_3^* are the I-nullclines of F_1, F_2 and F_3, denoted by L_{12}, L_{22} and L_{32}, respectively. That is, the curves

$$L_{12} := \left\{(S, I) \in \Gamma_1 : \gamma S \left(1 - \frac{S}{K}\right) - \beta S I = 0\right\},$$

$$L_{22} := \left\{(S, I) \in \Gamma_2 : \gamma S \left(1 - \frac{S}{K}\right) - \beta(1 - p) S I = 0\right\}$$

and

$$L_{32} := \left\{(S, I) \in \Gamma_3 : \gamma S \left(1 - \frac{S}{K}\right) - \beta(1 - p) S I - qS = 0\right\}$$

are the S-nullclines of systems F_1, F_2 and F_3, denoted by L_{11}, L_{21} and L_{31}, respectively.

3. Sliding dynamics and bifurcations of (2.4) under Case 1: $S_T < S_1^* < S_2^* = S_3^*$

In this part, we first consider sliding mode dynamics of (2.4) on Π_1 under Case 1: $S_T < S_1^* < S_2^* = S_3^*$. Second, the sliding mode dynamics on Π_2 are also given under Case 1: $S_T < S_1^* < S_2^* = S_3^*$. In addition, we investigate the bifurcations of (2.4) under Case 1: $S_T < S_1^* < S_2^* = S_3^*$. Finally, some numerical simulations are displayed to confirm the results.

3.1. *Sliding mode dynamics of (2.4) on* Π_1 *under Case 1:* $S_T < S_1^* < S_2^* = S_3^*$

This part investigates the existence of the sliding mode region on Π_1. Based on Definition 3, if $\langle n_1, F_1 \rangle > 0$ and $\langle n_1, F_3 \rangle < 0$ hold, then we know that there is the sliding mode region ℓ_1, which is expressed as

$$\ell_1 = \{(S, I) \in \Pi_1 : S_1^* < S < S_3^*\}.$$

Using the Filippov convex method [13, 14], we obtain

$$\begin{pmatrix} S_t \\ I_t \end{pmatrix} = \lambda_1 F_1(S, I) + (1 - \lambda_1) F_3(S, I), \text{where } \lambda_1 = \frac{\langle n_1, F_3 \rangle}{\langle n_1, F_3 - F_1 \rangle}.$$

So, we have the differential equations describing the sliding mode dynamics along the manifold ℓ_1 for system (2.4):

$$\begin{pmatrix} S_t \\ I_t \end{pmatrix} = \begin{pmatrix} \gamma S(1 - \frac{S}{K}) - \frac{q}{p}S - (d + \delta + r)I_T + \frac{(d+\delta+r)q}{\beta p} \\ 0 \end{pmatrix}. \quad (3.1)$$

Next, we analyze the existence of the positive equilibriums on ℓ_1 of (3.1). Let

$$\Delta_1 = (\gamma - \frac{q}{p})^2 - 4\frac{\gamma}{K}[(d + \delta + r)I_T - \frac{(d + \delta + r)q}{\beta p}], I_T^* = \frac{q}{\beta p} + \frac{(\gamma - \frac{q}{p})^2 K}{4\gamma(d + \delta + r)}.$$

Proposition 1. *For varied I_T, we have the following results.*

- *If $I_T > I_T^*$, then system (3.1) has no equilibrium.*
- *If $I_T^* > I_T > \frac{q}{\beta p}$, system (3.1) has two positive equilibria $E_{s1}^\pm = (S_{s1}^\pm, I_T)$, where $S_{s1}^\pm = \frac{(\gamma - \frac{q}{p}) \pm \sqrt{\Delta_1}}{2\gamma} K$.*
- *If $I_T < \frac{q}{\beta p}$, then system (3.1) has a unique positive equilibrium $E_{s2} = (S_{s2}, I_T)$, where $S_{s2} = \frac{(\gamma - \frac{q}{p}) + \sqrt{\Delta_1}}{2\gamma} K$.*

In addition, when the sliding mode ℓ_1 has a pseudo-equilibrium, we have

$$S_T^* = \frac{\gamma - \frac{q}{p}}{2\gamma} K. \quad (3.2)$$

Proposition 2. *Under the condition $S_T^* < S_1^*$, $E_{s1}^- \notin \ell_1$, and the following results are given.*

- *If $I_T < I_3^*$, we have $E_{s1}^+ \notin \ell_1$.*
- *If $I_3^* < I_T < I_1^*$, we have $E_{s1}^+ \in \ell_1$.*
- *If $I_T > I_1^*$, we have $E_{s1}^+ \notin \ell_1$.*

Proposition 3. *Under the condition $S_1^* < S_T^* < S_3^*$, the following results hold.*

(1) Assume that $I_1^* < I_3^*$, and then

- *if $I_1^* < I_T < I_3^*$, we have $E_{s1}^- \in \ell_1$, $E_{s1}^+ \notin \ell_1$;*
- *if $I_3^* < I_T < I_T^*$, we have $E_{s1}^- \in \ell_1$, $E_{s1}^+ \in \ell_1$.*

(2) Assume that $I_1^* > I_3^*$, and then

- *if $I_3^* < I_T < I_1^*$, we have $E_{s1}^- \notin \ell_1$, $E_{s1}^+ \in \ell_1$;*

- *if* $I_1^* < I_T < I_T^*$*, we have* $E_{s1}^- \in \ell_1$, $E_{s1}^+ \in \ell_1$.

Proposition 4. *Under the condition* $S_T^* > S_3^*$, $E_{s1}^+ \notin \ell_1$*, and the following conclusions hold.*

- *If* $I_T < I_1^*$*, we have* $E_{s1}^- \notin \ell_1$.
- *If* $I_1^* < I_T < I_3^*$*, we have* $E_{s1}^- \in \ell_1$.
- *If* $I_T > I_3^*$*, we have* $E_{s1}^- \notin \ell_1$.

Theorem 2. *If* $I_T^* > I_T > \frac{q}{\beta p}$*, then a stable pseudo-equilibriums* E_{s1}^+ *is located on the sliding mode* ℓ_1*, and the unstable pseudo-equilibriums* E_{s1}^- *is located on the sliding mode* ℓ_1.

Proof. Notice that

$$\frac{\partial}{\partial S}\left(rS(1-\frac{S}{K})-\frac{q}{p}S-(d+\delta+r)I_T+\frac{(d+\delta+r)q}{\beta p}\right)\Bigg|_{S_{s1}^{\pm}} = \mp\sqrt{\Delta_1}.$$

Therefore, the point E_{s1}^+ is attracting, and the point E_{s1}^- is repelling.

Proposition 5. *Assume that* $\frac{\gamma-\frac{q}{p}}{\gamma}K > \frac{d+\delta+r}{\beta(1-p)}$*, and then the pseudo-equilibrium* E_{s2} *is not located on* ℓ_1.

Proof. From Proposition 1, Eq (3.2) and the condition of Proposition 5, one has

$$S_{s2} = \frac{(\gamma-\frac{q}{p})+\sqrt{\Delta_1}}{2\gamma}K > 2S_T^* = 2\frac{\gamma-\frac{q}{p}}{\gamma}K > \frac{d+\delta+r}{\beta(1-p)} = S_3^*,$$

which implies that $S_{s2} \geq 2S_T^* > S_3^*$. Based on the definition of sliding mode region ℓ_1, we know that $S_{s2} \notin \ell_1$. Then, ℓ_1 does not have a pseudo-equilibrium E_{s2}. The proof is completed.

3.2. *Sliding mode dynamics on* Π_2 *under Case 1:* $S_T < S_1^* < S_2^* = S_3^*$

Let

$$J_1 = \frac{\gamma}{\beta(1-p)}(1-\frac{S_T}{K}-\frac{q}{\gamma}),\ J_2 = \frac{\gamma}{\beta(1-p)}(1-\frac{S_T}{K}).$$

Clearly $J_1 < J_2$. A subset of Π_2 is a sliding mode domain if $\langle F_2, n_2\rangle\langle F_3, n_2\rangle < 0$. If $I_T > J_2$, there is no sliding mode domain on Π_2. If $I_T < J_2$, then the sliding mode domain of (2.4) on Π_2 is given as

$$\ell_2 = \{(S, I) \in \Pi_2 : \max\{I_T, J_1\} < I < J_2\}. \tag{3.3}$$

Using the Filippov convex method [13], we have the differential equations describing the sliding mode dynamics along the manifold ℓ_2 for system (2.4) with (2.3):

$$\begin{pmatrix} S_t \\ I_t \end{pmatrix} = \begin{pmatrix} 0 \\ \beta(1-p)S_T I-(d+\delta+r)I \end{pmatrix}. \tag{3.4}$$

Clearly, there is not a positive equilibrium. Thus, if there exists a sliding domain ℓ_2 on Π_2, we know the system does not have a pseudo-equilibrium.

3.3. Bifurcations of (2.4) under Case 1: $S_T < S_1^* < S_2^* = S_3^*$

Due to $S_T < S_1^* < S_2^* = S_3^*$, we know that $E_2 \notin \Gamma_2$ is a virtual equilibrium, denoted by E_2^V. However, point E_1 and point E_3 may exist depending on I_T, and we have the following three Propositions:

Proposition 6. *Assume that $S_T^* < S_1^*$, and the following assertions hold.*

- *If $I_T < I_3^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$.*
- *If $I_3^* < I_T < I_1^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \in \ell_1, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$.*
- *If $I_1^* < I_T < I_T^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.*
- *If $I_T > I_T^*$, we have E_{s1}^- and E_{s1}^+ don't exist, $E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.*

Proposition 7. *Under the condition $S_1^* < S_T^* < S_3^*$, the following assertions hold.*

(1) Assume that $I_1^* < I_3^*$, and further,

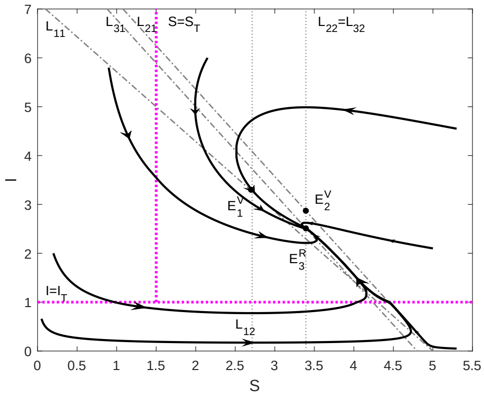
- *if $I_T < I_1^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;*
- *if $I_1^* < I_T < I_3^*$, we have $E_{s1}^- \in \ell_1, E_{s1}^+ \notin \ell_1, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;*
- *if $I_3^* < I_T < I_T^*$, we have $E_{s1}^- \in \ell_1$, $E_{s1}^+ \in \ell_1, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;*
- *if $I_T > I_T^*$, we have E_{s1}^- and E_{s1}^+ don't exist, $E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.*

(2) Assume that $I_1^* > I_3^*$, and further,

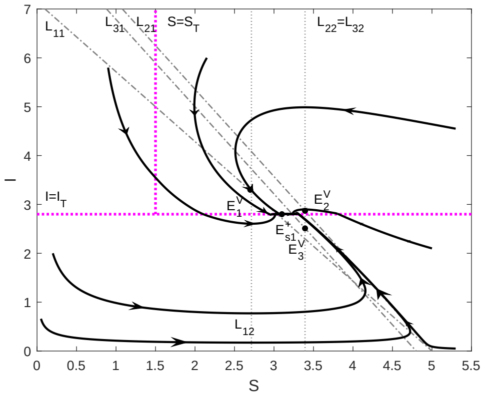
- *if $I_T < I_3^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;*
- *if $I_3^* < I_T < I_1^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \in \ell_1, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;*
- *if $I_1^* < I_T < I_T^*$, we have $E_{s1}^- \in \ell_1$, $E_{s1}^+ \in \ell_1, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;*
- *if $I_T > I_T^*$, we have E_{s1}^- and E_{s1}^+ don't exist, $E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.*

Proposition 8. *Under the condition $S_T^* > S_3^*$, the following assertions hold.*

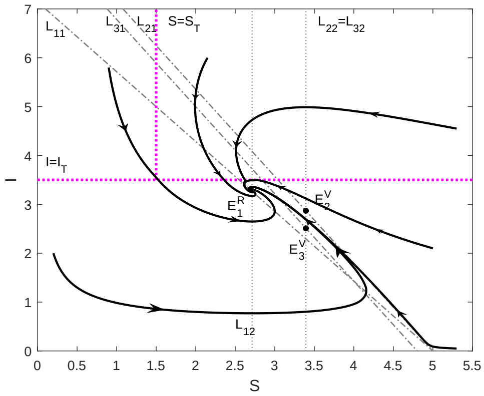
- *If $I_T < I_1^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$.*
- *If $I_1^* < I_T < I_3^*$, we have $E_{s1}^- \in \ell_1, E_{s1}^+ \notin \ell_1, E_1 \in \Gamma_1, E_3 \in \Gamma_3$.*
- *If $I_3^* < I_T < I_T^*$, we have $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.*
- *If $I_T > I_T^*$, we have E_{s1}^- and E_{s1}^+ don't exist, $E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.*



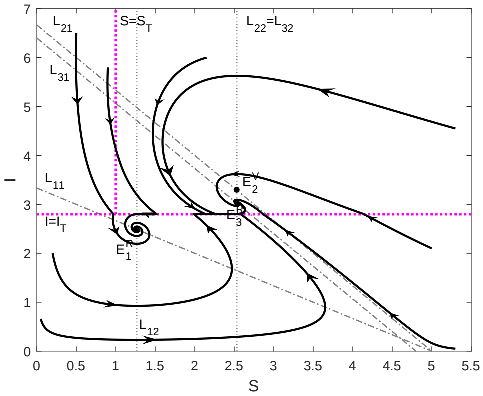
(a)



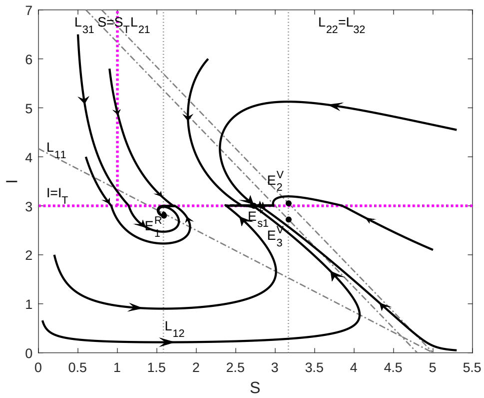
(b)



(c)



(d)



(e)

Figure 2. The trajectory of (2.4) under Case 1: $S_T < S_1^* < S_2^* = S_3^*$, $d = 0.01, \delta = 0.01, r = 0.01, \gamma = 0.2, K = 3$, (a) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 1.5, I_T = 1$; (b) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 1.5, I_T = 2.8$; (c) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 1.5, I_T = 3.5$; (d) where $\beta = 1.5, p = 0.5, q = 0.2, S_T = 1, I_T = 2.8$; (e) where $\beta = 1.2, p = 0.5, q = 0.2, S_T = 1, I_T = 3$.

Based on Propositions 6–8, we have the following summary:

B1. Let $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set B_{1-1}. Then, we conclude that there does not exist a pseudo-equilibrium, and all trajectories of the system (2.4) will converge to E_3^R, as shown in Figure 2(a), where

$$B_{1-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_T < S_1^*, I_T < \min\{I_1^*, I_3^*\}\right\}.$$

B2. Let $E_{s1}^- \notin \ell_1, E_{s1}^+ \in \ell_1, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set B_{2-1}. Then, we know that $E_{s1}^+ \in \ell_1 \subset \Pi_1$ is a stable pseudo-equilibrium, and all solutions of the system will approach E_{s1}^+, as shown in Figure 2(b), where

$$B_{2-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_T < S_1^*, I_3^* < I_T < I_1^*\right\}.$$

B3. Let $E_{s1}^- \notin \ell_1, E_{s1}^+ \notin \ell_1, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set $B_{3-1} \cup B_{3-2} \cup B_{3-3}$. Then, we conclude that the system (2.4) does not have a pseudo-equilibrium, and all trajectories of the system (2.4) will converge to the equilibrium point E_1^R, as shown in Figure 2(c), where

$$\begin{aligned} B_{3-1} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_T < S_1^*, I_1^* < I_T < I_T^*, \text{ if } S_T^* < S_1^*\right\}, \\ B_{3-2} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_T < S_1^*, I_T > I_T^*, \text{ if } S_1^* < S_T^* < S_3^*\right\}, \\ B_{3-3} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_T < S_1^*, I_3^* < I_T < I_T^*, \text{ if } S_T^* > S_3^*\right\}. \end{aligned}$$

B4. Let $E_{s1}^- \in \ell_1, E_{s1}^+ \notin \ell_1, E_1 \in \Gamma_1, E_3 \in \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set B_{4-1}. Then, we know that $E_{s1}^- \in \ell_1 \subset \Pi_1$ is an unstable pseudo-equilibrium, and all solutions will approach E_1^R or E_3^R. The result of this numerical simulation is shown in Figure 2(d), where

$$B_{4-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_T < S_1^*, I_1^* < I_T < I_3^*\right\}.$$

B5. Let $E_{s1}^- \in \ell_1, E_{s1}^+ \in \ell_1, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set B_{5-1}. Then, we conclude that $E_{s1}^- \in \ell_1 \subset \Pi_1$ is an unstable pseudo-equilibrium, and all solutions of the system (2.4) will tend to E_1^R or E_{s1}^+. The result of this numerical simulation is shown in Figure 2(e), where

$$B_{5-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_T < S_1^*, \max\{I_1^*, I_3^*\} < I_T < I_T^*, \text{ if } S_1^* < S_T^* < S_3^*\right\}.$$

For the B1 of case 1, when system (2.4) has a unique equilibrium E_3^R, we have the following.

Theorem 3. *If $S_T < S_1^* < S_3^* = S_2^*$ and $I_T <$ min$\{I_1^*, I_3^*\}$, then the point E_3^R of system (2.4) is globally asymptotically stable.*

Proof. Suppose that the system (2.4) has a closed orbit U (shown in Figure 3(a)) that surrounds the real equilibrium E_{32}^R and the sliding mode ℓ_1. Define $U = U_1 + U_2 + U_3$, where $U_i = U \cap \Gamma_i, i = 1, 2, 3$. Let Ω be the bounded region delimited by U and $\Omega_i = \Omega \cap \Gamma_i$ for $i = 1, 2, 3$. Considering the Dulac function $D = \frac{1}{SI}$. Three Steps are given as follows:

Step 1: System (2.4) does not have a closed orbit in region $\Gamma_i, i = 1, 2, 3$.

$$\iint_U \left(\frac{\partial (Df_1)}{\partial S} + \frac{\partial (Df_2)}{\partial I}\right) dS\,dI = \sum_{i=1}^{3} \iint_{U_i} \left(\frac{\partial (DF_{i1})}{\partial S} + \frac{\partial (DF_{i2})}{\partial I}\right) dS\,dI$$
$$= -\frac{3\gamma}{K} \sum_{i=1}^{3} \iint_{U_i} \frac{1}{I} dS\,dI < 0,$$

where f_1 is the first component of f, and f_2 is the second component of f. F_{i1} is the first component of F_i and F_{i2} is the second component of $F_i, i = 1, 2, 3$. Let $\tilde{\Omega}_i$ be the region bounded by $\tilde{U}_i, P_i$ and Q_i, where $\tilde{\Omega}_i$ and $\tilde{U}_i$ depend on ϵ and converge to Ω_i and U_i as ϵ approaches 0.

Step 2: System (2.4) does not have a closed trajectory in region Γ.

We can get

$$\iint_{\Omega_i} \left(\frac{\partial (DF_{i1})}{\partial S} + \frac{\partial (DF_{i2})}{\partial I}\right) dS\,dI = \lim_{\epsilon\to 0} \iint_{\tilde{\Omega}_i} \left(\frac{\partial (DF_{i1})}{\partial S} + \frac{\partial (DF_{i2})}{\partial I}\right) dS\,dI.$$

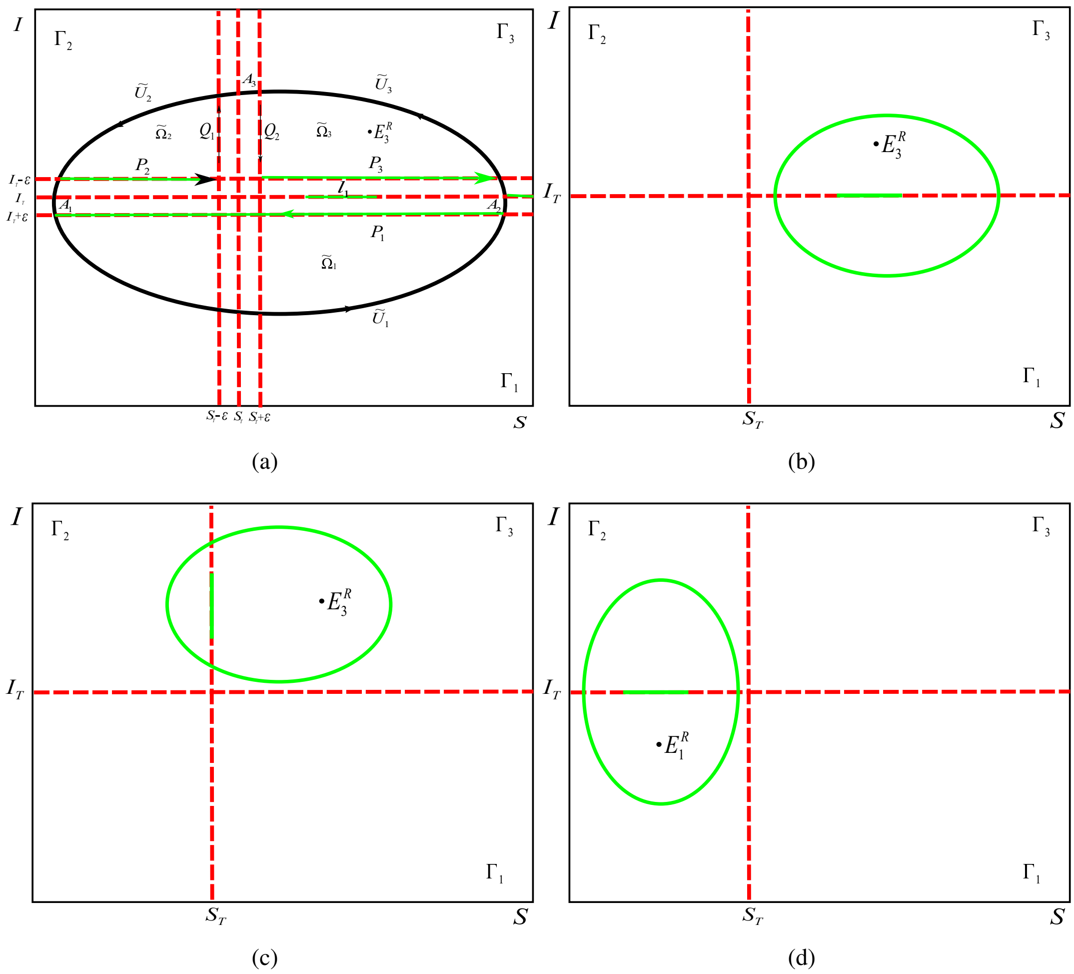


Figure 3. The schematic diagram of limit cycle.

Since $dS = F_{11}dt$ and $dI = F_{12}dt$ along $\tilde{U}_1$ and $dI = 0$ along P_1, by using Green's theorem, for region $\tilde{\Omega}_1$, we have

$$\begin{aligned}\iint_{\tilde{\Omega}_1}\left(\frac{\partial(DF_{11})}{\partial S}+\frac{\partial(DF_{12})}{\partial I}\right)dS\,dI &= \oint_{\partial\tilde{\Omega}_1} DF_{11}dI - DF_{12}dS \\ &= \int_{\tilde{U}_1} DF_{11}dI - DF_{12}dS + \int_{P_1} DF_{11}dI - DF_{12}dS \\ &= -\int_{P_1} DF_{12}dS.\end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\iint_{\tilde{\Omega}_2}\left(\frac{\partial(DF_{21})}{\partial S}+\frac{\partial(DF_{22})}{\partial I}\right)dS\,dI = -\int_{P_2} DF_{22}dS + \int_{Q_2} DF_{21}dI \tag{3.6}$$

and

$$\iint_{\tilde{\Omega}_3}\left(\frac{\partial(DF_{31})}{\partial S}+\frac{\partial(DF_{32})}{\partial I}\right)dS\,dI = -\int_{P_3} DF_{32}dS + \int_{Q_3} DF_{31}dI. \tag{3.7}$$

From Eqs (3.5)–(3.7), we have

$$\begin{aligned} 0 &> \sum_{i=1}^{3}\iint_{\Omega_i}\left(\frac{\partial(DF_{i1})}{\partial S}+\frac{\partial(DF_{i2})}{\partial I}\right)dS\,dI \\ &= \lim_{\epsilon\to 0}\sum_{i=1}^{3}\iint_{\tilde{\Omega}_i}\left(\frac{\partial DF_{i1}}{\partial S}+\frac{\partial DF_{i2}}{\partial I}\right)dS\,dI \\ &= \lim_{\epsilon\to 0}\left(-\int_{P_1} DF_{12}dS - \int_{P_2} DF_{22}dS + \int_{Q_2} DF_{21}dI - \int_{P_3} DF_{32}dS + \int_{Q_3} DF_{31}dI\right).\end{aligned} \tag{3.8}$$

Denote the intersection points of the closed trajectory U and the line $I = I_T$ by A_1 and A_2 and the intersection point of U and line $S = S_T$ if $I > I_T$ by A_3. In addition, denote the intersection point of the line $I = I_T$ and the line $S = S_T$ by E_T. Note that $A_{1S} < S_T < A_{2S}$ and $A_{3I} > I_T$. Then, Eq (3.6) becomes

$$\begin{aligned} 0 >& -\int_{A_{2S}}^{A_{1S}}\left(\beta-\frac{d+\delta+r}{S}\right)dS - \int_{A_{1S}}^{S_T}\left(\beta(1-p)-\frac{d+\delta+r}{S}\right)dS + \int_{I_T}^{A_{3I}}\left(\frac{\gamma(1-\frac{S}{K})}{I}-\beta(1-p)\right)dI \\ & - \int_{S_T}^{A_{2S}}\left(\beta(1-p)-\frac{d+\delta+r}{S}\right)dS + \int_{A_{3I}}^{I_T}\left(\frac{\gamma(1-\frac{S}{K})-q}{I}-\beta(1-p)\right)dI \\ =& -\int_{A_{1S}}^{A_{2S}} -p\,dS + \int_{A_{3I}}^{I_T}\left(-\frac{q}{I}\right)dI \\ =& p(A_{2S}-A_{1S}) + q\ln\left(\frac{A_{3I}}{I_T}\right) \\ >& 0, \end{aligned} \tag{3.9}$$

which is a contradiction. Therefore, we know that there does not have a closed orbit U surrounding the sliding mode ℓ_1 and the real equilibrium E_3^R.

Step 3: System (2.4) does not have a closed trajectory in regions Γ_i and $\Gamma_j (i \neq j)$. With a similar proof procedure to Step 2, it is easy to that there is no closed trajectory in Γ_1 and Γ_2 (see Figure 3(d)), Γ_1 and Γ_3 (see Figure 3(b)) , Γ_2 and Γ_3 (see Figure 3(c)), respectively.

Therefore, based on Steps 1–3, we know that the point E_3^R of (2.4) is globally asymptotically stable if $S_T < S_1^* < S_3^* = S_2^*$ and $I_T < \min\{I_1^*, I_3^*\}$. This completes this theorem.

With a similar proof procedure to Theorem 3, we have the following.

Theorem 4. *If $S_T < S_1^* < S_2^* = S_3^*$ and the conditions of (B3) hold, then the point E_1^R of system (2.4) is globally asymptotically stable.*

4. Sliding dynamics and bifurcations of (2.4) under Case 2: $S_1^* < S_T < S_2^* = S_3^*$

In this part, we first consider sliding mode dynamics of (2.4) on Π_1 under Case 2: $S_1^* < S_T < S_2^* = S_3^*$. Second, we study the sliding mode dynamics on Π_2 under Case 2: $S_1^* < S_T < S_2^* = S_3^*$. In addition, the bifurcations of (2.4) are investigated under Case 2: $S_1^* < S_T < S_2^* = S_3^*$. Finally, some numerical simulations are displayed to confirm the results.

4.1. Sliding mode dynamics of system (2.4) on Π_1 under Case 2: $S_1^* < S_T < S_2^* = S_3^*$

For Case 2, $S_1^* < S_T < S_2^* = S_3^*$, two sliding domains on Π_1 are given as

$$\ell_3 = \{(S, I) \in \Pi_1 : S_1^* < S < S_T\}, \qquad \ell_4 = \{(S, I) \in \Pi_1 : S_T < S < S_3^*\}.$$

The dynamics on ℓ_3 are governed by

$$\begin{pmatrix} S_t \\ I_t \end{pmatrix} = \begin{pmatrix} \gamma S(1 - \frac{S}{K}) - (d + \delta + r) I_T \\ 0 \end{pmatrix}. \tag{4.1}$$

Now, we investigate the existence of a positive equilibrium on ℓ_3 of system (4.1). Let

$$\Delta_2 = \gamma^2 - 4\frac{\gamma(d + \delta + r) I_T}{K}, I_T^{*'} = \frac{\gamma K}{4(d + \delta + r)}.$$

Proposition 9. *For varied I_T, we have the following results.*

(1) If $I_T > I_T^{*'}$, then system (4.1) does not have an equilibrium;

(2) If $0 < I_T < I_T^{*'}$, system (4.1) has two positive equilibria $E_{s3}^{\pm} = (S_{s3}^{\pm}, I_T)$, where $S_{s3}^{\pm} = \frac{\gamma \pm \sqrt{\Delta_2}}{2\gamma} K$.

Next, we find the conditions of the pseudo-equilibrium on the sliding mode ℓ_3. Let

$$H_1 = \frac{-\frac{\gamma}{K}}{d + \delta + r} {S_T}^2 + \frac{\gamma}{d + \delta + r} S_T, S_T^{*'} = \frac{K}{2}.$$

Proposition 10. *Under the condition $S_T^{*'} < S_1^* < S_T$, $E_{s3}^- \notin \ell_3$, and the following assertions hold.*

(1) If $I_T > I_1^*$, we have $E_{s3}^+ \notin \ell_3$;

(2) If $H_1 < I_T < I_1^*$, we have $E_{s3}^+ \in \ell_3$;

(3) If $I_T < H_1$, we have $E_{s3}^{+} \notin \ell_3$.

Proposition 11. *Under the condition $S_1^* < S_T^{*'} < S_T$, we have the following results.*

(1) If $I_T < min\{I_1^*, H_1\}$, we have $E_{s3}^{\pm} \notin \ell_3$.

(2) If $min\{I_1^*, H_1\} < I_T < max\{I_1^*, H_1\}$, we have

- $E_{s3}^{-} \in \ell_3, E_{s3}^{+} \notin \ell_3$ if $I_1^* < H_1$;
- $E_{s3}^{-} \notin \ell_3, E_{s3}^{+} \in \ell_3$ if $I_1^* > H_1$.

(3) If $max\{I_1^*, H_1\} < I_T < I_T^{*'}$, we have $E_{s3}^{\pm} \in \ell_3$.

Proposition 12. *Under the condition $S_1^* < S_T < S_T^{*'}$, $E_{s3}^{+} \notin \ell_3$, and the following assertions hold.*

(1) If $I_T < I_1^*$, we have $E_{s3}^{-} \notin \ell_3$;

(2) If $I_1^* < I_T < H_1$, we have $E_{s3}^{-} \in \ell_3$;

(3) If $I_T > H_1$, we have $E_{s3}^{-} \notin \ell_3$.

Theorem 5. *If $0 < I_T < I_T^{*'}$, then a stable pseudo-equilibrium E_{s3}^{+} of (2.4) is located on the sliding mode ℓ_3, and the unstable pseudo-equilibrium E_{s3}^{-} of (2.4) is located on the sliding mode ℓ_3.*

Proof. Notice that

$$\frac{\partial}{\partial S}\left(\gamma S(1-\frac{S}{K})-(d+\delta+r)I_T\right)\Big|_{E_{s3}^{\pm}} = \mp\sqrt{\Delta_2}.$$

Therefore, the point E_{s3}^{+} is attracting, and the point E_{s3}^{-} is repelling.

The dynamics on region ℓ_4 are described by (3.1). Let

$$H_2 = \frac{q}{\beta p} - \frac{\gamma}{K(d+\delta+r)}{S_T}^2 + (\frac{\gamma}{d+\delta+r} - \frac{q}{p(d+\delta+r)})S_T.$$

From Proposition 1, we have the following.

Proposition 13. *Under the condition $S_T^* < S_T < S_3^*$, $E_{s1}^{-} \notin \ell_4$, and the following assertions hold.*

(1) If $I_T < I_3^*$, we have $E_{s1}^{+} \notin \ell_4$;

(2) If $I_3^* < I_T < H_2$, we have $E_{s1}^{+} \in \ell_4$;

(3) If $I_T > H_2$, we have $E_{s1}^{+} \notin \ell_4$.

Proposition 14. *Under the condition $S_T < S_T^* < S_3^*$, we have the following results.*

(1) If $I_T < min\{I_3^*, H_2\}$, we have $E_{s1}^{\pm} \notin \ell_4$.

(2) If $min\{I_3^*, H_2\} < I_T < max\{I_3^*, H_2\}$, we have

- $E_{s1}^{-} \in \ell_4, E_{s1}^{+} \notin \ell_4$ if $I_3^* > H_2$;
- $E_{s1}^{-} \notin \ell_4, E_{s1}^{+} \in \ell_4$ if $I_3^* < H_2$.

(3) If $max\{I_3^*, H_2\} < I_T < I_T^*$, we have $E_{s1}^{\pm} \in \ell_4$.

Proposition 15. *Under the condition $S_T < S_3^* < S_T^*$, $E_{s1}^+ \notin \ell_4$, and the following assertions hold.*

(1) If $I_T < H_2$, we have $E_{s1}^- \notin \ell_4$;

(2) If $H_2 < I_T < I_3^*$, we have $E_{s1}^- \in \ell_4$;

(3) If $I_T > I_3^*$, we have $E_{s1}^- \notin \ell_4$.

Theorem 6. *If $0 < I_T < I_T^{*'}$, then a stable pseudo-equilibrium E_{s1}^+ is located on ℓ_4, and the unstable pseudo-equilibrium E_{s1}^- is located on ℓ_4.*

4.2. Sliding mode dynamics of (2.4) on Π_2 under Case 2: $S_1^* < S_T < S_2^* = S_3^*$

If $S_1^* < S_T < S_2^* = S_3^*$, the sliding mode dynamics on region Π_2 are the same as Section 3.2. We omit it here.

4.3. Bifurcations of system (2.4) under Case 2: $S_1^* < S_T < S_2^* = S_3^*$

For this case, we conclude that the point E_2 is a virtual equilibrium. E_1 and E_3 are changeable depending on I_T, and we have the following five situation.

Proposition 16. *Under the conditions $S_T^{*'} < S_1^* < S_T$ and $S_T^* < S_T < S_3^*$, the following assertions hold.*

(1) If $I_T < I_3^*$, we have $E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;

(2) If $I_3^* < I_T < H_2$, we have $E_{s3}^+ \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;

(3) If $H_2 < I_T < H_1$, we have $E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;

(4) If $H_1 < I_T < I_1^*$, we have $E_{s3}^+ \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;

(5) If $I_1^* < I_T < I_T^*$, we have $E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

Proposition 17. *Under the conditions $S_1^* < S_T^{*'} < S_T$ and $S_T^* < S_T < S_3^*$, the following assertions hold.*

(1) Assume that $I_1^* > H_1$, and further,

- *if $I_T < I_3^*$, we have $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;*
- *if $I_3^* < I_T < H_2$, we have $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;*
- *if $H_2 < I_T < H_1$, we have $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;*
- *if $H_1 < I_T < I_1^*$, we have $E_{s3}^- \notin \ell_3, E_{s3}^+ \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;*
- *if $I_1^* < I_T < I_T^{*'}$, we have $E_{s3}^- \in \ell_3, E_{s3}^+ \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.*

(2) Assume that $H_2 < I_1^* < H_1$, and further,

- *if $I_T < I_3^*$, we have $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;*
- *if $I_3^* < I_T < H_2$, we have $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;*
- *if $H_2 < I_T < I_1^*$, we have $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;*

- *if* $I_1^* < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(3) Assume that $I_3^* < I_1^* < H_2$, *and*

- *if* $I_T < I_3^*$, *then* $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $I_1^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(4) Assume that $I_1^* < I_3^*$, *and further,*

- *if* $I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_1^* < I_T < I_3^*$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \in \ell_3\ E_{s3}^+ \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

Proposition 18. *Under the conditions* $S_1^* < S_T < S_T^{*'}$ *and* $S_T^* < S_T < S_3^*$, *we have the following results.*

(1) Assume that $I_1^* > H_2$, *and further,*

- *if* $I_T < I_3^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < H_2$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_2 < I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $I_1^* < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(2) Assume that $I_3^* < I_1^* < H_2$, *and further,*

- *if* $I_T < I_3^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $I_1^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(3) Assume that $I_1^* < I_3^*$, *and further,*

- *if* $I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_1^* < I_T < I_3^*$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;

- *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

Proposition 19. *Under the conditions* $S_1^* < S_T < S_T^{*'}$ *and* $S_T < S_T^* < S_3^*$, *we have the following results.*

(1) Assume that $I_3^* > H_1$, $I_1^* < H_2 < H_1 < I_3^*$, *and*

- *if* $I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_1^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $H_1 < I_T < I_3^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < I_T^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(2) Assume that $H_2 < I_3^* < H_1$, $I_1^* < H_2 < I_3^* < H_1$, *and further,*

- *if* $I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_1^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $H_2 < I_T < I_3^*$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_T > I_3^*$, *then*
  - ◇ *if* $I_T^* > H_1$, *and further,*
    - ◇ *if* $I_3^* < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $H_1 < I_T < I_T^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.
  - ◇ *if* $I_T^* < H_1$, *and*
    - ◇ *if* $I_3^* < I_T < I_T^*$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $I_T^* < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(3) Assume that $I_1^* < I_3^* < H_2$, $I_1^* < I_3^* < H_2 < H_1$, *and*

- *if* $I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_1^* < I_T < I_3^*$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $I_T > H_2$,
  - ◇ *if* $I_T^* > H_1$, *and further,*
    - ◇ *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $H_1 < I_T < I_T^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.
  - ◇ *if* $I_T^* < H_1$, *and further,*
    - ◇ *if* $H_2 < I_T < I_T^*$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $I_T^* < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(4) Assume that $I_3^* < I_1^*$, *further,* $I_3^* < I_1^* < H_2 < H_1$, *and*

- *if* $I_T < I_3^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $I_1^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $I_T > H_2$, *and further,*
  - ◇ *if* $I_T^* > H_1$, *and*
    - ◇ *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $H_1 < I_T < I_T^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.
  - ◇ *if* $I_T^* < H_1$, *and further*
    - ◇ *if* $H_2 < I_T < I_T^*$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $I_T^* < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
    - ◇ *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

Proposition 20. *Suppose* $S_1^* < S_T < S_T^{*'}$ *and* $S_T < S_3^* < S_T^*$, $E_{s3}^+ \notin \ell_3$ *and* $E_{s1}^+ \notin \ell_4 \subset \Pi_1$, *we have the following results.*

(1) Assume that $I_3^* > H_1$, *and further,*

- *if* $I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_1^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $H_2 < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $H_1 < I_T < I_3^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \in \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3\ E_{s1}^- \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

(2) Assume that $I_3^* < H_1$, *and further,*

- *if* $I_T < I_1^*$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_1^* < I_T < H_2$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $H_2 < I_T < I_3^*$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \in \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$;
- *if* $I_3^* < I_T < H_1$, *we have* $E_{s3}^- \in \ell_3, E_{s1}^- \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$;
- *if* $H_1 < I_T < I_T^{*'}$, *we have* $E_{s3}^- \notin \ell_3, E_{s1}^- \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$.

Based on Propositions 16–20, the following summary is given.

C1. Let $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \notin \Gamma_1, E_3 \in \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{1-1}. Then, we conclude that the system (2.4) does not have a pseudo-equilibrium, and all trajectories of the system (2.4) will converge to the equilibrium point E_3^R. The result of this numerical simulation is shown in Figure 4(a), where

$$C_{1-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, I_T < \min\{I_1^*, I_3^*\}\right\}.$$

C2. Let $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \in \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{2-1}. Then, we know that $E_{s1}^+ \in \ell_4 \subset \Pi_1$ is a stable pseudo-equilibrium, and all solutions of the system (2.4) will approach the point E_{s1}^+, as shown in Figure 4(b), where

$$C_{2-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, I_3^* < I_T < \min\{H_2, I_1^*\}\right\}.$$

C3. Let $E_{s3}^{-} \notin \ell_3, E_{s3}^{+} \notin \ell_3, E_{s1}^{-} \notin \ell_4, E_{s1}^{+} \notin \ell_4, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{3-1}. Then, we conclude that the system (2.4) does not have a pseudo-equilibrium, and all trajectories of the system (2.4) will converge to the point $E_T = (S_T, I_T)$. The result of this numerical simulation is shown in Figure 4(c), where

$$C_{3-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_2 < I_T < \min\{H_1, I_1^*\}\right\}.$$

C4. Let $E_{s3}^{-} \notin \ell_3, E_{s3}^{+} \in \ell_3, E_{s1}^{-} \notin \ell_4, E_{s1}^{+} \notin \ell_4 \subset \Pi_1, E_1 \notin \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{4-1}. Then, we know that the point $E_{s3}^{+} \in \ell_3$ is a stable pseudo-equilibrium, and all solutions of the system (2.4) will approach the point E_{s3}^{+}, as shown in Figure 4(d), where

$$C_{4-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_1 < I_T < I_1^*\right\}.$$

C5. Let $E_{s3}^{-} \notin \ell_3, E_{s3}^{+} \notin \ell_3, E_{s1}^{-} \notin \ell_4, E_{s1}^{+} \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set $C_{5-1} \cup C_{5-2} \cup C_{5-3} \cup C_{5-4}$. Then, we know that the system (2.4) does not have a pseudo-equilibrium, and all trajectories of the system (2.4) will converge to the point E_1^R. The result of this numerical simulation is shown in Figure 4(e), where

$$\begin{aligned}
C_{5-1} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, I_1^* < I_T < I_T^*, \text{ if } S_T^{*'} < S_1^* < S_T \text{ and } S_T^* < S_T < S_3^*\right\},\\
C_{5-2} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_1 < I_T < I_T^{*'}, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T^* < S_T < S_3^*\right\},\\
C_{5-3} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_1 < I_T < I_T^{*'}, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_T^* < S_3^*, I_3^* < H_1\right\},\\
C_{5-4} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, \max\{H_1, I_3^*\} < I_T < I_T^{*'}, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_3^* < S_T^*\right\}.
\end{aligned}$$

C6. Let $E_{s3}^{-} \in \ell_3, E_{s3}^{+} \notin \ell_3, E_{s1}^{-} \notin \ell_4, E_{s1}^{+} \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{6-1}. Then, we conclude that $E_{s3}^{-} \in \ell_3$ is an unstable pseudo-equilibrium, and the solution of the system (2.4) will approach the point E_1^R or E_3^R or E_T. The result of this numerical simulation is shown in Figure 4(f), where

$$C_{6-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, I_1^* < I_T < \min\{H_2, I_3^*\}\right\}.$$

C7. Let $E_{s3}^{-} \in \ell_3, E_{s3}^{+} \notin \ell_3, E_{s1}^{-} \notin \ell_4, E_{s1}^{+} \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{7-1}. Then, we know that $E_{s3}^{-} \in \ell_3$ is an unstable pseudo-equilibrium, where $E_{s1}^{+} \in \ell_4$ is stable. The solution of the system (2.4) will tend to E_{s1}^{+} or E_1^R or E_T. The result of this numerical simulation is shown in Figure 5(a), where

$$C_{7-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, \max\{I_1^*, I_3^*\} < I_T < H_2\right\}.$$

C8. Let $E_{s3}^{-} \in \ell_3, E_{s3}^{+} \notin \ell_3, E_{s1}^{-} \notin \ell_4, E_{s1}^{+} \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set $C_{8-1} \cup C_{8-2} \cup C_{8-3} \cup C_{8-4}$. Then, we conclude that $E_{s3}^{-} \in \ell_3$ is an unstable pseudo-equilibrium. The solution of the system (2.4) will approach the equilibrium point E_1^R or E_T, as shown in Figure 5(b), where

$$\begin{aligned}
C_{8-1} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, \max\{I_1^*, H_2\} < I_T < H_1, \text{ if } S_1^* < S_T^{*'} < S_T \text{ and } S_T^* < S_T < S_3^*\right\},\\
C_{8-2} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, \max\{I_1^*, H_2\} < I_T < H_1, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T^* < S_T < S_3^*\right\},\\
C_{8-3} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, I_T^* < I_T < H_1, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_T^* < S_3^*\right\},\\
C_{8-4} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, I_3^* < I_T < H_1, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_3^* < S_T^*\right\}.
\end{aligned}$$

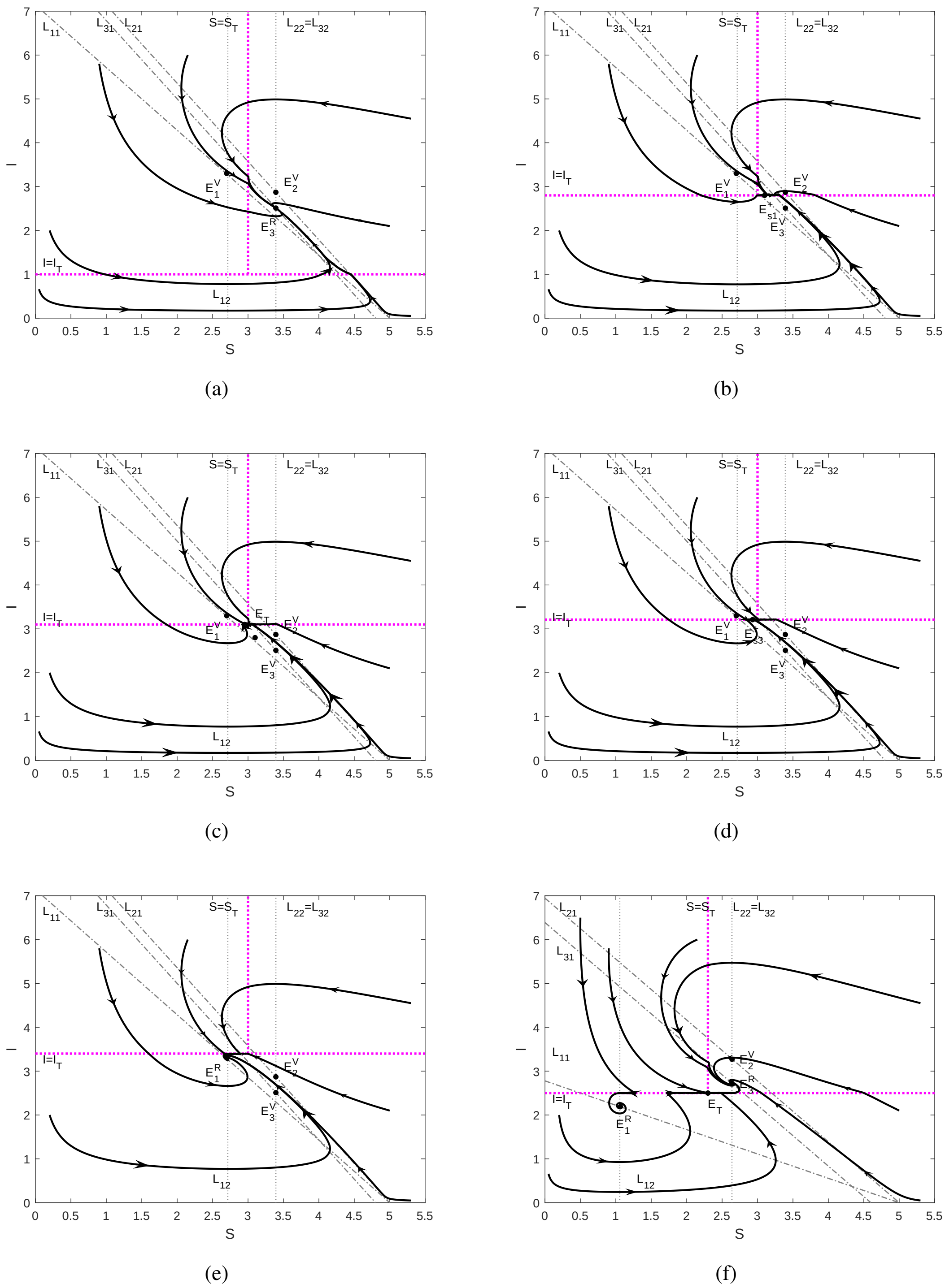


Figure 4. The trajectory of system (2.4) under Case 2: $S_1^* < S_T < S_2^* = S_3^*$, $d = 0.1, \delta = 0.1, r = 0.1, \gamma = 0.2, K = 1$, (a) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3, I_T = 1$; (b) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3, I_T = 2.8$; (c) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3, I_T = 3.1$; (d) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3, I_T = 3.21$; (e) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3, I_T = 3.4$; (f) where $\beta = 1.8, p = 0.6, q = 0.4, S_T = 2.3, I_T = 2.5$.

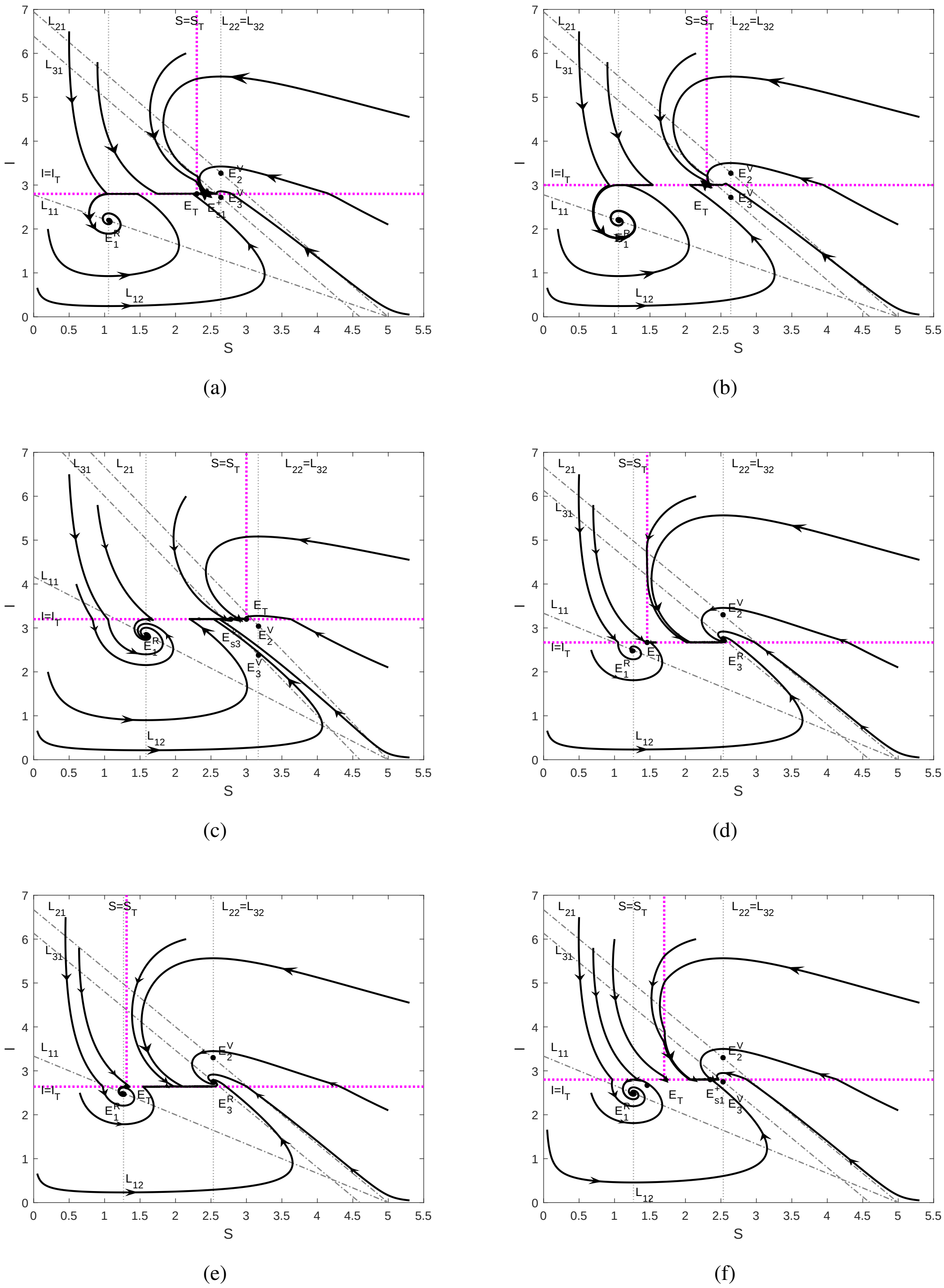


Figure 5. The trajectory of system (2.4) under Case 2: $S_1^* < S_T < S_2^* = S_3^*$, $d = 0.1, \delta = 0.1, r = 0.1, \gamma = 0.2, K = 3$, (a) where $\beta = 1.8, p = 0.6, q = 0.4, S_T = 2.3, I_T = 2.8$; (b) where $\beta = 1.8, p = 0.6, q = 0.4, S_T = 2.3, I_T = 3$; (c) where $\beta = 1.2, p = 0.5, q = 0.4, S_T = 3, I_T = 3.2$; (d) where $\beta = 1.5, p = 0.5, q = 0.4, S_T = 1.46, I_T = 2.67$; (e) where $\beta = 1.5, p = 0.5, q = 0.4, S_T = 1.31, I_T = 2.64$; (f) where $\beta = 1.5, p = 0.5, q = 0.4, S_T = 1.7, I_T = 2.8$.

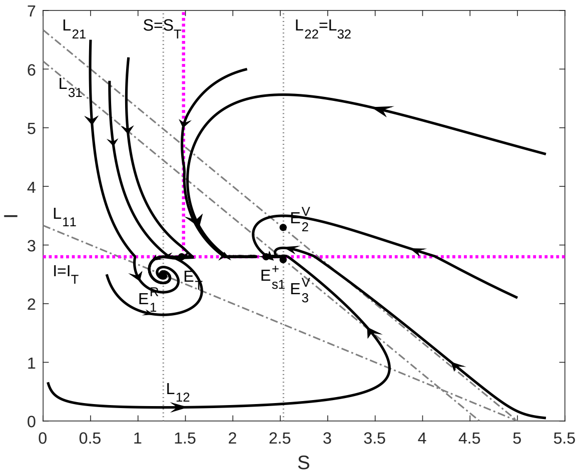


Figure 6. Dynamics of system (2.4) under Case 2: $S_1^* < S_T < S_2^* = S_3^*$. Here, the parameter values are $\beta = 1.5, d = 0.01, \delta = 0.01, r = 0.01, \gamma = 0.2, K = 3, p = 0.5, q = 0.4, S_T = 1.48, I_T = 2.8$ and $d + \delta + r = 0.03$.

C9. Let $E_{s3}^- \in \ell_3, E_{s3}^+ \in \ell_3, E_{s1}^- \notin \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to C_{9-1}. Then, we conclude that $E_{s3}^- \in \ell_3$ is a unstable pseudo-equilibrium, where $E_{s3}^+ \in \ell_3$ is stable. The solution of the system (2.4) will tend to the equilibrium point E_{s3}^+ or E_1^R or E_T. The results of this numerical simulation are shown in Figure 5(c), where

$$C_{9-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_1 < I_T < I_T^{*'}, \text{ if } S_1^* < S_T^{*'} < S_T \text{ and } S_T^* < S_T < S_3^*\right\}.$$

C10. Let $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set $C_{10-1} \cup C_{10-2}$. Then, we conclude that $E_{s3}^- \in \ell_3$ is an unstable pseudo-equilibrium. The solution of the system (2.4) will converge to E_1^R or E_3^R or E_T, as shown in Figure 5(d), where

$$C_{10-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_2 < I_T < \min\{H_1, I_3^*\}, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_T^* < S_3^*\right\},$$
$$C_{10-2} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_2 < I_T < \min\{H_1, I_3^*\}, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_3^* < S_T^*\right\}.$$

C11. Let $E_{s3}^- \notin \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \notin \ell_4, E_1 \in \Gamma_1, E_3 \in \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set $C_{11-1} \cup C_{11-2}$. Then, we show that $E_{s1}^- \in \ell_4$ is an unstable pseudo-equilibrium. The solution of the system (2.4) will tend to E_1^R or E_3^R or E_T, as shown in Figure 5(e), where

$$C_{11-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_1 < I_T < I_3^*, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_T^* < S_3^*\right\},$$
$$C_{11-2} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, H_1 < I_T < I_3^*, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_3^* < S_T^*\right\}.$$

C12. Let $E_{s3}^- \in \ell_3, E_{s3}^+ \notin \ell_3, E_{s1}^- \in \ell_4, E_{s1}^+ \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, if $S_1^* < S_T < S_T^{*'}$ and $S_T < S_T^* < S_3^*$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{12-1}. Then, we know that the point $E_{s3}^- \in \ell_3$ and the value $E_{s1}^- \in \ell_4$ are unstable pseudo-equilibriums, where $E_{s1}^+ \in \ell_4$ is stable. The solution of the system (2.4) will approach E_{s1}^+ or E_1^R or E_T. The result of this numerical simulation is shown in Figure 5(f), where

$$C_{12-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, \max\{I_3^*, H_2\} < I_T < \min\{H_1, I_T^*\}\right\}.$$

C13. Let $E_{s3}^{-} \notin \ell_3, E_{s3}^{+} \notin \ell_3, E_{s1}^{-} \in \ell_4, E_{s1}^{+} \in \ell_4, E_1 \in \Gamma_1, E_3 \notin \Gamma_3$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set C_{13-1}. Then, we conclude that $E_{s1}^{-} \in \ell_4$ is an unstable pseudo-equilibrium, where $E_{s1}^{+} \in \ell_4 \subset \Pi_1$ is stable. The solution of the system (2.4) will converge to E_{s1}^{+} or E_1^R or E_T, as shown in Figure 6, where

$$C_{13-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*, \max\{I_3^*, H_1\} < I_T < I_T^*, \text{ if } S_1^* < S_T < S_T^{*'} \text{ and } S_T < S_T^* < S_3^*\right\}.$$

5. Sliding dynamics and bifurcations of (2.4) under Case 3: $S_2^* = S_3^* < S_T$

In this part, we first consider sliding mode dynamics of (2.4) on Π_1 under Case 3: $S_2^* = S_3^* < S_T$. Second, the sliding mode dynamics on Π_2 are given under Case 3: $S_2^* = S_3^* < S_T$. In addition, we investigate the bifurcations of (2.4) under Case 3: $S_2^* = S_3^* < S_T$. Finally, some numerical simulations are displayed to confirm the results.

5.1. Sliding mode dynamics on Π_1 under Case 3: $S_2^* = S_3^* < S_T$

If $\langle n_1, F_1\rangle > 0$ and $\langle n_1, F_2\rangle < 0$ on ℓ_5, then ℓ_5 is described as

$$\ell_5 = \{(S, I) \in \Pi_1 : S_1^* < S < S_2^*\}.$$

Next, the conditions of a pseudo-equilibrium on the sliding mode ℓ_5 are given as follows.

Proposition 21. *Under the condition $S_T^{*'} > S_2^*$, $E_{s3}^{+} \notin \ell_5$ and the following assertions hold.*

(1) If $I_T < I_1^*$, we have $E_{s3}^{-} \notin \ell_5$;

(2) If $I_1^* < I_T < I_2^*$, we have $E_{s3}^{-} \in \ell_5$;

(3) If $I_T > I_2^*$, we have $E_{s3}^{-} \notin \ell_5$.

Proposition 22. *Under the condition $S_1^* < S_T^{*'} < S_2^*$, the following assertions hold.*

(1) Assume that $I_1^* < I_2^*$, and further,

- *if $I_1^* < I_T < I_2^*$, we have $E_{s3}^{-} \in \ell_5$, $E_{s3}^{+} \notin \ell_5$;*
- *if $I_2^* < I_T < I_T^{*'}$, we have $E_{s3}^{-} \in \ell_5$, $E_{s3}^{+} \in \ell_5$.*

(2) Assume that $I_1^* > I_2^*$, and further,

- *if $I_2^* < I_T < I_1^*$, we have $E_{s3}^{-} \notin \ell_5$, $E_{s3}^{+} \in \ell_5$;*
- *if $I_1^* < I_T < I_T^{*'}$, we have $E_{s3}^{-} \in \ell_5$, $E_{s3}^{+} \in \ell_5$.*

Proposition 23. *Under the condition $S_T^{*'} < S_1^*$, $E_{s3}^{-} \notin \ell_5$ and the following assertions hold.*

(1) If $I_T < I_2^*$, we have $E_{s3}^{+} \notin \ell_5$;

(2) If $I_2^* < I_T < I_1^*$, we have $E_{s3}^{+} \in \ell_5$;

(3) If $I_T > I_1^*$, we have $E_{s3}^{+} \notin \ell_5$.

Theorem 7. *If $0 < I_T < I_T^{*'}$, the sliding mode ℓ_5 has a stable pseudo-equilibrium E_{s3}^{+}, and the sliding mode ℓ_5 E_{s3}^{-} has an unstable pseudo-equilibrium.*

5.2. Sliding mode dynamics on Π_2 under Case 3: $S_2^* = S_3^* < S_T$

When $S_2^* = S_3^* < S_T$, the sliding mode dynamics on Π_2 are the same as Section 3.2. We omit it here.

5.3. Bifurcations of system (2.4) under Case 3: $S_2^* = S_3^* < S_T$

For this Case 3, the point E_3 is a virtual equilibrium, denoted by E_3^V. Points E_1 and E_2 are changeable depending on I_T, and then we have the following.

Proposition 24. *Under the condition $S_T^{*'} > S_2^*$, the following assertions hold.*

(1) If $I_T < I_1^*$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \notin \Gamma_1, E_2 \in \Gamma_2$;

(2) If $I_1^* < I_T < I_2^*$, we have $E_{s3}^- \in \ell_5, E_{s3}^+ \notin \ell_1, E_1 \in \Gamma_1, E_2 \in \Gamma_2$;

(3) If $I_2^* < I_T < I_T^{*'}$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \in \Gamma_1, E_2 \notin \Gamma_2$;

(4) If $I_T > I_T^{*'}$, we have that E_{s3}^- and E_{s3}^+ do not exist, $E_1 \in \Gamma_1, E_2 \notin \Gamma_2$.

Proposition 25. *Under the condition $S_1^* < S_T^{*'} < S_2^*$, the following assertions hold.*

(1) Assume that $I_1^* < I_2^*$, and further,

- *if $I_T < I_1^*$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \notin \Gamma_1, E_2 \in \Gamma_2$;*
- *if $I_1^* < I_T < I_2^*$, we have $E_{s3}^- \in \ell_5, E_{s3}^+ \notin \ell_5, E_1 \in \Gamma_1, E_2 \in \Gamma_2$;*
- *if $I_2^* < I_T < I_T^{*'}$, we have $E_{s3}^- \in \ell_1$, $E_{s3}^+ \in \ell_5, E_1 \in \Gamma_1, E_2 \notin \Gamma_2$;*
- *if $I_T > I_T^{*'}$, we have that E_{s1}^- and E_{s1}^+ do not exist, $E_1 \in \Gamma_1, E_2 \notin \Gamma_2$.*

(2) Assume that $I_1^* > I_2^*$, and further,

- *if $I_T < I_2^*$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \notin \Gamma_1, E_2 \in \Gamma_2$;*
- *if $I_2^* < I_T < I_1^*$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \in \ell_5, E_1 \notin \Gamma_1, E_2 \notin \Gamma_2$;*
- *if $I_1^* < I_T < I_T^{*'}$, we have $E_{s3}^- \in \ell_5$, $E_{s3}^+ \in \ell_5\, E_1 \in \Gamma_1, E_2 \notin \Gamma_2$;*
- *if $I_T > I_T^{*'}$, we have that E_{s3}^- and E_{s3}^+ do not exist, $E_1 \in \Gamma_1, E_2 \notin \Gamma_2$.*

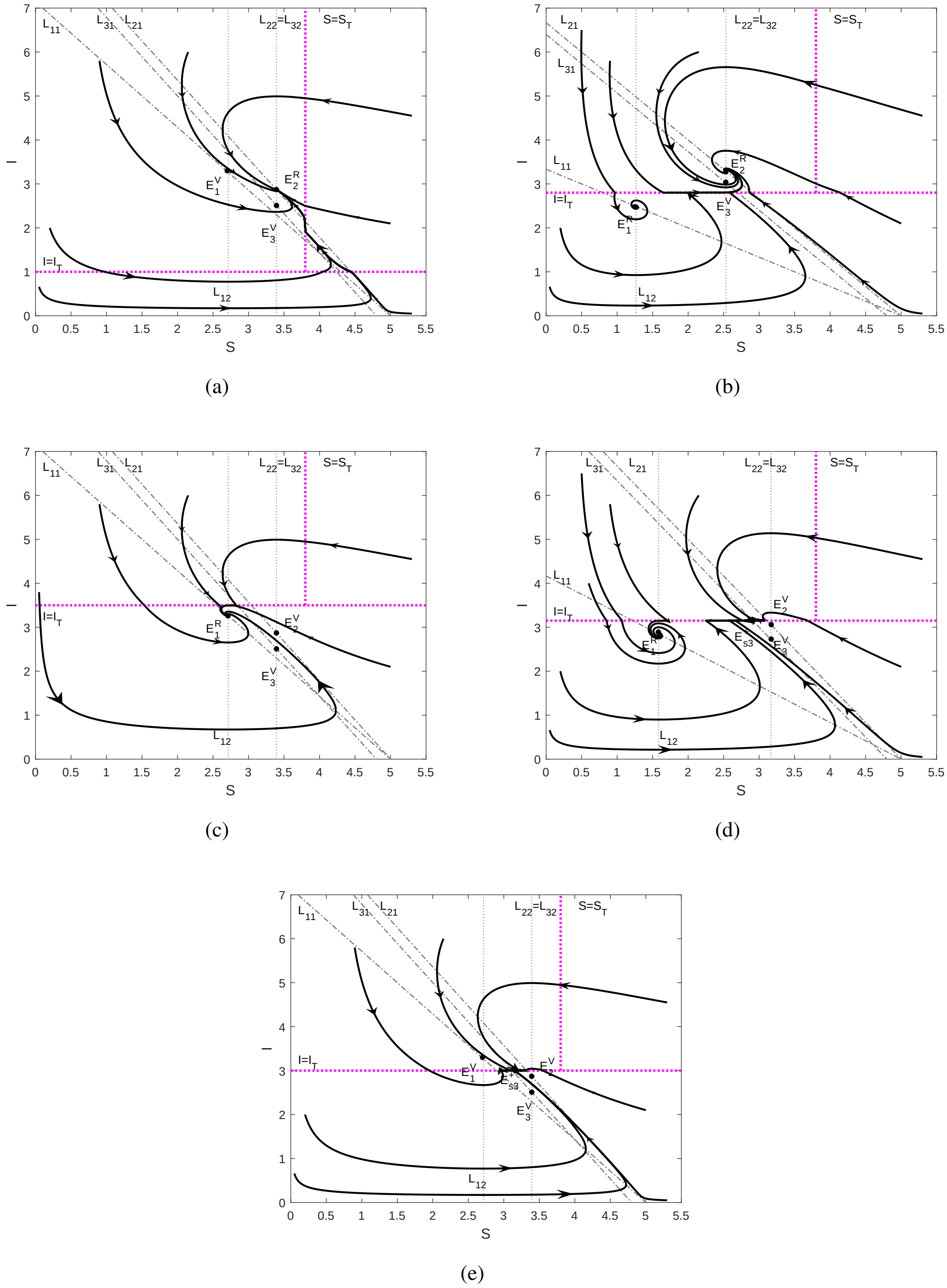


Figure 7. The trajectory of the system (2.4) under Case 3: $S_2^* = S_3^* < S_T$, $d = 0.01, \delta = 0.01, r = 0.01, \gamma = 0.2, K = 3$, (a) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3.8, I_T = 1$; (b) where $\beta = 1.5, p = 0.5, q = 0.2, S_T = 3.8, I_T = 2.8$; (c) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3.8, I_T = 3.5$; (d) where $\beta = 1.2, p = 0.5, q = 0.2, S_T = 3.8, I_T = 3.15$; (e) where $\beta = 0.7, p = 0.2, q = 0.2, S_T = 3.8, I_T = 3$.

Proposition 26. *Under the condition $S_T^{*'} < S_1^*$, the following assertions hold.*

(1) If $I_T < I_2^*$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \notin \Gamma_1, E_2 \in \Gamma_2$;

(2) If $I_2^* < I_T < I_1^*$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \in \ell_5, E_1 \notin \Gamma_1, E_2 \notin \Gamma_2$;

(3) If $I_1^* < I_T < I_T^{*'}$, we have $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \in \Gamma_1, E_2 \notin \Gamma_2$;

(4) If $I_T > I_T^{*'}$,we have that E_{s3}^- and E_{s3}^+ do not exist, $E_1 \in \Gamma_1, E_2 \notin \Gamma_2$.

Based on Propositions 24–26, the following summary is given.

D1. Let $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \notin \Gamma_1, E_2 \in \Gamma_2$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set D_{1-1}. Then, we conclude that the system (2.4) does not have a pseudo-equilibrium, and all trajectories of the system (2.4) will converge to E_2^R. The result of this numerical simulation is shown in Figure 7(a), where

$$D_{1-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_2^* < S_T, I_T < \min\{I_1^*, I_2^*\}\right\}.$$

D2. Let $E_{s3}^- \in \ell_5, E_{s3}^+ \notin \ell_5, E_1 \in \Gamma_1, E_2 \in \Gamma_2$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set D_{2-1}. Then, we show that $E_{s3}^- \in \ell_5$ is an unstable pseudo-equilibrium. The solution of the system (2.4) will approach E_1^R or E_2^R, as shown in Figure 7(b), where

$$D_{2-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_2^* < S_T, I_1^* < I_T < I_2^*\right\}.$$

D3. Let $E_{s3}^- \notin \ell_5, E_{s3}^+ \notin \ell_5, E_1 \in \Gamma_1, E_2 \notin \Gamma_2$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set $D_{3-1} \cup D_{3-2} \cup D_{3-3}$. Then, we know that the system (2.4) does not have a pseudo-equilibrium, and all trajectories of the system (2.4) will converge to E_1^R, as shown in Figure 7(c), where

$$\begin{aligned} D_{3-1} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_2^* < S_T, I_2^* < I_T < I_T^{*'}, \text{ if } S_T^{*'} > S_2^*\right\}, \\ D_{3-2} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_2^* < S_T, I_T > I_T^{*'} \text{ if } S_1^* < S_T^{*'} < S_2^*\right\}, \\ D_{3-3} &= \left\{\Upsilon \in \mathbb{R}_+^2 : S_2^* < S_T, I_1^* < I_T < I_T^{*'}, \text{ if } S_T^{*'} < S_1^*\right\}. \end{aligned}$$

D4. Let $E_{s3}^- \in \ell_5, E_{s3}^+ \in \ell_5, E_1 \in \Gamma_1, E_2 \notin \Gamma_2$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set D_{4-1}. Then, we show that $E_{s3}^- \in \ell_5$ is an unstable pseudo-equilibrium, and the solution of system (2.4) will approach E_1^R or E_{s3}^+, as shown in Figure 7(d), where

$$D_{4-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_2^* < S_T, \max\{I_1^*, I_2^*\} < I_T < I_T^{*'}, \text{ if } S_1^* < S_T^{*'} < S_2^*\right\}.$$

D5. Let $E_{s3}^- \notin \ell_5, E_{s3}^+ \in \ell_5, E_1 \notin \Gamma_1, E_2 \notin \Gamma_2$, and the value $\Upsilon = (S_T, I_T)$ belongs to the set D_{5-1}. Then, we conclude that $E_{s3}^+ \in \ell_5$ is a stable pseudo-equilibrium. All solutions of the system (2.4) will tend to E_{s3}^+. The result of this numerical simulation is shown in Figure 7(e), where

$$D_{5-1} = \left\{\Upsilon \in \mathbb{R}_+^2 : S_2^* < S_T, I_2^* < I_T < I_1^*\right\}.$$

Remark 1. *For smooth system (2.1), we have discussed the three equilibrium points, that is,* $(0, 0)$*, the disease free equilibrium point $E_{i1} (i = 1, 2, 3)$ and the endemic equilibrium point $E_i (i = 1, 2, 3)$ of (2.4). By constructing a Lyapunov function, we obtain the global stability of system (2.1) in Theorem 1.*

For the non-smooth system (2.4), we investigate the non-smooth system (2.4) with two threshold control strategies. Using a Filippov analysis method, Green's formula, the comparison theorem and numerical simulation method, the rich dynamics of the system are given, such as the bistability phenomenon, the globally stable pseudo-equilibrium and the regular/virtulal equilibrium bifurcations. Through the two control strategies, we can control the disease individuals to the appropriate balance. In particular, Theorems 3, 6 and 7 in this paper cannot appear in the smooth system (2.1); please see Figure 4(b),(c). There is bistability in the system (2.4).

Remark 2. *In [25], a Filippov model describing the effects of media coverage and quarantine on the spread of human influenza was considered, and the threshold conditions for stability switches were obtained analytically. The discontinuous system (2.4) considered in our paper is a logistic source, and [25] considered a linear source. Second, the dynamics are different. Our paper employs the Green's theorem and a Dulac function. Then, we show that two real equilibria occur simultaneously in our paper. Using numerical simulation methods, the sliding dynamics and bifurcations of a human influenza system under logistic source and broken line control strategy are given. The results of this paper are new with respect to [25].*

6. The regular/virtulal equilibrium bifurcations

Based on the previous discussion, it is shown that system (2.4) will exhibit multiple equilibriums and sliding modes. In order to better construct the bifurcation diagram, we choose γ and I_T as bifurcation parameters, and the other parameters are fixed as shown in Figure 8. With the expressions of equilibria found in Section 2.3, the lines to divide the relevant parameter plane are given as follows:

$$l_1 := \left\{(\gamma, I_T)|I_T = I_1^* = \frac{\gamma}{\beta}(1 - \frac{d+\delta+r}{K\beta})\right\},$$

$$l_2 := \left\{(\gamma, I_T)|I_T = I_2^* = \frac{\gamma}{\beta(1-p)}(1 - \frac{d+\delta+r}{K\beta(1-p)})\right\},$$

$$l_3 := \left\{(\gamma, I_T)|I_T = I_3^* = \frac{\gamma}{\beta(1-p)}(1 - \frac{d+\delta+r}{K\beta(1-p)} - \frac{q}{r})\right\}.$$

The three solid lines l_1, l_2 and l_3 divide the $\gamma - I_T$ two-dimensional plane space into four regions in the first quadrant. Suppose that the control value I_T satisfies $I_3^* < I_T < I_2^*$ and $I_1^* < I_3^*$ (that is, regions Ω_2^* and region Ω_3^*; see Figure 8), the points E_2 and E_3 are virtual equilibria points (denoted by E_2^v, and E_3^v, respectively), and E_{s1}^- exists with the sliding mode domain. If the control value I_T satisfies $I_T > I_2^*$ (that is, Ω_1^*, as shown Figure 8), E_2 is a regular equilibrium, while the point E_3 is a virtual equilibrium (denoted by the equilibrium E_2^R and the equilibrium E_3^v, respectively), and point E_{s1}^- does not exist with the sliding mode domain. If $I_T < I_3^*$ (that is, Ω_4^*; see Figure 8), E_3 is a regular equilibrium, while point E_2 is a virtual equilibrium (denoted by E_3^R and E_2^v), and point E_{s1}^- does not exist with the sliding mode domain.

Next, we choose q and I_T as bifurcation parameters, and the other parameters are fixed. From Proposition 1 in this paper, the lines to divide the relevant parameter plane are given as follows:

$$l_4 := \left\{(q, I_T)|I_T = I_T^* = \frac{q}{\beta p} + \frac{(\gamma - \frac{q}{p})^2 K}{4\gamma(d+\delta+r)}\right\},$$

$$l_5 := \left\{(q, I_T)|I_T = \frac{q}{\beta p}\right\}.$$

The two solid lines l_4 and l_5 divide the $q - I_T$ two-dimensional plane space into three regions in the first quadrant $\mathbb{R}^+$. Suppose that the control value I_T satisfies $I_T > I_T^*$ (that is, region Ω_7; see Figure 9(a)), then system (3.1) has no equilibrium. If the control value I_T satisfies $I_T^* > I_T > \frac{q}{\beta p}$ (that is, region Ω_6, as shown in Figure 9(a)), system (3.1) has two positive equilibria $E_{s1}^+ = (S_{s1}^+, I_T)$ and $E_{s1}^- = (S_{s1}^-, I_T)$, where $S_{s1}^{\pm} = \frac{(\gamma - \frac{q}{p}) \pm \sqrt{\Delta_1}}{2\gamma} K$. When the control value I_T satisfies $0 < I_T < \frac{q}{\beta p}$ (that is, region Ω_5; see Figure 9(a)), system (3.1) has a unique positive equilibrium $E_{s2} = (S_{s2}, I_T)$, where $S_{s2} = \frac{(\gamma - \frac{q}{p}) + \sqrt{\Delta_1}}{2\gamma} K$.

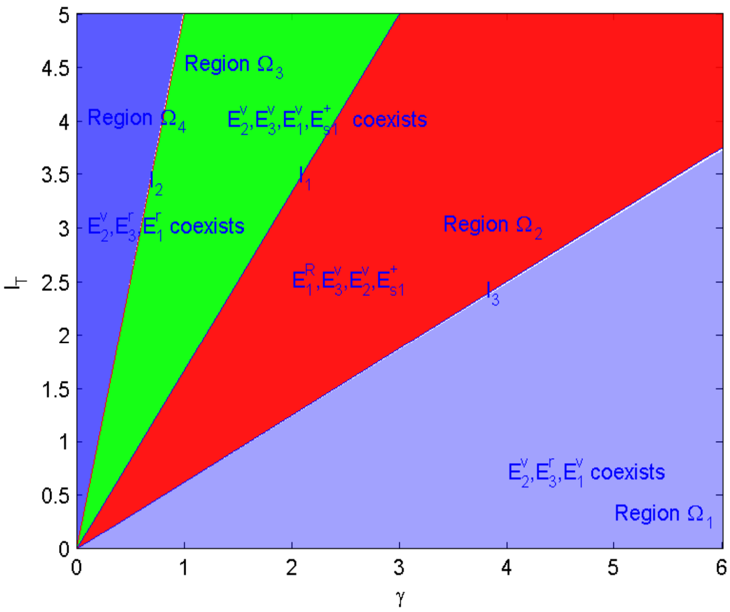
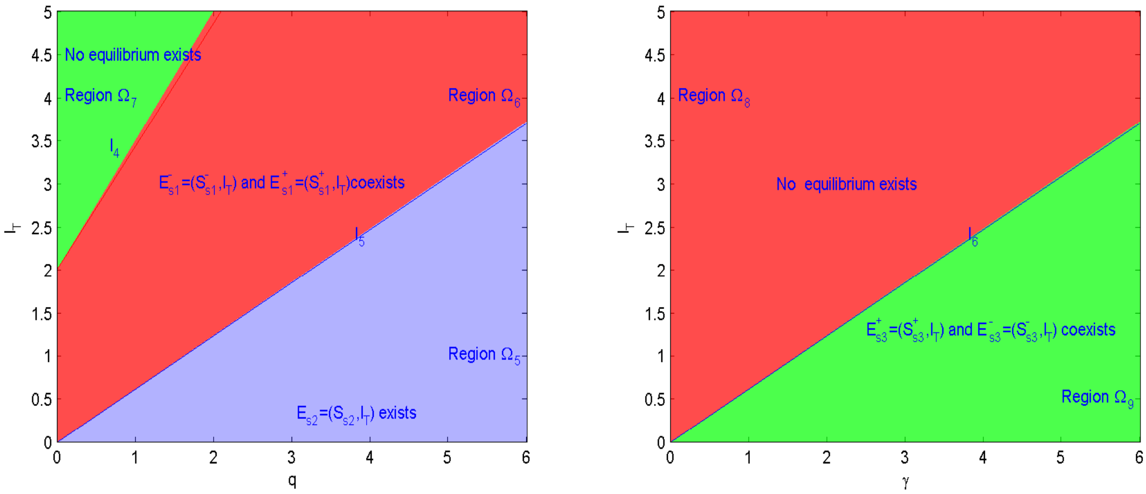


Figure 8. The regular/virtual equilibrium bifurcations where $\beta = 0.7, p = 0.2, q = 0.2, d = 0.01, \delta = 0.01, r = 0.01, K = 3$.

We choose γ and I_T as bifurcation parameters, and the other parameters are fixed. With Proposition 9 in this paper, the line to divide the relevant parameter plane is given as

$$l_6 := \left\{(\gamma, I_T)|I_T = I_T^{*'} = \frac{\gamma K}{4(d + \delta + r)}\right\}.$$

The solid line l_6 divides the $\gamma - I_T$ two-dimensional plane space into two regions in the first quadrant $\mathbb{R}^+$. Suppose that the control value I_T satisfies $I_T > I_T^{*'}$ (that is, region Ω_8, as shown in Figure 9(b)), and then system (4.1) does not have an equilibrium. If the control value I_T satisfies $0 < I_T < I_T^{*'}$ (that is, region Ω_9, as shown in Figure 9(b)), system (4.1) has two positive equilibria $E_{s3}^+ = (S_{s3}^+, I_T)$ and $E_{s3}^- = (S_{s3}^-, I_T)$, where $S_{s3}^{\pm} = \frac{\gamma \pm \sqrt{\Delta_2}}{2\gamma} K$.



(a) (b)

Figure 9. The pseudo-equilibrium bifurcations, (a) where $\beta = 0.7, p = 0.2, d = 0.01, \delta = 0.01, r = 0.01, \gamma = 0.2, K = 3$; (b) where $\beta = 0.7, p = 0.2, d = 0.01, \delta = 0.01, r = 0.01, q = 0.2, K = 3$.

Remark 3. *Notice that [30] considered the global dynamics of a Filippov predator-prey model with two thresholds for integrated pest management. By using Filippov theory, the sliding mode dynamics and global dynamics were established. Different from [30], our paper shows the dynamic behavior of the Filippov model with respect to all possible equilibria. It is shown that the Filippov system tends to the pseudo-equilibrium on sliding mode domain or one endemic equilibrium or two endemic equilibria under some conditions. Second, although both this paper and [30] discuss the global dynamics of a Filippov model with two thresholds, this paper first gives different control strategies. In particular, the two real equilibria occur simultaneously using methods such as Green's theorem and a Dulac function.*

7. Discussion

In this paper, we have established a non-smooth system to determine whether it is necessary to adopt the control strategy of media coverage and quarantine of susceptible individuals according to the number of infected and susceptible individuals. Media coverage changes the transmission mode of influenza. Further, in order to reduce the spread of influenza, when the number of cases exceeds the larger infection threshold I_T, and the number of susceptible individuals is greater than S_T, we will quarantine the susceptible individuals. It is worth noting that there are two difficulties in this paper. First, the traditional continuity theory cannot be applied due to the non-smooth system with the broken line control strategy. For example, when proving the global stability of discontinuous systems, the traditional Lyapunov function cannot be similarly constructed. Second, Green's formula of continuous systems cannot be used to prove the existence of global stability of the pseudo equilibriums in discontinuous systems. In this paper, by choosing different thresholds I_T and S_T and using Filippov theory, we study the dynamic behavior of the Filippov model with respect to all possible equilibria.

The regular/virtual equilibrium bifurcations are given. It is shown that the Filippov system tends to the pseudo-equilibrium on sliding mode domain or one endemic equilibrium or two endemic equilibria under some conditions.

Table 1. Dynamics of Filippov model (2.4).

Condition 1	Condition 2	Result
$S_T < S_1^*$	$(S_T, I_T) \in B_{1-1}$	Figure 2(a)
$S_1^* < S_T < S_2^*$	$(S_T, I_T) \in C_{1-1}$	Figure 4(a)
$S_2^* < S_T$	$(S_T, I_T) \in D_{1-1}$	Figure 7(a)

Table 2. Dynamics of Filippov model (2.4).

Condition 1	Condition 2	Result
$S_T < S_1^*$	$(S_T, I_T) \in B_{2-1}$	Figure 2(b)
	$(S_T, I_T) \in B_{3-1} \cup B_{3-2} \cup B_{3-3}$	Figure 2(c)
$S_1^* < S_T < S_2^*$	$(S_T, I_T) \in C_{2-1}$	Figure 4(b)
	$(S_T, I_T) \in C_{3-1}$	Figure 4(c)
	$(S_T, I_T) \in C_{4-1}$	Figure 4(d)
	$(S_T, I_T) \in C_{5-1} \cup C_{5-2} \cup C_{5-3} \cup C_{5-4}$	Figure 4(e)
$S_2^* < S_T$	$(S_T, I_T) \in D_{3-1} \cup D_{3-2} D_{3-3}$	Figure 7(c)
	$(S_T, I_T) \in D_{5-1}$	Figure 7(e)

Table 3. Dynamics of Filippov model (2.4).

Condition 1	Condition 2	Result
$S_T < S_1^*$	$(S_T, I_T) \in B_{4-1}$	Figure 2(d)
	$(S_T, I_T) \in B_{5-1}$	Figure 2(e)
$S_1^* < S_T < S_2^*$	$(S_T, I_T) \in C_{6-1}$	Figure 4(f)
	$(S_T, I_T) \in C_{7-1}$	Figure 5(a)
	$(S_T, I_T) \in C_{8-1} \cup C_{8-2} \cup C_{8-3} \cup C_{8-4}$	Figure 5(b)
	$(S_T, I_T) \in C_{9-1}$	Figure 5(c)
	$(S_T, I_T) \in C_{10-1} \cup C_{10-2}$	Figure 5(e)
	$(S_T, I_T) \in C_{11-1} \cup C_{11-2}$	Figure 5(e)
	$(S_T, I_T) \in C_{12-1}$	Figure 5(f)
	$(S_T, I_T) \in C_{13-1}$	Figure 6
$S_2^* < S_T$	$(S_T, I_T) \in D_{2-1}$	Figure 7(b)
	$(S_T, I_T) \in D_{4-1}$	Figure 7(d)

Next, we summarize the corresponding biological results of Tables 1–3. These results show that the choice of values of I_T and S_T is very important, and it determines whether to adopt control strategies.

- In Table 1, we know that the infection threshold value I_T is chosen to be small enough, i.e., $I \ll I_T$, and then the number of infected individuals will reach the equilibrium E_1^R of system (2.4).
- From Table 2, system (2.4) has a unique globally asymptotically stable pseudo-equilibrium E_{s1}^+ or E_{s3}^+ if $I = I_T$ or admits a unique globally asymptotically stable equilibrium when $I < I_T$. Our control goal can be achieved finally, and there is no need to adjust the threshold strategy.
- In Table 3, the solution of system (2.4) will converge to a locally asymptotically stable equilibrium if $I < I_T$ or tends to a locally asymptotically stable equilibrium E_3^R when $I > I_T$ or pseudo-equilibrium E_{s1}^+, E_{s3}^+ if $I = I_T$. We show that it may be necessary to adjust the threshold policy according to the initial number of susceptible individuals and infected individuals. The results obtained have certain guiding significance for choosing thresholds and designing a corresponding threshold strategy.

Next, we consider the effect of key parameters in the subsystem on the basic regeneration number R_{0i} as follows.

The three-dimensional diagram of the parameter space (δ, d, R_{01}) is shown in Figure 10(a) under the parameter values of $K = 4, \beta = 0.5, r = 0.2$. It observe when the parameter $\delta = 0.8, d$ increase from 0.83 to 1, the basic reproduction number R_{01} decreases correspondingly and is less than unit 1. The trajectory of the subsystem (2.4) will converge globally to the free equilibrium (see Theorem 1), implying that the infected individuals extinct and then a stable free steady state occurs.

The three-dimensional diagram of the parameter space (β, p, R_{02}) is shown in Figure 10(b) under the parameter values of $K = 2$, $d = 0.1$, $r = 0.05$, $\delta = 0.05$. It is easy to observe when fixing the $p = 0.5$, β increase from 0.78 to 1, R_{02} decreases correspondingly and is less than unit 1. By using Theorem 1, the infected individuals persist and the trajectory of the subsystem (2.4) will converge globally to the endemic steady state.

The three-dimensional diagram of the parameter space (β, δ, R_{01}) is shown in Figure 10(c) under the parameter values of $K = 2$, $d = 0.1$, $r = 0.1$. We observe when the parameters δ, β increase from 0.6 to 1, R_{01} also increases correspondingly and is greater than the unit 1. By using Theorem 1, the infected individuals persist and the trajectory of the subsystem (2.4) will converge globally to the endemic steady state.

The three-dimensional diagram of the parameter space (β, p, R_{03}) is shown in Figure 10(d) under the parameter values of $K = 1$, $\gamma = 1.8$, $r = \frac{1}{27}$, $\delta = \frac{1}{27}$, $d = \frac{1}{27}$. It is easy to observe when fixing the parameter $p = 0.2$, with a transmission rate β increase from 0.81 to 1, R_{03} increases correspondingly and is greater than the unit 1. By using Theorem 1, the infected individuals persist, and the trajectory of the subsystem (2.4) will converge globally to the endemic steady state. When fixing the parameters $p = 0.6$, transmission rate β increase from 0.81 to 1, R_{03} decreases correspondingly and is less than the unit 1. The trajectory of the subsystem (2.4) will converge globally to the free equilibrium (see Theorem 1 of our paper), implying that the infected individuals become extinct, and then a stable free steady state occurs.

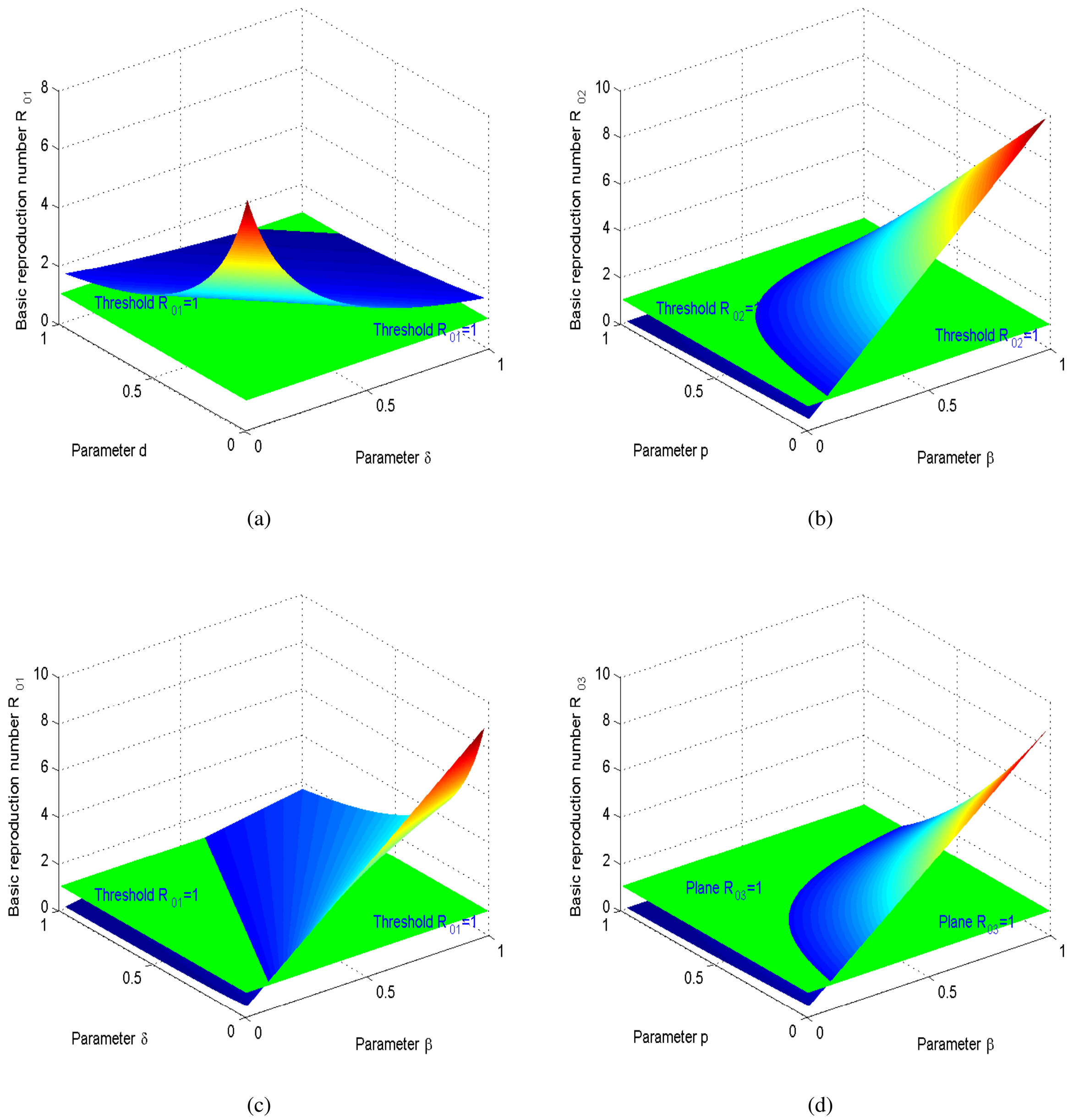


Figure 10. (a) Variation of the basic reproduction number R_{01} with the effect of parameters d and δ. (b) Variation of R_{02} with the effect of parameters β and p. (c) Variation of R_{01} with the effect of parameters β and δ. (d) Variation of R_{03} with the effect of parameters β and p.

In addition, this model has not been validated by actual influenza data, and we only have analyzed theoretically. In the next stage, we will verify and simulate the validity of the conclusions in actual time from some websites and statistics of health departments. However, the paper has studied the non-smooth system of two threshold control strategies and has validated the correctness of the theory through numerical simulation. Due to the serious lack of current influenza data from SARS-CoV-2 infections, the verification of the work is extremely difficult. However, the theory of this paper can provide appropriate guidance for the current influenza by SARS-CoV-2 infection. In this paper, we only consider the dynamics of the system (2.4) if the basic reproduction number $R_{0i} > 1$. However, under the saturation rate $\frac{\beta SI}{1+I}$, and the conditions $R_{02} < R_{01} < 1$ and $R_{02} < 1 < R_{01}$, the dynamical behaviors of system (2.4) and the method of proving global stability are not yet fully clear and would be our further topic.

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Conflict of interest

The authors have no conflict of interest to declare in carrying out this research work.

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