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Review

Impulsive strategies in nonlinear dynamical systems: A brief overview

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Abstract: The studies of impulsive dynamical systems have been thoroughly explored, and extensive publications have been made available. This study is mainly in the framework of continuous-time systems and aims to give an exhaustive review of several main kinds of impulsive strategies with different structures. Particularly, (i) two kinds of impulse-delay structures are discussed respectively according to the different parts where the time delay exists, and some potential effects of time delay in stability analysis are emphasized. (ii) The event-based impulsive control strategies are systematically introduced in the light of several novel event-triggered mechanisms determining the impulsive time sequences. (iii) The hybrid effects of impulses are emphatically stressed for nonlinear dynamical systems, and the constraint relationships between different impulses are revealed. (iv) The recent applications of impulses in the synchronization problem of dynamical networks are investigated. Based on the above several points, we make a detailed introduction for impulsive dynamical systems, and some significant stability results have been presented. Finally, several challenges are suggested for future works.

Keywords: impulsive dynamical systems; time delay; event-triggered impulsive control; hybrid impulses; impulsive dynamical networks; stability analysis

1. Introduction

In an engineering environment, some abrupt changes may interfere with system behavior at some certain instants, which are hard to consider continuously. These abrupt changes are usually called impulsive phenomena, and those systems with impulsive phenomena are called impulsive systems [1–9]. Generally, impulsive systems are usually composed of three elements: A continuous dynamics governing the continuous evolution of the system between impulses, which is typically described by a differential equation; a discrete dynamics governing the way that the system state is changed at an impulsive instant; and an impulsive law for determining when the impulse occurs. According to the different effects of impulses on the dynamical behavior of the system, the study of dynamics of impulsive systems can be divided into two classes: i) impulsive disturbance problem (IDP) and ii) impulsive control problem (ICP). If a given system without impulse possesses certain performances, such as periodic solution, attractor, stability, and boundedness, while the corresponding performances can be preserved when the system is subjected to sudden impulsive disturbance, then this phenomenon can be regarded as IDP. Lots of interesting works on IDP have been reported; see [10–14] and the references therein. Particularly, Zhu et al. studied the input-to-state stability (ISS) property of a class of nonlinear impulsive systems via saturated control strategy, where both the impulsive disturbance and external disturbance could severely destroy the dynamical behaviour, and the optimization design procedures were provided with the hope of obtaining the estimates of admissible external disturbance and domain of initial value [10]. Lu et al. extended the average impulsive interval (AII) to linearly coupled neural networks with time-varying delay and impulsive disturbance, and the globally exponential synchronization were obtained by referring to an impulsive delayed differential inequality [11]. If a given system without impulse does not possess a certain performance, but it may possess it via a proper impulsive control, then such phenomenon is generally regarded as ICP. Some interesting works on ICP can be found in [15–20], where Liu et al. established the sufficient criteria for incremental stability via impulsive control strategy by using the methods of Lyapunov-like function and average dwell-time (ADT) condition [17]. The saturation structure was first introduced in impulsive control strategy by Li and Zhu, and the optimization problem on designing controller was presented in order to maximize the domain of attraction [16]. Interestingly, in recent years, a class of impulses, named hybrid impulses, have gradually attracted the attention of researchers. Namely, this kind of impulse includes both impulsive disturbance and impulsive control, and the IDP and ICP can be characterized simultaneously in one system. In this case, the dynamical behavior will become more complex compared with those systems with single impulse. Hence, how to give a uniform criterion to reveal the relationship between the hybrid effects of impulses and dynamical behavior of systems is worth exploring deeply. Recently, some interesting works have been presented to analyze the effects of hybrid impulses; see [21–27] and the references therein. Particularly, Zhu et al. established the relationship between impulsive frequency and time delay existing in hybrid impulses, so that the ISS and iISS of impulsive systems with delayed hybrid impulses can be revealed [21]. Moreover, for impulsive systems with switching topology, Zhu et al. further considered the effects of hybrid impulses, and the incommensurate impulsive switching signals were firstly studied. With the help of Lyapunov method and ADT approach, some sufficient conditions were provided to cope with the problem of ISS for impulsive switched systems [22].

It is noted that time delay is often encountered in many practical cases, such as engineering, biological and economical systems [28–31]. As is known, time delay in a system makes the evolution of the system depend on the historical state heavily, and the performance of the closed-loop system may be degraded if time delay is not taken into consideration. The stability analysis of delayed systems usually includes time-domain and frequency-domain methods. In general, frequency-domain methods are often favored for their conceptual simplicity and computational ease, which typically can be checked in an efficient manner by plotting graphically a certain frequency-dependent measure. Different from the frequency-domain methods, the time-domain methods have more advantages in adapting to nonlinear time-delay systems. In this scene, a large number of stability results have been derived by using

Lyapunov-Razumikhin (L-R) approach, Lyapunov-Krasovskii (L-K) functional approach, Halanay-type inequalities, comparison principle, and so on. Owing to the important role in dynamical behavior estimation of engineering systems, time delay has been receiving increasing attention in various dynamical systems [32–38]. For instance, Zhu et al. considered the leakage delay existing in neural networks, and the delayed state-feedback control strategy was constructed for globally exponential stability, which showed the controller can be used in all states or in some state [32]. Yang et al. explored the state-dependent state delay making the value of delay dependent on the state change, which indicated that it was impossible to exactly know a priori how far the historical state information was needed, and the sufficient conditions for exponential stability of the zero equilibrium were derived by using the Lyapunov stability theory [33]. According to the different parts where the time delay exists, impulsive systems with time delay can be roughly divided into two classes: impulsive delayed systems (IDSs) [39–42] and delayed impulsive systems (DISs) [43–46]. In IDSs, time delay exists in the continuous dynamics of impulsive systems, and the continuous state evolution is affected by historical state while the impulsive jump excludes time delay. In DISs, impulsive jump depends on the historical state, and the continuous dynamics of the system may contain time delay. Interestingly, with the deepening of research, some potential effects of time delay on the dynamical behavior of impulsive systems have been slowly discovered. That is, in addition to destroying the original performance of the system to a certain extent, some unstable systems will become stable by properly applying time delay; see [47, 48] for more details.

Most of the existing results on ICP focus on a time-triggered mechanism (TTM), which means the instants when the impulses occur, normally called impulsive instants, are pre-scheduled [5, 49–52]. Nevertheless, sometimes unnecessary impulsive control tasks may be executed in TTM, which clearly is a waste of control efforts and communication resources. Different from TTM, an event-triggered mechanism (ETM) can avoid unnecessary waste of resources because the information transmissions are determined by the occurrence of some well-designed events which are related to the system state or output. Specifically, in ETM, a transmission only happens when an event is triggered; otherwise, the impulsive control signal will not be updated. Due to the advantages of ETM, many efforts have been made for various practical control systems [53–57]. Furthermore, by combining the advantages of impulsive control strategy and ETM, a kind of novel control strategy named event-triggered impulsive control (ETIC) has gradually obtained more and more attentions, and many meaningful results have been derived [58–63]. Shanmugasundaram et al. investigated the synchronization problem of inertial neural networks with time delays by virtue of ETIC, in which a Lyapunov function based ETM was used to determine impulsive instants [63]. In the presence of time delay, Li et al. studied the exponential stability of nonlinear delayed systems by means of ETIC approach, and the theoretical result has been applied to nonlinear delayed multi-agent systems [64]. However, one extreme case, where the event function cannot be activated while the state of system diverges in a relatively long time interval, should be mentioned. To solve this situation, some ETIC strategies containing the forced mechanism have been proposed, implying that the impulses may be forcibly triggered if the system status or output fails to trigger the event within a relatively long time interval, and some meaningful results have been obtained [65–67].

The mathematical descriptions of many evolution processes and hybrid dynamical systems can be characterized by impulsive systems, and the applications of the theory of impulsive systems to engineering or biosciences are increasing, including mechanical systems, automatic and remote control, secure communication, neural networks, epidemiology, forestry, vaccination, population management, etc. See [68–77] for more details. As a very important and meaningful application of impulsive systems, impulsive dynamical networks (IDNs) have received more and more attentions, and impulsive control has shown its advantages in control purpose on the clustering properties of IDNs, especially in the synchronization problems. Intuitively, dynamical networks can be regarded as a whole composed of a large number of systems with dynamical behaviors and their interactions. Generally, the system is regarded as a single node, and the interactions between systems are regarded as the connections or edges between the nodes. A single node is the identity element of the network, which has certain dynamical behavior. Edges represent the connections between these nodes. All edges combined together constitute the network topology, so that the networks show the clustering property. Now, there are many studies on the clustering behaviour of dynamical networks, and synchronization is the key. It can be found everywhere, such as applauding, the beating of the heart and so on. Synchronization is that all/some nodes in a network can reach a consistent state [78–86]. For many realistic networks, the state of nodes is often subject to instantaneous disturbance and experiences abrupt change at certain instants which may be caused by switching phenomena, frequency change or other sudden noise, that is, it exhibits impulsive effects [87, 88]. As we mentioned before, impulsive control has a simple structure and lower control cost, thus impulsive control strategy has been widely applied in the synchronization for dynamical networks, and many insightful results have been proposed [89–92]. For instance, the exponential synchronization for memristive networks with inertial and nonlinear coupling terms was studied by Fu et al. [91], and two novel hybrid mode-dependent pinning impulsive control approaches were proposed, where one was adaptive element-selection, and the other was fixed node-selection. Ji et al. introduced the concept of average delayed impulsive weight (ADIW) to explore the synchronization of IDNs with both system delay and coupled delay, and the certain flexible criteria were given as well as the estimated corresponding convergence rate [92].

By analyzing related studies, this paper provides a comprehensive and intuitive overview for nonlinear impulsive systems, involving time delay, event-triggered mechanism (ETM), hybrid impulses, and the applications in dynamical networks. Essentially, it focuses on the stability and stabilization problems, and some meaningful results in recent works have been presented. The rest of this paper is organized as follows. In Section 2, some fundamental notations and descriptions for impulsive systems are presented. Section 3 covers the stability problems of IDSs and DISs, which are divided according to the different parts where the time delay exists. The impulsive control strategy based on certain event, named ETIC, is introduced in Section 4, which involves several classical ETMs. Section 5 considers the analysis of dynamical behaviour of nonlinear systems in the presence of hybrid impulses. The effects of impulses existing in dynamical networks are illustrated in Section 6, and some synchronization results are given. Section 7 concludes the paper and discusses the future research directions on this topic.

Notation: Let $\mathbb{Z}_+$ denote the set of positive integer numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{R}_{>0}$ the set of positive real numbers, and $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. The symbol $||x||$ stands for the Euclidean norm of a real vector x, and I represents the identity matrix with compatible dimension. $a \vee b$ and $a \wedge b$ are the maximum and minimum of a and b, respectively. For any interval $J \subseteq \mathbb{R}$, set $S \subseteq \mathbb{R}^k (1 \leq k \leq n)$, $\mathbb{C}(J,S) = \{\varphi : J \to S \text{ is continuous}\}$ and $\mathbb{C}^1(J,S) = \{\varphi : J \to S \text{ is continuously differentiable}\}$. $\mathbb{PC}(J,S) = \{\varphi : J \to$

S is continuous everywhere except at finite number of points t, at which $\varphi(t^+), \varphi(t^-)$ exist and $\varphi(t^+) = \varphi(t)\}$. If a continuous function $\varphi(s) > 0$ for all $s > 0$ and $\varphi(0) = 0$, then $\varphi \in \mathbb{P}$. Moreover, if $\varphi \in \mathbb{P}$ and is globally Lipschitz continuous, then $\varphi \in \mathbb{P}_L$. For given $\mathfrak{I} > 0$, set $\mathbb{C}_{\mathfrak{I}} = \mathbb{C}([-\mathfrak{I}, 0], \mathbb{R}^n)$ and $\mathbb{PC}_{\mathfrak{I}} = \mathbb{PC}([-\mathfrak{I}, 0], \mathbb{R}^n)$ with the norm $\| \times \|$ defined by $\|\varphi\|_{\mathfrak{I}} = \sup\{\|\varphi(s)\| : s \in [-\mathfrak{I}, 0]\}$. $\mathcal{B}(\mathfrak{I}) = \{x \in \mathbb{R}^n : \|x\| < \mathfrak{I}\}$. Let $(\Omega, \mathcal{R}, \{\mathcal{R}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{R}_t\}_{t\geq 0}$ satisfying the usual condition (i.e., it is right continuous, and $\mathcal{R}_0$ satisfies the usual condition and contains all $\mathbb{P}$-null sets). $\mathbb{PC}^b_{\mathcal{R}_0}([-\mathfrak{I}, 0], \mathbb{R}^n)$ denotes the family of all bounded $\mathcal{R}_0$-measurable, and $\mathbb{PL}^p_{\mathcal{R}_0}([-\mathfrak{I}, 0], \mathbb{R}^n)$ denotes the family of all $\mathcal{R}_0$-measurable. $\mathbb{E}(\cdot)$ denotes the expectation operator with respect to the probability measure $\mathbb{P}$. Given a constant $M > 0$, set $\mathbb{C}^M_{\mathfrak{I}} = \{\varphi \in \mathbb{C}_{\mathfrak{I}} : 0 < \|\varphi\| \leq M\}$. A function $\vartheta : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$ if ϑ is continuous, strictly increasing, and $\vartheta(0) = 0$. In addition, if ϑ is unbounded, then it is of class $\mathcal{K}_\infty$. $\mathcal{KL} = \{\beta \in \mathbb{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)|\, \beta(r, t)$ is in class $\mathcal{K}$ w.r.t. r for each fixed $t \geq 0$, and $\beta(r, t)$ is decreasing to 0 as $t \to \infty$ for each fixed $r \geq 0\}$.

2. Preliminaries

Consider the general impulsive system

$$\begin{cases} \dot{x}(t) = f(t, x(t)),\ t \neq t_k,\ t \geq t_0, \\ x(t) = g_k(t^-, x(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $\dot{x}$ denotes the left-hand derivative of x, functions $f, g_k \in \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, with f locally Lipschitz, such that $f(t, 0) = 0$ and $g_k(t, 0) = 0$, so that system (2.1) admits a trivial solution $x \equiv 0$. The impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ describes that the continuous dynamics from equation $\dot{x}(t) = f(t, x(t))$ activated when $t \neq t_k$, and the discrete dynamics from the impulsive condition $x(t) = g_k(t^-, x(t^-))$ activated when $t = t_k$. As is usual for an impulsive system, we consider the impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ that is strictly increasing and has no accumulation points, i.e., $\lim_{k\to\infty} t_k = \infty$. Let the set $\mathcal{F}_0$ denote such kind of impulsive time sequences for later use. The state variables of system (2.1) are right continuous at each t_k, i.e., $x(t_k^+) = x(t_k)$. In other words, $x(t)$ is continuous at each interval $[t_{k-1}, t_k), k \in \mathbb{Z}_+$. Moreover, a solution of (2.1) is called a solution with initial value $x_0 := x(t_0)$, denoted by $x(t, t_0, x_0)$, if it satisfies initial value x_0.

From above model description, one can observe that impulsive system usually consists of three parts: i) an ordinary differential equation that characterizes the dynamical behaviour between two consecutive impulsive instants, ii) a difference equation that describes the instantaneous change of state at each impulsive instant, iii) a regulation that determines when the impulse occurs. Hence, the impulsive system typically produces solutions that are piecewise continuous. Namely, for system (2.1), a function $x \in \mathbb{PC}([t_0, t_0 + \ell), \mathbb{R}^n)$ is said to be a solution of system (2.1) on $[t_0, t_0 + \ell)$ for some $\ell > 0$, if $x(t)$ is piecewise absolutely continuous in $[t_0, t_0 + \ell)$, is continuous at each $t \neq t_k$ in $[t_0, t_0 + \ell)$, satisfies the differential equation of (2.1) for all $t \in [t_0, t_0 + \ell)$ except on a set of Lebesgue measure zero, and satisfies the difference equation of (2.1) for all $t_k \in [t_0, t_0 + \ell)$. Other necessary and sufficient conditions for function $x \in \mathbb{PC}([t_0, t_0 + \ell), \mathbb{R}^n)$ to be a solution of system (2.1) can be found in [1, 5]. In what follows, some main stability definitions for (2.1) will be briefly introduced.

Definition 1. *Suppose that an impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is given. Let $x(t) = x(t, t_0, x_0)$ be the solution of system (2.1) through (t_0, x_0). Then, system (2.1) is said to be*

(D_1) stable if for any $\varepsilon > 0, t_0 \geq 0$, there exists a $\sigma = \sigma(t_0, \varepsilon) > 0$ such that $\forall x_0 : \|x_0\| < \sigma$ implies that $\|x(t)\| < \varepsilon$ for all $t \geq t_0$;

(D_2) uniformly stable (US) if δ in (D_1) is independent of t_0;

(D_3) asymptotically stable (AS) if (D_1) holds, and there exists a $\sigma > 0$ such that $\forall x_0 : \|x_0\| < \sigma$ implies that $\lim_{t\to\infty} \|x(t)\| = 0$;

(D_4) exponentially stable (ES) if there exist $\sigma, \lambda > 0, \eta \geq 1$, such that $\forall x_0 : \|x_0\| < \sigma$ implies that $\|x(t)\| \leq \eta\|x_0\|e^{-\lambda(t-t_0)}$ for all $t \geq t_0$;

(D_5) globally exponentially stable (GES) if there exist $\lambda > 0, \eta \geq 1$, such that $\forall x_0$ implies that $\|x(t)\| \leq \eta\|x_0\|e^{-\lambda(t-t_0)}$ for all $t \geq t_0$;

(D_6) globally uniformly exponentially stable (GUES) if the GES property holds over the class Φ, where Φ denotes a class of impulsive time sequences in $\mathcal{F}_0$.

Definition 2. *$V(t, x) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ is said to be in $\mathcal{V}_0$ if*

(D_1) $V(t, x)$ is continuous in $[t_{k-1}, t_k) \times \mathbb{R}^n, k \in \mathbb{Z}_+$ and $\lim_{(t,z)\to(t_k,z')} V(t, z) = V(t_k, z')$;

(D_2) $V(t, x)$ is locally Lipschitz in x and $V(t, 0) = 0$ for all $t > 0$;

(D_3) there exist positive scalars ω_1, ω_2 and p such that $\omega_1\|x\|^p \leq V(t, x) \leq \omega_2\|x\|^p, \forall x \in \mathbb{R}^n$.

Definition 3. *Let $V(t, x) \in \mathcal{V}_0$, and then the left-upper Dini derivative of $V(t, x)$ along (2.1) is defined as*

$$D^+V(t, x) = \limsup_{h\to 0^+} \frac{1}{h}[V(t + h, x + hf(x)) - V(t, x)],$$

where $(t, x) : [t_{k-1}, t_k) \times \mathbb{R}^n, k \in \mathbb{Z}_+$.

Based on the Lyapunov method, the Lyapunov function $V(t, x)$ with proper form is usually constructed to reflect the evolution process of system state. By introducing indexes $\alpha \in \mathbb{R}, \beta \in \mathbb{R}_+$ named rate coefficients, it holds for any $k \in \mathbb{Z}_+$ that $D^+V(t, x(t)) \leq \alpha V(t, x(t)), t \neq t_k$ and $V(t, x(t)) \leq \beta V(t^-, x(t^-)), t = t_k$. Specially, β is usually called the impulsive gain. Due to the fact the impulsive gain in (2.1) may be different at the different impulsive instants, the dynamical behaviour $x(t) = g_k(t^-, x(t^-)), t = t_k$ implies that β can also be rewritten as β_k, that is, $V(t, x(t)) \leq \beta_k V(t^-, x(t^-)), t = t_k$. It should be mentioned that, β corresponds to different value ranges in terms of the different effects of impulses on the dynamic behaviour of the system, that is,

i) If the state value at t_k is lower than the state value at t_k^-, that is, the impulses are beneficial for the stability property of system (2.1), then we say that impulses are stabilizing. In this case, β can be chosen in $(0, 1)$ to characterize the positive effect of impulses. The stabilizing impulses can be considered as impulsive control input, and the relevant results have been shown in [15, 48, 92–94];

ii) If the state value at t_k is equal to the state value at t_k^-, that is, the impulses are neither harmful nor beneficial for the stability property of system (2.1), then these impulses are named as inactive impulses, under which β can usually be chosen as 1.

iii) If the state value at t_k is larger than the state value at t_k^-, that is, the impulses can potentially destroy the stability property of system (2.1), then we say that impulses are destabilizing. In this case, β can be chosen in $(1, \infty)$ to characterize the negative effect of impulses. The destabilizing impulses can be regarded as impulsive disturbance; see [10, 12, 13].

Remark 1. Clearly, constants α and β are used to characterize the evolution of the Lyapunov function along the solutions to an impulsive system, and such characterizations are referred to as linear rates (since the resulting inequalities on $D^+V(t, x(t)), t \neq t_k$ and $V(t, x(t)), t = t_k$ are linear). Interestingly, one can use arbitrary nonlinear functions instead of linear ones with slopes α and β. Specifically, assume there exist functions $\alpha_L \in \mathbb{P}$ and $\beta_L \in \mathbb{P}_L$, the dynamical behaviour of impulsive system can be described by $D^+V(t, x(t)) \leq \alpha_L(V(t, x(t))), t \neq t_k$ and $V(t, x(t)) \leq \beta_L(V(t^-, x(t^-))), t = t_k$. Such an approach with nonlinear rates is especially beneficial for nonlinear impulsive systems, and it is able to obtain less conservative stability conditions compared to the conditions based on linear rates. For more detailed discussions, see [95–99] and the references therein.

For impulsive systems, an important scheme is to consider the impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$, which determines when the impulse occurs. Generally, there are two mechanisms that have been used in existing works: Time-triggered mechanism (TTM) and event-triggered mechanism (ETM). Namely, the former means that the instants of impulses follow the certain preset rule as time goes by, and the conditions proposed to restrict the impulsive time sequences can be divided into the following several classes:

i) ArDT (arbitrary dwell-time) condition [100]: impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy ArDT condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_0$, if $t_k - t_{k-1} > 0$ for any $k \in \mathbb{Z}_+$;

ii) FDT (fixed dwell-time) condition [101]: impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy FDT condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_\rho$ for some $\rho > 0$, if $t_k - t_{k-1} \equiv \rho$ for any $k \in \mathbb{Z}_+$;

iii) MiDT (minimum dwell-time) condition [102]: impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy MiDT condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_\rho^+$ for some $\rho > 0$, if $t_k - t_{k-1} \geq \rho$ for any $k \in \mathbb{Z}_+$;

iv) MaDT (maximum dwell-time) condition [103]: impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy MaDT condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_\rho^-$ for some $\rho > 0$, if $t_k - t_{k-1} \leq \rho$ for any $k \in \mathbb{Z}_+$;

v) TsDT/PDT/RDT (two sided/persistent/range dwell-time) condition [103–105]: impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy TsDT/PDT/RDT condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_{\rho_1}^{\rho_2}$ for some $\rho_2 \geq \rho_1 > 0$, if $\rho_1 \leq t_k - t_{k-1} \leq \rho_2$ for any $k \in \mathbb{Z}_+$;

vi) AII (average impulsive interval) condition [88]: impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy AII condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_{AII}$, if there are positive constants τ_{AII}, N_0, such that $\frac{t-s}{\tau_{AII}} - N_0 \leq N(t, s) \leq \frac{t-s}{\tau_{AII}} + N_0, \forall t \geq s \geq t_0$, where $N(t, s)$ denotes the number of impulsive instants in the semi-open interval $[s, t)$, τ_{AII} represents the AII constant of $\mathcal{F}_{AII}$, and N_0 is called the elasticity number.

vii) EUB (eventually uniformly bounded) condition [95,96]: impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy eventually uniformly upper bounded (EUupB) condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_{B_{up}}$, if for $\forall \varepsilon > 0$, there are positive constants $T(\varepsilon), \delta$, such that $\frac{N(t+\Delta,t)}{\Delta} \leq \delta + \varepsilon, \forall \Delta \geq T(\varepsilon), \forall t \geq t_0$. Correspondingly, impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is said to satisfy eventually uniformly lower bounded (EUlowB) condition, denoted by $\{t_k, k \in \mathbb{Z}_+\} \subseteq \mathcal{F}_{B_{low}}$, if for $\forall \varepsilon > 0$, there are positive constants $T(\varepsilon), \delta$, such that $\delta - \varepsilon \leq \frac{N(t+\Delta,t)}{\Delta}, \forall \Delta \geq T(\varepsilon), \forall t \geq t_0$.

Remark 2. In [106, 107], the above described dwell-time conditions have been systematically introduced, and the corresponding stability and stabilization of impulsive systems have been summarized. Based on TsDT/PDT/RDT condition, Dashkovskiy and Slynko recently considered the nonlinear impulsive systems on Banach spaces subjected to disturbances and looked for dwell-time conditions guaranteeing the stability property [104, 105]. Interestingly, the presented results covered the case where both continuous and discrete dynamics could be unstable simultaneously. That is, the controlled stability performance of the system could still be obtained, even if the continuous and discrete behaviours of the system were both unstable, as long as the proposed TsDT/PDT/RDT condition could be met. In consideration of the fact that the occurrence of impulses may be not uniformly distributed, hence there may exist some consecutive impulsive signals separated by less than or greater than fixed bound ρ. Inspired by this phenomenon, Feketa and Dashkovskiy proposed the EUB conditions [95,96] by improving AII condition [88], which is able to characterize the wider range of impulsive time sequences. In particular, EUB conditions can be separated into

$$\mathbf{EU_{up}B}: \ \frac{N(t+\Delta, t)}{\Delta} \leq \delta + \varepsilon, \ \forall \Delta \geq T(\varepsilon), \ \forall \varepsilon > 0, \ \forall t \geq t_0, \tag{2.2}$$

$$\mathbf{EU_{low}B}: \ \frac{N(t+\Delta, t)}{\Delta} \geq \delta - \varepsilon, \ \forall \Delta \geq T(\varepsilon), \ \forall \varepsilon > 0, \ \forall t \geq t_0. \tag{2.3}$$

when the original system without impulsive disturbance is stable, and the impulses are harmful, in order to guarantee the stability, the impulses should not occur frequently. In this case, condition (2.2) enforces an upper bound on the number of impulses, and impulsive intervals should be longer. Conversely, when the original system without impulse is unstable and the impulses are beneficial, in order to ensure the stability of the system, it is usually assumed that the frequency of impulses should not be too low. Therefore, condition (2.3) enforces a lower bound on the number of impulses, and impulsive intervals will be not overly long in this case. Because EUB conditions are the improved criteria for AII conditions and are substantially broader than other previously considered classes, it is undoubted the above two discussions still correspond to the AII conditions. By comparison, conditions ii)–iv) may impose the relatively strong restrictions on the interval length of arbitrary two adjacent impulsive instants, while there are still some stability criteria based on these conditions that deserve attentions [95,96,108]. Based on the classification criterion proposed by Feketa et al., impulsive time sequences with conditions ii)–iv) can also be regarded as being defined within predefined time-windows [106]. Clearly, no matter by what standard, such classifications can be a basis to answer a question on the robustness of impulsive stabilization, w.r.t. the perturbations of the moments of jumps.

For the latter triggered mechanism, that is, the ETM, the instants of impulses are determined by an event, which is usually generated by some well-designed and output-based or state-based event conditions. The ETM is generally in the form

$$t_k = \inf_{k \in \mathbb{Z}_+} \{t \geq t_{k-1} : \Omega(t) \geq 0\}, \tag{2.4}$$

where function $\Omega(t)$ is named event function. In recent studies, impulsive control equipped with ETM has been well known as event-triggered impulsive control (ETIC), which has shown its advantages in the problems of control and communication, see [53, 61–63, 67]. The detailed introductions of ETIC will be given in Section 4, and the impulsive time sequences involved in other sections are mainly in the category of TTM.

3. Stability for IDSs and DISs

Time delay is ubiquitous in nature and exists widely in many practical systems [109, 110]. For those systems with many types of dynamical behaviours, the operation of the system will be affected differently according to the different parts where the time delay exists. It is possible that the time delay exists in continuous or discontinuous dynamics, thus impulsive systems with time delay can be roughly divided into impulsive delayed systems (IDSs) and delayed impulsive systems (DISs). More specifically, they are generally reviewed as:

i)

$$\textbf{IDSs}:\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau)),\ t \neq t_k,\ t \geq t_0, \\ x(t) = g_k(t^-, x(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \\ x(t_0+\theta) = \phi(\theta),\ \theta \in [-\tau, 0], \end{cases} \tag{3.1}$$

where τ is a positive constant representing the time delay existing in the continuous dynamics of (3.1). The function $\phi \in \mathbb{PC}([-\tau, 0], \mathbb{R}^n)$ denotes the initial value of the system state.

ii)

$$\textbf{DISs}:\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau)),\ t \neq t_k,\ t \geq t_0, \\ x(t) = g_k((t-\tau_k)^-, x((t-\tau_k)^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \\ x(t_0+\theta) = \phi(\theta),\ \theta \in [-\varrho, 0], \end{cases} \tag{3.2}$$

where the time delays in impulses meet $\tau_k \in \mathbb{R}_+, k \in \mathbb{Z}_+$, and sequence $\{\tau_k\}$ is said to be the impulsive delay sequence. The function $\phi \in \mathbb{PC}([-\varrho, 0], \mathbb{R}^n)$ denotes the initial value of the system (3.2), where $\varrho = \sup_{k\in\mathbb{Z}_+}\{\tau_k - t_k, \tau\}$.

For impulsive systems (3.1) and (3.2), due to the existing of time delay, initial value (t_0, x_0) in those systems without time delay will be transmitted to initial function (t_0, ϕ), and the solution to systems (3.1) and (3.2) through (t_0, ϕ) should be denoted by $x(t) = x(t, t_0, \phi)$. For simplicity, here we will not redescribe the relevant stability definitions of impulsive systems with time delay, which can be easily deduced on the basis of Definition 1 and 3. Particularly, some necessary definitions should be added here, which are presented as follows:

Definition 4. *Suppose that an impulsive time sequence $\{t_k, k \in \mathbb{Z}_+\}$ is given. Let $x(t) = x(t, t_0, \phi)$ be the solution of system (3.1) or (3.2) through (t_0, ϕ). Then, given constant $\Im > 0$, (3.1) or (3.2) is said to be*

(D_1) locally uniformly stable (LUS) in the region $\phi \in \mathbb{C}_{\Im}^{M}$ if there exists a constant $M > 0$, and if for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon, M) \in (0, M]$ such that $\forall \phi : \phi \in \mathbb{C}_{\Im}^{\delta}$ implies that $\|x(t, t_0, \phi)\| < \varepsilon$ for all $t \geq t_0$;

(D_2) locally uniformly asymptotically stable (LUAS) in the region $\phi \in \mathbb{C}_{\Im}^{M}$ if it is US and uniformly attractive;

(D_3) locally exponentially stable (LES) in the region $\phi \in \mathbb{C}_{\Im}^{M}$ if there exist constants $\lambda > 0, \eta \geq 1, M > 0$ such that $\forall \phi : \phi \in \mathbb{C}_{\Im}^{M}$ implies that $\|x(t)\| \leq \eta\|\phi\|_{\Im} e^{-\lambda(t-t_0)}$ for all $t \geq t_0$.

In what follows, we start with the stability of IDSs where the time delay exists in the continuous dynamics of impulsive systems.

3.1. Stability of IDSs

According to IDSs (3.1), it is shown the evolution of continuous dynamical behaviour depends on not only their current but also historical states of the system. For IDSs (3.1), Lu et al. have studied its ES property [48], which showed that some unstable IDSs may be stabilized by increasing time delay in continuous dynamics. More interestingly, it was proved that along with the increase of time delay within a certain range, the convergence rate of such IDSs also increased correspondingly. Then, by utilizing a comparison principle, it could be shown that for some stable IDSs, under certain conditions, the stability was robust against any large but bounded time delay. Compared with the previous results on delay-free impulsive systems, some potential impacts of time delay on the stability were investigated. Next we will emphasize the corresponding results obtained in [48].

Definition 5. *A function $V \in \mathcal{V}_0$ is called an exponential Lyapunov function (ELF) for system (3.1) with rate coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ if*

$$\begin{aligned} &D^+V(t,\varphi(0)) \le \alpha_1 V(t,\varphi(0)) + \alpha_2 V(t-\tau,\varphi(-\tau)),\ t \ne t_k,\ t \ge t_0,\\ &V(t, g_k(\varphi(0))) \le \beta V(t^-,\varphi(0)),\ t = t_k,\ k \in \mathbb{Z}_+, \end{aligned}$$

for any $\varphi \in \mathbb{PC}([-\tau,0],\mathbb{R}^n)$.

Theorem 1. *[48] Let $V \in \mathcal{V}_0$ be an ELF for system (3.1) with rate coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ such that $\alpha_1 + \alpha_2 > 0$. Set $\eta^* = \{\alpha_1, \alpha_2, \eta_0\}$ with $\alpha_1 + \frac{\alpha_2}{\beta} e^{-\eta_0 \tau} - \eta_0 = 0$, and then system (3.1) is GUES over the class $\mathcal{F}_{AII}$ if the following condition holds:*

$$\eta^* \tau_{AII} + \ln\beta < 0. \tag{3.3}$$

Remark 3. Condition (3.3) may not hold when time delay τ is small enough, but it may turn to be satisfied as τ becomes larger. It implies that for some unstable IDSs, one may achieve the stabilization by increasing time delay in continuous dynamics. Thus, it may be a hint that the increase of time delay has the potential stabilizing impact on the stability of impulsive systems, which is not well derived in the previous results. Further, one can observe that the estimate of convergence rate, i.e., $-(\eta^* + \frac{\ln\beta}{\tau_{AII}})/p$ turns to be larger as time delay τ increases. Namely, in a certain interval, the larger the time delay is, the faster the system state may converge. Note that in most of the existing results on the stability of IDSs, one may infer that there is a tendency to destroy the stability along with the increase of time delay.

In particular, when there is no time delay, i.e., $\tau = 0$, the following corollary has been derived.

Corollary 1. *[48] Let $V \in \mathcal{V}_0$ be an ELF for system (3.1) with rate coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ such that $\alpha_1 + \alpha_2 > 0$. Then, system (3.1) without time delay is GUES over the class $\mathcal{F}_{AII}$ if the following condition holds:*

$$(\alpha_1 + \alpha_2)\tau_{AII} + \ln\beta < 0.$$

For the case of system (3.1) with arbitrarily finite time delay, the following result has been obtained.

Theorem 2. *[48] Let $V \in \mathcal{V}_0$ be an ELF for system (3.1) with rate coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ such that $\alpha_1 + \alpha_2 > 0$. Then, system (3.1) with any large but bounded time delay τ is GUES over the class $\mathcal{F}_{AII}$ if the following condition holds:*

$$\alpha_1 + \frac{\alpha_2}{\beta^{N_0}} + \frac{\ln\beta}{\tau_{AII}} < 0. \tag{3.4}$$

Remark 4. One may see that in the case where $\beta \in (0,1)$ (both in Theorem 1 and Theorem 2), the impulsive intervals should be small enough on average to satisfy the conditions (3.3) and (3.4), respectively. Namely, the more frequent the impulsive control is, the more favorable it may be to the stabilization of delayed system. In particular, a valid framework has been established by Dashkovskiy and Feketa to study asymptotic behaviour of Zeno solutions [111], which may be applied to consider the stabilization under impulsive control with extremely high input frequency. Furthermore, it is worth noting that the stability criterion in Theorem 2 involves the elasticity number N_0, which is very different from the relevant results on delay-free systems (see [88, 112]). That is, in the previous results, the stability cannot be destroyed by any given parameter N_0 for impulsive systems without time delay. In addition, we can also observe that parameter N_0 is not involved in Theorem 1, i.e., the case for impulsive systems with small delay. But Theorem 2 implies that for an original stable impulsive control system with large time delay, it may turn to be unstable with the increase of the parameter N_0. Thus, Theorem 2 further reveals the potential impact of large time delay on the stability of impulsive control systems under AII condition.

Furthermore, time delay is very likely to be time-varying. Under this case, Li et al. addressed the IDSs model with time-varying time delay [40]:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau(t))),\ t \neq t_k,\ t \geq t_0, \\ \Delta x(t) = I_k(t, x(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \\ x(t_0+\theta) = \phi(\theta),\ \theta \in [-\tau^*, 0], \end{cases} \tag{3.5}$$

where $\tau : \mathbb{R}_+ \to [0, \tau^*]$ is the time-varying time delay. For IDSs (3.5), [40] has presented the following stability result:

Theorem 3. *[40] System (3.5) is GES if there exist constants $\omega_1 > 0, \omega_2 > 0, p > 0, \gamma > 1, \varsigma > 0, \sigma_k > 0, \aleph \in \mathbb{Z}_+$, functions $\alpha_1 \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}), \alpha_2 \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ and $V \in \mathcal{V}_0$ such that:*

i) $\omega_1 \|x\|^p \leq V(t,x) \leq \omega_2 \|x\|^p,\ (t,x) \in [t_0 - \tau^*, \infty) \times \mathbb{R}^n$;

ii) $D^+V(t, \varphi(0)) \leq \alpha_1(t) V(t, \varphi(0)) + \alpha_2 V(t-\tau(t), \varphi(-\tau(t))),\ (t, \varphi) \in \mathbb{R}_+ \times \mathbb{PC}_{\tau^*}$;

iii) $V(t_k, \varphi(0)) + I_k(t_k, \varphi) \leq \dfrac{1}{\sigma_k} V(t_k^-, \varphi(0)),\ (t_k, \varphi) \in \mathbb{R}_+ \times \mathbb{PC}_{\tau^*}$, where $\sigma_{\aleph+k} = \sigma_k,\ k \in \mathbb{Z}_+$;

iv) $\sup\limits_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} < \infty$ and $\sup\limits_{k \in \mathbb{Z}_+} \displaystyle\int_{t_{k-1}}^{t_k} \alpha_1(s) ds + \sigma_\aleph \sup\limits_{k \in \mathbb{Z}_+} \int_{t_{k-1}}^{t_k} |\alpha_2(s)| e^{\varsigma \tau(s)} ds < \ln \gamma$;

v) $\begin{cases} \prod_{1 \leq j \leq \aleph-1} (\frac{\gamma}{\sigma_j} \vee 1) \leq \frac{\sigma_\aleph}{\gamma},\ \aleph \geq 2, \\ \sigma_k \equiv \gamma,\ \aleph = 1. \end{cases}$

Remark 5. Note that when there is no impulsive control, it is possible that Lyapunov function V in Theorem 3 tends to infinity since $\alpha_1 \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ and $\alpha_2 \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$. That is, it implies that it is possible that the equilibrium solution of system (3.5) is unstable without impulsive control. However, under proper impulsive control, the equilibrium solution of system (3.5) can be stabilized and becomes GES. The impulsive controller $\{t_k, \sigma_k\}, k \in \mathbb{Z}_+$ is composed of impulsive instants t_k and impulsive weights σ_k, which is implicit in Theorem 3. Compared with the existing results in [113–117] in which a common threshold for the impulsive weights is needed at each impulsive instant, the proposed result in Theorem 3 removes the restriction on threshold of the impulsive weights. It provides an effective way to ensure

GES via impulsive control, despite the existence of impulsive disturbance which causes negative effect to the control.

Remark 6. If the time delay $\tau(t)$ in system (3.5) is bounded, then condition iv) in Theorem 3 can be replaced by

$$\sup_{k\in\mathbb{Z}_+}\int_{t_{k-1}}^{t_k}\alpha_1(s)ds+\sigma_{\aleph}\sup_{k\in\mathbb{Z}_+}\int_{t_{k-1}}^{t_k}|\alpha_2(s)|ds<\ln\gamma.$$

In addition, under conditions i)–iii) and v) in Theorem 3, Theorem 3 still holds if there exist constants $\gamma>1,\varsigma>0,\alpha_1^*\in\mathbb{R},\alpha_2^*>0$, such that

$$\alpha_1(t)\le\alpha_1^*,\ |\alpha_2(t)|e^{\varsigma\tau(t)}\le\alpha_2^*,\ t\ge t_0$$

and

$$[\alpha_1^*+\alpha_2^*\sigma_{\aleph}]\cdot\sup_{k\in\mathbb{Z}_+}\{t_k-t_{k-1}\}<\ln\gamma.$$

Corollary 2. *[40] System (3) without impulse is GES if there exist constants $\omega_1>0,\omega_2>0,p>0,\varsigma>0,$ functions $\alpha_1\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}),\alpha_2\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})$, and a differentiable function $V:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$ such that:*

i) $\omega_1\|x\|^p\le V(t,x)\le\omega_2\|x\|^p,\ (t,x)\in[t_0-\tau^*,\infty)\times\mathbb{R}^n;$

ii) $\dot V(t,\varphi(0))\le\alpha_1(t)V(t,\varphi(0))+\alpha_2(t)V(t-\tau(t),\varphi(-\tau(t))),\ (t,\varphi)\in\mathbb{R}_+\times\mathbb{PC}_{\tau^*};$

iii) $\sup_{t\ge t_0}\{\alpha_1(t)+|\alpha_2(t)|e^{\varsigma\tau(t)}\}<0.$

Remark 7. Corollary 2 is a special case of Theorem 3 (i.e., without impulse) which has been partially derived in [109] and [118]. Nevertheless, there are a lot of differences between the results of Corollary 2 and that of [109] and [118]. Corollary 2 can be applied to the case that the time delay is time-varying and unbounded, which is not covered in [109] and [118]. For example, it follows from Corollary 2 that system (3.5) without impulse is GES for the case that $\tau(t)=\ln(1+t),\alpha_2(t)=\frac{1}{(1+t)^{\varsigma}}$, and $\alpha_1(t)=-2$.

In the previous presentation, some sufficient conditions for stability are given in terms of L-R approach, which could typically be expressed as a Lyapunov function of the form $V(x)=x^TPx$ with $P>0$. When using the Lyapunov function, it is necessary to choose a suitable minimum functional class to estimate the derivative of the Lyapunov function, which is often called the Razumikhin condition. The concept of L-K functional method introduced by Krasovskii [119] provides less conservative results than the Razumikhin method since it takes advantage of the detailed information of the time delay, especially for the case involving small time delay. When using L-K functional, the corresponding derivative can be estimated without the need of minimal functional class. Results in this category are generally referred to as theorems of L-K type, see [120–123]. In what follows, some stability results of L-K type for IDSs will be presented.

Theorem 4. *[124] If there exist functions $V_1(t,x)\in\mathcal{V}_0,V_2(t,\varphi)\in\mathcal{V}_0^*(\cdot)$, positive constants ω_1, $\omega_2,\omega_3,\alpha,l,p,p_1,p_2$, and $\{t_k\}\in\mathcal{F}_1$, such that for every $k\in\mathbb{Z}_+$,*

i) $\omega_1\|x\|^{p_1}\le V_1(t,x)\le\omega_2\|x\|^{p_2},\ 0\le V_2(t,\varphi)\le\omega_3\|\varphi\|_\tau^{p_2},\ (t,x)\in[t_0-\tau,\infty)\times\mathbb{R}^n;$

ii) $D^+V(t,\varphi) \le pV(t,\varphi),\ t \ne t_k,$ where $V(t,\varphi) = V_1(t,x) + V_2(t,\varphi)$;

iii) $V_1(t, x + g_k(t,x)) \le \beta_k V_1(t^-, x),\ t = t_k,$ where $\beta_k \ge 0$;

iv) $\ln(\beta_k + \dfrac{\omega_3}{\omega_1} e^{(\frac{p_2}{p_1}-1)pkl}) \le -(\alpha + p)l$;

then system (3.1) is UES over the class $\mathcal{F}_1$*, where* $\mathcal{F}_1 \in \mathcal{F}_0$ *denotes a class of* $\{t_k\}$ *satisfying*

$$\tau \le t_k - t_{k-1} \le l,\ \forall t \ge t_0.$$

Note that Theorem 4 is given from the impulsive control point of view, namely, impulses may be used as a control to stabilize the underlying continuous system. While from the impulsive disturbance point of view, combining the advantages of L-K and L-R approaches, the following result can be given.

Theorem 5. *[125] Given constants* $\ell > 0$ *and* $L > 0$*. If there exist functions* $V_1(t,x) \in \mathcal{V}_0$, $V_2(t,\varphi) \in \mathcal{V}_0^*(\cdot)$, $W_i \in \mathcal{K}, i = 1,2,...,7, \Psi : [t_0 - \ell, \infty) \to [0, A]$ *which is continuous with* $A > 0, \Upsilon \in C(\mathbb{R}_+, \mathbb{R}_+), \Upsilon \ge L, p \in C(\mathbb{R}_+, \mathbb{R}_+), q \in C(\mathbb{R}_+, \mathbb{R}_+)$ *which is nonincreasing,* $q(s) > s, s > 0$, *and* $\{t_k\} \in \mathcal{F}_0$*, such that for every* $k \in \mathbb{Z}_+$,

i) $W_1(\|x\|) \le V(t,\varphi) \le W_2(\|x\|) + W_3(\int_{t-\ell}^{t} \Psi(s) W_4(\|\varphi(s)\|)ds) + W_5(\int_{-\tau}^{t} \Upsilon(t-s) W_6(\|\varphi(s)\|)ds)$,
where $V(t,\varphi) = V_1(t,x) + V_2(t,\varphi)$;

ii) $D^+V(t,\varphi) \le -W_7(\|x(t)\|),\ t \ne t_k,$ whenever $p(V(t,x(\cdot))) > V(s,x(\cdot))$
for $\max\{-\tau, t - q(V(t,x(\cdot)))\} \le s \le t,$ where $p(s) > Ms,\ M = \prod_{k=1}^{\infty}(1+\beta_k)$;

iii) $V_1(t, x + g_k(t,x)) \le (1+\beta_k)V_1(t^-, x)\ t = t_k,$ where $\beta_k \ge 0$ with $\Sigma_{k=1}^{\infty}\beta_k < \infty$;

then system (3.1) is UAS over the class $\mathcal{F}_0$.

Remark 8. Since the method of L-K functional is sometimes more general than the L-R method, it has been receiving increasing attention in various dynamical systems [35,101,103,126–129]. Interestingly, Briat and Seuret developed a new functional-based approach including looped functionals to consider non-monotonic Lyapunov functions, leading to LMI conditions devoid of exponential terms [103]. Furthermore, in the looped-functional and clock-dependent Lyapunov function frameworks, Briat studied the stability of uncertain periodic and pseudo-periodic systems with impulses [129]. In addition to L-R and L-K methods, the Halanay-type inequalities and comparison principle are also the useful tools to analyze the stability properties of nonlinear delay systems; see [50, 130–132] and the references therein.

When it comes to the stochastic factors, that is, the impulsive stochastic delay systems (ISDSs), there are still a lot of meaningful works [133–137]. Generally, the general ISDSs can be modeled as follows:

$$\begin{cases} dx(t) = f(t,x_t)dt + h(t,x_t)dB(t),\ t \ne t_k,\ t \ge t_0, \\ x(t) = g_k(t^-, x(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{3.6}$$

where $x_{t_0} = \phi \in \mathcal{PC}^b_{\mathcal{R}_0}([-\tau,0],\mathbb{R}^n)$, $f : [0,\infty) \times \mathbb{PL}^p_{\mathcal{R}_t}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, and $h : [0,\infty) \times \mathbb{PL}^p_{\mathcal{R}_t}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^{n\times m}$, x_t is regarded as a stochastic process such that $x_t(\theta) = x(t+\theta), \theta \in [-\tau,0]$,

$B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$ is an m-dimensional Brownian motion defined on a complete probability space. Before we introduce these works, the following definitions need to be supplied.

Definition 6. *The trivial solution (3.6) is called pth moment exponentially stable (p-ES), if there are positive constants λ and η such that for all $\phi \in \mathcal{PC}^b_{\mathcal{R}_0}([-\infty, 0], \mathbb{R}^n)$, $\mathbb{E}\|x(t, \phi)\|^p \le \eta\|\phi\|^p e^{-\lambda(t-t_0)}$ holds.*

Definition 7. *The function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to class Ψ if it is continuously twice differentiable with respect to x and once differentiable with respect to t.*

We first introduce the results derived by Hu et al., which studied the Razumikhin stability for a class of ISDSs with which the time-derivatives of the Razumikhin functions were allowed to be indefinite [138].

Theorem 6. *[138] Let $p, \omega_1, \omega_2, q > 1$ be all positive constants. If there exist a function $V \in \Psi$ and constants $\rho, \beta \in (0, 1)$ such that the conditions:*

$$i)\ \omega_1\|x\|^p \le V(t, x) \le \omega_2\|x\|^p,\ (t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n;$$
$$ii)\ \mathbb{E}\mathcal{L}V(t, x_t) \le \mu(t)\mathbb{E}V(t, x(t)) \text{ if } \mathbb{E}V(t + \theta, x(t + \theta)) \ge q\mathbb{E}V(t, x(t)),\ t \ne t_k;$$
$$iii)\ \mathbb{E}V(t, x(t)) \le \beta\mathbb{E}V(t^-, x(t^-)),\ t = t_k;$$
$$iv)\ (\beta \vee q^{-1})e^{\tilde{\mu}} \le \rho, \text{ where } \tilde{\mu} = \sup_{k\in\mathbb{Z}_+} \int_{t_{k-1}}^{t_k} \max\{0, \mu(t)\}dt,$$

then system (3.6) is p-ES.

Obviously, the β reflecting the effect of impulse in Theorem 6 is in (0, 1), that is, the impulses are conducive to the stability of the system. Next, the result under case $\beta \ge 1$ will be introduced.

Theorem 7. *[138] Let $p, \omega_1, \omega_2, q > 1$ be all positive constants. If there exist a function $V \in \Psi$, a uniformly exponentially stable function $\mu(t)$ with parameters α and γ, constants $\rho \in (0, 1), \beta \ge 1$ such that the conditions:*

$$i)\ \omega_1\|x\|^p \le V(t, x) \le \omega_2\|x\|^p,\ (t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n;$$
$$ii)\ \mathbb{E}\mathcal{L}V(t, x_t) \le \mu(t)\mathbb{E}V(t, x(t)) \text{ if } \mathbb{E}V(t + \theta, x(t + \theta)) \ge q\mathbb{E}V(t, x(t)),\ t \ne t_k;$$
$$iii)\ \mathbb{E}V(t, x(t)) \le \beta\mathbb{E}V(t^-, x(t^-)),\ t = t_k;$$
$$iv)\ \beta^{N_0}e^{\gamma}q^{-1} \le \rho,\ \tau_{AII} > \frac{\ln\beta}{\alpha},$$

then system (3.6) is p-ES.

Moreover, [139] investigated p-ES of ISDSs with infinite delays. The main idea was to construct a positive function $q(t)$ that was determined by the infinite delay. Thereby, model (3.6) can be rewritten as

$$\begin{cases} dx(t) = f(t, x(t), x(t - \tau(t)))dt + h(t, x(t), x(t - \tau(t)))dB(t),\ t \ne t_k,\ t \ge t_0, \\ x(t) = g_k(t^-, x(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{3.7}$$

where $\tau \in \mathbb{PC}(\mathbb{R}_+, \mathbb{R}_+)$ is the infinite delay which satisfies $(t - \tau(t)) \to \infty$ when $t \to \infty$. Furthermore, define $\check{t} = \inf_{t\ge t_0}\{t - \tau(t)\}$.

Theorem 8. *[139] Suppose that there exist functions $V \in \Psi, m(t), q(t)$ defined from $(\check{t}, +\infty)$ to $\mathbb{R}_+$ and some positive constants $p, \omega_1, \omega_1, \tilde{c}, \gamma$ such that the following conditions hold:*

i) $\omega_1\|x\|^p \leq V(t, x) \leq \omega_2\|x\|^p,\ (t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n$;

ii) $\mathbb{E}\mathcal{L}V(t, x(t), x(t - \tau(t))) \leq m(t)\mathbb{E}V(t, x(t))$ if $\mathbb{E}V(t - \tau(t), x(t - \tau(t))) \geq q(t)\mathbb{E}V(t, x(t)),\ t \neq t_k$;

iii) $\mathbb{E}V(t, x(t)) \leq (1 + \nu_k)\mathbb{E}V(t^-, x(t^-)),\ \nu_k \geq 0$ and $\Sigma_{k=1}^{\infty}(1 + \nu_k) < \infty,\ t = t_k$;

iv) $\sup_{t \geq t_0} \int_t^{t+z} m(s)ds,$ where $z = \max_{k \in \mathbb{Z}_+}\{z_k = t_k - t_{k-1}\}$.

Then, system (3.7) is p-ES.

Since the IDSs (ISDSs) model was proposed, the related studies on their stability analysis have made great progress during the past few years, for their wide applications in real engineering practice. For example, an early study can be traced back to 2001, where Liu and Ballinger applied the impulse (especially impulsive control) in delayed differential equations and established the sufficient criteria on uniform asymptotic stability (UAS) for impulsive delayed differential equations by using Lyapunov functions and Razumikhin techniques, which showed that impulses did contribute to yield stability properties even when the underlying system did not enjoy any stability behavior [140]. Furthermore, for those differential equations with infinite time delay, by using the inequality technique given in [141], the more general results on stability of infinite delayed differential equations were presented via impulsive control. Since time delay is very likely to be time-varying, Li and Cao addressed the impulsive systems with unbounded time-varying time delay and introduced a new impulsive delayed inequality that involved unbounded and non-differentiable time-varying time delay [142], where some sufficient conditions ensuring stability and stabilization of impulsive time-unvarying and time-varying systems were derived, respectively. Moreover, when the change of time lag is related to the system state, which is usually called the state-dependent time delay, some interesting works can be found in [33, 143]. In [144], for the case that impulsive strengths were stochastic and impulsive intervals were confined by the AII and the case that both the impulsive intensity and density were stochastic, the ISS problems were considered for nonlinear impulsive systems, respectively. Zhang et al. considered the semi-Markov jump and stochastic mixed impulses simultaneously in the stochastic delayed systems, and the p-ES property was derived by using the methods of graph theory, stochastic analysis technique, and a new impulsive differential inequality [145]. Furthermore, an output-feedback control problem was investigated for a class of stochastic systems with impulsive effects under Round-Robin protocol [146], and the designed problem was deduced by solving a set of semi-definite programming problem. For the synchronization problem, Zhang et al. investigated the synchronization problem of coupled switched neural networks (SNNs) with mode-dependent impulsive effects and time delays [147], where the involved impulses included those that suppressed synchronization or enhanced synchronization. Based on switching analysis techniques and comparison principle, the exponential synchronization criteria were derived for coupled delayed SNNs with mode-dependent impulses. For more related works on IDSs (ISDSs) and the potential impacts of time delay, see [26, 148–152] and the references therein.

3.2. Stability of DISs

DISs (3.2) describe a phenomenon where impulsive transients depend on not only their current but also historical states of the system; see [45, 153–158]. For instance, in population dynamics such as in the fishing industry [45], effective impulsive control such as harvesting and re-leasing can keep the balance of fishing, and the quantities of every impulsive harvesting or releasing are not only measured by the current numbers of fish but also depend on the numbers in recent history due to the fact that the immature fishes need some time to grow. As another example, in communication security systems based on impulsive synchronization [157,158], there exist transmission and sampling delays during the information transmission process, where the sampling delays created from sampling the impulses at some discrete instants causes the impulsive transients to depend on their historical states. In previous results, the time delays in impulses are confined by some strong constraints, e.g., time delays are required to be fixed, or their upper bound should be small enough (see [44, 159–161]) and thus the corresponding criteria may not be flexible to some degree. However, in practice, the occurrence of delays is not always changeless. On the contrary, time delays may be flexible or even larger than the length of impulsive interval. Motivated by the above-mentioned current situation, the concept of average impulsive delay (AID) is proposed [162], which will be introduced in the following. Note that this concept allows that time delays in impulses can exist flexibly and even be larger than the length of impulsive interval. Namely, the upper bound of the sizes of delays in impulses is allowed to be very large. In particular, if $\tau_k \equiv 0$, then system (3.2) becomes a well-known case, that is, $x(t_k) = g_k(t_k^-, x(t_k^-))$, which implies that the state jump at an impulsive instant depends on its current state.

Definition 8. *A function $V \in \mathcal{V}_0$ is called an ELF for system (3.2) with rate coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ if*

$$i)\ D^+V(t, \varphi(0)) \le \alpha_1 V(t, \varphi(0)) + \alpha_2 V(t - \tau, \varphi(-\tau)),\ t \ne t_k,\ t \ge t_0,$$
$$ii)\ V(t, g_k(\varphi(0))) \le \beta V((t - \tau_k)^-, \varphi(0)),\ t = t_k,\ k \in \mathbb{Z}_+,$$

for any $\varphi \in \mathbb{PC}([-\varrho, 0], \mathbb{R}^n)$.

Definition 9. *[162] Assume that there exist positive numbers τ_{AID} and Δ such that*

$$\tau_{AID} N(t, t_0) - \Delta \le \sum_{j=1}^{N(t,t_0)} \tau_j \le \tau_{AID} N(t, t_0) + \Delta,\ \forall t \ge t_0,$$

where Δ is called the preset value, and τ_{AID} is the AID constant of impulsive delay sequence $\{\tau_k\}$.

Let $H[\tau_{AII}, \tau_{AID}]$ denote the class composed of impulsive time sequence $\{t_k\}$ and impulsive delay sequence $\{\tau_k\}$. The main results in [162] have been presented from the following two cases: i) stability of delayed systems with destabilizing delayed impulses, where time delays in impulses can be flexible and even larger than the length of impulsive interval, and ii) stability of delayed systems with stabilizing delayed impulses, where time delays in impulses are flexible between two consecutive impulsive instants. For the former case, the following result has been obtained.

Theorem 9. *[162] Let V be the ELF for system (3.2) with constants $\alpha_1, \alpha_2 > 0$ and $\beta > 1$. Then, system (3.2) is GUES over the class $H[\tau_{AII}, \tau_{AID}]$ if the following condition holds:*

$$\epsilon_1(\tau_{AID} - \tau_{AII}) + \ln\beta < 0, \tag{3.8}$$

where $\epsilon_1 > 0$ satisfies that $\alpha_1 + \alpha_2 e^{\epsilon_1 \tau} + \epsilon_1 = 0$.

Remark 9. One can observe that when the rate coefficients of an ELF for system (3.2) satisfy that $\alpha_1 > \alpha_2 > 0$ and $\beta > 1$, the continuous dynamics are stable, but the impulses potentially destroy the stability. Hence, the impulses cannot happen too frequently if system (3.2) is required to be stable. Consequently, condition (3.8) indicates that $\tau_{AII} > \tau_{AID} + \frac{\ln\beta}{\epsilon_1}$. Note that this condition also implies that AII constant is supposed to be larger when AID constant becomes larger. In other words, the time delay in destabilizing impulses brings a destructive tendency to the stability, and further, the larger the delay is, the stronger the destructiveness it may bring. In addition, one can check that ϵ_1 will be smaller if delay τ in continuous dynamics becomes larger and condition (3.8) may not hold when (3.8) is sufficient small. Thus, it is required that delay τ in continuous dynamics cannot be too large in the above results.

For the latter case, that is, the delayed impulse is stabilizing, and [162] presented the following result:

Theorem 10. *[162] Let V be the ELF for system (3.2) with constants* $\alpha_1 \in \mathbb{R}, \alpha_2 > 0$ *and* $\beta \in (0, 1)$. *Then system (3.2) is GUES over the class* $H[\tau_{AII}, \tau_{AID}]$ *if the following condition holds:*

$$\eta^*(\tau_{AII} - \tau_{AID}) + \ln\beta < 0, \tag{3.9}$$

where $\eta^* = \max\{-\alpha_1 + \frac{\alpha_2}{\beta}, 0\}$.

Remark 10. For delayed systems with stabilizing delayed impulses, from $\alpha_1 \in \mathbb{R}, \alpha_2 \geq 0$ and $\beta \in (0, 1)$, it is clear that the continuous dynamics may be unstable, and the impulses are beneficial for the stability. Hence, stabilizing impulses are supposed to occur persistently, namely, the AII constant cannot be too large if system (3.2) is required to be stable. Therefore, condition (3.9) implies that the AII holds $\tau_{AII} < \tau_{AID} - \frac{\ln\beta}{\eta^*}$. One can observe that, owing to the existence of time delays in stabilizing impulses, τ_{AII} is permitted to be greater than the delay-free case, i.e., $\tau_{AII} < -\frac{\ln\beta}{\eta^*}$. It illustrates that the time delay in stabilizing impulses may bring a stabilizing effect to the stability of system (3.2). Further, from Theorem 10 one can also observe that there are no extra restriction on time delay τ in continuous dynamics. Thus, we can conclude that Theorem 10 holds for all time delays τ such that $\tau_k < \tau \leq t_k - t_{k-1}$. Namely, for some stable delayed systems with stabilizing delayed impulses, under some conditions, the stability can always be ensured regardless of the size of delay in continuous dynamics.

It should be noted that most of the results focusing on DISs mainly consider the time delay existing in controller-sensor pair. For those delays existing in the controller-actuator pair, which known as actuation delays, there is little literature to explore. Recently, the newest results in [163] considered this case, and hence the impulsive systems with actuation delay can be shown as

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \ t \neq s_k + \tau, \ t \geq t_0, \\ x(t + \tau) = g_k(t^-, x(t^-)), \ t = s_k + \tau, \ k \in \mathbb{Z}_+. \end{cases} \tag{3.10}$$

Compared (3.2) with (3.10), one may amazingly find that their differential equation expression are same mathematically, that is, system (3.10) can be transformed to (3.2) as long as one denotes $s_k + \tau$ by t_k. Nevertheless, the dynamical behaviour they describe are actually very different. For system (3.2), the impulsive jump at t_k is dependence of the system state at $t_k - \tau$, while (3.10) shows the impulsive jump generated at t_k can not be implemented successfully at impulsive instant s_k until time goes to $s_k+\tau$. Hence, the impulse mechanisms in (3.2) and (3.10) can be considered as the forward and the backward process, respectively, even through their differential equation expressions are same in mathematics.

Furthermore, it is possible that time delay is dependent on both time and system state. In this case, [164] has considered the following model:

$$\begin{cases} \dot{x}(t) = f(t, x(t)),\ t \neq t_k,\ t \geq t_0, \\ x(t) = g_k((t-\tilde{\tau})^-, x((t-\tilde{\tau})^-)), \tilde{\tau} = \tilde{\tau}(t^-, x(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \\ x(t_0+\theta) = \phi(\theta),\ \theta \in [-\tau^*, 0], \end{cases} \tag{3.11}$$

where $\tilde{\tau} \in \mathbb{C}(\mathbb{R}_+ \times \mathbb{R}^n, [0, \tau^*])$ represents the state-dependent delay. Some sufficient conditions under which the stability property of (3.11) can be guaranteed were also derived in [164].

Theorem 11. *[164] Assume that there exist constants $\gamma > 0, \theta \in (0,1), M > 0, \beta_k \geq 1, k \in \mathbb{Z}_+$, functions $\omega_1, \omega_2 \in \mathcal{K}, \Xi \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and $V \in \mathcal{V}_0$ such that*

$$i)\ \omega_1(\|x\|) \leq V(t,x) \leq \omega_2(\|x\|),\ (t,x) \in [t_0-\tau^*, \infty) \times \mathbb{R}^n;$$
$$ii)\ D^+V(t,x) \leq -\Xi(t, V(t,x)),\ t \neq t_k,\ t \geq t_0;$$
$$iii)\ V(t, x(t)) \leq \beta_k V((t-\tilde{\tau})^-, x((t-\tilde{\tau})^-)),\ t = t_k,\ k \in \mathbb{Z}_+;$$
$$iv)\ |\tilde{\tau}(s, \mathbf{u}) - \tilde{\tau}(s, \mathbf{0})| \leq \gamma |u|,\ s \in \mathbb{R}_+,\ \mathbf{u} \in \mathbb{R}^n;$$
$$v)\ \hat{\tau} = \sup_{t \geq t_0} \tilde{\tau}(t, \mathbf{0}) < \infty;$$
$$vi)\ \forall k \in \mathbb{Z}_+,\ \ln \beta_k + \int_{t_k-\zeta}^{t_k} \sup_{u \in (0, \omega_2(M))} \frac{\Xi(s,u)}{u} ds \leq \theta \int_{t_k-\varpi}^{t_k} \inf_{u \in (0, \omega_2(M))} \frac{\Xi(s,u)}{u} ds.$$

Then system (3.11) is LUS in the region $\phi \in \mathbb{C}^M_{\tau^*}$, where $\zeta = \gamma \omega_1^{-1}[\omega_2(M)] + \hat{\tau} < \varpi, \varpi = \inf_{k \in \mathbb{Z}_+}\{t_k - t_{k-1}\} > 0$. Furthermore, if for any $\kappa > 0$, there exist $\Omega = \Omega(\kappa) > 0$ such that

$$vii)\ \int_{t_0}^{t} \inf_{u \in (0, \omega_2(M))} \frac{\Xi(s,u)}{u} ds > \kappa,\ t \geq t_0 + \Omega,$$

then system (3.11) is LUAS in the region $\phi \in \mathbb{C}^M_{\tau^*}$.

Remark 11. Theorem 11 presents some conditions for US and UAS of systems with impulses involving state-dependent time delay. One may observe that these kinds of impulses are more complicated than the ones in [157–159, 165] that are only dependent on current states or past states in given time interval. Even if $\beta_k \leq 1$, it is possible that function V has state-dependent increase at different impulse points. Thus, more conditions such as restrictions iv) and v) on state-dependent time delay $\tilde{\tau}$ must be imposed on these kinds of impulsive systems. In fact, one may note that the local stability of system (3.11) implies the local boundedness of system states, which leads to the boundedness of the state delay. Thus, the time delay in Theorem 11 actually is bounded, but it is not required a priori. In other words, one can utilize the stability criteria to know the boundedness of the state-dependent time delay, but do not assume the boundedness of time delay a priori. Mnt impulsive disturbance ($\beta_k > 1$), it is possible that system (3.11) is unbounded if the impulsive interval is small oreover, due to the existence of impulses, especially for persisteenough.

In addition to the latest results described above, Chen and Zheng have dealt with both destabilizing delayed impulses and stabilizing delayed impulses, and the corresponding Lyapunov-type sufficient conditions for exponential stability have been derived [166]. In [164], Li and Wu focused on stability problem of nonlinear differential systems with impulses involving state-dependent time delay based on Lyapunov methods, and some general and applicable results for US, UAS and ES of systems were derived by using the impulsive control theory and some comparison arguments. It has been shown how restrictions on the change rates of states and impulses should be imposed to achieve systems stability, in comparison with general impulsive delayed differential systems with state-dependent time delay in the nonlinearity, or the differential systems with constant time delays. Furthermore, Li and Wu studied the delayed impulsive control of nonlinear differential systems, where the impulsive control involved the delayed state of the system for which time delay was state-dependent [167]. With the development of impulsive control theory, some recent works have focused on ISS property of delayed control system under the delayed impulsive control. For example, Zhang and Li addressed the ISS and integral ISS (iISS) of nonlinear systems with distributed delayed impulses [168]. Meanwhile, Li et al. studied the ISS property of nonlinear systems with delayed impulses and external input affecting both the continuous dynamics and the state impulse map [43]. Correspondingly, when it comes to the stochastic factor in delayed impulses, i.e., the stochastic delayed impulses, there are some (but very little) results [169], which deserve to be deeply explored. In addition, it seems that there have been few results that consider the effect of delayed impulses on ISS property for nonlinear systems, which still remains as an important direction in research fields. In the application of networked control systems, due to the finite speed of computation, a type of delayed impulses which were called sensor-to-controller time delay and controller-to-actuator time delay do exist in a working network; see [156, 160, 170, 171] for more details. Clearly, the above mentioned results are mainly focused on the discrete delays and time-varying delays. There are still many important results on the stability of impulsive systems with distributed delays that deserve attentions. For instance, Rubbioni showed the asymptotic stability of the solutions of some differential equations with distributed delay and subject to impulses [172], and the established criteria were prodromes of the mild solutions to a semi-linear differential equation with functional delay and impulses in Banach spaces and of its application to a parametric differential equation driving a population dynamics model. Liu and Zhang applied the distributed-delay dependent impulsive control on the stabilization of general nonlinear delay systems [173], where the derived sufficient conditions on the system parameters, impulsive control gains, impulsive instants and distributed delays were in the form of an inequality for GES. Furthermore, Zhao et al. studied the infinite distributed delay in nonlinear impulsive system [174], and the GES property was derived based on flexible impulsive frequency. Note that, the distributed delays are more common in network models, so some novel results about distributed delays will be given in Section 6.

4. Stability under ETIC strategy

Generally speaking, TTM strictly gives the instants that the impulses are occurring, and then there may exist some unnecessary impulsive control tasks in TTM even though system has achieved its desired performance, which may cause a waste of control efforts and communication resources. To overcome this disadvantage, researchers propose the activation strategy based on a designed event, that is, the so-called ETM. By designing the proper event function, the triggered instants can be adaptively determined according to what is currently happening within the system. The general form of ETM is $t_k = \inf_{k\in\mathbb{Z}_+}\{t \geq t_{k-1} : \Omega(t) \geq 0\}$, where $\Omega(t)$ is the deigned event function, t_{k-1} is the latest impulsive instant or the initial instant, and t_k is the impulsive instant to be determined. Note that, due to the fact that the information transmissions in ETM are determined by the occurrence of some well-designed events which are related to the system state or output, then there may be infinite triggered instants in a very small time interval, which is the so-called Zeno phenomenon. Hence, before verifying the feasibility of the designed ETM, the possible Zeno phenomenon should be excluded first.

Definition 10. *[5] For the closed-loop system (2.1) with triggered instants determined by (2.4), a solution with initial condition x_0 is said to exhibit Zeno behaviour if there exists $T > 0$ such that $t_k \leq T$ for all $k \in \mathbb{Z}_+$. If Zeno behaviour does not occur along any solution of system (2.1), then we say system (2.1) does not exhibit Zeno behaviour.*

It should be mentioned that in general Zeno behavior is not always possible to avoid (think of a bouncing ball). In this case, still approaches to study asymptotic stability even exist [111], in which Dashkovskiy and Feketa proposed a method to prolong solutions to the hybrid dynamical system beyond its Zeno time and moreover, gave a way for asymptotic characterization of the prolonged solutions. Hence the asymptotic stability with respect to a closed set for those hybrid systems with Zeno solutions could be verified. For the case that a triggered mechanism has no Zeno behaviour, there are generally two main ways to confirm it: The first one is to ensure the existence of a uniform lower bound of any two adjacent triggered instants, and the second one is based on the contradiction argument to show the exclusion of Zeno behaviour by direct use of Definition 10. It is worth mentioning that ensuring a lower bound of any two adjacent triggered instants is stronger than ruling out Zeno behaviour. For example, suppose system (2.1) has a solution with triggered instants $t_k = \Sigma_{i=1}^{k} 1/i$ for $k \in \mathbb{Z}_+$. It can be seen that $t_k \to \infty$ as $k \to \infty$, which means such a solution does not exhibit Zeno behaviour. However, a lower bound of any two adjacent triggered instants is not ensured because $t_k - t_{k-1} \to 0$ as $k \to \infty$. The detailed discussion can be found in [55]. Next, we will first introduce an ETM proposed by Li et al. in [175]:

$$t_k = \inf_{k\in\mathbb{Z}_+}\{t \geq t_{k-1} : V(t, x) \geq e^{a_k} V(t_{k-1}, x(t_{k-1}))\}, \tag{4.1}$$

where $a_k \in \mathbb{R}_+$ are event parameters satisfying $\sum_{k=1}^{m} a_k \to \infty$ as $m \to \infty$. Based on (4.1), the following result has been derived.

Theorem 12. *[175] System (2.1) is US under ETM (4.1) if there exist functions $\omega_1, \omega_2 \in \mathcal{K}_\infty$, a locally Lipschitz continuous function $V \in \mathcal{V}_0$, and positive constants α, a_k, β_k, M, such that*

$$i)\ \omega_1(\|x\|) \leq V(t, x) \leq \omega_2(\|x\|),\ x \in \mathbb{R}^n; \tag{4.2}$$

$$ii)\ D^+V(t, x) \leq \alpha V(t, x),\ t \neq t_k,\ t \geq t_0; \tag{4.3}$$

$$iii)\ V(t, x(t)) \leq e^{-\beta_k} V(t^-, x(t^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \tag{4.4}$$

where $\{t_k, k \in \mathbb{Z}_+\}$ are triggered instants generated by (4.1), and a_k, β_k satisfy

$$iv)\ \sum_{k=1}^{m}(a_k - \beta_k) + a_{m+1} \leq M,\ \forall m \in \mathbb{Z}_+. \tag{4.5}$$

Remark 12. Theorem 12 presents some Lyapunov-based sufficient conditions for US of impulsive system (2.1) according to ETM (4.1), where condition (4.3) governs the continuous dynamics of the system between impulsive instants, condition (4.4) governs discrete dynamics of the system when impulses occur, which can be regarded as an impulse generator, and condition (4.5) establishes the relationship between event parameters a_k and impulse rate coefficients β_k, which is crucial to the proposed ETM (4.1). In addition, event parameters a_k in ETM (4.1) can be appropriately changed to adjust the possible triggered instants. Observe that choosing a small value of a_k can potentially reduce the value of the inter-event bound and, thus, increase the frequency of the impulses, which will speed up the convergence rate. Conversely, choosing a large one will potentially increase the value of the inter-event bound and, thus, decrease the frequency of the impulses, which will slow down the convergence rate. As a special case, when the stabilizing effect of the impulsive control vanish gradually as time goes, i.e., $\beta_k \to 0$ with increasing k, one must require continuous flows to be persistently interrupted by impulses in order to guarantee the stability. In this case, condition (4.3) enforces a restriction between a_k and β_k such that impulsive control appears more and more frequently, but still the Zeno behaviour is avoided under the help of condition $\sum_{k=1}^{m} a_k \to \infty$ as $m \to \infty$.

It is not hard to find that (4.1) needs to memorize the system information at the previous triggered instant to determine the next triggered instant, which increases the burden of information storage to a certain extent. Hence, an improved ETM has provided in the form of $t_k = \inf_{k\in\mathbb{Z}_+}\{t \geq t_{k-1} : V(x(t)) \geq ae^{-bt}\}$ by Zhang et al. in [163]. Particularly, Zhang and Braverman also considered an interesting time delay phenomenon named actuator delay, under which the dynamical behaviour in (2.1) at impulsive instants can be described by $x(t_k + \tau) = g(t_k^-, x(t_k^-))$, and the extended ETM can be written as

$$t_k = \begin{cases} \inf_{k\in\mathbb{Z}_+}\{t \geq t_{k-1} : V(t,x) \geq ae^{-bt}\}, & \text{if } k = 1, \\ \inf_{k\in\mathbb{Z}_+}\{t \geq t_{k-1} + \tau : V(t,x) \geq ae^{-bt}\}, & \text{if } k \geq 2, \end{cases} \tag{4.6}$$

where $a > 0, b > 0$ are event parameters. Based on the following assumptions, the main results in [163] can be derived.

Assumption 1. *Given a positive constant $\Im$, there exist positive constants L_1, L_2 such that*

$$\|f(t,x)\| \leq L_1\|x\| \text{ and } \|x + g(t,x)\| \leq L_2\|x\|$$

holds for any $x \in \mathcal{B}(\Im)$.

Assumption 2. *There exist functions $V \in \mathcal{V}_0, \omega_1, \omega_2 \in \mathcal{K}_\infty$, positive constants α and β such that, for any $x \in \mathcal{B}(\Im)$,*

i) $\omega_1(\|x\|) \leq V(t,x) \leq \omega_2(\|x\|)$;
ii) $D^+V(t,x) \leq \alpha V(t,x)$;
iii) if y and $y + g(t,x) \in \mathcal{B}(\Im)$, then $V(y + g(t,x)) \leq \beta V(t)$,

where $y = z(s)$, and $z(s)$ is the solution of the following initial value problem:

$$\begin{cases} z(t) = f(z(t)), \\ z(0) = x. \end{cases}$$

Theorem 13. *[163] Consider system (2.1) with actuator delay in impulsive control, and the triggered instants are determined by (4.6). Supposed that there exists $\mathfrak{I} > 0$ so that Assumption 2 holds for all $x \in \mathcal{B}(\mathfrak{I})$. If $\beta < 1, a < \omega_1(\mathfrak{I})$ and $0 \leq \tau < \min\{\varepsilon_1, \varepsilon_2\}$ with*

$$\varepsilon_1 = \frac{1}{\alpha} \ln(\frac{\omega_1(\mathfrak{I})}{a}) \text{ and } \varepsilon_2 = \frac{1}{b} \ln(\frac{1}{\beta})$$

satisfy

$$L_2 + \sqrt{\frac{\tau L_1(e^{2L_1\tau} - 1)}{2}} < \frac{\mathfrak{I}}{\omega_1^{-1}(a)}, \tag{4.7}$$

then, for any initial value $x_0 \in \mathcal{B}(\omega_2^{-1}(a))$, $\{t_k - t_{k-1}, k \in \mathbb{Z}_+\}$ can be lower bounded by $\mathcal{D} = \tau - \frac{\ln(\beta e^{b\tau})}{b+\alpha}\tau$. Moreover, the trivial solution of system (2.1) with actuator delay in impulsive control is AS.

Remark 13. To ensure the stability, two types of conditions on τ are included in Theorem 13: i) $0 \leq \tau < \varepsilon_1$ with (4.7) satisfied; ii) $\tau \leq \varepsilon_2$. The type i) condition guarantees that the system trajectory stays in $\mathcal{B}(\mathfrak{I})$ for all t, so that Assumption 1 can be applied, and the type ii) condition ensures the validity of event-triggered algorithm with triggered instants determined by (4.6). If f and g are globally Lipschitz, i.e., $\mathfrak{I} = \infty$, then both $0 \leq \tau < \varepsilon_1$ and (4.7) hold for all τ, and the only requirement on the actuation delay is $\tau \leq \varepsilon_2$, which implies $\beta e^{b\tau} \leq 1$. Large delay τ allows the Lyapunov function to go over and deviate far from the threshold, then big jump of the Lyapunov function at each impulsive instant is expected such that the Lyapunov function can be smaller than the threshold after the impulse. For f and/or g being locally Lipschitz on $\mathcal{B}(\mathfrak{I})$, large $\mathfrak{I}$ may lead to large L_1 and L_2, but the largest admissible delay τ is not directly conclusive from (4.7).

However, one extreme case, where the event function can not be activated while the state of system diverges in a relatively long time interval, should be mentioned. In this case, the above mentioned ETMs may be invalid, and hence the desired property of system can not be guaranteed. To deal with this extreme case, the ETM with forced mechanism gradually attracts people's attention, where impulses may be forcibly triggered if the system status or output fails to trigger the event within a relatively long time interval. By defining forced period $\mathfrak{R} > 0$ (a relatively larger real number), Zhu et al. have constructed an improved ETM [66, 67]:

$$\begin{cases} \text{if } \mho_{1k} = \{\exists t \in [t_{k-1}, t_{k-1} + \mathfrak{R}) : \Omega(t) \geq 0\} \neq \emptyset, \\ \text{then } t_k = \inf\{t : t \in \mho_{1k}\}; \\ \text{if } \mho_{2k} = \{\forall t \in [t_{k-1}, t_{k-1} + \mathfrak{R}) : \Omega(t) \geq 0\} = \emptyset, \\ \text{then } t_k = t_{k-1} + \mathfrak{R}. \end{cases} \tag{4.8}$$

Remark 14. Under ETM (4.8), we can see that the triggered instants are determined by two kinds of mechanisms: triggered mechanism and forced mechanism. Specifically, after the last triggered instant t_{k-1}, if the event function can be triggered at t^*, where $t^* \in [t_{k-1}, t_{k-1} + \mathfrak{R})$, then the next triggered instant $t_k = t^*$. Instead, if the event function fails to be triggered in $[t_{k-1}, t_{k-1} + \mathfrak{R})$, then the next triggered instant is chosen as $t_{k-1} + \mathfrak{R}$, i.e., $t_k = t_{k-1} + \mathfrak{R}$. This mechanism is able to effectively deal with a class of extreme cases where the event function can not be activated while the state of system diverges in a relatively long time interval, which may degrade the system controlled performance seriously, and the necessity of forced mechanism has been detailed discussed in [65–67].

Both impulsive control and event-triggered control (ETC) have been widely explored in the past decade and years, and many meaningful works have been presented. In view of their wide applications and great advantages in the field of control, the presentation of ETIC strategy well combines the advantages of impulsive control and ETC strategy. Roughly speaking, ETIC strategy was first introduced by Du et al. to change the search performance of the population in a positive way after revising the positions of some individuals at certain moments, and the proposed ETIC strategy was flexible to be incorporated into several state-of-the-art DE variants [176]. Immediately after, Zhu et al. investigated the exponential stabilization of continuous-time dynamical systems (CDSs) via ETIC approach, and then the developed ETIC was applied to the synchronization problem of master and slave memristive neural networks [53]. Tan et al. applied ETIC into the leader-following consensus problem of multi-agent systems in the sense of distributed strategy, which showed that continuous communication of neighboring agents can be avoided, and Zeno behavior can be excluded [61]. Furthermore, based on the ETIC strategy designed in [65], Li et al. considered the case that the states of the system are not fully available [66]. By designing an improved ETM and constructing the proper observer, the ISS problem for a class of nonlinear systems with external disturbance was solved. Different from [65, 66], Li et al. further proposed a more flexible ETIC approach in [177], which excluded the forced mechanism, with guaranteeing the ISS of nonlinear systems by establishing the relationship between impulsive strength and ETIC. More interestingly, Li et al. gave an improved triggered strategy named self-triggered impulsive control (STIC) for stabilization purpose in [178], which was based on the comparison approach. Different from ETIC strategies mentioned above in which the triggering conditions need to be continuously or periodically monitored to determine whether an impulse should be generated, the STIC does not require those monitoring because it can utilize the measurable information to predict the next impulsive instant.

5. Stability under hybrid impulses

Generally, most results on impulsive systems consider the single kind of impulses. For those systems with unstable continuous dynamics, the stabilizing impulse (impulsive control) is introduced to for control purpose, and hence the desired stability property can be achieved. For those systems with stable continuous dynamics, it is meaningful to consider the possible destabilizing impulse (impulsive disturbance) so as to explore the robustness of systems against impulsive disturbance. Different from the above descriptions, it is possible that both two kinds of impulses (stabilizing impulse and destabilizing impulse) are included in one system simultaneously. Specifically, in addition to establishing the constraint relationship between impulse and continuous dynamics of system, the coupling relationship between stabilizing impulse and destabilizing impulse is also very important. Based on this basic thinking, many efforts had been paid to the stability of dynamical systems with hybrid impulses, especially for ISS [23, 24, 26, 27]. When there existed some external disturbance, ISS was proposed by Sontag in 1989 so that the effect caused by external disturbance on dynamical behavior of systems can be effectively depicted [179]. Consider the following IDSs with locally continuous bounded external input $u(t) \in \mathbb{R}^m$, which is in the form

$$\begin{cases} \dot{x}(t) = f(t, x_t, u(t)), \ t \neq t_k, \ t \geq t_0, \\ x(t) = g_k(t^-, x(t^-), u(t^-)), \ t = t_k, \ k \in \mathbb{Z}_+, \\ x(t_0 + \theta) = \phi(\theta), \ \theta \in [-\tau, 0]. \end{cases} \tag{5.1}$$

Definition 11. *For the prescribed sequence $\{t_k, k \in \mathbb{Z}_+\}$, system (5.1) is said to be input-to-state stable (ISS) if there exist functions $\vartheta \in \mathcal{KL}$ and $\chi \in \mathcal{K}$ such that for every initial condition (t_0, x_0) and each bounded external input $u(t)$, the solution $x(t)$ satisfies*

$$\|x(t)\| \leq \vartheta(\|\phi\|_\tau, t - t_0) + \chi(\|u\|_{[t_0,t]}),\ t \geq t_0.$$

Furthermore, it is said to be uniformly ISS (UISS) over a given class of time sequences Φ if the ISS property expressed by the above inequality holds for every sequence in Φ, with functions ϑ and χ that are independent of the choice of the time sequences.

Particularly, Li et al. investigated the ISS property of delayed systems with hybrid impulses [27]. In terms of the Razumikhin-type conditions, the sufficient conditions, under which the ISS can be guaranteed via exponential ISS-Lyapunov functions, were given from two cases: i) the continuous dynamics were ISS; ii) the continuous dynamics were non-ISS. First, for the former case, it was shown that when the continuous dynamics were ISS, and the discrete dynamics that governed the jumps involved multiple impulses, the ISS property could be retained for arbitrary impulsive time sequences if the cumulative strength of hybrid impulses satisfied the conditions in the following result.

Theorem 14. *[27] Assume that there exist functions $V \in \mathcal{V}_0$ and $\omega_1, \omega_2, \mathcal{X}_\infty, \mathcal{X}_2 \in \mathcal{K}_\infty$, constants $\aleph \in \mathbb{Z}_+, \alpha > 0, \lambda \in (0, \alpha)$, and $\beta_k, k \in \mathbb{Z}_+$ such that $\beta_{\aleph+k} = \beta_k$ for every $k \in \mathbb{Z}_+, E(1, \aleph) \leq 1$, and the following hold:*

$$\begin{aligned}
&i)\ \omega_1(\|x\|) \leq V(t, x) \leq \omega_2(\|x\|),\ (t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n;\\
&ii)\ V(t, x) \leq \beta_k V(t^-, x(t^-)) + \mathcal{X}_2(\|u\|),\ t = t_k,\ k \in \mathbb{Z}_+;\\
&iii)\ D^+V(t, \varphi(0)) \leq -\alpha V(t, \varphi(0)) + \mathcal{X}_1(\|u\|),\ t \neq t_k,\ \varphi \in \mathbb{PC}_\tau,\\
&\qquad \text{whenever } BV(t + s, \varphi(s)) \leq e^{\lambda\tau} V(t, \varphi(0)),\ s \in [-\tau, 0],
\end{aligned}$$

where

$$B = B_{\min} \wedge \frac{B_{\min}}{B_{\max}}, B_{max} = \max_{1\leq m,l\leq\aleph} E(m, l), B_{\min} = \min_{1\leq m,l\leq\aleph} E(m, l),$$
$$E(m, l) = \begin{cases} \prod_{k=1}^{l-1} \beta_{m+k} \text{ for } 1 \leq l, m \leq \aleph, \\ 1 \text{ for } l = 0 \text{ and } 1 \leq m \leq \aleph. \end{cases}$$

Then, the system (5.1) is UISS over the class $\mathcal{F}_0$.

For the latter case where the continuous dynamics were not ISS, then the multiple impulses that satisfied the following conditions could stabilize the system in ISS sense if there was no overly long interval between impulses.

Theorem 15. *[27] Assume that there exist functions $V \in \mathcal{V}_0$ and $\omega_1, \omega_2, \mathcal{X}_\infty, \mathcal{X}_2 \in \mathcal{K}_\infty$, constants $\aleph \in \mathbb{Z}_+, \delta > 0, \alpha > 0, q \in (0, e^{-\alpha\delta}), \lambda \in (0, -\frac{\ln q}{\delta} - \alpha)$, $\beta_k, k \in \mathbb{Z}_+$ such that $\beta_{\aleph+k} = \beta_k$ for every $k \in \mathbb{Z}_+$, $\sqrt[\aleph]{E(1, \aleph)} \leq q$, and the following hold:*

$$\begin{aligned}
&i)\ \omega_1(\|x\|) \leq V(t, x) \leq \omega_2(\|x\|),\ (t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n;\\
&ii)\ V(t, x) \leq \beta_k V(t^-, x(t^-)) + \mathcal{X}_2(\|u\|),\ t = t_k,\ k \in \mathbb{Z}_+;
\end{aligned}$$

$iii)\ D^+V(t,\varphi(0)) \le -\alpha V(t,\varphi(0)) + \mathcal{X}_1(\|u\|),\ t \ne t_k,\ \varphi \in \mathbb{PC}_\tau,$
whenever $q^{\aleph}BV(t+s,\varphi(s)) \le e^{\lambda\tau}V(t,\varphi(0)),\ s \in [-\tau,0],$
where B is same as in Theorem 14.

Then, the system (5.1) is UISS over the class $\mathcal{F}_\delta^-$.

Remark 15. In [127, 159], the authors presented some sufficient conditions for ISS of impulsive delayed systems, where [127, 159] considered the case that the delayed continuous dynamics of the system was ISS, but all of the discrete dynamics of the impulses were destabilizing. In order to achieve the ISS property, they required that the impulses did not happen too frequently, i.e., there was a restriction on lower bound of the impulsive intervals. Meanwhile, in Theorem 14, Li et al. presented the sufficient condition for ISS in which the multiple impulses were involved. It showed that under the stabilizing continuous dynamics and the stabilizing cumulative influence of the jumps (i.e., $E(1,\aleph) \le 1$), the restriction on lower bound of impulsive intervals (i.e., the frequency of the impulses) could be removed.

Furthermore, Liu et al. extended the hybrid effects of impulses to (3.2) with $\tau = 0, \tau_k \equiv h, \forall k \in \mathbb{Z}_+$ in [180]. Based on the results in [27], Li et al. illustrated that impulsive system was ISS provided that the combined action of time delay existing in impulses, continuous dynamics, and the cumulative strength of hybrid impulses satisfied some conditions, even if the hybrid delayed impulses played destabilizing effects on ISS.

Theorem 16. *[180] Assume that there exist functions* $V \in \mathcal{V}_0, \omega_1, \omega_2, \rho_1, \rho_2 \in \mathcal{K}_\infty$, *continuous functions* $U_1 : \mathbb{R} \to \mathbb{R}, U_2 : \mathbb{R}_+ \to \mathbb{R}_{>0}$, *constants* $M, \lambda > 0$, *such that for all* $t \ge t_0, x \in \mathbb{R}^n, u \in R^m$,

$i)\ \omega_1(\|x\|) \le V(t,x) \le \omega_2(\|x\|),\ (t,x) \in [t_0 - h, \infty) \times \mathbb{R}^n;$
$ii)\ D^+V(t,x) \le U_1(t)V(t,x)$ whenever $V(t) \ge \rho_1(\|u\|),\ t \ne t_k,\ t \ge t_0;$
$iii)\ V(t,x) \le U_2(t)V(t^- - h, x(t^-))$ whenever $V(t,x) \ge \rho_2(\|u\|),\ t = t_k,\ k \in \mathbb{Z}_+;$
$iv)\ \mathcal{W}(t) \ge 1.$

Then system (3.2) with $\tau = 0, \tau_k \equiv h$ *is UISS over the class* $\int_{\mathcal{F}}$, *where* $\int_{\mathcal{F}}$ *denotes a class of impulsive time sequences in* $\mathcal{F}_0$ *satisfying*

$$\int_{t'}^{t} U_1(s)ds + \sum_{t_k \in [s,t)} \ln(\mathcal{W}(t_k)) \le -\lambda(t-s) + M,\ \forall t_0 - h \le s \le t, \tag{5.2}$$

where $\mathcal{W}(t) = U_2(t)e^{\int_t^{t-h} U_1(s)ds}$.

Remark 16. Conditions ii) and iii) in Theorem 16 imply that the continuous dynamics of impulsive system are indefinite, and discrete dynamics (i.e., hybrid impulse) contain multiple effects, respectively. Condition (5.2) establishes a relationship between the continuous dynamical behaviour of system and hybrid delayed impulses, which is crucial to ensure the ISS property. Note that, if there is no impulse during the interval $[s,t)$, then the term $\sum_{t_k \in [s,t)} \ln(\mathcal{W}(t_k))$ is understood to be zero. Besides, condition iv) can be removed if the time delay is not too large (i.e., $t_k - t_{k-1} \ge h, \forall k \in \mathbb{Z}_+$).

Remark 17. Compared with the existing results in [6, 27, 181], one of the main contributions of Theorem 16 is to consider the effect of time delay in impulses on ISS property. Especially, if there is no time delay, i.e., $h = 0$ and the description of impulse dynamics iii) is replaced by $V(t, x) \leq U_1(t)V(t^-, x(t^-))$ for $V(t, x) \geq \rho_2(\|u\|)$, while considering the linear rates of the continuous dynamics and the discrete dynamics, i.e., $U_1(t) = \alpha, U_2(t) = \beta$, then the requirement $\ln(\mathcal{W}(t_k)) \geq 0$ can be discarded, and condition (5.2) reduces to $N(t, s) \ln\beta + (\alpha + \lambda)(t - s) \leq M$, which has been sufficiently studied in [49]. Furthermore, Ning et al. studied the ISS property of delay-free impulsive systems based on the Lyapunov function with indefinite derivative [182]. However, only a single effect of the impulse, i.e., the stabilizing impulse or the destabilizing impulse, was taken into account. Hence the results obtained in [180] have wilder applications. In addition, in [182], the authors required that the corresponding function $U_1(t)$ should satisfy $\int_{t_0}^{\infty}(U_1(s) \vee 0)ds = \infty$, which was quite strict. Observe that in result of Theorem 16, such restriction on the continuous dynamics is completely dropped. Thus, the result derived in [180] is less conservative than the results proposed in [182] even in the delay-free case.

Since they were first systematically studied by Dashkovskiy and Feketa [6, 183], the hybrid impulses have been proved that they are likely to appear in various engineering systems with complex operating environment, and many works have been given. For instance, Feketa and Bajcinca studied a class of impulsive system with multiple impulsive time sequences and a distinct jump map for each sequence, and a set of less conservative criteria were proposed by introducing the multiple nonlinear rate functions to characterize system behaviour during flows and jumps [184]. Recently, Wang et al. proposed two new concepts on AII and average impulsive gain (AIG) to deal with the difficulties coming from hybrid impulses so that the problem of globally exponential synchronization of coupled neural networks with hybrid impulses could be solved, and the derived method and criteria were proved to be effective for impulsively coupled neural networks simultaneously with synchronizing impulses and desynchronizing impulses [24]. Particularly, these two kinds of impulses did not need to be discussed separately. Furthermore, the pinning synchronization problem of impulsive Lur'e networks with nonlinear and asymmetrical coupling was studied by Wang et al. in [23], where the synchronizing impulses and desynchronizing impulses were allowed to occur simultaneously. In the presence of external disturbance, Liu et al. considered the hybrid impulses affected by time delay, and the ISS and iISS properties were investigated via the Lyapunov method, where the time derivative of Lyapunov function was indefinite [180]. Then, considering the stochastic factor, the stochastic hybrid impulses possessing stochastic impulsive instants and impulsive gains were studied by Zhang et al. in [185]. By the use of Dupire Itô's formula, based on Lyapunov method, graph theory and stochastic analysis techniques, two sufficient criteria for the mean-square exponential stability were derived, which were closely related to average stochastic impulsive gain, stochastic disturbance strength as well as the topological structure of the network itself.

6. Synchronization for IDNs

As a very important and meaningful application of impulsive systems, IDNs have received more and more attentions, and impulsive control has shown its advantages in control purpose on the clustering properties of IDNs, especially in the synchronization problems. For example, the latest results proposed by Ji et al. considered the synchronization of dynamical networks with system and coupling delays under the distributed delayed impulsive control [92]. It has been shown that impulsive delay and impulsive weight could be combined, for which the concept of limiting average delayed impulsive weight (ADIW) was put forward. Noted that although impulse may produce inhibition of synchronization, but the networks under the given criterion could still achieve synchronization, which implied time delay in impulsive control could potentially promote network synchronization. Consider the following IDNs consisting of N coupled identical nodes:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + f(t, x_i(t), x_i(t-\tau)) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t) + c^{\tau}\sum_{j=1}^{N} b_{ij}^{\tau}\Gamma^{\tau} x_j(t-\tau), \\ x_i(t_0+\theta) = \phi_i(\theta),\ \theta \in [-\tau, 0],\ i = 1, 2, ...N, \end{cases} \tag{6.1}$$

where A is a constant n-dimensional square matrix, τ denotes constant delay. $c > 0(c^{\tau} > 0)$ and $\Gamma \in \mathbb{R}^{n\times n}(\Gamma^{\tau} \in \mathbb{R}^{n\times n})$ represent the delay-free (delayed) inner coupling strength and matrices, respectively. Then, the distributed delayed impulsive controller of ith node is designed as

$$x_i(t) = \sum_{k=1}^{\infty} C_k \left[x_i\left((t-d_k)^-\right) - \sum_{j=1}^{N} \xi_{ij} x_j\left((t-d_k)^-\right) \right], \tag{6.2}$$

where $t = t_k,\ x\left(t_k^+\right) = x(t_k)$, $x_i\left((t_k-d_k)^-\right) = \lim_{t\to t_k^-} x_i(t-d_k),\ i = 1, 2, ..., N,\ k \in \mathbb{Z}_+$. $\{d_k, k \in \mathbb{Z}_+\}$ is the impulsive delay sequence, and $d_k \in \mathbb{R}_+$. $\xi_{ij} \geq 0$ satisfies that $\sum_{j=1}^{N} \xi_{ij} = 1$, for all $i = 1, 2, ..., N$. Then, (6.2) can be easily converted into:

$$x_i(t) = \sum_{k=1}^{\infty} C_k \sum_{j=1}^{N} \xi_{ij} x_j\left((t-d_k)^-\right),\ t = t_k,\ i = 1, 2, ..., N.$$

Given function $V \in \mathcal{V}_0$, the following impulsive delayed inequality with rate coefficients $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}_+$, $\beta_k \in \mathbb{R}$ holds:

$$\begin{cases} D^+V(t,x) \leq \alpha_1 V(t,x) + \alpha_2 V(t-\tau, x(t-\tau)),\ t \neq t_k,\ t \geq t_0, \\ V(t,x) \leq e^{-\beta_k} V\left(t-d_k, x((t-d_k)^-)\right),\ t = t_k,\ k \in \mathbb{Z}_+. \end{cases} \tag{6.3}$$

Definition 12. *[186] The IDNs (6.1) is called globally exponentially synchronized (GES), if there exist scalars $\lambda > 0$, $T \geq 0$ and $M > 0$ such that*

$$\|x_i - x_j\| \leq Me^{-\lambda t},\ t \geq T,$$

for all $i, j = 1, 2, \cdots, N$.

Lemma 1. *[186] Let $M \in M_2$ be $m\times n$ matrix and $A \in T(\varepsilon)$ be $n\times n$ matrix. Then, the $n\times p$ matrix G_M can be calculated such that $MA = \hat{A}M$, where $\hat{A} = MAG_M$. In addition, if M is a $(n-1)\times n$ matrix, then $MG_M = I_{n-1}$. Correspondingly, $\mathbf{M}\mathbf{A}_{\Gamma} = \hat{\mathbf{A}}_{\Gamma}\mathbf{M}$, where $\mathbf{M} = M \otimes I_n$ and Γ is a constant $n\times n$ matrix such that $\mathbf{A}_{\Gamma} = A \otimes \Gamma$, $\hat{\mathbf{A}}_{\Gamma} = \hat{A} \otimes \Gamma$.*

In what follows, [92] proposed a new concept named average delayed impulsive weight (ADIW), which contained both impulsive weights and impulsive delays, as follows:

Definition 13. *The ADIW for impulsive delayed inequality (6.3) is defined as*

$$\sigma = \lim_{t\to\infty} \frac{\frac{\beta_1}{\eta} + d_1 + \cdots + \frac{\beta_{N(t,t_0)}}{\eta} + d_{N(t,t_0)}}{N(t,t_0)},$$

where $\eta > 0$ such that $\alpha_1 + \alpha_2\left(1 \vee e^{\beta}\right) - \eta < 0$.

Remark 18. In some previous papers, the upper bound of time delay was relatively small. In this way, the corresponding criterion was a bit conservative. In [187], the average sense of time delay in impulses was put forward to analyze delayed impulsive control, which enabled time delays to exist flexibly and not be limited to a fixed value. But it ignored the effects of impulsive weights. Thus, without the one-by-one constraints, the new definition of ADIW that considered impulsive delays and impulsive weights simultaneously was proposed.

Based on above discussions, the synchronization criterion through Lyapunov function and LMI-based method has been derived in [92].

Theorem 17. *[92] If there are $n-$dimensional matrices $Q_1 > 0, Q_2 > 0$ and constants $\alpha_1 \in \mathbb{R},\ \alpha_2 \in \mathbb{R}_+, \beta_k \in \mathbb{R}$ such that the conditions:*

i) $Q_1 + Q_2 \leq \alpha_2 I$;

ii) $\begin{pmatrix} \Pi & \hat{\boldsymbol{B}}_\Gamma^\tau & LI \\ * & -Q_1 & 0 \\ * & * & -Q_2 \end{pmatrix} < 0,$ where $\Pi = -\alpha_1 I + 2KI + \left(\boldsymbol{A}^P + \hat{\boldsymbol{B}}_\Gamma\right)^T + \left(\boldsymbol{A}^P + \hat{\boldsymbol{B}}_\Gamma\right)$;

iii) $\begin{pmatrix} -e^{-\beta_k} \cdot I_{p_n} & -(I_p \otimes C_k)(\hat{\upsilon} \otimes I_n) \\ * & -I_{p_n} \end{pmatrix} < 0,\ k \in \mathbb{Z}_+$;

iv) $\dfrac{\sigma}{\tau_{AII}} - 1 > 0$

then networks (6.1) is GES via delayed impulsive control (6.2), and the exponential convergence rate is $\dfrac{\gamma}{2}$, where $\gamma = \frac{\sigma}{\tau_{AII}} - 1 > 0$.

Remark 19. Technically speaking, impulsive control is an advanced discontinuous control protocol, which provides instantaneous change. Distributed impulsive control considers the coupling topology to obtain a more general criterion to realize synchronization. The matrix C_k corresponds to impulsive weight matrix whose quantitative value has a great impact on network synchronization. In previous studies, network synchronization will be destroyed for the most part when impulses work negatively. If synchronization is required, a strong restriction is always added to impulsive delays d_k, which makes the criterion conservative to some extent. However, the above result requires the inequalities $\frac{\sigma}{\tau_{AII}} - 1 > 0$ and $(I_p \otimes C_k)(\hat{\upsilon} \otimes I_n))^T (I_p \otimes C_k)(\hat{\upsilon} \otimes I_n)) \leq e^{\beta_k} I_{p_n}$. That is to say, it takes the comprehensive effects of impulsive weights and impulsive delays on synchronization, rather than restrict them separately.

Although there are also many interesting results for synchronization problem with different kinds of time delays, only discrete time delay is considered in most previous studies. In view of the fact that dynamical networks often have spatial properties due to the differences in parallel paths and axon lengths, which means that state changes cannot be regarded as discrete time delays, it is more valuable to introduce distributed delay into dynamical networks [188]. For this consideration, [189] has made use of distributed impulsive control to further study the synchronization of dynamical networks with multiple delays, where the coupling delay considered both discrete and distributed delay. The limiting ADIW was extended to a more general form, and the comparison principle was used to eliminate the limitation that the system delay must be less than impulsive interval. Specifically, Ji et al. defined

IDNs with distributed delay composed of N identical nodes as

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + f(t, x_i(t), x_i(t-\tau_0(t))) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t) \\ \qquad + c^{\tau_1}\sum_{j=1}^{N} b_{ij}^{\tau_1}\Gamma^{\tau_1} x_j(t-\tau_1(t)) + c^{\tau_2}\sum_{j=1}^{N} b_{ij}^{\tau_2}\Gamma^{\tau_2}\int_{t-\tau_2(t)}^{t} x_j(s)\,ds, \\ x_i(t_0+\theta) = \phi_i(\theta),\ \theta \in [-\varrho, 0], \end{cases} \tag{6.4}$$

where delays $\tau_0(t), \tau_1(t), \tau_2(t)$ have the common upper bound $\tau^*, \varrho = \max_{k\in\mathbb{Z}_+}\{d_k - t_k, \tau^*\}$.

Lemma 2. *[189] Let $0 \le \tau_i(t) \le h, i = 0,1,2$. For fixed t and x, select $G(t, x, x_{\tau_0}, x_{\tau_1}) : \mathbb{R}^+ \times \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ as a nondecreasing function with respect to x_{τ_0} and x_{τ_1}. Let $\Upsilon_k(x) : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function with respect to x. Denote a function class $\mathcal{S}(s) \subseteq \mathbb{PC}([-\varrho,\infty], \mathbb{R}^n)$. Assume that $\mu(t), \nu(t) \in \mathcal{S}(s)$ satisfy for all $k \in \mathbb{Z}_+$*

$$\begin{cases} D^+\mu(t) \le G(t, \mu(t), \mu(t-\tau_0(t)), \mu(t-\tau_1(t))) \\ \qquad +p\int_{t-\tau_2(t)}^{t}\mu(s)\,ds,\ t \ne t_k,\ t \ge t_0, \\ \mu(t) \le \Upsilon_k(\mu((t-d_k)^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \end{cases}$$

$$\begin{cases} D^+\nu(t) > G(t, \nu(t), \nu(t-\tau_0(t)), \nu(t-\tau_1(t))) \\ \qquad +p\int_{t-\tau_2(t)}^{t}\nu(s)\,ds,\ t \ne t_k,\ t \ge t_0, \\ \nu(t) \ge \Upsilon_k(\nu((t-d_k)^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \end{cases}$$

where $p > 0$, and $D^+\mu(t) = \limsup_{\epsilon\to 0^+}\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon}$. If the inequality $\mu(t) \le \nu(t)$ is satisfied when $t_0 - \varrho \le t \le t_0$, then it can be derived that $\mu(t) \le \nu(t)$ for all $t > 0$.

Consider an impulsive delayed equation with rate coefficients $\alpha \in \mathbb{R}_+$, $\beta_k \in \mathbb{R}$:

$$\begin{cases} D^+V(t,x) = \alpha V(t,x),\ t \ne t_k,\ t \ge t_0, \\ V(t,x) = e^{-\beta_k}V((t-d_k)^-, x((t-d_k)^-)),\ t = t_k,\ k \in \mathbb{Z}_+. \end{cases} \tag{6.5}$$

To comprehensively consider the delayed impulsive control, the previous Definition 13 can be extended into the following form:

Definition 14. *For $V \in \mathcal{V}_0$ defined in (6.5), suppose that there exist scalars $\sigma^* \ge 0$ and $\sigma > 0$ satisfy*

$$\sigma N(t,s) - \sigma^* \le \sum_{i=1}^{N(t,s)}\sigma_i \le \sigma N(t,s) + \sigma^*, \forall t \ge s \ge 0,$$

where $\sigma_i = \frac{\beta_i}{\alpha} + d_i$. Then, the σ is defined as the ADIW.

Remark 20. It is clear that the above concepts describe impulsive frequency and other impulsive parameters, respectively. Then, these two concepts can be combined into

$$\sigma\frac{t-s}{\tau_{AII}} - \sigma N_0 - \sigma^* \le \sum_{i=1}^{N(t,s)}\sigma_i \le \sigma\frac{t-s}{\tau_{AII}} + \sigma N_0 + \sigma^*,\ \forall t \ge s \ge 0.$$

Theorem 18. *[189] If there are n-dimensional matrices $Q_0 > 0$, $Q_1 > 0$, $Q_2 > 0$ and positive scalars $\alpha, \alpha_0, \alpha_1, \alpha_2$ and $\beta_k \in \mathbb{R}, k \in \mathbb{Z}_+$ such that:*

i) $Q_0 \le \alpha_0 I,\ Q_1 \le \alpha_1 I,\ Q_2\tau_2 \le \alpha_2 I$;

$$ii)\ \begin{pmatrix} \Pi & LI & \hat{\boldsymbol{B}}_{\Gamma}^{\tau_1} & \hat{\boldsymbol{B}}_{\Gamma}^{\tau_2} \\ * & -Q_0 & 0 & 0 \\ * & * & -Q_1 & 0 \\ * & * & * & -Q_2 \end{pmatrix} < 0,\ \text{where } \Pi = -\alpha I + 2KI + \left(\boldsymbol{A}^P + \hat{\boldsymbol{B}}_{\Gamma}\right)^T + \left(\boldsymbol{A}^P + \hat{\boldsymbol{B}}_{\Gamma}\right);$$

$$iii)\ \begin{pmatrix} -e^{-\beta_k} \cdot I_{p_n} & -\left(I_p \otimes C_k\right)(\hat{\upsilon} \otimes I_n) \\ * & -I_{p_n} \end{pmatrix} < 0;$$

$$iv)\ -\gamma + (\alpha_0 + \alpha_1 + \alpha_2\tau_2)\, e^{\gamma_0} < 0,\ \text{where } \gamma = \alpha\left(\frac{\sigma}{\tau_{AII}} - 1\right), \gamma_0 = \alpha\left(\sigma^* + \sigma N_0\right),$$

then system (6.4) is GES in the following sense:

$$\|x_i - x_j\| \le \sqrt{\hat{J}(t_0)} \cdot e^{-\frac{\delta}{2}t},\ t \ge t_0,\ i, j = 1, 2, ..., N.$$

where $\hat{J}(t_0) = e^{\gamma_0} \sup_{t_0-\varrho\le s\le t_0} J(s)$*, and the convergence rate is* $\frac{\delta}{2}$*.* δ *is the unique solution of* $\delta - \gamma + e^{\gamma_0}\left(\alpha_0 e^{\tau_0\delta} + \alpha_1 e^{\tau_1\delta} + \alpha_2 \cdot \frac{e^{\tau_2\delta}-1}{\delta}\right) = 0.$

Remark 21. In fact, the frequency of impulse generation plays a vital role in impulsive control. Relevant previous studies usually simply set the impulsive interval as $\tau_{AII} = \max_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$ or $\tau_{AII} = \min_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$. If this maximum value is used in Theorem 18, then the constraints of impulsive delays or impulsive weights in $-\gamma + (\alpha_0 + \alpha_1 + \alpha_2\tau_2)e^{\gamma_0}$ would be strengthened, and hence the control cost of controller will be greatly augmented. Otherwise, it can not control the network to achieve synchronization. If it uses the minimum value, it may cause a waste of resources. Therefore, the concept of AII does not limit the upper or lower bounds of impulsive interval. In this regard, the impulsive interval satisfies $\min_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} \le \tau_{AII} \le \max_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$. One can find that AII effectively releases the limitations on the range of impulsive effects and control gains and reduces the conservatism of criteria.

Remark 22. A distributed delayed impulsive controller is proposed in (6.2), and the structure is $\Sigma_{k=1}^{\infty} C_k \Sigma_{j=1}^{N} \xi_{ij} x_j(t - d_k)$ where C_k represents the impulsive weight matrix, d_k denotes the impulsive delays, and ξ_{ij} represents the topological structure that can be the same as or independent of the system coupling matrix. Compared with previous studies [44, 187], Theorem 18 eliminates the limitation that impulsive delays d_k should satisfy $d_k \le \tau_0$. The impulsive delays can be larger than the system delays. In addition, if the impulsive weights work positively to network synchronization, the impulsive delays d_k could be designed as some small scalars to save control costs. If the impulsive weights are so large that will hinder the synchronization, then the impulsive delays would work positively through increasing the delay values. Based on those discussion, Theorem 18 actually considers a more comprehensive case, without any limitation for the upper bounds of multiple time-varying delays or for the magnitude of impulsive weights.

In fact, many researchers have studied network synchronization with impulsive control. As with in [39], Guan et al. investigated synchronization of dynamical networks with multiple coupling time delay via distributed impulsive control. In particular, the control topology could be designed either to be the same as the non-delayed coupling topology of the network, or to be independent of the intrinsic network topology. Then, He et al. extended it to the coupled networks with distributed coupling delay by constructing a novel impulsive pinning strategy involving pinning ratio, under which a general criterion was derived to ensure an array of neural networks with two different topologies synchronized with the desired trajactory [190]. Immediately after, Huang et al. were concerned with quasi-synchronization issue of neural networks which took parameter mismatches and a delayed impulsive controller into account on time scales in [191], and the relationships between mismatching parameters and delayed impulses were revealed using a novel inherent quasi-synchronization mechanism. Based on the ETM, Ding et al. investigated the synchronization of IDNs with nonlinear couplings and distributed time-varying delays [192], and the dynamic self-triggered impulsive controller was devoted to predict the available instants of impulsive inputs. In networked control systems (NCSs), Antunes et al. considered the case that sensors, actuators, and controller transmitted through asynchronous communication links, each introducing independent and identically distributed intervals between transmission [193]. By introducing an impulsive systems approach, they proved the origin of a non-linear impulsive system was (locally) stable with probability one if its local linearization about the zero equilibrium was mean exponentially stable. Under the integral quadratic constraint framework, Yuan and Wu presented a new design approach for NCSs containing measurement delay and actuation delay via a novel delay scheduled impulsive controller [194]. All in all, IDNs have been extensively studied, and they have been expanded to many fields, such as bioengineering, vaccination, forestry governance, etc., but there are still many topics that deserve to be explored in depth [170, 171, 195–197].

As mentioned in [1, 5, 198], the mathematical descriptions of many evolution processes and hybrid dynamical systems can be characterized by impulsive systems, and the applications of the theory of impulsive systems to various fields are increasing. Next, some examples of impulsive systems will be briefly introduced.

Example 1. (Satellite rendezvous) The proximity operations between two spacecrafts are characterized by the use of relative navigation, since the separation between the spacecrafts is sufficiently small. In this framework, the relative motion of the chaser is described in the local-vertical local-horizontal (LVLH) frame attached to the target [199]. The origin of the coordinate frame is located at the center of mass of the target, and the space is spanned by (x, y, z), where the z-axis is in the radial direction (R-bar) oriented toward the center of the Earth, and the y-axis is perpendicular to the target orbital plane and pointing in the opposite direction of the angular momentum (H-bar), while the x-axis is chosen such that $x = y \times z$; see Figure 1.

Under Keplerian assumptions (no orbital perturbations are considered) and an elliptic reference orbit, [200] considers the impulsive thrusts which mean that instantaneous velocity jumps are applied to the chaser when firing, whereas its position is continuous. The mathematical model for the relative motion in the LVLH frame between the chaser and the target is given by

$$\begin{aligned} &\dot{X}(t) = A(t)X(t),\ t \neq t_k, \\ &\Delta X(t) = \begin{bmatrix} 0 \\ I \end{bmatrix} \int_{t^-}^{t} \frac{f(s)}{m_F} ds,\ t = t_k,\ k \in \mathbb{Z}_+, \end{aligned}$$

where $X(t) = col(x(t), y(t), z(t), dx(t)/dt, dy(t)/dt, dz(t)/dt)$ represents positions and velocities in the three fundamental axes of the LVLH frame, $\Delta X(t)$ represents the change of the state vector after the impulsive thrust (essentially equivalent to velocity jumps in the three axes), $f(\cdot)$ is the thrust vector, m_F is the mass of the chaser, t_k is the firing time, and matrix $A(t)$ is a suitable periodic function of time t

given by

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_1(t) & 0 & \ddot{\upsilon} & 0 & 0 & 2\dot{\upsilon} \\ 0 & a_2(t) & 0 & 0 & 0 & 0 \\ -\ddot{\upsilon} & 0 & a_3(t) & -2\dot{\upsilon} & 0 & 0 \end{pmatrix}$$

in which some related parameters can be found in [200]. Since the impulsive thrust plan only needs to be implemented at discrete times, the complexity of guidance and control design can be reduced. Moreover, such control scheme has been widely used in the literature [201, 202] dedicated to rendezvous.

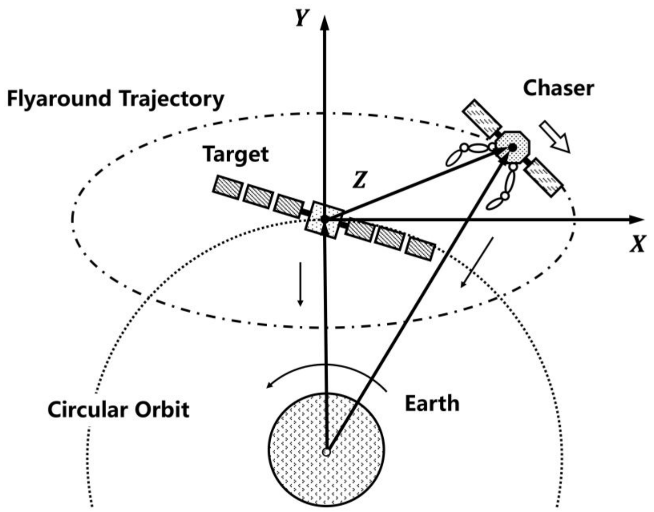

Figure 1. Satellite rendezvous.

Example 2. (HIV dynamics) For the control and parameter estimation based on clinical data, the dynamics of HIV-1 virus infection are often modeled by simple ordinary differential equations for the interactions of healthy $CD4^+$ cells (T), infected $CD4^+$ cells (y), free viruses (z), see [203]. According to [204], one can see that the problem of drug administration is classically divided into two phases: A so-called pharmacodynamic phase that relates the concentration of drugs at the site of action to the magnitude of the effect produced; a pharmacokinetic phase that relates dose, frequency, and route of administration to drug level-time relationships in the body. Then, the addressed model in [203, 205] takes into account the interaction of the intake of drugs and its concentration in blood, in which drug administration can be characterized by impulsive behavior throughout the treatment process; see Figure 2. Consequently, the HIV dynamics model is given as follows:

$$\dot{T}(t) = s - \delta T(t) - \beta T(t)z(t),$$
$$\dot{y}(y) = \beta T(t)z(t) - \mu y(t),$$

$$\dot{z}(t) = (1-\eta)ky(t) - cz(t),$$
$$\dot{\omega}(t) = -K\omega(t) + u(t),$$

with the impulsive control input

$$u(t) = \Sigma_{k=1}^{\infty} d(t)\delta(t - t_k),$$

where $\delta(\cdot)$ is the Dirac impulse function, $d(t)$ is the amount of drug expressed in milligrams, and t_k is the time at which the patient takes the drug. The healthy $CD4^+$ cells (T) are produced from the thymus at a constant rate s and die with a half life time equal to $\frac{1}{\delta}$. The healthy cells are infected by the virus at a rate that is proportional to the product of their population and the amount of free virus particles. The proportionality constant β is an indication of the effectiveness of the infection process. The infected $CD4^+$ cells (y) result from the infection of healthy $CD4^+$ cells and die at a rate μ. Free viruses (z) are produced from infected $CD4^+$ cells at a rate k and die with a half life time equal to $\frac{1}{c}$. The pharmacodynamic phase is modeled as $\eta = \frac{\omega(t)}{\omega(t)+\omega_{50}}$, $\omega(t)$ is the amounts of drug, and ω_{50} represents the concentration of drug that lowers the viral load by 50%, and K is the elimination rate constant of the drug. The above analysis based on impulsive control systems allows variations of the control variable in amounts of drug (milligrams) instead of in terms of their efficiency and describes in a realistic way how the patient follows the therapy.

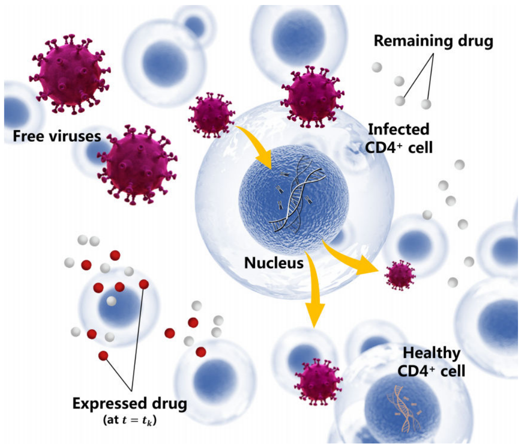

Figure 2. HIV virus infection.

The other applications of the theory of impulsive systems in engineering or biosciences include mechanical systems, automatic and remote control, secure communication, neural networks, epidemiology, forestry, vaccination, and population management, etc.; See [68–77] for more details.

7. Conclusion and further works

Impulsive dynamical systems are a very important research area with wide applications, and stability analysis is one of the fundamental issues. This paper has given a brief overview on the research area of impulsive systems with emphasis on the following several topics:

1) Some recent results of IDSs and DISs, which are divided according to the different dynamical parts where the time delay exists, are separately given. Particularly, the potential impacts of time delay on stability of impulsive systems are highlighted.

2) Several classical ETMs are introduced, which are cooperated with impulsive control to form the ETIC strategy. An extreme case, where the event function can not be activated while the state of system diverges in a relatively long time interval, is fully considered by introducing forced mechanism. Furthermore, the improved ETIC strategy with actuator delay is presented.

3) The hybrid effects of multiple impulses in dynamical systems are detailed and described, and some interesting results on ISS property involving hybrid impulses have been given.

4) The applications of impulses in complex networks are systematically introduced, involving distributed impulsive control and delayed impulsive control, and the corresponding synchronization results are derived.

Although impulsive dynamical systems and their control theory have been developed for many years, there are still some shortcomings and problems to be solved:

1) When designing the impulsive control for stabilization purpose, one may observe the control gain is relatively large, which means that the system state needs to be quickly pulled down at impulsive instant. In this case, the requirement for actuator is so high that it is difficult to implement. Hence, the saturation structure of actuator in controller should be taken into account in engineering practice, especially for impulsive control.

2) Packet loss phenomenon is common in engineering practice. For a class of impulsive control systems, if the impulsive control suffers from the effect of packet loss, it is still unknown whether the desired controlled performance of system can be achieved. In this case, it is important to estimate the admissible bound of packet loss so that the controlled performance of systems is robust against the possible inactive impulses.

3) The actuator in engineering systems often has its own dynamical behaviour. When we apply the impulsive control strategy in practical systems, the actuator dynamics should not be ignored. In this case, what will happen to the implementation mechanism of impulse needs to be explored in depth.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. W. Haddad, V. Chellaboina, S. Nersesov, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*, Princeton University Press, 2004. https://doi.org/10.1515/9781400865246
2. A. M. Samoilenko, N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, 1995. https://doi.org/10.1142/2892

3. I. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, De Gruyter, 2009. https://doi.org/10.1515/9783110221824
4. T. Yang, *Impulsive Control Theory*, Springer Science & Business Media, 2001.
5. V. Lakshmikantham, D. Bainov, P. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, 1989. https://doi.org/10.1142/0906
6. S. Dashkovskiy, P. Feketa, Input-to-state stability of impulsive systems and their networks, *Nonlinear Anal. Hybrid Syst.*, **26** (2017),190–200. https://doi.org/10.1016/j.nahs.2017.06.004
7. A. Ignatyev, A. Soliman, Asymptotic stability and instability of the solutions of systems with impulse action, *Math. Notes*, **80** (2006), 491–499. https://doi.org/10.1007/s11006-006-0167-7
8. A. N. Michel, L. Hou, D. Liu, *Stability of Dynamical Systems*, Birkhäuser Cham, 2008. https://doi.org/10.1007/978-3-319-15275-2
9. R. Goebel, R. G. Sanfelice, A. R. Teel, *Hybrid Dynamical Systems: Modeling Stability, and Robustness*, Princeton University Press, 2012. https://doi.org/10.23943/princeton/9780691153896.001.0001
10. H. Zhu, X. Li, S. Song, Input-to-state stability of nonlinear impulsive systems subjects to actuator saturation and external disturbance, *IEEE Trans. Cyber.*, **2021** (2021). https://doi.org/10.1109/TCYB.2021.3090803
11. J. Lu, D. W. C. Ho, J. Cao, J. Kurths, Exponential synchronization of linearly coupled neural networks with impulsive disturbances, *IEEE Trans. Neural Networks*, **22** (2011), 329–336. https://doi.org/10.1109/TNN.2010.2101081
12. Y. Guo, Y. Shen, W. Gui, Asymptotical stability of logic dynamical systems with random impulsive disturbances, *IEEE Trans. Automa. Control*, **66** (2021), 513–525. https://doi.org/10.1109/TAC.2020.2985302
13. Z. He, C. Li, Z. Cao, H. Li, Periodicity and global exponential periodic synchronization of delayed neural networks with discontinuous activations and impulsive perturbations, *Neurocomputing*, **431** (2021), 111–127. https://doi.org/10.1016/j.neucom.2020.09.080
14. J. Yang, J. Lu, J. Lou, Y. Liu, Synchronization of drive-response boolean control networks with impulsive disturbances, *Appl. Math. Comput.*, **364** (2020), 124679. https://doi.org/10.1016/j.amc.2019.124679
15. B. Jiang, J. Lu, X. Li, K. Shi, Impulsive control for attitude stabilization in the presence of unknown bounded external disturbances, *Int. J. Robust Nonlinear Control*, **32** (2022), 1316–1330. https://doi.org/10.1002/rnc.5889
16. X. Li, C. Zhu, Saturated impulsive control of nonlinear systems with applications, *Automatica*, **142** (2022), 110375. https://doi.org/10.1016/j.automatica.2022.110375
17. B. Liu, B. Xu, Z. Sun, Incremental stability and contraction via impulsive control for continuous-time dynamical systems, *Nonlinear Anal. Hybrid Syst.*, **39** (2021), 100981. https://doi.org/10.1016/j.nahs.2020.100981
18. H. Li, A. Liu, Asymptotic stability analysis via indefinite lyapunov functions and design of nonlinear impulsive control systems, *Nonlinear Anal. Hybrid Syst.*, **38** (2020), 100936. https://doi.org/10.1016/j.nahs.2020.100936

19. H. Ren, P. Shi, F. Deng, Y. Peng, Fixed-time synchronization of delayed complex dynamical systems with stochastic perturbation via impulsive pinning control, *J. Franklin Inst.*, **357** (2020), 12308–12325. https://doi.org/10.1016/j.jfranklin.2020.09.016

20. Y. Wang, X. Li, S. Song, Input-to-state stabilization of nonlinear impulsive delayed systems: An observer-based control approach, *IEEE/CAA J. Autom. Sin.*, **9** (2022), 1273–1283. https://doi.org/10.1109/JAS.2022.105422

21. H. Zhu, P. Li, X. Li, Input-to-state stability of impulsive systems with hybrid delayed impulse effects, *J. Appl. Anal. Comput.*, **9** (2019), 777–795. https://doi.org/10.11948/2156-907X.20180182

22. H. Zhu, P. Li, X. Li, H. Akca, Input-to-state stability for impulsive switched systems with incommensurate impulsive switching signals, *Commun. Nonlinear Sci. Numer. Simul.*, **80** (2020), 104969. https://doi.org/10.1016/j.cnsns.2019.104969

23. Y. Wang, J. Lu, J. Liang, J. Cao, M. Perc, Pinning synchronization of nonlinear coupled Lur'e networks under hybrid impulses, *IEEE Trans. Circuits Syst. II Express Briefs*, **66** (2018), 432–436. https://doi.org/10.1109/TCSII.2018.2844883

24. N. Wang, X. Li, J. Lu, F. Alsaadi, Unified synchronization criteria in an array of coupled neural networks with hybrid impulses, *Neural Networks*, **101** (2018), 25–32. https://doi.org/10.1016/j.neunet.2018.01.017

25. W. Wang, C. Huang, C. Huang, J. Cao, J. Lu, L. Wang, Bipartite formation problem of second-order nonlinear multi-agent systems with hybrid impulses, *Appl. Math. Comput.*, **370** (2020), 124926. https://doi.org/10.1016/j.amc.2019.124926

26. B. Jiang, B. Li, J. Lu, Complex systems with impulsive effects and logical dynamics: A brief overview, *Discrete Contin. Dyn. Syst. Ser. S*, **14** (2021), 1273–1299. https://doi.org/10.3934/dcdss.2020369

27. P. Li, X. Li, J. Lu, Input-to-state stability of impulsive delay systems with multiple impulses, *IEEE Trans. Autom. Control*, **66** (2021), 362–368. https://doi.org/10.1109/TAC.2020.2982156

28. T. Ensari, S. Arik, Global stability analysis of neural networks with multiple time varying delays, *IEEE Trans. Autom. Control*, **50** (2005), 1781–1785. https://doi.org/10.1109/TAC.2005.858634

29. C. Murguia, R. Fey, H. Nijmeijer, Network synchronization using invariant-manifold-based diffusive dynamic couplings with time-delay, *Automatica*, **57** (2015), 34–44. https://doi.org/10.1016/j.automatica.2015.03.031

30. H. Freedman, J. Wu, Periodic solutions of single-species models with periodic delay, *SIAM J. Math. Anal.*, **23** (1992), 689–701. https://doi.org/10.1137/0523035

31. A. Martin, S. Ruan, Predator-prey models with delay and prey harvesting, *J. Math. Biol.*, **43** (2001), 247–267. https://doi.org/10.1007/s002850100095

32. H. Zhu, R. Rakkiyappan, X. Li, Delayed state-feedback control for stabilization of neural networks with leakage delay, *Neural Networks*, **105** (2018), 249–255. https://doi.org/10.1016/j.neunet.2018.05.013

33. X. Li, X. Yang, Lyapunov stability analysis for nonlinear systems with state-dependent state delay, *Automatica*, **112** (2020), 108674. https://doi.org/10.1016/j.automatica.2019.108674

34. X. Wu, W. Zhang, Y. Tang, pth moment stability of impulsive stochastic delay differential systems with markovian switching, *Commun. Nonlinear Sci. Numer. Simul.*, **18** (2013), 1870–1879. https://doi.org/10.1016/j.cnsns.2012.12.001

35. S. Dashkovskiy, M. Kosmykov, A. Mironchenko, L. Naujok, Stability of interconnected impulsive systems with and without time delays, using lyapunov methods, *Nonlinear Anal. Hybrid Syst.*, **6** (2012), 899–915. https://doi.org/10.1016/j.nahs.2012.02.001

36. X. Liu, K. Zhang, Input-to-state stability of time-delay systems with delay-dependent impulses, *IEEE Trans. Autom. Control*, **65** (2020), 1676–1682. https://doi.org/10.1109/TAC.2019.2930239

37. H. Yang, X. Wang, S. Zhong, L. Shu, Synchronization of nonlinear complex dynamical systems via delayed impulsive distributed control, *Appl. Math. Comput.*, **320** (2018), 75–85. https://doi.org/10.1016/j.amc.2017.09.019

38. G. Wang, Y. Liu, J. Lu, Z. Wan, Stability analysis of totally positive switched linear systems with average dwell time switching, *Nonlinear Anal. Hybrid Syst.*, **36** (2020), 100877. https://doi.org/10.1016/j.nahs.2020.100877

39. Z. H. Guan, Z. W. Liu, G. Feng, Y. W. Wang, Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control, *IEEE Trans. Circuits Syst. I Regular Papers*, **57** (2010), 2182–2195. https://doi.org/10.1109/TCSI.2009.2037848

40. X. Li, J. Cao, D. W. C. Ho, Impulsive control of nonlinear systems with time-varying delay and applications, *IEEE Trans. Cybern.*, **50** (2020), 2661–2673. https://doi.org/10.1109/TCYB.2019.2896340

41. B. Jiang, J. Lu, X. Li, J. Qiu, Input/output-to-state stability of nonlinear impulsive delay systems based on a new impulsive inequality, *Int. J. Robust Nonlinear Control*, **29** (2020), 6164–6178. https://doi.org/10.1002/rnc.4712

42. X. Li, X. Yang, S. Song, Lyapunov conditions for finite-time stability of time-varying time-delay systems, *Automatica*, **103** (2019), 135–140. https://doi.org/10.1016/j.automatica.2019.01.031

43. X. Li, X. Zhang, S. Song, Effect of delayed impulses on input-to-state stability of nonlinear systems, *Automatica*, **76** (2017), 378–382. https://doi.org/10.1016/j.automatica.2016.08.009

44. X. Li, S. Song, Stabilization of delay systems: Delay-dependent impulsive control, *IEEE Trans. Autom. Control*, **62** (2017), 406–411. https://doi.org/10.1109/TAC.2016.2530041

45. H. Wang, S. Duan, C. Li, L. Wang, T. Huang, Stability of impulsive delayed linear differential systems with delayed impulses, *J. Franklin Inst.*, **352** (2015), 3044–3068. https://doi.org/10.1016/j.jfranklin.2014.12.009

46. X. Li, S. Song, J. Wu, Exponential stability of nonlinear systems with delayed impulses and applications, *IEEE Trans. Autom. Control*, **64** (2019), 4024–4034. https://doi.org/10.1109/TAC.2019.2905271

47. X. Li, P. Li, Stability of time-delay systems with impulsive control involving stabilizing delays, *Automatica*, **124** (2021), 109336. https://doi.org/10.1016/j.automatica.2020.109336

48. J. Lu, B. Jiang, W. X. Zheng, Potential impact of delay on stability of impulsive control systems, *IEEE Trans. Autom. Control*, **67** (2021), 5179–5190. https://doi.org/10.1109/TAC.2021.3120672

49. J. P. Hespanha, D. Liberzon, A. R. Teel, Lyapunov conditions for input-to-state stability of impulsive systems, *Automatica*, **44** (2008), 2735–2744. https://doi.org/10.1016/j.automatica.2008.03.021

50. Y. Wang, J. Lu, Y. Lou, Halanay-type inequality with delayed impulses and its applications, *Sci. China Inf. Sci.*, **62** (2019), 192–206. https://doi.org/10.1007/s11432-018-9809-y

51. S. Luo, F. Deng, W. H. Chen, Stability and stabilization of linear impulsive systems with large impulse-delays: A stabilizing delay perspective, *Automatica*, **127** (2021), 109533. https://doi.org/10.1016/j.automatica.2021.109533

52. T. Jiao, W. Zheng, S. Xu, Unified stability criteria of random nonlinear time-varying impulsive switched systems, *IEEE Trans. Circuits Syst. I Regular Papers*, **67** (2020), 3099–3112. https://doi.org/10.1109/TCSI.2020.2983324

53. W. Zhu, D. Wang, L. Liu, G. Feng, Event-based impulsive control of continuous-time dynamic systems and its application to synchronization of memristive neural networks, *IEEE Trans. Neural Networks Learn. Syst.*, **29** (2017), 3599–3609. https://doi.org/10.1109/TNNLS.2017.2731865

54. B. Li, Z. Wang, Q. L. Han, Input-to-state stabilization of delayed differential systems with exogenous disturbances: The event-triggered case, *IEEE Trans. Syst. Man Cybern. Syst.*, **49** (2019), 1099–1109. https://doi.org/10.1109/TSMC.2017.2719960

55. K. Zhang, B. Gharesifard, E. Braverman, Event-triggered control for nonlinear time-delay systems, *IEEE Trans. Autom. Control*, **67** (2022), 1031–1037. https://doi.org/10.1109/TAC.2021.3062577

56. L. Meng, H. Bao, Synchronization of delayed complex dynamical networks with actuator failure by event-triggered pinning control, *Phys. A Stat. Mech. Appl.*, **606** (2022), 128138. https://doi.org/10.1016/j.physa.2022.128138

57. Y. Bao, Y. Zhang, B. Zhang, Fixed-time synchronization of coupled memristive neural networks via event-triggered control, *Appl. Math. Comput.*, **411** (2021), :126542. https://doi.org/10.1016/j.amc.2021.126542

58. P. Li, W. Zhao, J. Cheng, Event-triggered control for exponential stabilization of impulsive dynamical systems, *Appl. Math. Comput.*, **413** (2022), 126608. https://doi.org/10.1016/j.amc.2021.126608

59. K. Zhang, B. Gharesifard, Hybrid event-triggered and impulsive control for time-delay systems, *Nonlinear Anal. Hybrid Syst.*, **43** (2021), 101109. https://doi.org/10.1016/j.nahs.2021.101109

60. Y. Zou, Z. Zeng, Event-triggered impulsive control on quasi-synchronization of memristive neural networks with time-varying delays, *Neural Networks*, **110** (2019), 55–65. https://doi.org/10.1016/j.neunet.2018.09.014

61. X. Tan, J. Cao, X. Li, Consensus of leader-following multiagent systems: A distributed event-triggered impulsive control strategy, *IEEE Trans. Cybern.*, **49** (2019), 792–801. https://doi.org/10.1109/TCYB.2017.2786474

62. B. Jiang, J. Lu, X. Li, J. Qiu, Event-triggered impulsive stabilization of systems with external disturbances, *IEEE Trans. Autom. Control*, **67** (2022), 2116–2122. https://doi.org/10.1109/TAC.2021.3108123

63. S. Shanmugasundaram, K. Udhayakumar, D. Gunasekaran, R. Rakkiyappan, Event-triggered impulsive control design for synchronization of inertial neural networks with time delays, *Neurocomputing*, **483** (2022), 322–332. https://doi.org/10.1016/j.neucom.2022.02.023

64. X. Li, X. Yang, J. Cao, Event-triggered impulsive control for nonlinear delay systems, *Automatica*, **117** (2020), 108981. https://doi.org/10.1016/j.automatica.2020.108981

65. B. Liu, D. J. Hill, Z. Sun, Stabilisation to input-to-state stability for continuous-time dynamical systems via event-triggered impulsive control with three levels of events, *IET Control Theory Appl.*, **12** (2018), 1167–1179. https://doi.org/10.1049/iet-cta.2017.0820

66. X. Li, H. Zhu, S. Song, Input-to-state stability of nonlinear systems using observer-based event-triggered impulsive control, *IEEE Trans. Syst. Man Cybern. Syst.*, **51** (2021), 6892–6900. https://doi.org/10.1109/TSMC.2020.2964172

67. H. Zhu, J. Lu, J. Lou, Event-triggered impulsive control for nonlinear systems: The control packet loss case, *IEEE Trans. Circuits Syst. II Express Briefs*, **69** (2022), 3204–3208. https://doi.org/10.1109/TCSII.2022.3140346

68. E. I. Verriest, F. Delmotte, M. Egerstedt, Control of epidemics by vaccination, in *Proceedings of the 2005 American Control Conference*, (2005), 985–990. https://doi.org/10.1109/ACC.2005.1470088

69. C. Briata, E. I. Verriest, A new delay-sir model for pulse vaccination, *Biomed. Signal Process. Control*, **4** (2009), 272–277. https://doi.org/10.3182/20080706-5-KR-1001.01742 https://doi.org/10.1016/j.bspc.2009.06.003

70. Y. V. Orlov, *Discontinuous Systems: Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions*, Springer Science & Business Media, 2008.

71. X. Yang, X. Cao, A new approach to autonomous rendezvous for spacecraft with limited impulsive thrust: Based on switching control strategy, *Aerosp. Sci. Technol.*, **43** (2015), 454–462. https://doi.org/10.1016/j.ast.2015.04.007

72. L. A. Sobiesiak, C. J. Damaren, Optimal continuous/impulsive control for lorentz-augmented spacecraft formations, *J. Guid. Control Dyn.*, **38** (2015), 151–157. https://doi.org/10.2514/1.G000334

73. D. Auckly, L. Kapitanski, W. White, Control of nonlinear underactuated systems, *Commun. Pure Appl. Math.*, **53** (2000), 354–369. https://doi.org/10.1002/(SICI)1097-0312(200003)53:3354::AID-CPA33.0.CO;2-U

74. N. Kant, R. Mukherjee, D. Chowdhury, H. K. Khalil, Estimation of the region of attraction of underactuated systems and its enlargement using impulsive inputs, *IEEE Trans. Rob.*, **35** (2019), 618–632. https://doi.org/10.1109/TRO.2019.2893599

75. A. Churilov, A. Medvedev, A. Shepeljavyi, Mathematical model of non-basal testosterone regulation in the male by pulse modulated feedback, *Automatica*, **45** (2009), 78–85. https://doi.org/10.1016/j.automatica.2008.06.016

76. V. Badri, M. J. Yazdanpanah, M. S. Tavazoei, Global stabilization of lotka-volterra systems with interval uncertainty, *IEEE Trans. Autom. Control*, **64** (2018), 1209–1213. https://doi.org/10.1109/TAC.2018.2845659

77. E. I. Verriest, P. Pepe, Time optimal and optimal impulsive control for coupled differential difference point delay systems with an application in forestry, in *Topics in Time Delay Systems*, (2009), 255–265. https://doi.org/10.1007/978-3-642-02897-7_22

78. J. Zhang, J. Lu, M. Xing, J. Liang, Synchronization of finite field networks with switching multiple communication channels, *IEEE Trans. Network Sci. Eng.*, **8** (2021), 2160–2169. https://doi.org/10.1109/TNSE.2021.3079631

79. A. Arenas, A. Diaz-Guilera, C. J. Pérez-Vicente, Synchronization processes in complex networks, *Phys. D Nonlinear Phenom.*, **224** (2006), 27–34. https://doi.org/10.1016/j.physd.2006.09.029

80. A. Arenas, A. Diaz-Guilera, C. J. Pérez-Vicente, Synchronization reveals topological scales in complex networks, *Phys. Rev. Lett.*, **96** (2006), 114102. https://doi.org/10.1103/PhysRevLett.96.114102

81. V. Belykh, I. Belykh, M. Hasler, K. Nevidin, Cluster synchronization in three-dimensional lattices of diffusively coupled oscillators, *Int. J. Bifurcation Chaos*, **13** (2003), 755–779. https://doi.org/10.1142/S0218127403006923

82. W. Lu, T. Chen, Quad-condition, synchronization, consensus of multiagents, and antisynchronization of complex networks, *IEEE Trans. Cybern.*, **51** (2021), 3384–3388. https://doi.org/10.1109/TCYB.2019.2939273

83. L. Zhang, J. Zhong, J. Lu, Intermittent control for finite-time synchronization of fractional-order complex networks, *Neural Networks*, **144** (2021), 11–20. https://doi.org/10.1016/j.neunet.2021.08.004

84. L. Zhang, Y. Li, J. Lou, J. Qiu, Bipartite asynchronous impulsive tracking consensus for multi-agent systems, *Front. Inf. Technol. Electron. Eng.*, **2022** (2022), 1–11. https://doi.org/10.1631/FITEE.2100122

85. H. Fan, K. Shi, Y. Zhao, Global μ-synchronization for nonlinear complex networks with unbounded multiple time delays and uncertainties via impulsive control, *Phys. A Stat. Mech. Appl.*, **599** (2022), 127484. https://doi.org/10.1016/j.physa.2022.127484

86. Q. Cui, L. Li, J. Lu, A. Alofi, Finite-time synchronization of complex dynamical networks under delayed impulsive effects, *Appl. Math. Comput.*, **430** (2022), 127290. https://doi.org/10.1016/j.amc.2022.127290

87. T. Chen, X. Liu, W. Lu, Pinning complex networks by a single controller, *IEEE Trans. Syst. Man Cybern. Syst.*, **54** (2007), 1317–1326. https://doi.org/10.1109/TCSI.2007.895383

88. J. Lu, D. W. C. Ho, J. Cao, A unified synchronization criterion for impulsive dynamical networks, *Automatica*, **46** (2010), 1215–1221. https://doi.org/10.1016/j.automatica.2010.04.005

89. X. Ji, J. Lu, J. Lou, J. Qiu, K. Shi, A unified criterion for global exponential stability of quaternion-valued neural networks with hybrid impulses, *Int. J. Robust Nonlinear Control*, **30** (2020), 8098–8116. https://doi.org/10.1002/rnc.5210

90. R. Li, H. Wu, J. Cao, Exponential synchronization for variable-order fractional discontinuous complex dynamical networks with short memory via impulsive control, *Neural Networks*, **148** (2022), 13–22. https://doi.org/10.1016/j.neunet.2021.12.021

91. Q. Fu, S. Zhong, K. Shi, Exponential synchronization of memristive neural networks with inertial and nonlinear coupling terms: Pinning impulsive control approaches, *Appl. Math. Comput.*, **402** (2021), 126169. https://doi.org/10.1016/j.amc.2021.126169

92. X. Ji, J. Lu, B. Jiang, J. Zhong, Network synchronization under distributed delayed impulsive control: Average delayed impulsive weight approach, *Nonlinear Anal. Hybrid Syst.*, **44** (2022), 101148. https://doi.org/10.1016/j.nahs.2021.101148

93. A. D'Jorgea, A. Andersona, A. Ferramoscab, A. H. Gonz*á*leza, M. Actis, On stability of nonzero set-point for nonlinear impulsive control systems, *Syst. Control Lett.*, **165** (2022), 105244. https://doi.org/10.1016/j.sysconle.2022.105244

94. J. Liu, L. Guo, M. Hu, Z. Xu, Y. Yang, Leader-following consensus of multi-agent systems with delayed impulsive control, *IMA J. Math. Control Inf.*, **33** (2016), 137–146. https://doi.org/10.1093/imamci/dnu037

95. S. Dashkovskiy, A. Mironchenko, Input-to-state stability of nonlinear impulsive systems, *SIAM J. Math. Anal.*, **51** (2013), 1962–1987. https://doi.org/10.1137/120881993

96. P. Feketa, N. Bajcinca, On robustness of impulsive stabilization, *Automatica*, **104** (2019), 48–56. https://doi.org/10.1016/j.automatica.2019.02.056

97. P. Feketa, S. Bogomolov, T. Meurer, Safety verification for impulsive systems, *IFAC-Papers OnLine*, **53** (2020), 1949–1954. https://doi.org/10.1016/j.ifacol.2020.12.2589

98. P. Feketa, V. Klinshov, L. L*ü*cken, A survey on the modeling of hybrid behaviors: How to account for impulsive jumps properly, *Commun. Nonlinear Sci. Numer. Simul.*, **103** (2021), 105955. https://doi.org/10.1016/j.cnsns.2021.105955

99. P. Feketa, N. Bajcinca, Average dwell-time for impulsive control systems possessing iss-lyapunov function with nonlinear rates, in *Proceedings of the 18th European Control Conference*, (2019), 3686–3691. https://doi.org/10.23919/ECC.2019.8796238

100. A. S. Morse, Supervisory control of families of linear set-point controllers-part i. exact matching, *IEEE Trans. Autom. Control*, **41** (1996), 1413–1431. https://doi.org/10.1109/9.539424

101. C. Briat, A. Seuret, Convex dwell-time characterizations for uncertain linear impulsive systems, *IEEE Trans. Autom. Control*, **57** (2012), 3241–3246. https://doi.org/10.1109/TAC.2012.2200379

102. J. C. Geromel, P. Colaneri, Stability and stabilization of continuous-time switched linear systems, *SIAM J. Control Optim.*, **45** (2006), 1915–1930. https://doi.org/10.1137/050646366

103. C. Briat, A. Seuret, A looped-functional approach for robust stability analysis of linear impulsive systems, *Syst. Control Lett.*, **61** (2012), 980–988. https://doi.org/10.1016/j.sysconle.2012.07.008

104. S. Dashkovskiy, V. Slynko, Stability conditions for impulsive dynamical systems, *Math. Control Signals Syst.*, **34** (2022), 95–128, 2022. https://doi.org/10.1007/s00498-021-00305-y

105. S. Dashkovskiy, V. Slynko, Dwell-time stability conditions for infinite dimensional impulsive systems, *Automatica*, **147** (2023), 110695. https://doi.org/10.1016/j.automatica.2022.110695

106. J .L. Mancilla-Aguilar, H. Haimovich, P. Feketa, Uniform stability of nonlinear time-varying impulsive systems with eventually uniformly bounded impulse frequency, *Nonlinear Anal. Hybrid Syst.*, **38** (2020), 100933. https://doi.org/10.1016/j.nahs.2020.100933

107. C. Briat, Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems, *Nonlinear Anal. Hybrid Syst.*, **24** (2017), 198–226. https://doi.org/10.1016/j.nahs.2017.01.004

108. J. Tan, C. Li, T. Huang, Stability of impulsive systems with time window via comparison method, *Int. J. Control Autom. Syst.*, **13** (2015), 1346–1350. https://doi.org/10.1007/s12555-014-0197-y

109. E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*, Springer, 2014. https://doi.org/10.1007/978-3-319-09393-2

110. J. K. Hale, *Theory of Functional Differential Equations*, Springer, 1971. https://doi.org/10.1007/BFb0060406

111. S. Dashkovskiy, P. Feketa, Asymptotic properties of zeno solutions, *Nonlinear Anal. Hybrid Syst.*, **30** (2018), 256–265. https://doi.org/10.1016/j.nahs.2018.06.005

112. J. P. Hespanha, A. S. Morse, Stability of switched systems with average dwell-time, in *Proceedings of the 38th IEEE Conference on Decision and Control*, (1999), 2655–2660. https://doi.org/10.1109/CDC.1999.831330

113. T. Yang, L. Chua, Impulsive stabilization for control and synchronization of chaotic systems: Theory and application to secure communication, *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.*, **44** (1997), 976–988. https://doi.org/10.1109/81.633887

114. E. Kaslik, S. Sivasundaram, Impulsive hybrid discrete-time hopfield neural networks with delays and multistability analysis, *Neural Networks*, **24** (2011), 370–377. https://doi.org/10.1016/j.neunet.2010.12.008

115. S. Duan, H. Wang, L. Wang, T. Huang, C. Li, Impulsive effects and stability analysis on memristive neural networks with variable delays, *IEEE Trans. Neural Networks Learn. Syst.*, **28** (2017), 476–481. https://doi.org/10.1109/TNNLS.2015.2497319

116. R. Rakkiyappan, A. Chandrasekar, S. Lakshmanan, J. H. Park, H. Y. Jung, Effects of leakage time-varying delays in markovian jump neural networks with impulse control, *Neurocomputing*, **121** (2013), 365–378. https://doi.org/10.1016/j.neucom.2013.05.018

117. Z. Tang, J. Park, Impulsive effects on quasi-synchronization of neural networks with parameter mismatches and time-varying delay, *IEEE Trans. Neural Networks Learn. Syst.*, **29** (2018), 908–919. https://doi.org/10.1109/TNNLS.2017.2651024

118. K. Gu, J. Chen, V. L. Kharitonov, *Stability of Time-Delay Systems*, Springer Science & Business Media, 2003. https://doi.org/10.1007/978-1-4612-0039-0

119. N. N. Krasovskii, *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*, Stanford University Press, 1963.

120. B. Zhou, Construction of strict lyapunov-krasovskii functionals for time-varying time delay systems, *Automatica*, **107** (2019), 382–397. https://doi.org/10.1016/j.automatica.2019.05.058 https://doi.org/10.1016/j.jfranklin.2020.05.051

121. Q. L. Han, On stability of linear neutral systems with mixed time delays: A discretized lyapunov functional approach, *Automatica*, **41** (2005), 1209–1218. https://doi.org/10.1016/j.automatica.2005.01.014

122. I. Haidar, P. Pepe, Lyapunov-krasovskii characterizations of stability notions for switching retarded systems, *IEEE Trans. Autom. Control*, **66** (2021), 437–443. https://doi.org/10.1109/TAC.2020.2979754

123. T. H. Lee, H. M. Trinh, J. H. Park, Stability analysis of neural networks with time-varying delay by constructing novel lyapunov functionals, *IEEE Trans. Neural Networks Learn. Syst.*, **29** (2018), 4238–4247. https://doi.org/10.1109/TNNLS.2017.2760979

124. X. Liu, Q. Wang, The method of lyapunov functionals and exponential stability of impulsive systems with time delay, *Nonlinear Anal. Theory Methods Appl.*, **66** (2007), 1465–1484. https://doi.org/10.1016/j.na.2006.02.004

125. Z. Luo, J. Shen, Stability of impulsive functional differential equations via the liapunov functional, *Appl. Math. Lett.*, **22** (2009), 163–169. https://doi.org/10.1016/j.aml.2008.03.004

126. M. A. Davo, A. Banos, F. Gouaisbaut, S. Tarbouriech, A. Seuret, Stability analysis of linear impulsive delay dynamical systems via looped-functionals, *Automatica*, **81** (2017), 107–114. https://doi.org/10.1016/j.automatica.2017.03.029

127. J. Liu, X. Liu, W. Xie, Input-to-state stability of impulsive and switching hybrid systems with time-delay, *Automatica*, **47** (2011), 899–908. https://doi.org/10.1016/j.automatica.2011.01.061

128. X. Sun, W. Wang, Integral input-to-state stability for hybrid delayed systems with unstable continuous dynamics, *Automatica*, **48** (2012), 2359–2364. https://doi.org/10.1016/j.automatica.2012.06.056

129. C. Briat, Theoretical and numerical comparisons of looped functionals and clock-dependent Lyapunov functions???The case of periodic and pseudo-periodic systems with impulses, *Int. J. Robust Nonlinear Control*, **26** (2016), 2232–2255. https://doi.org/10.1002/rnc.3405

130. J. J Nieto, R. R. Lopez, New comparison results for impulsive integro-differential equations and applications, *J. Math. Anal. Appl.*, **328** (2007), 1343–1368. https://doi.org/10.1016/j.jmaa.2006.06.029

131. Q. Wu, H. Zhang, L. Xiang, J. Zhou, A generalized halanay inequality on impulsive delayed dynamical systems and its applications, *Chaos Solitons Fractals*, **45** (2012), 56–62. https://doi.org/10.1016/j.chaos.2011.09.010

132. R. Kumar, U. Kumar, S. Das, J. Qiu, J. Lu, Effects of heterogeneous impulses on synchronization of complex-valued neural networks with mixed time-varying delays, *Inf. Sci.*, **551** (2021), 228–244. https://doi.org/10.1016/j.ins.2020.10.064

133. A. R. Teel, A. Subbaraman, A. Sferlazza, Stability analysis for stochastic hybrid systems: a survey, *Automatica*, **50** (2014), 2435–2456. https://doi.org/10.1016/j.automatica.2014.08.006

134. W. Hu, Q. Zhu, Stability criteria for impulsive stochastic functional differential systems with distributed-delay dependent impulsive effects, *IEEE Trans. Syst. Man Cybern. Syst.*, **51** (2021), 2027–2032. https://doi.org/10.1109/TSMC.2019.2905007

135. K. Rengamannar, G. P. Balakrishnan, M. Palanisamy, M. Niezabitowski, Exponential stability of non-linear stochastic delay differential system with generalized delay-dependent impulsive points, *Appl. Math. Comput.*, **382** (2020), 125344. https://doi.org/10.1016/j.amc.2020.125344

136. H. Chen, P. Shi, C. C. Lim, Synchronization control for neutral stochastic delay markov networks via single pinning impulsive strategy, *IEEE Trans. Syst. Man Cybern. Syst.*, **50** (2020), 5406–5419. https://doi.org/10.1109/TSMC.2018.2882836

137. X. Wu, Y. Tang, W. Zhang, Input-to-state stability of impulsive stochastic delayed systems under linear assumptions, *Automatica*, **66** (2016), 195–204. https://doi.org/10.1016/j.automatica.2016.01.002

138. W. Hu, Q. Zhu, H. R. Karimi, Some improved razumikhin stability criteria for impulsive stochastic delay differential systems, *IEEE Trans. Autom. Control*, **64** (2019), 5207–5213. https://doi.org/10.1109/TAC.2019.2911182

139. H. Xu, Q. Zhu, New criteria on exponential stability of impulsive stochastic delayed differential systems with infinite delays, *Commun. Nonlinear Sci. Numer. Simul.*, **111** (2022), 106460. https://doi.org/10.1016/j.cnsns.2022.106460

140. X. Liu, G. Ballinger, Uniform asymptotic stability of impulsive delay differential equations, *Comput. Math. Appl.*, **41** (2001), 903–915. https://doi.org/10.1016/S0898-1221(00)00328-X

141. S. Zhang, A new technique in stability of infinite delay differential equations, *Comput. Math. Appl.*, **44** (2002), 1275–1287. https://doi.org/10.1016/S0898-1221(02)00255-9

142. X. Li, J. Cao, An impulsive delay inequality involving unbounded time-varying delay and applications, *IEEE Trans. Autom. Control*, **62** (2017), 3618–3625. https://doi.org/10.1109/TAC.2017.2669580

143. X. Li, D. Peng, Uniform stability of nonlinear systems with state-dependent delay, *Automatica*, **137** (2011), 110098. https://doi.org/10.1016/j.automatica.2021.110098

144. Y. Tang, X. Wu, P. Shi, F. Qian, Input-to-state stability for nonlinear systems with stochastic impulses, *Automatica*, **113** (2020), 108766. https://doi.org/10.1016/j.automatica.2019.108766

145. N. Zhang, S. Huang, W. Li, Stability of stochastic delayed semi-markov jump systems with stochastic mixed impulses: A novel stochastic impulsive differential inequality, *J. Franklin Inst.*, **2022** (2022). https://doi.org/10.1016/j.jfranklin.2022.06.033

146. M. Yao, G. Wei, D. Ding, W. Li, Output-feedback control for stochastic impulsive systems under round-robin protocol, *Automatica*, **143** (2022), 110394. https://doi.org/10.1016/j.automatica.2022.110394

147. W. Zhang, Y. Tang, Q. Miao, W. Du, Exponential synchronization of coupled switched neural networks with mode-dependent impulsive effects, *IEEE Trans. Neural Networks Learn. Syst.*, **24** (2013), 1316–1326. https://doi.org/10.1109/TNNLS.2013.2257842

148. X. Yang, X. Li, Q. Xi, P. Duan, Review of stability and stabilization for impulsive delayed systems, *Math. Biosci. Eng.*, **15** (2018), 1495–1515. https://doi.org/10.3934/mbe.2018069

149. A. Vinodkumar, T. Senthilkumar, S. Hariharan, J. Alzabut, Exponential stabilization of fixed and random time impulsive delay differential system with applications, *Math. Biosci. Eng.*, **18** (2021), 2384–2400. https://doi.org/10.3934/mbe.2021121

150. B. Hu, Z. Wang, M. Xu, D. Wang, Quasilinearization method for an impulsive integro-differential system with delay, *Math. Biosci. Eng.*, **19** (2022), 612–623. https://doi.org/10.3934/mbe.2022027

151. Z. Xiong, X. Li, M. Ye, Q. Zhang, Finite-time stability and optimal control of an impulsive stochastic reaction-diffusion vegetation-water system driven by lévy process with time-varying delay, *Math. Biosci. Eng.*, **18** (2021), 8462–8498. https://doi.org/10.3934/mbe.2021419

152. C. Lu, B. Li, L. Zhou, L. Zhang, Survival analysis of an impulsive stochastic delay logistic model with lévy jumps, *Math. Biosci. Eng.*, **16** (2019), 3251–3271. https://doi.org/10.3934/mbe.2019162

153. L. Gao, D. Wang, G. Wang, Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects, *Appl. Math. Comput.*, **268** (2015), 186–200. https://doi.org/10.1016/j.amc.2015.06.023

154. J. Sun, Q. L. Han, X. Jiang, Impulsive control of time-delay systems using delayed impulse and its application to impulsive master–slave synchronization, *Phys. Lett. A*, **372** (2008), 6375–6380. https://doi.org/10.1016/j.physleta.2008.08.067

155. K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Springer Science & Business Media, 2013.

156. N. Wouw, P. Naghshtabrizi, M. Cloosterman, J. P. Hespanha, Tracking control for sampled-data systems with uncertain timevarying sampling intervals and delays, *Int. J. Robust Nonlinear Control*, **20** (2010), 387–411. https://doi.org/10.1002/rnc.1433

157. W. H. Chen, D. Wei, X. Lu, Exponential stability of a class of nonlinear singularly perturbed systems with delayed impulses, *J. Franklin Inst.*, **350** (2013), 2678–2709. https://doi.org/10.1016/j.jfranklin.2013.06.012

158. A. Khadra, X. Liu, X. Shen, Impulsively synchronizing chaotic systems with delay and applications to secure communication, *Automatica*, **41** (2005), 1491–1502. https://doi.org/10.1016/j.automatica.2005.04.012

159. W. H. Chen, W. X. Zheng, Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays, *Automatica*, **45** (2009), 1481–1488. https://doi.org/10.1016/j.automatica.2009.02.005

160. W. H. Chen, W. X. Zheng, Exponential stability of nonlinear time-delay systems with delayed impulse effects, *Automatica*, **47** (2011), 1075–1083. https://doi.org/10.1016/j.automatica.2011.02.031

161. X. Liu, K. Zhang, Synchronization of linear dynamical networks on time scales: Pinning control via delayed impulses, *Automatica*, **72** (2016), 147–152. https://doi.org/10.1016/j.automatica.2016.06.001

162. B. Jiang, J. Lu, Y. Liu, Exponential stability of delayed systems with average-delay impulses, *SIAM J. Control Optim.*, **58** (2020), 3763–3784. https://doi.org/10.1137/20M1317037

163. K. Zhang, E. Braverman, Event-triggered impulsive control for nonlinear systems with actuation delays, *IEEE Trans. Autom. Control*, **2022** (2022). https://doi.org/10.1109/TAC.2022.3142127

164. X. Li, J. Wu, Stability of nonlinear differential systems with state-dependent delayed impulses, *Automatica*, **64** (2016), 63–69. https://doi.org/10.1016/j.automatica.2015.10.002

165. H. Akca, R. Alassar, V. Covachev, Z. Covacheva, E. Al-Zahrani, Continuous-time additive hopfield-type neural networks with impulses, *J. Math. Anal. Appl.*, **290** (2004), 436–451. https://doi.org/10.1016/j.jmaa.2003.10.005

166. W. H. Chen, W. X. Zheng, The effect of delayed impulses on stability of impulsive time-delay systems, *IFAC Proc. Volumes*, **44** (2011), 6307–6312. https://doi.org/10.3182/20110828-6-IT-1002.02984

167. X. Li, J. Wu, Sufficient stability conditions of nonlinear differential systems under impulsive control with state-dependent delay, *IEEE Trans. Autom. Control*, **63** (2018), 306–311. https://doi.org/10.1109/TAC.2016.2639819

168. X. Zhang, X. Li, Input-to-state stability of non-linear systems with distributed delayed impulses, *IET Control Theory Appl.*, **11** (2017), 81–89. https://doi.org/10.1049/iet-cta.2016.0469

169. Q. Cui, L. Li, J. Cao, Stability of inertial delayed neural networks with stochastic delayed impulses via matrix measure method, *Neurocomputing*, **471** (2022), 70–78. https://doi.org/10.1016/j.neucom.2021.10.113

170. X. Yang, Z. Yang, Synchronization of ts fuzzy complex dynamical networks with time-varying impulsive delays and stochastic effects, *Fuzzy Sets Syst.*, **235** (2014), 25–43. https://doi.org/10.1016/j.fss.2013.06.008

171. X. Yang, X. Li, J. Lu, Z. Cheng, Synchronization of time-delayed complex networks with switching topology via hybrid actuator fault and impulsive effects control, *IEEE Trans. Cybern.*, **50** (2020), 4043–4052. https://doi.org/10.1109/TCYB.2019.2938217

172. P. Rubbioni, Asymptotic stability of solutions for some classes of impulsive differential equations with distributed delay, *Nonlinear Anal. Real World Appl.*, **61** (2021), 103324. https://doi.org/10.1016/j.nonrwa.2021.103324

173. X. Liu, K. Zhang, Stabilization of nonlinear time-delay systems: Distributed-delay dependent impulsive control, *Syst. Control Lett.*, **120** (2018), 17–22. https://doi.org/10.1016/j.sysconle.2018.07.012

174. Y. Zhao, X. Li, J. Cao, Global exponential stability for impulsive systems with infinite distributed delay based on flexible impulse frequency, *Appl. Math. Comput.*, **386** (2020), 125467. https://doi.org/10.1016/j.amc.2020.125467

175. X. Li, D. Peng, J. Cao, Lyapunov stability for impulsive systems via event-triggered impulsive control, *IEEE Trans. Autom. Control*, **65** (2020), 4908–4913. https://doi.org/10.1109/TAC.2020.2964558

176. W. Du, S. Leung, Y. Tang, A. Vasilakos, Differential evolution with event-triggered impulsive control, *IEEE Trans. Cybern.*, **47** (2017), 244–257. https://doi.org/10.1109/TCYB.2015.2512942

177. X. Li, P. Li, Input-to-state stability of nonlinear systems: Event-triggered impulsive control, *IEEE Trans. Autom. Control*, **67** (2022), 1460–1465. https://doi.org/10.1109/TAC.2021.3063227

178. X. Li, Y. Wang, S. Song, Stability of nonlinear impulsive systems: Self-triggered comparison system approach, *IEEE Trans. Autom. Control*, **2022** (2011). https://doi.org/10.1109/TAC.2022.3209441

179. E. D. Sontag, Smooth stabilization implies coprime factorization, *IEEE Trans. Autom. Control*, **34** (1989), 435–443. https://doi.org/10.1109/9.28018

180. W. Liu, P. Li, X. Li, Impulsive systems with hybrid delayed impulses: Input-to-state stability, *Nonlinear Anal. Hybrid Syst.*, **46** (2022), 101248. https://doi.org/10.1016/j.nahs.2022.101248

181. J. Mancilla-Aguilar, H. Haimovich, Uniform input-to-state stability for switched and time-varying impulsive systems, *IEEE Trans. Autom. Control*, **65** (2020), 5028–5042. https://doi.org/10.1109/TAC.2020.2968580

182. C. Ning, Y. He, M. Wu, S. Zhou, Indefinite lyapunov functions for input-to-state stability of impulsive systems, *Inf. Sci.*, **436** (2018), 343–351. https://doi.org/10.23919/ChiCC.2018.8483927 https://doi.org/10.1016/j.ins.2018.01.016

183. S. Dashkovskiy, P. Feketa, Input-to-state stability of impulsive systems with different jump maps, *IFAC-Papers OnLine*, **49** (2016), 1073–1078.

184. P. Feketa, N. Bajcinca, Stability of nonlinear impulsive differential equations with non-fixed moments of jumps, in *Proceedings of the 17th European Control Conference*, (2018), 900–905. https://doi.org/10.23919/ECC.2018.8550434

185. N. Zhang, X. Wang, W. Li, Stability for multi-linked stochastic delayed complex networks with stochastic hybrid impulses by dupire itô's formula, *Nonlinear Anal. Hybrid Syst.*, **45** (2022), 101200. https://doi.org/10.1016/j.nahs.2022.101200

186. C. W. Wu, L. Chua, Synchronization in an array of linearly coupled dynamical systems, *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.*, **42** (1995), 430–447.

187. B. Jiang, J. Lu, J. Lou, J. Qiu, Synchronization in an array of coupled neural networks with delayed impulses: Average impulsive delay method, *Neural Networks*, **121** (2020), 452–460. https://doi.org/10.1016/j.neunet.2019.09.019

188. Y. Fiagbedzi, A. Pearson, A multistage reduction technique for feedback stabilizing distributed time-lag systems, *Automatica*, **23** (1987), 311–326. https://doi.org/10.1016/0005-1098(87)90005-7

189. X. Ji, J. Lu, B. Jiang, K. Shi, Distributed synchronization of delayed neural networks: Delay-dependent hybrid impulsive control, *IEEE Trans. Network Sci. Eng.*, **9** (2021), 634–647. https://doi.org/10.1109/TNSE.2021.3128244

190. W. He, F. Qian, J. Cao, Pinning-controlled synchronization of delayed neural networks with distributed-delay coupling via impulsive control, *Neural Networks*, **85** (2017), 1–9. https://doi.org/10.1016/j.neunet.2016.09.002

191. Z. Huang, J. Cao, J. Li, H. Bin, Quasi-synchronization of neural networks with parameter mismatches and delayed impulsive controller on time scales, *Nonlinear Anal. Hybrid Syst.*, **33** (2019), 104–115. https://doi.org/10.1016/j.nahs.2019.02.005

192. D. Ding, Z. Tang, J. H. Park, Y. Wang, Z. Ji, Dynamic self-triggered impulsive synchronization of complex networks with mismatched parameters and distributed delay, *IEEE Trans. Cybern.*, **2022** (2022). https://doi.org/10.1109/TCYB.2022.3168854
193. D. Antunes, J. P. Hespanha, C. Silvestre, Stability of networked control systems with asynchronous renewal links: An impulsive systems approach, *Automatica*, **49** (2013), 402–413. https://doi.org/10.1016/j.automatica.2012.11.033
194. C. Yuan, F. Wu, Delay scheduled impulsive control for networked control systems, *IEEE Trans. Control Network Syst.*, **4** (2017), 587–597. https://doi.org/10.1109/TCNS.2016.2541341
195. X. Yang, J. Lu, D. W. C. Ho, Q. Song, Synchronization of uncertain hybrid switching and impulsive complex networks, *Appl. Math. Modell.*, **59** (2018), 379–392. https://doi.org/10.1016/j.apm.2018.01.046
196. J. Hu, G. Sui, X. Lv, X. Li, Fixed-time control of delayed neural networks with impulsive perturbations, *Nonlinear Anal. Modell. Control*, **23** (2018), 904–920. https://doi.org/10.15388/NA.2018.6.6
197. J. Lu, L. Li, D. W. C. Ho, J. Cao, *Collective Behavior in Complex Networked Systems under Imperfect Communication*, Springer, 2021. https://doi.org/10.1007/978-981-16-1506-1
198. X. Li, S. Song, *Impulsive Systems with Delays*, Springer, 2022. https://doi.org/10.1007/978-981-16-4687-4
199. C. Louembet, D. Arzelier, G. Deaconu, Robust rendezvous planning under maneuver execution errors, *J. Guid. Control Dyn.*, **38** (2015), 76–93. https://doi.org/10.2514/1.G000391
200. M. Brentari, S. Urbina, D. Arzelier, C. Louembet, L. Zaccarian, A hybrid control framework for impulsive control of satellite rendezvous, *IEEE Trans. Control Syst. Technol.*, **27** (2019), 1537–1551. https://doi.org/10.1109/ACC.2016.7526843 https://doi.org/10.1109/TCST.2018.2812197
201. G. Deaconu, C. Louembet, A. Théron, A two-impulse method for stabilizing the spacecraft relative motion with respect to a periodic trajectory, in *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)*, (2012), 6541–6546. https://doi.org/10.1109/CDC.2012.6426542
202. W. Fehse, *Automated Rendezvous and Docking of Spacecraft*, Cambridge University Press, 2003.
203. P. S. Rivadeneira, C. H. Moog, Impulsive control of single-input nonlinear systems with application to hiv dynamics, *Appl. Math. Comput.*, **218** (2012), 8462–8474. https://doi.org/10.1016/j.amc.2012.01.071
204. M. Legrand, E. Comets, G. Aymard, R. Tubiana, C. Katlama, B. Diquet, An in vivo pharmacokinetic/pharmacodynamic model for antiretroviral combination, *HIV Clin. Trials*, **4** (2003), 170–183. https://doi.org/10.1310/77YN-GDMU-95W3-RWT7
205. P. S. Rivadeneira, C. H. Moog, Observability criteria for impulsive control systems with applications to biomedical engineering processes, *Automatica*, **55** (2015), 125–131. https://doi.org/10.1016/j.automatica.2015.02.042

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