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Research article

Optimal harvesting strategy for stochastic hybrid delay Lotka-Volterra systems with Lévy noise in a polluted environment

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Abstract: This paper concerns the dynamics of two stochastic hybrid delay Lotka-Volterra systems with harvesting and Lévy noise in a polluted environment (i.e., predator-prey system and competitive system). For every system, sufficient and necessary conditions for persistence in mean and extinction of each species are established. Then, sufficient conditions for global attractivity of the systems are obtained. Finally, sufficient and necessary conditions for the existence of optimal harvesting strategy are provided. The accurate expressions for the optimal harvesting effort (OHE) and the maximum of expectation of sustainable yield (MESY) are given. Our results show that the dynamic behaviors and optimal harvesting strategy are closely correlated with both time delays and three types of environmental noises (namely white Gaussian noises, telephone noises and Lévy noises).

Keywords: optimal harvesting; Lotka-Volterra system; Markovian switching; time delay; Lévy noise

1. Introduction

Optimal harvesting problem is an important and interesting topic from both biological and mathematical point of view. The classical two-species deterministic Lotka-Volterra system with harvesting under catch-per-unit-effort hypothesis [1] can be expressed as follows:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[\overline{r_1} - \overline{a_{11}}x_1(t) - \overline{a_{12}}x_2(t)\right]\mathrm{d}t - h_1x_1(t)\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[\overline{r_2} - \overline{a_{21}}x_1(t) - \overline{a_{22}}x_2(t)\right]\mathrm{d}t - h_2x_2(t)\mathrm{d}t, \end{cases} \tag{1.1}$$

where $x_i(t)$ represent the population densities of species i at time t. $\overline{r_i}$ and $\overline{a_{ij}}$ are constants. $h_i \geq 0$ represents the harvesting effort of $x_i(t)$ $(i, j = 1, 2)$.

However, the deterministic system has its limitation in mathematical modeling of ecosystems since the parameters involved in the system are unable to capture the influence of environmental noises [2,3].

Hence, it is of enormous importance to study the effects of environmental noises on the dynamics of population systems. Introducing white Gaussian noises into the deterministic system is the most common way to characterize environmental noises [4,5]. Assume that $\overline{r_i}$ are affected by white Gaussian noises, i.e., $\overline{r_i} \hookrightarrow \overline{r_i} + \sigma_i \dot{W}_i(t)$, where $W_i(t)$ are standard Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Then, system (1.1) becomes

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[\overline{r_1} - h_1 - \overline{a_{11}}x_1(t) - \overline{a_{12}}x_2(t)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[\overline{r_2} - h_2 - \overline{a_{21}}x_1(t) - \overline{a_{22}}x_2(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}W_2(t). \end{cases} \tag{1.2}$$

On the other hand, many academics argue that parameters in ecosystems often switch because of environmental changes, for example, some species have different growth rates at different temperatures, and these changes can be well described by a continuous-time Markov chain $\rho(t)$ with finite-state space, instead of white Gaussian noises [6–13]. System (1.2) under regime switching can be expressed as follows:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[\overline{r_1(\rho(t))} - h_1 - \overline{a_{11}(\rho(t))}x_1(t) - \overline{a_{12}(\rho(t))}x_2(t)\right]\mathrm{d}t + \sigma_1(\rho(t))x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[\overline{r_2(\rho(t))} - h_2 - \overline{a_{21}(\rho(t))}x_1(t) - \overline{a_{22}(\rho(t))}x_2(t)\right]\mathrm{d}t + \sigma_2(\rho(t))x_2(t)\mathrm{d}W_2(t), \end{cases} \tag{1.3}$$

where $\rho(t)$ is a right-continuous Markov chain with finite values $\mathbb{S} = \{1, 2, ..., S\}$, $\overline{r_i(\rho(t))}$ and $\overline{a_{ij}(\rho(t))}$ are functions with finite values. Furthermore, population systems may suffer sudden environmental perturbations, such as earthquake, torrential flood, typhoon and infectious disease. Some scholars claimed that Lévy noise can be used to describe these sudden environmental perturbations [14–19]. Introducing Lévy noise into system (1.2) yields

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[\overline{r_1(\rho(t))} - h_1 - \overline{a_{11}(\rho(t))}x_1(t) - \overline{a_{12}(\rho(t))}x_2(t)\right]\mathrm{d}t \\ \qquad\qquad + \sigma_1(\rho(t))x_1(t)\mathrm{d}W_1(t) + \int_{\mathbb{Z}} x_1(t)\gamma_1(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu), \\ \mathrm{d}x_2(t) = x_2(t)\left[\overline{r_2(\rho(t))} - h_2 - \overline{a_{21}(\rho(t))}x_1(t) - \overline{a_{22}(\rho(t))}x_2(t)\right]\mathrm{d}t \\ \qquad\qquad + \sigma_2(\rho(t))x_2(t)\mathrm{d}W_2(t) + \int_{\mathbb{Z}} x_2(t)\gamma_2(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu), \end{cases} \tag{1.4}$$

where N is a Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Z} \subseteq [0, +\infty)$ with $\lambda(\mathbb{Z}) < +\infty$ and $\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu) = N(\mathrm{d}t, \mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t$, $\gamma_j(\mu, \rho(t))$ are bounded functions. For the sake of simplicity, we define

$$\mathcal{S}_i(t, \rho(t)) = \sigma_i(\rho(t))\mathrm{d}W_i(t) + \int_{\mathbb{Z}} \gamma_i(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu) \quad (i = 1, 2). \tag{1.5}$$

Hence, system (1.4) can be rewritten into

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left\{\left[\overline{r_1(\rho(t))} - h_1 - \overline{a_{11}(\rho(t))}x_1(t) - \overline{a_{12}(\rho(t))}x_2(t)\right]\mathrm{d}t + \mathcal{S}_1(t, \rho(t))\right\}, \\ \mathrm{d}x_2(t) = x_2(t)\left\{\left[\overline{r_2(\rho(t))} - h_2 - \overline{a_{21}(\rho(t))}x_1(t) - \overline{a_{22}(\rho(t))}x_2(t)\right]\mathrm{d}t + \mathcal{S}_2(t, \rho(t))\right\}. \end{cases} \tag{1.6}$$

Given the growing importance of environmental noises in the dynamics of complex physical and biological systems, interdisciplinary stochastic systems driven by two different types of environment noises have attracted great attention in the last few decades [20–32]. Particularly, Giorgio Parisi's Nobel Prize in Physics (2021) expounded on the importance of fluctuations on physics and systems from microscopic to macroscopic physics.

In the natural environment, in addition to being subject to environmental noises, the trends of biological systems depend not only on the present state but also on the past state, such as the growth period from juvenile to adult in the growth model of biological populations. Such phenomena are called time-delay phenomena. "All species should exhibit time delay" in the real world [33] and incorporating time delay into biological systems makes them much more realistic than those without delay, since a species growth rate relies on not only the current state, but also the past state [34–36]. As we all know, systems with discrete time delays and those with continuously distributed time delays do not contain each other. However, systems with S-type distributed time delays contain both [37, 38].

Furthermore, with a growing number of toxicant entering into the ecosystem, many species have been extinctive and some of them are on the verge of extinction, environmental pollution has received much attention in international society. Naturally, it is meaningful to estimate environmental toxicity so as to develop optimal harvesting policies.

In the past few decades, stochastic population systems driven by different types of environment noises have received great attention and have been studied extensively. For example, Abbas et al. [39] studied the effect of stochastic perturbation on a two-species competitive system by constructing a suitable Lyapunov functional. Han et al. [40] investigated two-species Lotka-Volterra delayed stochastic predator-prey systems, with and without pollution. Liu and Chen [41] investigated a stochastic delay predator-prey system with Lévy noise in a polluted environment. Zhao and Yuan [42] considered the optimal harvesting policy of a stochastic two-species competitive model with Lévy noise in a polluted environment. Liu et al. [43] studied the dynamics of a stochastic regime-switching predator-prey system with harvesting and distributed delays.

However, to the best of our knowledge to date, results about stochastic time-delay population system driven by three different types of environment noises have rarely been report. Hence, in this paper we consider the optimization problems of harvesting for the following two stochastic hybrid delay Lotka-Volterra systems with Lévy noise in a polluted environment:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[(r_1(\rho(t)) - h_1 - r_{11}C_1(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))], \\ \mathrm{d}x_2(t) = x_2(t)[(-r_2(\rho(t)) - h_2 - r_{22}C_2(t) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))], \\ \mathrm{d}C_1(t) = [k_1C_E(t) - (g_1 + m_1)C_1(t)]\mathrm{d}t, \\ \mathrm{d}C_2(t) = [k_2C_E(t) - (g_2 + m_2)C_2(t)]\mathrm{d}t, \\ \mathrm{d}C_E(t) = [-hC_E(t) + u(t)]\mathrm{d}t, \end{cases} \tag{1.7}$$

and

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[(r_1(\rho(t)) - h_1 - r_{11}C_1(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))], \\ \mathrm{d}x_2(t) = x_2(t)[(r_2(\rho(t)) - h_2 - r_{22}C_2(t) - \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))], \\ \mathrm{d}C_1(t) = [k_1C_E(t) - (g_1 + m_1)C_1(t)]\mathrm{d}t, \\ \mathrm{d}C_2(t) = [k_2C_E(t) - (g_2 + m_2)C_2(t)]\mathrm{d}t, \\ \mathrm{d}C_E(t) = [-hC_E(t) + u(t)]\mathrm{d}t, \end{cases} \tag{1.8}$$

where

$$\mathcal{D}_{ji}(x_i)(t) = a_{ji}x_i(t) + \int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta) \quad (i, j = 1, 2),$$

$\int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta)$ are Lebesgue-Stieltjes integrals, $\tau_{ji} > 0$ are time delays, $\mu_{ji}(\theta)$, $\theta \in [-\tau, 0]$ are nondecreasing bounded variation functions, $\tau = \max\left\{\tau_{ji}\right\}$. For other parameters in systems (1.7) and (1.8), see Table 1.

Table 1. Definition of some parameters in system (1.7).

Parameter	Definitions
$C_i(t)$	the toxicant concentration in the organism of species i at time t
$C_E(t)$	the toxicant concentration in the environment at time t
r_{ii}	the dose-response rate of species i to the organismal toxicant
k_i	the toxin uptake rate per unit biomass
g_i	the organismal net ingestion rate of toxin
m_i	the organismal deportation rate of toxin
h	the rate of toxin loss in the environment
$u(t)$	the exogenous total toxicant input into environment at time t

As fundamental assumptions, we assume that $W_1(t)$, $W_2(t)$, $\rho(t)$ and N are independent and $\rho(t)$ is irreducible. Hence, $\rho(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, ..., \pi_S)$. Our aim is, for each system of (1.7) and (1.8), to get the optimal harvesting effort $H^* = \left(h_1^*, h_2^*\right)^{\mathrm{T}}$ such that
① Both $x_1(t)$ and $x_2(t)$ are not extinct;
② The expectation of sustained yield $Y(H) = \lim_{t\to+\infty} \mathbb{E}\left[\sum_{i=1}^{2} h_i x_i(t)\right]$ is maximal.

The rest of this paper is arranged as follows. In Section 2, we study the existence and uniqueness of global positive solution to systems (1.7) and (1.8). For every system, sufficient and necessary conditions for persistence in mean and extinction of each species are obtained in Section 3. We discuss the conditions for global attractivity of the systems in Section 4. In Section 5, sufficient and necessary conditions for the existence of optimal harvesting strategy are established. Furthermore, we give the accurate expressions for the OHE and MESY. Finally, some brief conclusions and discussions are shown in Section 6.

2. Existence and uniqueness of global positive solution

In this paper, we have three fundamental assumptions for systems (1.7) and (1.8).

Assumption 1. *$r_j(i) > 0$, $a_{jk}(i) > 0$ and there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that $\gamma_{j*}(i) \leq \gamma_j(\mu, i) \leq \gamma_j^*(i)$ ($\mu \in \mathbb{Z}$), $\forall i \in \mathbb{S}$, $j, k = 1, 2$. Hence, for any constant $p > 0$, there exists $C_j(p) > 0$ such that*

$$\max_{i\in\mathbb{S}}\left\{\int_{\mathbb{Z}}\left[\ln\left(1+\gamma_j(\mu, i)\right)\right]^2 \lambda(\mathrm{d}\mu)\right\} \leq C_j(p) < +\infty. \tag{2.1}$$

Remark 1. *Assumption 1 implies that the intensity of Lévy noise is not too big to ensure that the solution will not explode in finite time (see, e.g., [42, 44–47]).*

Assumption 2. $0 < k_i \le g_i + m_i$ $(i = 1, 2)$, $\sup_{t\in\mathbb{R}_+} u(t) \le h$.

Remark 2. *Assumption 2 means* $0 \le C_i(t) < 1$ $(i = 1, 2)$ *and* $0 \le C_E(t) < 1$*, which must be satisfied to be realistic because* $C_1(t)$*,* $C_2(t)$ *and* $C_E(t)$ *are concentrations of the toxicant (see Lemma 2.1 in [48]).*

Assumption 3. *The limit of* $u(t)$ *when* $t \to +\infty$ *exists, i.e.,* $\lim_{t\to+\infty} u(t) = u^E$.

Lemma 1. *(Lemma 4.2 in [49]) If Assumption 3 holds, then*

$$\lim_{t\to+\infty} C_E(t) = \frac{u^E}{h}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t C_i(s)\mathrm{d}s = \frac{k_i u^E}{(g_i + m_i)h} \triangleq C_i^E \ (i = 1, 2). \tag{2.2}$$

To study the long-term dynamics of a stochastic population system, we first study the existence and uniqueness of global positive solution to the system.

Theorem 1. *For any initial condition* $(\xi_1, \xi_2)^{\mathrm{T}} \in C([-\tau, 0], \mathbb{R}_+^2)$*, system (1.7) (or system (1.8)) has a unique global solution* $(x_1(t), x_2(t))^{\mathrm{T}} \in \mathbb{R}_+^2$ *on* $t \in [-\tau, +\infty)$ *a.s. Moreover, for any constant* $p > 0$*, there exists* $K_i(p) > 0$ *such that*

$$\sup_{t\ge-\tau} \mathbb{E}\left[x_i^p(t)\right] \le K_i(p) \quad (i = 1, 2). \tag{2.3}$$

Proof. The proof is standard and hence is omitted (see e.g., [50]). □

3. Persistence in mean and extinction

Before studying the persistence in mean and extinction of systems (1.7) and (1.8), we first present the following lemma.

Lemma 2. *Denote* $o(t) = \left\{f(t) \mid \lim_{t\to+\infty} \frac{f(t)}{t} = 0\right\}$*. Suppose* $Z(t) \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$*([51]).*

(i) *If there exists constant* $\delta_0 > 0$ *such that for* $t \gg 1$*,*

$$\ln Z(t) \le \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + o(t), \tag{3.1}$$

then

$$\begin{cases} \limsup_{t\to+\infty} t^{-1}\int_0^t Z(s)\mathrm{d}s \le \dfrac{\delta}{\delta_0} \ a.s. & (\delta \ge 0)\,; \\ \lim_{t\to+\infty} Z(t) = 0 \ a.s. & (\delta < 0)\,. \end{cases} \tag{3.2}$$

(ii) *If there exist constants* $\delta > 0$ *and* $\delta_0 > 0$ *such that for* $t \gg 1$*,*

$$\ln Z(t) \ge \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + o(t), \tag{3.3}$$

then

$$\liminf_{t\to+\infty} t^{-1}\int_0^t Z(s)\mathrm{d}s \ge \frac{\delta}{\delta_0} \ a.s. \tag{3.4}$$

3.1. Predator-prey system

Denote

$$
\begin{cases}
B_1(\cdot) = r_1(\cdot) - \dfrac{\sigma_1^2(\cdot)}{2} - \displaystyle\int_{\mathbb{Z}} [\gamma_1(\mu,\cdot) - \ln(1+\gamma_1(\mu,\cdot))]\,\lambda(\mathrm{d}\mu),\\
B_2(\cdot) = r_2(\cdot) + \dfrac{\sigma_2^2(\cdot)}{2} + \displaystyle\int_{\mathbb{Z}} [\gamma_2(\mu,\cdot) - \ln(1+\gamma_2(\mu,\cdot))]\,\lambda(\mathrm{d}\mu),\\
A_{ij} = a_{ij} + \displaystyle\int_{-\tau_{ij}}^{0} \mathrm{d}\mu_{ij}(\theta)\ \ (i,j=1,2),\\
\Sigma_1 = \displaystyle\sum_{i=1}^{S} \pi_i B_1(i),\ \ \Sigma_2 = -\sum_{i=1}^{S} \pi_i B_2(i) + \frac{A_{21}}{A_{11}}\Sigma_1,\ \ |A| = \begin{vmatrix} A_{11} & A_{12} \\ -A_{21} & A_{22} \end{vmatrix},\\
|A_1| = \begin{vmatrix} \Sigma_1 - r_{11}C_1^E - h_1 & A_{12} \\ \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E - h_2 & A_{22} \end{vmatrix},\\
|A_2| = \begin{vmatrix} A_{11} & \Sigma_1 - r_{11}C_1^E - h_1 \\ -A_{21} & \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E - h_2 \end{vmatrix},\\
\Delta_1 = \begin{vmatrix} \Sigma_1 - r_{11}C_1^E & A_{12} \\ \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E & A_{22} \end{vmatrix},\ \ \Delta_2 = \begin{vmatrix} A_{11} & \Sigma_1 - r_{11}C_1^E \\ -A_{21} & \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E \end{vmatrix}.
\end{cases}
\tag{3.5}
$$

To begin with, let us consider the following stochastic auxiliary system:

$$
\begin{cases}
\mathrm{d}X_1(t) = X_1(t)\left[(r_1(\rho(t)) - h_1 - r_{11}C_1(t) - \mathcal{D}_{11}(X_1)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right],\\
\mathrm{d}X_2(t) = X_2(t)\left[(-r_2(\rho(t)) - h_2 - r_{22}C_2(t) + \mathcal{D}_{21}(X_1)(t) - \mathcal{D}_{22}(X_2)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right],\\
\mathrm{d}C_1(t) = [k_1C_E(t) - (g_1+m_1)C_1(t)]\mathrm{d}t,\\
\mathrm{d}C_2(t) = [k_2C_E(t) - (g_2+m_2)C_2(t)]\mathrm{d}t,\\
\mathrm{d}C_E(t) = [-hC_E(t) + u(t)]\mathrm{d}t.
\end{cases}
\tag{3.6}
$$

Lemma 3. *For system (3.6):*
(a) *If* $\Sigma_1 - r_{11}C_1^E - h_1 < 0$, *then* $\lim_{t\to+\infty} X_i(t) = 0$ *a.s.* $(i=1,2)$.
(b) *If* $\Sigma_1 - r_{11}C_1^E - h_1 \geq 0$, $\Sigma_2 - \frac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \frac{A_{21}}{A_{11}}h_1 - h_2 < 0$, *then*

$$
\lim_{t\to+\infty} t^{-1}\int_0^t X_1(s)\mathrm{d}s = \frac{\Sigma_1 - r_{11}C_1^E - h_1}{A_{11}},\quad \lim_{t\to+\infty} X_2(t) = 0\ \ a.s.
\tag{3.7}
$$

(c) *If* $\Sigma_1 - r_{11}C_1^E - h_1 \geq 0$, $\Sigma_2 - \frac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \frac{A_{21}}{A_{11}}h_1 - h_2 \geq 0$, *then*

$$
\begin{aligned}
&\lim_{t\to+\infty} t^{-1}\int_0^t X_1(s)\mathrm{d}s = \frac{\Sigma_1 - r_{11}C_1^E - h_1}{A_{11}},\\
&\lim_{t\to+\infty} t^{-1}\int_0^t X_2(s)\mathrm{d}s = A_{22}^{-1}\left(\Sigma_2 - \frac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \frac{A_{21}}{A_{11}}h_1 - h_2\right)\ \ a.s.
\end{aligned}
\tag{3.8}
$$

Proof. By Itô's formula and the strong law of large numbers, we compute

$$\begin{cases} \ln X_1(t) = \left(\Sigma_1 - r_{11}C_1^E - h_1\right)t - A_{11}\int_0^t X_1(s)\mathrm{d}s - \mathcal{T}_{11}(X_1)(t) + o(t), \\ \ln X_2(t) = \left(\Sigma_2 - \dfrac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E - h_2\right)t + A_{21}\int_0^t X_1(s)\mathrm{d}s - A_{22}\int_0^t X_2(s)\mathrm{d}s \\ \qquad + \mathcal{T}_{21}(X_1)(t) - \mathcal{T}_{22}(X_2)(t) + o(t), \end{cases} \tag{3.9}$$

where

$$\mathcal{T}_{ji}(X_i)(t) = \int_{-\tau_{ji}}^0 \int_\theta^0 X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta) - \int_{-\tau_{ji}}^0 \int_{t+\theta}^t X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta). \tag{3.10}$$

$\mathfrak{Case}$ (i) : $\Sigma_1 - r_{11}C_1^E - h_1 < 0$. Then $\lim_{t\to+\infty} X_1(t) = 0$ a.s. Hence, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln X_2(t) \leq \left(\Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E - h_2 + \epsilon\right)t - a_{22}\int_0^t X_2(s)\mathrm{d}s, \tag{3.11}$$

which implies $\lim_{t\to+\infty} X_2(t) = 0$ a.s.

$\mathfrak{Case}$ (ii) : $\Sigma_1 - r_{11}C_1^E - h_1 \geq 0$. Consider the following auxiliary function:

$$\mathrm{d}\widetilde{X_2(t)} = \widetilde{X_2(t)}\left[\left(-r_2(\rho(t)) - h_2 - r_{22}C_2(t) + \mathcal{D}_{21}(X_1)(t) - a_{22}\widetilde{X_2(t)}\right)\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right]. \tag{3.12}$$

Then $X_2(t) \leq \widetilde{X_2(t)}$ a.s. By Itô's formula, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln \widetilde{X_2(t)} \leq \left(\Sigma_2 - \dfrac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \dfrac{A_{21}}{A_{11}}h_1 - h_2 + \epsilon\right)t - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s, \\ \ln \widetilde{X_2(t)} \geq \left(\Sigma_2 - \dfrac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \dfrac{A_{21}}{A_{11}}h_1 - h_2 - \epsilon\right)t - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s. \end{cases} \tag{3.13}$$

Thanks to Lemma 2 and the arbitrariness of ϵ, for arbitrary $\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\gamma}^t X_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1,2). \tag{3.14}$$

According to (3.14) and system (3.9), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln X_2(t) \leq \left(\Sigma_2 - \dfrac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \dfrac{A_{21}}{A_{11}}h_1 - h_2 + \epsilon\right)t - A_{22}\int_0^t X_2(s)\mathrm{d}s, \\ \ln X_2(t) \geq \left(\Sigma_2 - \dfrac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \dfrac{A_{21}}{A_{11}}h_1 - h_2 - \epsilon\right)t - A_{22}\int_0^t X_2(s)\mathrm{d}s. \end{cases} \tag{3.15}$$

Based on Lemma 2 and the arbitrariness of ϵ, we obtain:

$(1)^\ddagger$ If $\Sigma_1 - r_{11}C_1^E - h_1 \geq 0,\ \Sigma_2 - \dfrac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \dfrac{A_{21}}{A_{11}}h_1 - h_2 < 0$, then $\lim\limits_{t\to+\infty} X_2(t) = 0$ *a.s.*

$(2)^\ddagger$ If $\Sigma_1 - r_{11}C_1^E - h_1 \geq 0,\ \Sigma_2 - \dfrac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \dfrac{A_{21}}{A_{11}}h_1 - h_2 \geq 0$, then $\lim\limits_{t\to+\infty} t^{-1}\int_0^t X_2(s)\mathrm{d}s = A_{22}^{-1}\left(\Sigma_2 - \dfrac{A_{21}}{A_{11}}r_{11}C_1^E - r_{22}C_2^E - \dfrac{A_{21}}{A_{11}}h_1 - h_2\right)$ *a.s.*

□

Lemma 4. *For system (1.7),* $\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \le 0$ *a.s.* $(i = 1, 2)$.

Proof. Thanks to Lemma 3 and (3.9), system (3.6) satisfies $\lim_{t\to+\infty} t^{-1}\ln X_i(t) = 0$ a.s. $(i = 1, 2)$. From the stochastic comparison theorem, we obtain the desired assertion. □

Lemma 5. *For system (1.7), if* $\lim_{t\to+\infty} x_1(t) = 0$ *a.s., then* $\lim_{t\to+\infty} x_2(t) = 0$ *a.s.*

Proof. The proof of Lemma 5 is similar to that of Lemma 3 (a) and here is omitted. □

Theorem 2. *For system (1.7), define* $\Theta_1 = \Sigma_1 - r_{11}C_1^E - h_1$, $\Theta_2 = \frac{|A_2|}{A_{21}}$.

(i) *If* $\Theta_2 > 0$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s = \frac{|A_i|}{|A|} \quad a.s. \quad (i = 1, 2). \tag{3.16}$$

(ii) *If* $\Theta_1 > 0 > \Theta_2$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Theta_1}{A_{11}}, \quad \lim_{t\to+\infty} x_2(t) = 0 \quad a.s. \tag{3.17}$$

(iii) *If* $0 > \Theta_1$, *then* $\lim_{t\to+\infty} x_i(t) = 0$ *a.s.* $(i = 1, 2)$.

Proof. Clearly, $\Theta_1 > \Theta_2$. Thanks to (3.14), for $\forall\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\gamma}^t x_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1, 2). \tag{3.18}$$

By Itô's formula and (3.18), we deduce

$$\begin{cases} \ln x_1(t) = \left(\Sigma_1 - r_{11}C_1^E - h_1\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s - A_{12}\int_0^t x_2(s)\mathrm{d}s) + o(t), \\ \ln x_2(t) = \left(\Sigma_2 - \dfrac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E - h_2\right)t + A_{21}\displaystyle\int_0^t x_1(s)\mathrm{d}s - A_{22}\int_0^t x_2(s)\mathrm{d}s + o(t). \end{cases} \tag{3.19}$$

$\mathfrak{Case}$ (i) : $\Theta_2 > 0$. According to system (3.19), we compute

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\left(A_{22}\ln x_1(t) - A_{12}\ln x_2(t) + |A|\displaystyle\int_0^t x_1(s)\mathrm{d}s\right) = |A_1|, \\ \lim\limits_{t\to+\infty} t^{-1}\left(A_{21}\ln x_1(t) + A_{11}\ln x_2(t) + |A|\displaystyle\int_0^t x_2(s)\mathrm{d}s\right) = |A_2|. \end{cases} \tag{3.20}$$

Based on Lemma 4, for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$\begin{cases} A_{22}\ln x_1(t) \le (|A_1| + \epsilon)\,t - |A|\displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ A_{11}\ln x_2(t) \ge (|A_2| - \epsilon)\,t - |A|\displaystyle\int_0^t x_2(s)\mathrm{d}s. \end{cases} \tag{3.21}$$

In view of Lemma 2 and the arbitrariness of ϵ, we obtain

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \ge \frac{|A_2|}{|A|} \quad a.s. \tag{3.22}$$

By (3.22), $x_2(t)$ is not extinct. Based on Lemma 5, $x_1(t)$ is not extinct either. In view of Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le \frac{|A_1|}{|A|} \quad a.s. \tag{3.23}$$

According to (3.23) and system (3.19), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \le \left(\Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 - r_{22}C_2^E - h_2 + A_{21}\frac{|A_1|}{|A|} + \epsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \tag{3.24}$$

Thanks to Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le \frac{|A_2|}{|A|} \quad a.s. \tag{3.25}$$

Combining (3.22) with (3.25) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{|A_2|}{|A|} \quad a.s. \tag{3.26}$$

Combining (3.26) with system (3.19) yields that for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \ge \left(\Sigma_1 - r_{11}C_1^E - h_1 - A_{12}\dfrac{|A_2|}{|A|} - \epsilon\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_1(t) \le \left(\Sigma_1 - r_{11}C_1^E - h_1 - A_{12}\dfrac{|A_2|}{|A|} + \epsilon\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s. \end{cases} \tag{3.27}$$

Based on Lemma 2 and the arbitrariness of ϵ, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{|A_1|}{|A|} \quad a.s. \tag{3.28}$$

$\mathfrak{Case}$ (ii) : $\Theta_1 > 0 > \Theta_2$. In view of (3.20), we deduce

$$\limsup_{t\to+\infty} t^{-1}\ln\left[x_1^{A_{21}}(t)x_2^{A_{11}}(t)\right] \le |A_2| < 0 \quad a.s. \tag{3.29}$$

By Lemma 5, $\lim_{t\to+\infty} x_2(t) = 0$ a.s. Thus, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \ge \left(\Sigma_1 - r_{11}C_1^E - h_1 - \epsilon\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_1(t) \le \left(\Sigma_1 - r_{11}C_1^E - h_1 + \epsilon\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s. \end{cases} \tag{3.30}$$

Thanks to Lemma 2 and the arbitrariness of ϵ, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Theta_1}{A_{11}} \quad a.s. \tag{3.31}$$

$\mathfrak{Case}$ (iii) : $0 > \Theta_1$. By system (3.19), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_1(t) \le (\Theta_1 + \epsilon)t - A_{11}\int_0^t x_1(s)\mathrm{d}s. \tag{3.32}$$

So, $\lim_{t\to+\infty} x_1(t) = 0$ a.s. From Lemma 5, $\lim_{t\to+\infty} x_2(t) = 0$ a.s.

□

Remark 3. *If* $\mathbb{S}=\{1\}$, $h_i=0$, $r_{ii}=0$, $\mu_{ii}(\theta)$ *are constant functions defined on* $[-\tau,0]$, $a_{ij}=0$ *(*$i\neq j$*) and* $\mu_{ij}(\theta)$ *are defined as follows:*

$$\mu_{12}(\theta)=\begin{cases}a_{12}^*, & -\tau_{12}^*\le\theta\le 0,\\ 0, & -\tau_{12}\le\theta<-\tau_{12}^*,\end{cases}\quad \mu_{21}(\theta)=\begin{cases}a_{21}^*, & -\tau_{21}^*\le\theta\le 0,\\ 0, & -\tau_{21}\le\theta<-\tau_{21}^*,\end{cases}$$

then system (1.7) becomes

$$\begin{cases}\mathrm{d}x_1(t)=x_1(t)\left[\left(r_1-a_{11}x_1(t)-a_{12}^*x_2\left(t-\tau_{12}^*\right)\right)\mathrm{d}t+\mathcal{S}_1(t)\right],\\ \mathrm{d}x_2(t)=x_2(t)\left[\left(-r_2+a_{21}^*x_1\left(t-\tau_{21}^*\right)-a_{22}x_2(t)\right)\mathrm{d}t+\mathcal{S}_2(t)\right].\end{cases}\tag{3.33}$$

Hence, Theorem 2 contains Theorem 1 in [52] and Theorem 2 in [53] as a special case.

Remark 4. *If* $\mathbb{S}=\{1\}$, $h_i=0$, $\gamma_i(\mu,1)=0$, $\mu_{ii}(\theta)$ *are constant functions defined on* $[-\tau,0]$, $a_{ij}=0$ *(*$i\neq j$*) and* $\mu_{ij}(\theta)$ *are defined as follows:*

$$\mu_{12}(\theta)=\begin{cases}a_{12}^*, & -\tau_{12}^*\le\theta\le 0,\\ 0, & -\tau_{12}\le\theta<-\tau_{12}^*,\end{cases}\quad \mu_{21}(\theta)=\begin{cases}a_{21}^*, & -\tau_{21}^*\le\theta\le 0,\\ 0, & -\tau_{21}\le\theta<-\tau_{21}^*,\end{cases}$$

then system (1.7) becomes

$$\begin{cases}\mathrm{d}x_1(t)=x_1(t)\left[\left(r_1-r_{11}C_1(t)-a_{11}x_1(t)-a_{12}^*x_2\left(t-\tau_{12}^*\right)\right)\mathrm{d}t+\sigma_1\mathrm{d}W_1(t)\right],\\ \mathrm{d}x_2(t)=x_2(t)\left[\left(-r_2-r_{22}C_2(t)+a_{21}^*x_1\left(t-\tau_{21}^*\right)-a_{22}x_2(t)\right)\mathrm{d}t+\sigma_2\mathrm{d}W_2(t)\right],\\ \mathrm{d}C_1(t)=[k_1C_E(t)-(g_1+m_1)C_1(t)]\mathrm{d}t,\\ \mathrm{d}C_2(t)=[k_2C_E(t)-(g_2+m_2)C_2(t)]\mathrm{d}t,\\ \mathrm{d}C_E(t)=[-hC_E(t)+u(t)]\mathrm{d}t.\end{cases}\tag{3.34}$$

Hence, Theorem 2 contains Theorems 4.1, 4.2, 5.1 and 5.2 in [40] as a special case.

3.2. Competitive system

Denote

$$\begin{cases}B_i(\cdot)=r_i(\cdot)-\dfrac{\sigma_i^2(\cdot)}{2}-\displaystyle\int_{\mathbb{Z}}\left[\gamma_i(\mu,\cdot)-\ln(1+\gamma_i(\mu,\cdot))\right]\lambda(\mathrm{d}\mu),\\ A_{ij}=a_{ij}+\displaystyle\int_{-\tau_{ij}}^0\mathrm{d}\mu_{ij}(\theta),\ \Sigma_j=\sum_{i=1}^S\pi_iB_j(i)-r_{jj}C_j^E,\ \Xi_j=\Sigma_j-h_j\ \ (i,j=1,2),\\ \Delta=\begin{vmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix},\ \Delta_1=\begin{vmatrix}\Xi_1 & A_{12}\\ \Xi_2 & A_{22}\end{vmatrix},\ \Delta_2=\begin{vmatrix}A_{11} & \Xi_1\\ A_{21} & \Xi_2\end{vmatrix},\ \Gamma_1=\begin{vmatrix}\Sigma_1 & A_{12}\\ \Sigma_2 & A_{22}\end{vmatrix},\ \Gamma_2=\begin{vmatrix}A_{11} & \Sigma_1\\ A_{21} & \Sigma_2\end{vmatrix}.\end{cases}\tag{3.35}$$

To begin with, let us consider the following stochastic auxiliary system:

$$\begin{cases}\mathrm{d}X_1(t)=X_1(t)[(r_1(\rho(t))-h_1-r_{11}C_1(t)-\mathcal{D}_{11}(X_1)(t))\,\mathrm{d}t+\mathcal{S}_1(t,\rho(t))],\\ \mathrm{d}X_2(t)=X_2(t)[(r_2(\rho(t))-h_2-r_{22}C_2(t)-\mathcal{D}_{21}(X_1)(t)-\mathcal{D}_{22}(X_2)(t))\,\mathrm{d}t+\mathcal{S}_2(t,\rho(t))],\\ \mathrm{d}C_1(t)=[k_1C_E(t)-(g_1+m_1)C_1(t)]\mathrm{d}t,\\ \mathrm{d}C_2(t)=[k_2C_E(t)-(g_2+m_2)C_2(t)]\mathrm{d}t,\\ \mathrm{d}C_E(t)=[-hC_E(t)+u(t)]\mathrm{d}t.\end{cases}\tag{3.36}$$

Lemma 6. *For system (3.36):*

(1) *If* $\Xi_1 < 0,\ \Xi_2 < 0,$ *then* $\lim_{t\to+\infty} X_i(t) = 0$ *a.s.* $(i = 1, 2)$.

(2) *If* $\Xi_1 < 0,\ \Xi_2 \geq 0,$ *then* $\lim_{t\to+\infty} X_1(t) = 0,\ \lim_{t\to+\infty} t^{-1}\int_0^t X_2(s)\mathrm{d}s = \frac{\Xi_2}{A_{22}}$ *a.s.*

(3) *If* $\Xi_1 \geq 0,\ \Xi_2 - A_{21}\frac{\Xi_1}{A_{11}} < 0,$

then $\lim_{t\to+\infty} t^{-1}\int_0^t X_1(s)\mathrm{d}s = \frac{\Xi_1}{A_{11}},\ \lim_{t\to+\infty} X_2(t) = 0$ *a.s.*

(4) *If* $\Xi_1 \geq 0,\ \Xi_2 - A_{21}\frac{\Xi_1}{A_{11}} \geq 0,$

then $\lim_{t\to+\infty} t^{-1}\int_0^t X_1(s)\mathrm{d}s = \frac{\Xi_1}{A_{11}},\ \lim_{t\to+\infty} t^{-1}\int_0^t X_2(s)\mathrm{d}s = A_{22}^{-1}\left(\Xi_2 - A_{21}\frac{\Xi_1}{A_{11}}\right)$ *a.s.*

Proof. Thanks to Itô's formula and the strong law of large numbers, we obtain

$$\begin{cases} \ln X_1(t) = \Xi_1 t - A_{11}\int_0^t X_1(s)\mathrm{d}s - \mathcal{T}_{11}(X_1)(t) + o(t), \\ \ln X_2(t) = \Xi_2 t - A_{21}\int_0^t X_1(s)\mathrm{d}s - A_{22}\int_0^t X_2(s)\mathrm{d}s - \mathcal{T}_{21}(X_1)(t) - \mathcal{T}_{22}(X_2)(t) + o(t). \end{cases} \tag{3.37}$$

$\mathfrak{Case}$ $(\mathfrak{i})$: $\Xi_1 < 0$. Then $\lim_{t\to+\infty} X_1(t) = 0$ a.s. Consider the following auxiliary system:

$$\mathrm{d}\widetilde{X_2(t)} = \widetilde{X_2(t)}\left[\left(r_2(\rho(t)) - h_2 - r_{22}C_2(t) - \mathcal{D}_{21}(X_1)(t) - a_{22}\widetilde{X_2(t)}\right)\mathrm{d}t + \mathcal{S}_2(t, \rho(t))\right]. \tag{3.38}$$

Then $X_2(t) \leq \widetilde{X_2(t)}$ a.s. By Itô's formula, we obtain

$$\ln\widetilde{X_2(t)} = \Xi_2 t - A_{21}\int_0^t X_1(s)\mathrm{d}s - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s - \mathcal{T}_{21}(X_1)(t) + o(t). \tag{3.39}$$

Therefore, for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$\ln\widetilde{X_2(t)} \leq (\Xi_2 + \epsilon)\, t - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s, \quad \ln\widetilde{X_2(t)} \geq (\Xi_2 - \epsilon)\, t - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s. \tag{3.40}$$

In view of Lemma 2 and the arbitrariness of ϵ, we obtain:

$(1)^\dagger$ If $\Xi_1 < 0,\ \Xi_2 < 0,$ then $\lim_{t\to+\infty} \widetilde{X_2(t)} = 0$ *a.s.*

$(2)^\dagger$ If $\Xi_1 < 0,\ \Xi_2 \geq 0,$ then $\lim_{t\to+\infty} t^{-1}\int_0^t \widetilde{X_2(s)}\mathrm{d}s = \frac{\Xi_2}{a_{22}}$ *a.s.*

So, for $\forall\gamma > 0$, we have

$$\lim_{t\to+\infty} t^{-1}\int_{t-\gamma}^t X_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1, 2). \tag{3.41}$$

Combining (3.37) with (3.41) yields that for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\ln X_2(t)\le(\Xi_2+\epsilon)\,t-A_{22}\int_0^t X_2(s)\mathrm{d}s,\quad \ln X_2(t)\ge(\Xi_2-\epsilon)\,t-A_{22}\int_0^t X_2(s)\mathrm{d}s. \tag{3.42}$$

Thanks to Lemma 2 and the arbitrariness of ϵ, we obtain:

$(1)^{\ddagger}$ If $\Xi_1<0,\ \Xi_2<0,$ then $\lim\limits_{t\to+\infty}X_i(t)=0$ *a.s.* $(i=1,2)$.

$(2)^{\ddagger}$ If $\Xi_1<0,\ \Xi_2\ge 0,$ then $\lim\limits_{t\to+\infty}X_1(t)=0,\ \lim\limits_{t\to+\infty}t^{-1}\int_0^t X_2(s)\mathrm{d}s=\frac{\Xi_2}{A_{22}}$ *a.s.*

$\mathfrak{Case}$ (ii) : $\Xi_1\ge 0$. Then,

$$\lim_{t\to+\infty}t^{-1}\int_0^t X_1(s)\mathrm{d}s=\frac{\Xi_1}{A_{11}}\quad a.s. \tag{3.43}$$

Combining (3.37) with (3.43) yields that for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\begin{cases}\ln\widetilde{X_2(t)}\le\left(\Xi_2-A_{21}\frac{\Xi_1}{A_{11}}+\epsilon\right)t-a_{22}\int_0^t\widetilde{X_2(s)}\mathrm{d}s,\\ \ln\widetilde{X_2(t)}\ge\left(\Xi_2-A_{21}\frac{\Xi_1}{A_{11}}-\epsilon\right)t-a_{22}\int_0^t\widetilde{X_2(s)}\mathrm{d}s.\end{cases} \tag{3.44}$$

In view of Lemma 2 and the arbitrariness of ϵ, we obtain:

$(1)^{\natural}$ If $\Xi_1\ge 0,\ \Xi_2-A_{21}\frac{\Xi_1}{A_{11}}<0,$ then $\lim\limits_{t\to+\infty}\widetilde{X_2(t)}=0$ *a.s.*

$(2)^{\natural}$ If $\Xi_1\ge 0,\ \Xi_2-A_{21}\frac{\Xi_1}{A_{11}}\ge 0,$ then $\lim\limits_{t\to+\infty}t^{-1}\int_0^t\widetilde{X_2(s)}\mathrm{d}s=a_{22}^{-1}\left(\Xi_2-A_{21}\frac{\Xi_1}{A_{11}}\right)$ *a.s.*

Hence, for $\forall\gamma>0$, (3.41) is true. According to system (3.37) and (3.41), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\begin{cases}\ln X_2(t)\le\left(\Xi_2-A_{21}\frac{\Xi_1}{A_{11}}+\epsilon\right)t-A_{22}\int_0^t X_2(s)\mathrm{d}s,\\ \ln X_2(t)\ge\left(\Xi_2-A_{21}\frac{\Xi_1}{A_{11}}-\epsilon\right)t-A_{22}\int_0^t X_2(s)\mathrm{d}s.\end{cases} \tag{3.45}$$

Based on Lemma 2 and the arbitrariness of ϵ, we obtain:

$(1)^{\sharp}$ If $\Xi_1\ge 0,\ \Xi_2-A_{21}\frac{\Xi_1}{A_{11}}<0,$ then $\lim\limits_{t\to+\infty}t^{-1}\int_0^t X_1(s)\mathrm{d}s=\frac{\Xi_1}{A_{11}},\ \lim\limits_{t\to+\infty}X_2(t)=0$ *a.s.*

$(2)^{\sharp}$ If $\Xi_1\ge 0,\ \Xi_2-A_{21}\frac{\Xi_1}{A_{11}}\ge 0,$

then $\lim\limits_{t\to+\infty}t^{-1}\int_0^t X_1(s)\mathrm{d}s=\frac{\Xi_1}{A_{11}},\ \lim\limits_{t\to+\infty}t^{-1}\int_0^t X_2(s)\mathrm{d}s=A_{22}^{-1}\left(\Xi_2-A_{21}\frac{\Xi_1}{A_{11}}\right)$ *a.s.*

□

Lemma 7. *For system (1.8),* $\limsup_{t\to+\infty}t^{-1}\ln x_i(t)\le 0$ *a.s.* $(i=1,2)$.

Proof. Thanks to Lemma 6 and (3.37), system (3.36) satisfies $\lim_{t\to+\infty} t^{-1}\ln X_i(t) = 0$ a.s. ($i = 1, 2$). From the stochastic comparison theorem, we obtain the desired assertion. □

Theorem 3. *For system (1.8):*

(1) *If* $\Xi_1 < 0,\ \Xi_2 < 0;\ \Delta \geq 0,\ \Delta_1 < 0,\ \Xi_2 < 0;\ \Delta \geq 0,\ \Delta_2 < 0,\ \Xi_1 < 0$, *then* $\lim_{t\to+\infty} x_i(t) = 0$ *a.s.* ($i = 1, 2$).

(2) *If* $\Xi_1 < 0,\ \Xi_2 \geq 0;\ \Delta \geq 0,\ \Delta_1 < 0,\ \Xi_2 \geq 0$,
then $\lim_{t\to+\infty} x_1(t) = 0,\ \lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{\Xi_2}{A_{22}}$ *a.s.*

(3) *If* $\Xi_1 \geq 0,\ \Xi_2 < 0;\ \Delta \geq 0,\ \Delta_2 < 0,\ \Xi_1 \geq 0$,
then $\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Xi_1}{A_{11}},\ \lim_{t\to+\infty} x_2(t) = 0$ *a.s.*

(4) *If* $\Delta > 0,\ \Delta_1 > 0,\ \Delta_2 \geq 0$, *then* $\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Delta_1}{\Delta}$ *a.s.*

(5) *If* $\Delta > 0,\ \Delta_1 \geq 0,\ \Delta_2 > 0$, *then* $\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{\Delta_2}{\Delta}$ *a.s.*

Proof. By Itô's formula, we compute

$$\begin{cases} \ln x_1(t) = \Xi_1 t - A_{11}\int_0^t x_1(s)\mathrm{d}s - A_{12}\int_0^t x_2(s)\mathrm{d}s + o(t), \\ \ln x_2(t) = \Xi_2 t - A_{21}\int_0^t x_1(s)\mathrm{d}s - A_{22}\int_0^t x_2(s)\mathrm{d}s + o(t). \end{cases} \tag{3.46}$$

According to system (3.46) and Lemma 2, we obtain:

(1)‡ If $\Xi_1 < 0,\ \Xi_2 < 0$, then $\lim_{t\to+\infty} x_1(t) = 0,\ \lim_{t\to+\infty} x_2(t) = 0$ *a.s.*

(2)‡ If $\Xi_1 < 0,\ \Xi_2 \geq 0$, then $\lim_{t\to+\infty} x_1(t) = 0,\ \lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{\Xi_2}{A_{22}}$ *a.s.*

(3)‡ If $\Xi_1 \geq 0,\ \Xi_2 < 0$, then $\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Xi_1}{A_{11}},\ \lim_{t\to+\infty} x_2(t) = 0$ *a.s.*

By system (3.46), we compute

$$\begin{cases} A_{22}\ln x_1(t) - A_{12}\ln x_2(t) = \Delta_1 t - \Delta\int_0^t x_1(s)\mathrm{d}s + o(t), \\ A_{11}\ln x_2(t) - A_{21}\ln x_1(t) = \Delta_2 t - \Delta\int_0^t x_2(s)\mathrm{d}s + o(t). \end{cases} \tag{3.47}$$

By Lemma 7 and (3.47), we obtain that for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$A_{22}\ln x_1(t) \leq (\Delta_1 + \epsilon)\, t - \Delta\int_0^t x_1(s)\mathrm{d}s,\quad A_{11}\ln x_2(t) \leq (\Delta_2 + \epsilon)\, t - \Delta\int_0^t x_2(s)\mathrm{d}s. \tag{3.48}$$

Making use of Lemma 2 yields

$$\begin{cases} \lim\limits_{t\to+\infty} x_1(t) = 0 \quad a.s. \quad (\Delta \ge 0,\ \Delta_1 < 0); \\ \limsup\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t x_1(s)\mathrm{d}s \le \frac{\Delta_1}{\Delta} \quad a.s. \quad (\Delta > 0,\ \Delta_1 \ge 0), \\ \lim\limits_{t\to+\infty} x_2(t) = 0 \quad a.s. \quad (\Delta \ge 0,\ \Delta_2 < 0); \\ \limsup\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t x_2(s)\mathrm{d}s \le \frac{\Delta_2}{\Delta} \quad a.s. \quad (\Delta > 0,\ \Delta_2 \ge 0). \end{cases} \tag{3.49}$$

On the one hand, combining system (3.46) with (3.49) yields

$(1)^\natural$ If $\Delta \ge 0,\ \Delta_1 < 0,\ \Xi_2 < 0,$ then $\lim\limits_{t\to+\infty} x_1(t) = 0,\ \lim\limits_{t\to+\infty} x_2(t) = 0\ a.s.$

$(2)^\natural$ If $\Delta \ge 0,\ \Delta_1 < 0,\ \Xi_2 \ge 0,$

then $\lim\limits_{t\to+\infty} x_1(t) = 0,\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{\Xi_2}{A_{22}}\ a.s.$

$(3)^\natural$ If $\Delta \ge 0,\ \Delta_2 < 0,\ \Xi_1 < 0,$ then $\lim\limits_{t\to+\infty} x_1(t) = 0,\ \lim\limits_{t\to+\infty} x_2(t) = 0\ a.s.$

$(4)^\natural$ If $\Delta \ge 0,\ \Delta_2 < 0,\ \Xi_1 \ge 0,$

then $\lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{\Xi_1}{A_{11}},\ \lim\limits_{t\to+\infty} x_2(t) = 0\ a.s.$

On the other hand, in view of systems (3.46) and (3.49), we deduce that if $\Delta > 0$, $\Delta_1 \ge 0$ and $\Delta_2 \ge 0$, then for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \ge \left(\Xi_1 - A_{12}\dfrac{\Delta_2}{\Delta} - \epsilon\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_2(t) \ge \left(\Xi_2 - A_{21}\dfrac{\Delta_1}{\Delta} - \epsilon\right)t - A_{22}\displaystyle\int_0^t x_2(s)\mathrm{d}s. \end{cases} \tag{3.50}$$

According to Lemma 2 and the arbitrariness of ϵ, we obtain

$$\begin{cases} \liminf\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s \ge \frac{\Delta_1}{\Delta}\ a.s. \quad (\Delta > 0,\ \Delta_1 > 0,\ \Delta_2 \ge 0); \\ \liminf\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s \ge \frac{\Delta_2}{\Delta}\ a.s. \quad (\Delta > 0,\ \Delta_1 \ge 0,\ \Delta_2 > 0). \end{cases} \tag{3.51}$$

Thus, Theorem 3 (4)–(5) follows from combining (3.49) with (3.51). □

Remark 5. *If $\mathbb{S} = \{1\}$, $h_i = 0$, $r_{ii} = 0$ and $\mu_{ij}(\theta)$ are constant functions defined on $[-\tau, 0]$, then system (1.8) becomes*

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1 - a_{11}x_1(t) - a_{12}x_2(t))\,\mathrm{d}t + \mathcal{S}_1(t)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(r_2 - a_{21}x_1(t) - a_{22}x_2(t))\,\mathrm{d}t + \mathcal{S}_2(t)\right]. \end{cases} \tag{3.52}$$

Hence, Theorem 3 contains Theorem 4 in [17] as a special case.

Remark 6. *If* $\mathbb{S} = \{1\}$, $h_i = 0$, $r_{ii} = 0$, $\mu_{ii}(\theta)$ *are constant functions defined on* $[-\tau, 0]$, $a_{ij} = 0$ *(*$i \neq j$*) and* $\mu_{ij}(\theta)$ *are defined as follows:*

$$\mu_{12}(\theta) = \begin{cases} a_{12}^*, & -\tau_{12}^* \le \theta \le 0, \\ 0, & -\tau_{12} \le \theta < -\tau_{12}^*, \end{cases} \quad \mu_{21}(\theta) = \begin{cases} a_{21}^*, & -\tau_{21}^* \le \theta \le 0, \\ 0, & -\tau_{21} \le \theta < -\tau_{21}^*, \end{cases}$$

then system (1.8) becomes

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1 - a_{11}x_1(t) - a_{12}^* x_2\left(t - \tau_{12}^*\right))\,\mathrm{d}t + \mathcal{S}_1(t)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(r_2 - a_{21}^* x_1\left(t - \tau_{21}^*\right) - a_{22}x_2(t))\,\mathrm{d}t + \mathcal{S}_2(t)\right]. \end{cases} \tag{3.53}$$

Hence, Theorem 3 contains Theorem 1 in [53] as a special case.

4. Global attractivity

Assumption 4. $2a_{jj} > \sum_{i=1}^{2} A_{ij}$ $(j = 1, 2)$.

Theorem 4. *Under Assumption 4, system (1.7) (or system (1.8)) is globally attractive.*

Proof. Let $(x_1(t;\phi), x_2(t;\phi))^{\mathrm{T}}$ and $(x_1(t;\phi^*), x_2(t;\phi^*))^{\mathrm{T}}$ be, respectively, the solution to system (1.7) (or system (1.8)) with ϕ and $\phi^* \in C([-\tau, 0], \mathbb{R}_+^2)$, we only need to show

$$\lim_{t \to +\infty} \mathbb{E}\,|x_i(t;\phi) - x_i(t;\phi^*)| = 0 \ (i = 1, 2). \tag{4.1}$$

Define

$$W(t;\phi,\phi^*) = \sum_{i=1}^{2} \left| \ln\left(\frac{x_i(t;\phi^*)}{x_i(t;\phi)}\right) \right| + \sum_{i,j=1}^{2} \int_{-\tau_{ji}}^{0} \int_{t+\theta}^{t} |x_i(s;\phi^*) - x_i(s;\phi)|\,\mathrm{d}s\mathrm{d}\mu_{ji}(\theta). \tag{4.2}$$

By Itô's formula, we derive

$$\mathcal{L}[W(t;\phi,\phi^*)] \le -\sum_{j=1}^{2} \left(2a_{jj} - \sum_{i=1}^{2} A_{ij}\right) \left|x_j(t;\phi^*) - x_j(t;\phi)\right|. \tag{4.3}$$

Based on (4.3), we obtain

$$\mathbb{E}[W(t;\phi,\phi^*)] - \mathbb{E}[W(0;\phi,\phi^*)] \le -\sum_{j=1}^{2} \left(2a_{jj} - \sum_{i=1}^{2} A_{ij}\right) \int_0^t \mathbb{E}\left[\left|x_j(s;\phi^*) - x_j(s;\phi)\right|\right] \mathrm{d}s. \tag{4.4}$$

By (4.4), we deduce

$$\int_0^{+\infty} \mathbb{E}\left[\left|x_j(t;\phi^*) - x_j(t;\phi)\right|\right] \mathrm{d}t \le \frac{\mathbb{E}[W(0;\phi,\phi^*)]}{2a_{jj} - \sum_{i=1}^{2} A_{ij}} \quad (j = 1, 2). \tag{4.5}$$

Define $H_i(t) = \mathbb{E}\left[|x_i(t;\phi^*) - x_i(t;\phi)|\right]$ $(i = 1, 2)$. Then for any $t_1, t_2 \in [0, +\infty)$,

$$|H_i(t_2) - H_i(t_1)| \le \mathbb{E}\left[|x_i(t_2;\phi^*) - x_i(t_1;\phi^*)|\right] + \mathbb{E}\left[|x_i(t_2;\phi) - x_i(t_1;\phi)|\right]. \tag{4.6}$$

Denote $\max_{i\in\mathbb{S}} r_j(i) = r_j^*$, $\max_{i\in\mathbb{S}} |\sigma_j(i)| = \sigma_j^*$, $\sup_{s\geq-\tau} C_j(s) = C_j^*$, $\max_{i\in\mathbb{S}} \sup_{\mu\in\mathbb{Z}} |\gamma_j(\mu,i)| = \gamma_j^*$. Based on Hölder's inequality, for $t_2 > t_1$ and $p > 1$, we deduce

$$\begin{aligned}
&\left(\mathbb{E}\left[|x_j(t_2) - x_j(t_1)|\right]\right)^p \leq \mathbb{E}\left[|x_j(t_2) - x_j(t_1)|^p\right] \\
&\leq 3^{p-1}\mathbb{E}\left[\left(\int_{t_1}^{t_2} x_j(s)\left(r_j^* + h_j + r_{jj}C_j^* + \sum_{i=1}^{2} D_{ji}(x_i)(s)\right)\mathrm{d}s\right)^p\right] \\
&\quad + 3^{p-1}\mathbb{E}\left[\left|\int_{t_1}^{t_2} \sigma_j(\rho(s))x_j(s)\mathrm{d}W_j(s)\right|^p\right] + 3^{p-1}\mathbb{E}\left[\left|\int_{t_1}^{t_2}\int_{\mathbb{Z}} x_j(s)\gamma_j(\mu,\rho(s))\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right|^p\right] \\
&\triangleq 3^{p-1}\Upsilon_1 + 3^{p-1}\Upsilon_2 + 3^{p-1}\Upsilon_3 \quad (j = 1,2).
\end{aligned} \tag{4.7}$$

In view of Theorem 7.1 in [54], for $p \geq 2$, we obtain

$$\Upsilon_2 \leq \left(\sigma_j^*\right)^p \left(\tfrac{p(p-1)}{2}\right)^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} \mathbb{E}\left[x_j^p(s)\right]\mathrm{d}s. \tag{4.8}$$

From Hölder's inequality, we derive

$$\begin{aligned}
\Upsilon_1 \leq\ & 7^{p-1}\left(r_j^*\right)^p (t_2 - t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[x_j^p(s)\right]\mathrm{d}s + 7^{p-1}h_j^p (t_2 - t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[x_j^p(s)\right]\mathrm{d}s \\
&+ 7^{p-1}\left(r_{jj}C_j^*\right)^p (t_2 - t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[x_j^p(s)\right]\mathrm{d}s + 7^{p-1}\sum_{i=1}^{2} a_{ji}^p (t_2 - t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[x_i^p(s)x_j^p(s)\right]\mathrm{d}s \\
&+ 7^{p-1}\sum_{i=1}^{2}(t_2 - t_1)^{p-1}\mathbb{E}\left[\int_{t_1}^{t_2}\left|\int_{-\tau_{ji}}^{0} x_i(s+\theta)x_j(s)\mathrm{d}\mu_{ji}(\theta)\right|^p\mathrm{d}s\right].
\end{aligned} \tag{4.9}$$

According to Hölder's inequality, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_{-\tau_{ji}}^{0} x_j(s)x_i(s+\theta)\mathrm{d}\mu_{ji}(\theta)\right)^p\mathrm{d}s\right] \\
&\leq \frac{1}{2}\left(\int_{-\tau_{ji}}^{0}\mathrm{d}\mu_{ji}(\theta)\right)^p\int_{t_1}^{t_2}\mathbb{E}\left[x_j^{2p}(s)\right]\mathrm{d}s + \frac{1}{2}\left(\int_{-\tau_{ji}}^{0}\mathrm{d}\mu_{ji}(\theta)\right)^{p-1}\int_{t_1}^{t_2}\int_{-\tau_{ji}}^{0}\mathbb{E}\left[x_i^{2p}(s+\theta)\right]\mathrm{d}\mu_{ji}(\theta)\mathrm{d}s.
\end{aligned} \tag{4.10}$$

According to the Kunita's first inequality in [55], for $p > 2$, we get

$$\begin{aligned}
\Upsilon_3 &\leq D(p)\left\{\mathbb{E}\left[\left(\int_{t_1}^{t_2}\int_{\mathbb{Z}}\left|x_j(s)\gamma_j(\mu,\rho(s))\right|^2\lambda(\mathrm{d}\mu)\mathrm{d}s\right)^{\frac{p}{2}}\right] + \mathbb{E}\left[\int_{t_1}^{t_2}\int_{\mathbb{Z}}\left|x_j(s)\gamma_j(\mu,\rho(s))\right|^p\lambda(\mathrm{d}\mu)\mathrm{d}s\right]\right\} \\
&\leq D(p)\left\{\mathbb{E}\left[\left(\gamma_j^*\right)^p\left(\int_{t_1}^{t_2}\int_{\mathbb{Z}} x_j^2(s)\lambda(\mathrm{d}\mu)\mathrm{d}s\right)^{\frac{p}{2}}\right] + \mathbb{E}\left[\left(\gamma_j^*\right)^p\int_{t_1}^{t_2}\int_{\mathbb{Z}} x_j^p(s)\lambda(\mathrm{d}\mu)\mathrm{d}s\right]\right\} \\
&\leq D(p)\left\{\left(\gamma_j^*\right)^p\left(\int_{\mathbb{Z}}\lambda(\mathrm{d}\mu)\right)^{\frac{p}{2}}|t_2 - t_1|^{\frac{p-2}{2}}\int_{t_1}^{t_2}\mathbb{E}[x_j^p(s)]\mathrm{d}s + \left(\gamma_j^*\right)^p\int_{\mathbb{Z}}\lambda(\mathrm{d}\mu)\int_{t_1}^{t_2}\mathbb{E}[x_j^p(s)]\mathrm{d}s\right\}.
\end{aligned} \tag{4.11}$$

By Theorem 1 and (4.7)–(4.11), we deduce that for $p > 2$ and $|t_2 - t_1| \leq \frac{1}{2}$,

$$\left(\mathbb{E}\left[|x_j(t_2) - x_j(t_1)|\right]\right)^p \leq M\,|t_2 - t_1|, \tag{4.12}$$

where

$$M_1 = 21^{p-1}\left\{\left(r_j^*\right)^p K_j(p) + h_j^p K_j(p) + \left(r_{jj}C_j^*\right)^p K_j(p) + \sum_{i=1}^{2}\left[\frac{a_{ji}^p}{2} + \frac{1}{2}\left(\int_{-\tau_{ji}}^{0} \mathrm{d}\mu_{ji}(\theta)\right)^p\right]\left[K_i(2p) + K_j(2p)\right]\right\};$$

$$M_2 = 3^{p-1}\left(\sigma_j^*\right)^p\left(\tfrac{p(p-1)}{2}\right)^{\frac{p}{2}} K_j(p) + 3^{p-1}D(p)\left(\gamma_j^*\right)^p\left(\int_{\mathbb{Z}} \lambda(\mathrm{d}\mu)\right)^p K_j(p); \tag{4.13}$$

$$M_3 = 3^{p-1}D(p)\left(\gamma_j^*\right)^p \int_Z \lambda(\mathrm{d}\mu) K_j(p);$$

$$M = M_1\left(\frac{1}{2}\right)^{p-1} + M_2\left(\frac{1}{2}\right)^{\frac{p-2}{2}} + M_3.$$

Combining (4.7) with (4.12) yields

$$\left|H_j(t_2) - H_j(t_1)\right| \le 2(M|t_2 - t_1|)^{\frac{1}{p}}. \tag{4.14}$$

Then, for any $\epsilon > 0$, there exists $\delta(\epsilon) = \min\left\{\frac{\epsilon^p}{2^p M}, \frac{1}{2}\right\}$ such that for any $t_2 > t_1$ satisfying $|t_2 - t_1| < \delta(\epsilon)$, we have $\left|H_j(t_2) - H_j(t_1)\right| < \epsilon$. Therefore, (4.1) follows from (4.5), (4.14) and Barbalat's conclusion in [56]. □

5. Optimal harvesting strategy

Now, let consider the optimal harvesting problem of systems (1.7) and (1.8).

5.1. Predator-prey system

Theorem 5. *For system (1.7), define*

$$\begin{aligned} h_1^* &= \frac{2A_{11}\Delta_1 + (A_{12} - A_{21})\Delta_2}{4A_{11}A_{22} - (A_{12} - A_{21})^2}, \\ h_2^* &= \frac{(A_{12} - A_{21})\Delta_1 + 2A_{22}\Delta_2}{4A_{11}A_{22} - (A_{12} - A_{21})^2}, \\ Y^*(H) &= -A_{22}h_1^2 + (A_{12} - A_{21})h_1h_2 - A_{11}h_2^2 + \Delta_1 h_1 + \Delta_2 h_2. \end{aligned} \tag{5.1}$$

(i) *If*

$$\begin{cases} \Theta_2|_{h_1 = h_1^* \ge 0,\ h_2 = h_2^* \ge 0} > 0, \\ 4A_{11}A_{22} - (A_{12} - A_{21})^2 > 0, \end{cases} \tag{5.2}$$

then the optimal harvesting strategy exist. Moreover, $H^* = \left(h_1^*, h_2^*\right)^{\mathrm{T}}$ *and*

$$MESY = \frac{Y^*(H^*)}{|A|}. \tag{5.3}$$

(ii) *If one of the following conditions holds, then the optimal harvesting strategy does not exist:*

(a) $\Theta_1|_{h_1=h_1^*} < 0$;

(b) $\Theta_1|_{h_1=h_1^*} > 0 > \Theta_2|_{h_1=h_1^*,\ h_2=h_2^*}$;

(c) $h_1^* < 0$ *or* $h_2^* < 0$;

(d) $4A_{11}A_{22} - (A_{12} - A_{21})^2 < 0$.

Proof. Thanks to (2.3), there exists $\mathrm{C} = \sum_{i=1}^{2} K_i(p) > 0$ such that

$$t^{-1}\int_0^t \mathbb{E}\left[x_1^p(s) + x_2^p(s)\right]\mathrm{d}s \le \mathrm{C}. \tag{5.4}$$

By Theorem 3.1.1 in [57], $(x_1(t), x_2(t), \rho(t))^{\mathrm{T}}$ has an invariant measure $\nu(\cdot\times\cdot) \in \mathbb{R}_+^2 \times \mathbb{S}$. From Theorem 3.1 in [58], $\nu(\cdot\times\cdot)$ is unique. Thanks to Theorem 3.2.6 in [59], $\nu(\cdot\times\cdot)$ is ergodic. Hence, we have

$$\sum_{k=1}^{S}\int_{\mathbb{R}_+^2}\theta_i\nu\left(\mathrm{d}\theta_1, \mathrm{d}\theta_2, k\right) = \lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s \quad a.s. \quad (i = 1, 2). \tag{5.5}$$

Let

$$\mathcal{U} = \left\{H = (h_1, h_2)^{\mathrm{T}} \in \mathbb{R}^2 \,|\, \Theta_2 > 0,\ h_1 \ge 0,\ h_2 \ge 0\right\}. \tag{5.6}$$

On the one hand, from Theorem 2 (i), for every $H \in \mathcal{U}$, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s = \frac{|A_i|}{|A|}. \tag{5.7}$$

On the other hand, if the OHE H^* exists, then $H^* \in \mathcal{U}$.

Proof of (i). Based on the first condition of (5.2), we obtain that $\mathcal{U}$ is not empty. According to (5.7), for $H = (h_1, h_2)^{\mathrm{T}} \in \mathcal{U}$, we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t H^{\mathrm{T}}x(s)\mathrm{d}s = \sum_{i=1}^{2} h_i \lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s = \frac{Y^*(H)}{|A|}. \tag{5.8}$$

Let $\varrho(\cdot\times\cdot)$ be the stationary probability density of system (1.7), then we get

$$Y(H) = \lim_{t\to+\infty}\mathbb{E}\left[H^{\mathrm{T}}x(t)\right] = \sum_{k=1}^{S}\int_{\mathbb{R}_+^2} H^{\mathrm{T}}\theta\varrho(\theta, k)\mathrm{d}\theta. \tag{5.9}$$

Noting that system (1.7) has a unique ergodic invariant measure $\nu(\cdot\times\cdot)$ and that there exists a one-to-one correspondence between $\varrho(\cdot\times\cdot)$ and $\nu(\cdot\times\cdot)$, we deduce

$$\sum_{k=1}^{S}\int_{\mathbb{R}_+^2} H^{\mathrm{T}}\theta\varrho(\theta, k)\mathrm{d}\theta = \sum_{k=1}^{S}\int_{\mathbb{R}_+^2} H^{\mathrm{T}}\theta\nu(\mathrm{d}\theta, k). \tag{5.10}$$

In view of Eqs (5.5), (5.8), (5.9) and (5.10), we deduce

$$Y(H) = \frac{Y^*(H)}{|A|}. \tag{5.11}$$

Solving $\frac{\partial Y^*(H)}{\partial h_1} = \frac{\partial Y^*(H)}{\partial h_2} = 0$ yields

$$\begin{cases} h_1^* = \dfrac{2A_{11}\Delta_1 + (A_{12} - A_{21})\Delta_2}{4A_{11}A_{22} - (A_{12} - A_{21})^2}, \\ h_2^* = \dfrac{(A_{12} - A_{21})\Delta_1 + 2A_{22}\Delta_2}{4A_{11}A_{22} - (A_{12} - A_{21})^2}. \end{cases} \tag{5.12}$$

Define the Hessian matrix Λ as follows:

$$\Lambda = \begin{pmatrix} -2A_{22} & A_{12} - A_{21} \\ A_{12} - A_{21} & -2A_{11} \end{pmatrix}. \tag{5.13}$$

Thanks to $-2A_{22} < 0$ and $4A_{11}A_{22} - (A_{12} - A_{21})^2 > 0$, Λ is negative definite. Thus, $Y^*(H)$ has a unique maximum, and the unique maximum value point of $Y^*(H)$ is $H^* = (h_1^*, h_2^*)^{\mathrm{T}}$. Hence, (5.3) follows from (5.11).

Proof of (ii). First, from Theorem 2 (iii), under condition ($\mathfrak{a}$), the optimal harvesting strategy does not exist. Next, let us show that the optimal harvesting strategy does not exist, provided that either ($\mathfrak{b}$) or ($\mathfrak{c}$) holds. The proof is by contradiction. Suppose that the OHE is $\widetilde{H^*} = \left(\widetilde{h_1^*}, \widetilde{h_2^*}\right)^{\mathrm{T}}$. Then $\widetilde{H^*} \in \mathcal{U}$. In other words, we have

$$\Theta_2|_{h_1=\widetilde{h_1^*},\ h_2=\widetilde{h_2^*}} > 0, \quad \widetilde{h_1^*} \geq 0, \quad \widetilde{h_2^*} \geq 0. \tag{5.14}$$

On the other hand, since $\widetilde{H^*} = \left(\widetilde{h_1^*}, \widetilde{h_2^*}\right)^{\mathrm{T}} \in \mathcal{U}$ is the OHE, then $\left(\widetilde{h_1^*}, \widetilde{h_2^*}\right)^{\mathrm{T}}$ must be the unique solution to system $\frac{\partial Y^*(H)}{\partial h_1} = \frac{\partial Y^*(H)}{\partial h_2} = 0$. Hence, $\left(h_1^*, h_2^*\right)^{\mathrm{T}} = \left(\widetilde{h_1^*}, \widetilde{h_2^*}\right)^{\mathrm{T}}$. Therefore, the Eq (5.14) becomes

$$\Theta_2|_{h_1=h_1^*,\ h_2=h_2^*} > 0, \quad h_1^* \geq 0, \quad h_2^* \geq 0, \tag{5.15}$$

which contradicts with both ($\mathfrak{b}$) and ($\mathfrak{c}$).

Now we are in the position to prove that if the following condition holds, then the optimal harvesting strategy does not exist (i.e., prove ($\mathfrak{d}$)):

$$\begin{cases} \Theta_2|_{h_1=h_1^*\geq 0,\ h_2=h_2^*\geq 0} > 0, \\ 4A_{11}A_{22} - (A_{12} - A_{21})^2 < 0. \end{cases} \tag{5.16}$$

From the first condition of (5.16), we obtain that $\mathcal{U}$ is not empty. Hence (5.11) is true. $-2A_{22} < 0$ implies that Λ is not positive semidefinite. The second condition of (5.16) indicates that Λ is not negative semidefinite. Namely, Λ is indefinite. Thus, $Y^*(H)$ does not exist extreme point. So the OHE does not exist.

The proof is complete. □

Remark 7. *If* $\mathbb{S} = \{1\}$*,* $r_{ii} = 0$*,* $\gamma_i(\mu, 1) = 0$*,* $\mu_{ii}(\theta)$ *are constant functions defined on* $[-\tau, 0]$*,* $a_{ij} = 0$ *(*$i \neq j$*) and* $\mu_{ij}(\theta)$ *are defined as follows:*

$$\mu_{12}(\theta) = \begin{cases} a_{12}^*, & -\tau_{12}^* \leq \theta \leq 0, \\ 0, & -\tau_{12} \leq \theta < -\tau_{12}^*, \end{cases} \quad \mu_{21}(\theta) = \begin{cases} a_{21}^*, & -\tau_{21}^* \leq \theta \leq 0, \\ 0, & -\tau_{21} \leq \theta < -\tau_{21}^*, \end{cases}$$

then system (1.7) becomes

$$\begin{cases}\mathrm{d}x_1(t) = x_1(t)\left[(r_1 - h_1 - a_{11}x_1(t) - a^*_{12}x_2\left(t-\tau^*_{12}\right))\,\mathrm{d}t + \sigma_1\mathrm{d}W_1(t)\right],\\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2 - h_2 + a^*_{21}x_1\left(t-\tau^*_{21}\right) - a_{22}x_2(t))\,\mathrm{d}t + \sigma_2\mathrm{d}W_2(t)\right].\end{cases} \tag{5.17}$$

Hence, Theorem 5 contains Theorem 1 in [60] as a special case.

Remark 8. *If* $r_{ii} = 0$, $\gamma_i(\mu,\rho(t)) = 0$, $\mu_{ii}(\theta)$ *are constant functions defined on* $[-\tau, 0]$ *and* $a_{ij} = 0$ *(*$i \neq j$*), then system (1.7) becomes*

$$\begin{cases}\mathrm{d}x_1(t) = x_1(t)\left[\left(r_1(\rho(t)) - h_1 - a_{11}x_1(t) - \int_{-\tau_{12}}^{0} x_2(t+\theta)\mathrm{d}\mu_{12}(\theta)\right)\mathrm{d}t + \sigma_1(\rho(t))\mathrm{d}W_1(t)\right],\\ \mathrm{d}x_2(t) = x_2(t)\left[\left(-r_2(\rho(t)) - h_2 + \int_{-\tau_{21}}^{0} x_1(t+\theta)\mathrm{d}\mu_{21}(\theta) - a_{22}x_2(t)\right)\mathrm{d}t + \sigma_2(\rho(t))\mathrm{d}W_2(t)\right].\end{cases} \tag{5.18}$$

Therefore, Theorem 2 and Theorem 5 contains, respectively, Theorem 1 and Theorem 2 in [12] as a special case.

5.2. Competitive system

Theorem 6. *For system (1.8), define*

$$\begin{aligned} h^*_1 &= \frac{2A_{11}\Gamma_1 + (A_{12}+A_{21})\Gamma_2}{4A_{11}A_{22} - (A_{12}+A_{21})^2},\\ h^*_2 &= \frac{(A_{12}+A_{21})\Gamma_1 + 2A_{22}\Gamma_2}{4A_{11}A_{22} - (A_{12}+A_{21})^2},\\ Y^*(H) &= -A_{22}h_1^2 + (A_{12}+A_{21})h_1h_2 - A_{11}h_2^2 + \Gamma_1h_1 + \Gamma_2h_2.\end{aligned} \tag{5.19}$$

($\mathcal{A}_1$*) If*

$$4A_{11}A_{22} - (A_{12}+A_{21})^2 > 0,\ \ \Delta_i|_{h_1=h_1^*\geq 0,\, h_2=h_2^*\geq 0} > 0\ (i = 1, 2), \tag{5.20}$$

then the optimal harvesting strategy exists. Moreover, $H^* = \left(h^*_1, h^*_2\right)^{\mathrm{T}}$ *and*

$$MESY = \frac{Y^*(H^*)}{\Delta}. \tag{5.21}$$

($\mathcal{A}_2$*) If one of the following conditions holds, then the optimal harvesting strategy does not exist:*

(B$_1$**)** $\Xi_1|_{h_1=h_1^*} < 0$, $\Xi_2|_{h_2=h_2^*} < 0$*;*

(B$_2$**)** $\Xi_1|_{h_1=h_1^*} < 0$, $\Xi_2|_{h_2=h_2^*} \geq 0$*;*

(B$_3$**)** $\Xi_1|_{h_1=h_1^*} \geq 0$, $\Xi_2|_{h_2=h_2^*} < 0$*;*

(B$_4$**)** $\Delta \geq 0$, $\Delta_1|_{h_1=h_1^*,\, h_2=h_2^*} < 0$, $\Xi_2|_{h_2=h_2^*} < 0$*;*

($\mathbf{B}_5$) $\Delta \geq 0$, $\Delta_2|_{h_1=h_1^*,\, h_2=h_2^*} < 0$, $\Xi_1|_{h_1=h_1^*} < 0$*;*

($\mathbf{B}_6$) $\Delta \geq 0$, $\Delta_1|_{h_1=h_1^*,\, h_2=h_2^*} < 0$, $\Xi_2|_{h_2=h_2^*} \geq 0$*;*

($\mathbf{B}_7$) $\Delta \geq 0$, $\Delta_2|_{h_1=h_1^*,\, h_2=h_2^*} < 0$, $\Xi_1|_{h_1=h_1^*} \geq 0$*;*

($\mathbf{B}_8$) $h_1^* < 0$ *or* $h_2^* < 0$*;*

($\mathbf{B}_9$) $4A_{11}A_{22} - (A_{12} + A_{21})^2 < 0$.

Proof. The proof of Theorem 6 is similar to that of Theorem 5 and hence is omitted. □

Remark 9. *If* $\mathbb{S} = \{1\}$, $r_{ii} = 0$, $\mu_{ii}(\theta)$ *are constant functions defined on* $[-\tau, 0]$, $a_{ij} = 0$ $(i \neq j)$ *and* $\mu_{ij}(\theta)$ *are defined as follows:*

$$\mu_{12}(\theta) = \begin{cases} a_{12}^*, & -\tau_{12}^* \leq \theta \leq 0, \\ 0, & -\tau_{12} \leq \theta < -\tau_{12}^*, \end{cases} \quad \mu_{21}(\theta) = \begin{cases} a_{21}^*, & -\tau_{21}^* \leq \theta \leq 0, \\ 0, & -\tau_{21} \leq \theta < -\tau_{21}^*, \end{cases}$$

then system (1.8) becomes

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1 - h_1 - a_{11}x_1(t) - a_{12}^* x_2(t - \tau_{12}^*))\,\mathrm{d}t + \mathcal{S}_1(t)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(r_2 - h_2 - a_{21}^* x_1(t - \tau_{21}^*) - a_{22}x_2(t))\,\mathrm{d}t + \mathcal{S}_2(t)\right]. \end{cases} \tag{5.22}$$

Hence, Theorem 3 and Theorem 6 contain, respectively, Lemma 2.3 and Theorem 4.1 in [47] as a special case.

6. Conclusions and discussion

In this paper, we study the stochastic dynamics of two hybrid delay Lotka-Volterra systems with harvesting and jumps in a polluted environment. The main results include five theorems. Theorem 2 and Theorem 3 establish sufficient and necessary conditions for persistence in mean and extinction of each species. In Theorem 4, sufficient conditions for global attractivity of the systems are obtained. Theorems 5 and 6 provide sufficient and necessary conditions for the existence of optimal harvesting strategy. Furthermore, we obtain the accurate expressions for the OHE and MESY in Theorems 5 and 6. Our results show that the dynamic behaviors and optimal harvesting strategy are closely correlated with both time delays and environmental noises.

Some interesting questions deserve further investigation. On the one hand, it would be interesting to consider the stochastic hybrid delay food chain model with harvesting and jumps in a polluted environment. On the other hand, it is interesting to investigate the influences of impulsive perturbations on the systems. One may also propose some more realistic systems, such as considering the generalized functional response. We will leave these investigations for future work.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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