Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(2): 3047–3069.
DOI: 10.3934/mbe.2023144
Received: 26 October 2022
Revised: 11 November 2022
Accepted: 22 November 2022
Published: 01 December 2022

Research article

New finite-time synchronization conditions of delayed multinonidentical coupled complex dynamical networks

Zhen Yang[1,*], **Zhengqiu Zhang**[2] **and Xiaoli Wang**[3]

[1] School of Science, Hubei University of Technology, Wuhan 430068, China
[2] School of Mathematics, Hunan University, Changsha 410082, China
[3] School of Science, Henan University of Technology, Zhengzhou 450001, China

* **Correspondence:** Email: zixinyang808@163.com; Tel: +8617862988009.

Abstract: In this article, we mainly focus on the finite-time synchronization of delayed multinonidentical coupled complex dynamical networks. By applying the Zero-point theorem, novel differential inequalities, and designing three novel controllers, we obtain three new criteria to assure the finite-time synchronization between the drive system and the response system. The inequalities occurred in this paper are absolutely different from those in other papers. And the controllers provided here are fully novel. We also illustrate the theoretical results through some examples.

Keywords: multinonidentical coupled complex networks; finite-time synchronization; Zero-point theorem; novel controllers

1. Introduction

The coupled complex networks have aroused great interests of researchers, due to their extensive application in a wide range of observable systems, including electrical distribution networks, social networks, the Internet, scientific citation networks, and etc. Over the past few decades, complex networks have received much attention due to their potential applications in many real-world systems, such as biological systems, chemical systems, image processing, social systems, engineering and technological systems. The dynamical behaviors of complex networks are worthy of our consideration, and many useful theoretical results have been acquired in recent years, such as asymptotic/exponenial synchronization, such as [1–5] and H_∞ synchronization for markov jump chaotic systems [6].

Some researchers designed the synchronization, and the adaptive exponential synchronization problem of complex networks with nondifferentiable delays through analyzing the boundary of the adaptive control gain and the extended Halanay inequality, or a delay differential inequality, see [1,3]. In [2] and [4], the authors applied feedback control to investigate the issue of the asymptotic synchronization for complex networks. In [5], the synchronization of multiweighted and directed complex networks was studied. In the process, they decompose inner coupling matrices into diagonal matrices and residual matrices, and measure the similarity between outer coupling matrices. As opposed to the existing correlative studies, other professors considered the synchronization problems of complex networks by employing stochastic analysis techniques, for example, [7–11]. In [12], [13] and [14], from the prospective of integral inequality techniques, new delay-dependent global asymptotic synchronization conditions for dynamical systems and complex networks were established. In [8,11] and [15–19], the authors focused on the asymptotic synchronization of complex dynamical networks with coupling delays. Using impulsive effects control and linear matrix inequalities, they systematically concerned the synchronization analysis of complex networks in [20–22]. The global synchronization of fractional-order complex delayed networks was also explored in [23–28]. In actuality, the phenomenon of synchronization happens frequently in nature. There exists many wonderful results on controlling the asymptotic synchronization of complex networks, we can see [14,29–31] and [32–34].

The finite-time synchronization (FTS) plays an important role in the dynamic analysis of complex neural networks, for instance, [35–39]. Cui et al. [35] extended the existing criterion of local FTS to global FTS for complex dynamical networks with delayed impulses by means of establishing proper Lyapunov function. Zhang et al. [36] took advantage of the finite-time stability theorem to control the FTS for coupling delayed complex networks. Wang et al. [37] applied proportional-derivative (PD) control and inequality techniques to realize the finite-time passivity and FTS of complex dynamical networks with multiple state/derivative couplings. In [38], authors concentrated on switched complex dynamical networks with distributed coupling delays, and proposed the finite-time synchronized conditions by combining linear matrix inequalities with integral inequality. Yuan and Ma [39] took external disturbances into account in the delayed complex dynamical networks with unknown internal coupling matrices, and further investigated their finite-time $\mathcal{H}_\infty$ synchronization by utilizing the adaptive control method.

The synchronization issue for all dynamical nodes in the network is studied when the dynamical networks are first characterized in terms of a differential equation with a coupling term between dynamical nodes. Up to present, the results for finite-time synchronization of complex dynamic networks have been obtained chiefly by applying Lyapunov or Lyapunov-Krasovskii functional method [35, 37, 39–42, 51], Lyapunov stability theory [36, 44, 45], analysis theory [40, 43, 47–49, 51], graph theory [45], inequality techniques [37, 38, 41, 50], finite-time stability theorem [52, 53], the matrix inequality method [36, 37, 42, 44], and feedback control [46, 60]. Thus, it is urgent for us to find new study approaches to study the finite-time synchronization for drive-response complex dynamical systems. It inspires us to construct new inequalities to obtain the finite-time t of finite-time synchronization. In this paper, firstly, we construct three novel inequalities; then by combining the Zero-point theorem with these three novel differential inequalities and devising some different controllers related to time variable t, we obtain the finite-time synchronization criteria of drive-response delayed multinonidentical coupled complex dynamical networks. The difficulty of the proof in our paper is how to find novel inequalities related to the time variable t ($t > a$, a is a positive number) to design the expected controllers. Accordingly, the master contribution of this paper includes the following two aspects:

1) Three new inequalities are constructed to study the finite-time synchronization;

2) The novel controllers are designed in our paper which are completely different from those in the existing papers;

3) We give three novel criteria to make sure the finite-time synchronization between the drive system and the response system applying the Zero-point theorem and novel differential inequality.

The rest of this paper is arranged as follows: In Section 2, we introduce some necessary preliminaries. In Section 3, we adopt the Zero-point theorem and inequality techniques and prove that the finite-time synchronization between drive system (2.2) and response system (2.3) under three novel established criteria. And in Section 4, three examples are provided to illustrate the effectiveness and feasibility of the attained results intuitively.

2. Preliminaries

In our paper, we are interested in a kind of drive-response coupled heterogeneous complex networks. Generally, a Duffing-type oscillator is described as follows:

$$x(t) + bx'(t) + h(x(t)) = p(t),$$

which b is the damping constant, $p(t)$ is continuous and periodic, which is called driving force, and $h : \mathbb{R} \to \mathbb{R}$ is the nonlinear restoring force. With regard to n nonidentical Duffing-type oscillator respectively, they are represented as following:

$$x_i''(t) + bx_i'(t) + h(x_i(t)) = p_i(t),$$

where $p_i(i = 1, 2, \cdots, n)$ are not identical. In the field of neural networks, the property of delay plays a vital role. Naturally, we refer the description about the delayed complex dynamical network coupled with the n nonidentical nodes in [11]:

$$x_i''(t) + bx_i'(t) + h(x_i(t), x_i(t - \sigma(t))) = p_i(t) + c \sum_{j=1}^{n} a_{ij} x_j(t), \tag{2.1}$$

where $i = 1, 2, \cdots, n$, $b > 0$ is the damping constant,$c > 0$ is the coupling strength, and $p_i(t)$ are the continuous periodic driving forces. Here, $\sigma(t) : \mathbb{R} \to \mathbb{R}$ is a continuous function, which satisfies $\sigma'(t) < \sigma$. Besides,$a_{ij} = a_{ji} > 0$ if and only if there is a coupling between nodes i and $j(i \neq j)$, and $a_{ij} = 0$ otherwise. Then (2.1) is a coupled system, and we suppose that $a_{ij} = a_{ji} > 0$.

The initial values of system (2.1) are

$$x_i(s) = \varphi_i^x(s), \frac{dx_i(s)}{dt} = \psi_i^x(s).$$

where $\varphi_i^x(s), \psi_i^x(s)$ are continuous functions with real-valued bounds defined on $[-\alpha, 0]$, $\alpha = \max_{t\in\mathbb{R}}\{\sigma(t)\}$.

We make the linear transformation for the system (2.1), which is $y_i(t) = x_i'(t) + \lambda_i x_i(t)$. Here the parameter λ_i is a chosen constant. On account of this linear transformation, system (2.1) can be rewritten as

$$\begin{cases} x_i'(t) = -\lambda_i x_i(t) + y_i(t), \\ y_i'(t) = (b - \lambda_i)\lambda_i x_i(t) + (\lambda_i - b)y_i(t) - h(x_i(t), x_i(t - \sigma(t))) \\ \qquad\qquad + c \sum_{j=1}^{n} a_{ij} x_j(t) + p_i(t). \end{cases} \tag{2.2}$$

Supposing that system (2.2) is called as the drive system, and correspondingly, the response system can be represented as follows:

$$\begin{cases} u_i'(t) = -\lambda_i u_i(t) + v_i(t) + P_i(t), \\ v_i'(t) = (b-\lambda_i)\lambda_i u_i(t) + (\lambda_i - b)v_i(t) - h(u_i(t), u_i(t-\sigma(t))) \\ \qquad + c\sum_{j=1}^{n} a_{ij}u_j(t) + p_i(t) + Q_i(t), \end{cases} \tag{2.3}$$

where $P_i(t)$ and $Q_i(t)$ are the controllers which need to design later.
Assumption (H_1)**.** $\forall x(t), y(t), x(t-\sigma(t)), y(t-\sigma(t)) \in \mathbb{R}$, there exist positive constants M_1, M_2 such that the following inequality holds:

$$\begin{aligned} & |h(x(t), x(t-\sigma(t)) - f(y(t), y(t-\sigma(t)))| \\ \leq\ & M_1|x(t)-y(t)| + M_2|x(t-\sigma(t)) - y(t-\sigma(t))|, \end{aligned}$$

where $|\cdot|$ is the norm of Euclidean space $\mathbb{R}$.
Definition 2.1. Drive-response systems (2.2) and (2.3) are said to be finite-time synchronized, if for arbitrary solutions of systems (2.2) and (2.3) denoted by $[x_1(t), x_2(t), \cdots,$ $x_n(t), y_1(t), y_2(t), \cdots, y_n(t)]^T$ and $[u_1(t), u_2(t), \cdots, u_n(t), v_1(t), v_2(t), \cdots, v_n(t)]^T$, under a proper controller, there exists a time $T > 0$ which is related to the initial condition, such that for $i = 1, 2, \cdots, n$,

$$\lim_{t\to T}|u_i(t) - x_i(t)| = 0; \quad \lim_{t\to T}|v_i(t) - y_i(t)| = 0;$$

$$|u_i(t) - x_i(t)| = 0, t > T; \quad |v_i(t) - y_i(t)| = 0, t > T.$$

Lemma 2.1 If $t > \frac{1}{2}$, then $e^{t+\frac{1}{2}} - \ln(t+\frac{1}{2}) > 0$.
Proof. Due to $e^t > t > \ln t, (t > 1)$, so it is obvious that $e^{t+\frac{1}{2}} > \ln(t+\frac{1}{2}), (t > \frac{1}{2})$.
Lemma 2.2 If $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then when $t > \frac{2}{3}$, $\frac{1}{p}(t+\frac{1}{3})^p - \ln(t+\frac{1}{3}) > -\frac{1}{q}$.
Proof. Let $f(t) = \frac{1}{p}(t+\frac{1}{3})^p - \ln(t+\frac{1}{3}) + \frac{1}{q}$.

It is apparent that $f'(t) = \frac{(t+\frac{1}{3})^p}{t+\frac{1}{3}}$.

Since $t > \frac{2}{3}$, i.e., $t + \frac{1}{3} > 1$, we can know that when $t > \frac{2}{3}, p > 1, f'(t) > 0$, which means that $f(t)$ is a monotonically increasing function when $t > \frac{2}{3}$.

Therefore, $f(t) > f(\frac{2}{3}) = \frac{1}{p} + \frac{1}{q} = 1 > 0, \quad t > \frac{2}{3}$.

Furthermore, $\frac{1}{p}(t+\frac{1}{3})^p - \ln(t+\frac{1}{3}) > -\frac{1}{q}$ holds.
Lemma 2.3 If $t > \frac{1}{2}$, then $\frac{4t-2}{2t+3} - \ln(2t+1) + \ln 2 < 0$.
Proof. Let $f(t) = \frac{4t-2}{2t+3} - \ln(2t+1) + \ln 2, \quad t > \frac{1}{2}$.

Then, we can obtain $f'(t) = -\frac{2(2t-1)^2}{(2t+3)^2(2t+1)} < 0, \quad t > \frac{1}{2}$.

As a result, $f(t)$ is a monotonically decreasing function when $t > \frac{1}{2}$.

Thus, we have $f(t) < f(\frac{1}{2}) = 0$. So, $\frac{4t-2}{2t+3} - \ln(2t+1) + \ln 2 < 0, t > \frac{1}{2}$.
Lemma 2.4 Assume that $z = f(x,y)$ is defined on $(0,+\infty)\times(0,+\infty)$ and (x_0, y_0) is a unique local maximum value point. Then $\max\limits_{(x,y)\in(0,+\infty)\times(0,+\infty)} f(x,y) = \max\{f(x_0,y_0); f(0,+\infty);$ $f(+\infty,+\infty); f(+\infty,0)\}$.
Proof. The proof is known well and it is omitted.

3. Main results

Denote $e_i(t) = u_i(t) - x_i(t), r_j(t) = v_j(t) - y_j(t)$, then we can get the following error system of the drive system (2.2) and the response system (2.3) for $i = 1, 2, \cdots, n$:

$$\begin{cases} e_i'(t) = -\lambda_i e_i(t) + r_i(t) + P_i(t) \\ r_i'(t) = (b - \lambda_i)\lambda_i e_i(t) + (\lambda_i - a)r_i(t) + h[x_i(t), x_i(t - \sigma(t))] - \\ \qquad h[u_i(t), u_i(t - \sigma(t))] + c\sum_{j=1}^{n} a_{ij}e_j(t) + Q_i(t). \end{cases} \tag{3.1}$$

The controllers in system (3.1) are devised as follows for $i = 1, 2, \cdots, n$:

$$\begin{cases} P_i(t) = \text{sign}[e_i(t)]\Big[k_2|e_i(t)| - k_1 \exp\{|e_i(t)|\}\Big], \\ Q_i(t) = \text{sign}[r_i(t)]\Big[l_1|r_i(t)|^2 + l_2 \exp\{|r_i(t)|\} + \phi(t) + l_3 r_i(t)\Big], \end{cases} \tag{3.2}$$

$$\begin{cases} P_i(t) = [e_i(t)]^{-1}\Big[w_1 e_i^2(t) - \exp\{2|e_i(t)|\} + \theta_1|e_i(t)|\Big], \\ Q_i(t) = [r_i(t)]^{-1}\Big[w_2 r_i^2(t) + \theta_3|r_i(t)| + \theta_2 + \frac{\rho(t)}{2}\Big], \end{cases} \tag{3.3}$$

and

$$\begin{cases} P_i(t) = \text{sign}[e_i(t)]\Big[k_4|e_i(t)| - k_3|e_i(t)|^3\Big], \\ Q_i(t) = \text{sign}[r_i(t)]\Big[l_4|r_i(t)| + l_6 \sin(|r_i(t)|) + l_5 \cos(|r_i(t)|) + \psi(t)\Big], \end{cases} \tag{3.4}$$

where $\phi(t) = \frac{2}{2t+1}\mathrm{e}^{t+\frac{1}{2}} - 1, t > \frac{1}{2}, k_1 > 0, l_2 < 0, 2l_1 + l_2 < 0; \rho(t) = \frac{3}{3t+1} - (t + \frac{1}{3})^{p-1} - 1, t > \frac{2}{3}, w_1 < 0, w_2 < 0, \theta_1 > 2, \theta_3 > 0, \theta_2 < \frac{\theta_3^2}{2R_i}; \psi(t) = \frac{16}{(2t+3)^2} - \frac{2}{2t+1} - 3V(0), t > \frac{1}{2}, k_3 < 0, l_5 < 0, k_2, l_3, k_4, l_4, l_6$ are constants, $V(t)$ are defined in Theorem 3.1 and $V(0)$ is the value when $t = 0$ in $V(t)$. Here, the signs $e_i(t)$ and e represent the errors and the natural constant respectively.

Remark 1. For controllers (3.2), (3.3), $\phi(t), \rho(t)$ is independent of the initial values of error system, while in (3.4), $\psi(t)$ is related to the initial values of error system. Hence, there is a difference among the controllers (3.4) and (3.2), (3.3). The finite-time synchronization between the drive system (2.2) and the response system (2.3) is obtained under these three controllers and some conditions.

Theorem 3.1. Suppose that the condition (H_1) is satisfied. Then the drive system (2.2) and the response system (2.3) are finite-time synchronized under the controllers (3.2) in a finite time t_1, where $t_1 = \max\left\{\frac{1}{2}, \frac{V(0)}{n} + \ln 2 + \mathrm{e}^{\frac{1}{2}}\right\}$, if the following conditions hold:

(H_2)

$$0 < A_i = (k_2 - \lambda_i) + |b - \lambda_i||\lambda_i| + M_1 + c\sum_{j=1}^{n} a_{ji} + \frac{M_2}{1 - \sigma} < \mathrm{e}k_1;$$

(H_3)

$$B_i = l_3 + 1 + \lambda_i - b < -l_2.$$

Proof. We set up the following Lyapunov function:

$$V(t) = V_1(t) + V_2(t),$$

where, $V_1(t) = \sum_{i=1}^{n} |e_i(t)| + \sum_{i=1}^{n} |r_i(t)|, V_2(t) = \sum_{i=1}^{n} \frac{M_2}{1-\sigma} \int_{t-\sigma(t)}^{t} |e_i(s)|ds.$

According the system (3.1), Lemma 2.1, assumption (H_1) and the controllers (3.2), we have

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \sum_{i=1}^{n} \frac{d(\text{sign}[e_i(t)]e_i(t))}{dt} + \sum_{i=1}^{n} \frac{d(\text{sign}[r_j(t)]r_j(t))}{dt} \\
&= \sum_{i=1}^{n} \text{sign}[e_i(t)]\Big\{-\lambda_i e_i(t) + r_i(t) + P_i(t)\Big\} + \sum_{i=1}^{n} \text{sign}[r_j(t)]\Big\{(b-\lambda_i)\lambda_i e_i(t) \\
&\quad +(\lambda_i - b)r_i(t) + h[x_i(t), x_i(t-\sigma(t)))] - h[u_i(t), u_i(t-\sigma(t)))] \\
&\quad +c\sum_{j=1}^{n} a_{ij}e_j(t) + Q_i(t)\Big\} \\
&\leq \sum_{i=1}^{n} \Big\{(k_2 - \lambda_i)|e_i(t)| - k_1 \exp\{|e_i(t)|\} + |r_i(t)|\Big\} + \sum_{i=1}^{n} \Big\{(l_3 + \lambda_i - b)|r_j(t)| \\
&\quad +|b-\lambda_i||\lambda_i||e_i(t)| + |h(x_i(t), x_i(t-\sigma(t))) - h(u_i(t), u_i(t-\sigma(t)))| \\
&\quad +c\sum_{j=1}^{n} a_{ij}|e_j(t)| + l_1|r_j(t)|^2 + l_2 \exp\{|r_j(t)|\} + \phi(t)\Big\} \\
&\leq \sum_{i=1}^{n} \Big\{(k_2 - \lambda_i)|e_i(t)| - k_1 \exp\{|e_i(t)|\} + |r_i(t)|\Big\} + \sum_{i=1}^{n} \Big\{(l_3 + \lambda_i - b)|r_j(t)| \\
&\quad +|b-\lambda_i||\lambda_i||e_i(t)| + M_1|e_i(t)| + M_2|e_i(t-\sigma(t))| \\
&\quad +c\sum_{j=1}^{n} a_{ij}|e_j(t)| + l_1|r_j(t)|^2 + l_2 \exp\{|r_j(t)|\} + \phi(t)\Big\} \\
&\leq \sum_{i=1}^{n} \Big\{\Big[(k_2 - \lambda_i) + |b-\lambda_i||\lambda_i| + M_1 + c\sum_{j=1}^{n} a_{ji}\Big]|e_i(t)| + M_2|e_i(t-\sigma(t))| + (l_3 + 1 \\
&\quad +\lambda_i - b)|r_i(t)| - k_1 \exp\{|e_j(t)|\} + l_1|r_j(t)|^2 + l_2 \exp\{|r_j(t)|\} + \phi(t)\Big\}. \qquad (3.5)
\end{aligned}$$

On the other side, we have

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= \sum_{i=1}^{n} \frac{M_2}{1-\sigma} \int_{t-\sigma(t)}^{t} |e_i(s)|ds \\
&= \sum_{i=1}^{n} \frac{M_2}{1-\sigma}\Big[|e_i(t)| - (1-\sigma^{'}(t))|e_i(t-\sigma(t))|\Big] \\
&\leq \sum_{i=1}^{n} \Big\{\frac{M_2}{1-\sigma}|e_i(t)| - M_2|e_i(t-\sigma(t))|\Big\}. \qquad (3.6)
\end{aligned}$$

From the Eqs (3.5) and (3.6), one has

$$\begin{aligned}
& V'(t) = V_1'(t) + V_2'(t) \\
\leq & \sum_{i=1}^{n} \Big\{\Big[(k_2 - \lambda_i) + |b-\lambda_i||\lambda_i| + M_1 + c\sum_{j=1}^{n} a_{ji}\Big]|e_i(t)| + \frac{M_2}{1-\sigma}|e_i(t)| \\
& +(l_3 + 1 + \lambda_i - b)|r_i(t)| - k_1 \exp\{|e_i(t)|\} + l_1|r_i(t)|^2 + l_2 \exp\{|r_i(t)|\} + \phi(t)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\Big\{\Big[(k_2-\lambda_i)+|b-\lambda_i||\lambda_i|+M_1+c\sum_{j=1}^{n}a_{ji}+\frac{M_2}{1-\sigma}\Big]|e_i(t)| \\
&\quad +(l_3+1+\lambda_i-b)|r_i(t)|-k_1\exp\{|e_i(t)|\}+l_1|r_i(t)|^2+l_2\exp\{|r_i(t)|\}+\phi(t)\Big\} \\
&= \sum_{i=1}^{n}\Big[A_i|e_i(t)|-k_1\exp\{|e_i(t)|\}\Big]+\sum_{i=1}^{n}\Big[B_i|r_i(t)|+l_1|r_i(t)|^2 \\
&\quad +l_2\exp\{|r_i(t)|\}\Big]+n\phi(t).
\end{aligned} \tag{3.7}
$$

Denote

$$
\begin{aligned}
K_1(|e_i(t)|) &= A_i|e_i(t)|-k_1\exp\{|e_i(t)|\}, \\
G_1(|r_i(t)|) &= B_i|r_i(t)|+l_1|r_i(t)|^2+l_2\exp\{|r_i(t)|\}.
\end{aligned}
$$

Then $\frac{d[K_1(|e_i(t)|)]}{d|e_i(t)|} = A_i - k_1\exp\{|e_i(t)|\}$. From $\frac{d[K_1(|e_i(t)|)]}{d|e_i(t)|} = A_i - k_1\exp\{|e_i(t)|\}$, from $\frac{d[K_1(|e_i(t)|)]}{d|e_i(t)|} = 0$, it follows that when $|e_i(t)| < \ln\frac{A_i}{k_1}, K_1'(|e_i(t)| > 0$; when $|e_i(t)| > \ln\frac{A_i}{k_1}, F'(|e_i(t)|$ < 0. As a result, we attain $\max\{K_1(|e_i(t)|)\} = K_1(\ln\frac{A_i}{k_1}) = A_i\Big[\ln\frac{A_i}{k_1}-1\Big] < 0$. Thus

$$
A_i|e_i(t)|-k_1\exp\{|e_i(t)|\} \le 0. \tag{3.8}
$$

From $G_1(|r_i(t)|) = B_i|r_i(t)| + l_1|r_i(t)|^2 + l_2\exp\{|r_i(t)|\}$, one attains $G_1'(|r_j(t)|) = B_i + 2l_1|r_i(t)| + l_2\exp\{|r_i(t)|\}, G_1''(|r_i(t)|) = 2l_1 + l_2\exp\{|r_i(t)|\}$. Since $l_2 < 0$, thus, $G_1^{(3)}(|r_i(t)|) = l_2\exp\{|r_i(t)|\} < 0$. Then $G_1''(|r_i(t)|)$ is deceasing. Hence $G''(|r_i(t)|) < G''(0) = 2l_1 + l_2 < 0$. Then $G'(|r_i(t)|)$ is decreasing. As a result, $G_1'(|r_i(t)|) < G_1'(0) = B_i + l_2 < 0$. So, $G_1(|r_i(t)|)$ is decreasing. Therefore, $G_1(|r_i(t)|) < G_1(0) = l_2 < 0$. That is

$$
B_j|r_i(t)|+l_1|r_i(t)|^2+l_2\exp\{|r_i(t)|\} < 0. \tag{3.9}
$$

Substituting (3.8) and (3.9) into (3.7), it yields

$$
V'(t) < n\phi(t). \tag{3.10}
$$

Integrating (3.10) over $[0,t]$ gives

$$
\begin{aligned}
V(t) &< V(0)+n\Big[\ln(t+\frac{1}{2})-e^{t+\frac{1}{2}}+\ln 2+e^{\frac{1}{2}}-t\Big](t>\frac{1}{2}) \\
&= V(0)+n(\ln(t+\frac{1}{2})-e^{t+\frac{1}{2}})+n(\ln 2+e^{\frac{1}{2}}-t).
\end{aligned} \tag{3.11}
$$

It is not difficult to compute that when $t > \frac{V(0)}{n} + \ln 2 + e^{\frac{1}{2}}$,

$$
V(0)+n(\ln 2+e^{\frac{1}{2}}-t) < 0. \tag{3.12}
$$

According to Lemma 2.1, we have

$$
\ln(t+\frac{1}{2})-e^{t+\frac{1}{2}} < 0, t > \frac{1}{2}. \tag{3.13}
$$

Substituting (3.12) and (3.13) into (3.11), it follows that when $t > t_1 = \max\left\{\frac{1}{2}, \frac{V(0)}{n} + \ln 2 + e^{\frac{1}{2}}\right\}$,

$$0 \le V(t) \le 0, t \ge t_1.$$

That is $\lim_{t\to t_2} |e_i(t)| = 0, \lim_{t\to t_1} |r_i(t)| = 0, |e_i(t)| = 0, |r_i(t)| = 0, t \ge t_1$. In other words,

$$\lim_{t\to t_1} |u_i(t) - x_i(t)| = 0, |u_i(t) - x_i(t)| = 0, t \ge t_1;$$

$$\lim_{t\to t_1} |v_i(t) - y_i(t)| = 0, |v_i(t) - y_i(t)| = 0, t \ge t_1.$$

The proof of Theorem 3.1 is finished.

Theorem 3.2. Suppose that the condition (H_1) is satisfied. Then the drive system (2.2) and the response system (2.3) are finite-time synchronized under the controllers (3.3) in a finite time t_2, where $t_2 = \max\left\{\frac{2}{3}, \frac{U(0)}{n} + \ln 3 + \frac{1}{q} + \frac{1}{p3^p}\right\}$, if the following conditions hold:

(H_4)

$$0 < L_i = 2(w_1 - \lambda_i) + |b - \lambda_i||\lambda_i| + 1 + M_1 + \frac{M_2}{1-\sigma} + c\sum_{j=1}^{n} a_{ji} < \min\{4, 4e - 2\theta_1\};$$

(H_5)

$$R_i = 2(w_2 + \lambda_i - b) + 1 + |b - \lambda_i||\lambda_i| + M_1 + M_2 + c\sum_{j=1}^{n} a_{ij} < 0.$$

Proof. We set up the following Lyapunov function:

$$U(t) = U_1(t) + U_2(t),$$

where $U_1(t) = \sum_{i=1}^{n} e_i^2(t) + \sum_{i=1}^{n} r_i^2(t), U_2(t) = \sum_{i=1}^{n} \frac{M_2}{1-\sigma}\int_{t-\sigma(t)}^{t} e_i^2(s)ds$.

According the system (3.1), assumption (H_1) and the controllers (3.3), we have

$$\begin{aligned}
\frac{dU_1(t)}{dt} &= \sum_{i=1}^{n}\Big\{2e_i(t)e_i'(t) + 2r_i(t)r_i'(t)\Big\} \\
&= \sum_{i=1}^{n} 2e_i(t)\Big\{-\lambda_i e_i(t) + r_i(t) + P_i(t)\Big\} + \sum_{i=1}^{n} 2r_i(t)\Big\{(b-\lambda_i)\lambda_i e_i(t) + (\lambda_i - a)r_i(t) \\
&\quad + h[x_i(t), x_i(t-\sigma(t)))] - h[u_i(t), u_i(t-\sigma(t)))] + c\sum_{j=1}^{n} a_{ij}e_j(t) + Q_i(t)\Big\} \\
&\le \sum_{i=1}^{n}\Big\{2(w_1-\lambda_i)e_i^2(t) - 2\exp\{2|e_i(t)|\} + 2\theta_1|e_i(t)| + e_i^2(t) + r_i^2(t) + 2(w_2 + \\
&\quad \lambda_i - b)r_i^2(t) + |b-\lambda_i||\lambda_i|[e_i^2(t) + r_i^2(t)] + 2|r_i(t)|\cdot\Big|h(x_i(t), x_i(t-\sigma(t))) \\
&\quad - h(u_i(t), u_i(t-\sigma(t)))\Big| + c\sum_{j=1}^{n} a_{ij}[e_j^2(t) + r_i^2(t)] + 2\theta_3|r_i(t)| + 2\theta_2 + \rho(t)\Big\} \\
&\le \sum_{i=1}^{n}\Big\{2(w_1-\lambda_i)e_i^2(t) - 2\exp\{2|e_i(t)|\} + 2\theta_1|e_i(t)| + e_i^2(t) + r_i^2(t) + 2(w_2 + \lambda_i
\end{aligned}$$

$$-b)r_i^2(t)+|b-\lambda_i||\lambda_i|[e_i^2(t)+r_i^2(t)]+M_1[e_i^2(t)+r_i^2(t)]+M_2[e_i^2(t-\sigma(t))$$
$$+r_i^2(t)]+c\sum_{j=1}^{n}a_{ji}[e_i^2(t)+r_i^2(t)]+2\theta_3|r_i(t)|+2\theta_2+\rho(t)\Big\} \quad (3.14)$$

On the other side, we have

$$\begin{aligned}\frac{dU_2(t)}{dt} &= \sum_{i=1}^{n}\frac{M_2}{1-\sigma}\int_{t-\sigma(t)}^{t}e_i^2(s)ds\\ &= \sum_{i=1}^{n}\frac{M_2}{1-\sigma}\Big[e_i^2(t)-(1-\sigma^{'}(t))e_i^2(t-\sigma(t))\Big]\\ &\leq \sum_{i=1}^{n}\Big[\frac{M_2}{1-\sigma}e_i^2(t)-M_2e_i^2(t-\sigma(t))\Big]. \end{aligned} \quad (3.15)$$

From the Eqs (3.14) and (3.15), one has

$$\begin{aligned} & U'(t)=U_1'(t)+U_2'(t)\\ \leq & \sum_{i=1}^{n}\Big\{2(w_1-\lambda_i)e_i^2(t)-2\exp\{2|e_i(t)|\}+2\theta_1|e_i(t)|+e_i^2(t)+r_i^2(t)\\ & +2(w_2+\lambda_i-b)r_i^2(t)+|b-\lambda_i||\lambda_i|[e_i^2(t)+r_i^2(t)]+M_1[e_i^2(t)+r_i^2(t)]\\ & +M_2[\frac{1}{1-\sigma}e_i^2(t)+r_i^2(t)]+c\sum_{j=1}^{n}a_{ij}[e_j^2(t)+r_i^2(t)]+2\theta_3|r_i(t)|+2\theta_2+\rho(t)\Big\}\\ \\ = & \sum_{i=1}^{n}\Big\{\Big[2(w_1-\lambda_i)+|b-\lambda_i||\lambda_i|+1+M_1+\frac{M_2}{1-\sigma}+c\sum_{j=1}^{n}a_{ji}\Big]e_i^2(t)\\ & +\Big[2(w_2+\lambda_i-b)+1+|b-\lambda_i||\lambda_i|+M_1+M_2+c\sum_{j=1}^{n}a_{ij}\Big]r_i^2(t)\\ & -2\exp\{2|e_i(t)|\}+2\theta_1|e_i(t)|+2\theta_3|r_i(t)|+2\theta_2+\rho(t)\Big\}\\ = & \sum_{i=1}^{n}\Big[L_ie_i^2(t)-2\exp\{2|e_i(t)|\}+2\theta_1|e_i(t)|\}\Big]+\sum_{i=1}^{n}\Big[R_ir_i^2(t)+2\theta_3|r_i(t)|+2\theta_2\Big]\\ & +n\rho(t). \end{aligned} \quad (3.16)$$

Denote

$$\begin{aligned} K_2(|e_i(t)|) &= L_i|e_i(t)|^2-2\exp\{2|e_i(t)|\}+2\theta_1|e_i(t)|,\\ G_2(|r_i(t)|) &= R_i|r_i(t)|^2+2\theta_3|r_i(t)|+2\theta_2. \end{aligned}$$

where $|e_i(t)| \in [0,+\infty), |r_i(t)| \in [0,+\infty)$. It is obvious that $K_2'(|e_i(t)|)=\frac{dK_2(|e_i(t)|)}{d|e_i(t)|}=2L_i|e_i(t)|-4\exp\{2|e_i(t)|\}+2\theta_1$ and $K_2''(|e_i(t)|)=\frac{dK_2^2(|e_i(t)|)}{d|e_i(t)|^2}=2L_i-8\exp\{2|e_i(t)|\}$. Because of $0<L_i<4$, then $K_2''(|e_i(t)|)<0$. Thus $K_2'(|e_i(t)|)$ is a monotonically decreasing continuous function in $[0,\infty)$.

Because of $\theta_1 > 2, 0 < L_i < \min\{4, 4\mathrm{e}-2\theta_1\}$, there is $K_2'(0) = -4+2\theta_1 > 0$ and $F_2'(\frac{1}{2}) = L_i - 4e + 2\theta_1 < 0$. As a result, based on Zero-point theorem of monotonically continuous function, there exists a unique zero point $|e_i^*(t)| \in (0, \frac{1}{2})$ of $K_2'(|e_i(t)|)$ such that $K_2'(|e_i^*(t)|) = 2L_i|e_i^*(t)| - 4\exp\{2|e_i^*(t)|\} + 2\theta_1 = 0$. Regarding this equation of $|e_i^*(t)|$, we have

$$2\theta_1 = 4\exp\{2|e_i^*(t)|\} - 2L_i|e_i^*(t)|. \tag{3.17}$$

Furthermore, from $0 < |e_i^*(t)| < \frac{1}{2}$, (3.9) and the assumption (H_4), we have

$$\begin{aligned} K_2(|e_i^*(t)|) &= L_i|e_i^*(t)|^2 - 2\exp\{2|e_i^*(t)|\} + 2\theta_1|e_i^*(t)| \\ &= L_i|e_i^*(t)|^2 - 2\exp\{2|e_i^*(t)|\} + (4\exp\{2|e_i^*(t)|\} - 2L_i|e_i^*(t)|)|e_i^*(t)| \\ &= 2(2|e_i^*(t)| - 1)\exp\{2|e_i^*(t)|\} - L_i|e_i^*(t)|^2 < 0. \end{aligned} \tag{3.18}$$

Apparently,

$$K_2(0) = -2 < 0; \lim_{|e_i(t)|\to+\infty} K_2(|e_i(t)|) = -\infty < 0. \tag{3.19}$$

From (3.18), (3.19), and Lemma 2.4, one has

$$\max\{K_2(|e_i(t)|)\} < 0.$$

In addition, $G_2'(|r_i(t)|) = \frac{dG_2(|r_i(t)|)}{d|r_i(t)|} = 2R_i|r_i(t)| + 2\theta_3$. Based on $G_2'(|r_i(t)|) = 0$, it is easy to obtain $|r_i^*(t)| = -\frac{\theta_3}{R_i}$. In other words, when $0 < |r_i(t)| < |r_i^*(t)|, G_2'(|r_i(t)|) > 0, G_2(|r_i(t)|)$ is monotone increasing, and when $|r_i^*(t)| < |r_i(t)| < +\infty, G_2'(|r_i(t)|) < 0, G_2(|r_i(t)|)$ is monotone decreasing. Therefore, we conclude that $|r_i^*(t)|$ is the unique extreme point of $G_2(|r_i(t)|)$, and the extreme value is

$$\begin{aligned} G_2(|r_i^*(t)|) &= R_i|r_i^*(t)|^2 + 2\theta_3|r_i^*(t)| + 2\theta_2 \\ &= 2\theta_2 - \frac{\theta_3^2}{R_i} < 0. \end{aligned} \tag{3.20}$$

Furthermore,

$$G_2(0) = 2\theta_2 < 0; \lim_{|r_i(t)|\to+\infty} G_2(|r_i(t)|) = -\infty < 0. \tag{3.21}$$

Similarity, based on (3.20), (3.21) and Lemma 2.4, one has

$$\max\{G_2(|r_i(t)|)\} < 0.$$

Substituting $K_2(|e_i(t)|) < 0$ and $G_2(|r_i(t)|) < 0$ into (3.16), it yields

$$U'(t) < n\rho(t). \tag{3.22}$$

Integrating (3.22) over $[0, t]$ gives

$$U(t) < U(0) + n\Big[\ln(t + \frac{1}{3}) - \frac{1}{p}(t + \frac{1}{3})^p - \frac{1}{q} + \ln 3 + \frac{1}{q} + \frac{1}{p3^p} - t\Big]$$

$$= \quad U(0) + n\left[\ln(t+\frac{1}{3}) - \frac{1}{p}(t+\frac{1}{3})^p - \frac{1}{q}\right] + n\left(\ln 3 + \frac{1}{q} + \frac{1}{p3^p} - t\right). \tag{3.23}$$

Evidently, when $t > \frac{U(0)}{n} + \ln 3 + \frac{1}{q} + \frac{1}{p3^p}$,

$$U(0) + n(\ln 3 + \frac{1}{q} + \frac{1}{p3^p} - t) < 0. \tag{3.24}$$

According to Lemma 2.2, when $t > \frac{2}{3}$, the following inequality holds:

$$\ln(t+\frac{1}{3}) - \frac{1}{p}(t+\frac{1}{3})^p - \frac{1}{q} < 0. \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23), it follows that when $t > t_2 = \max\left\{\frac{2}{3}, \frac{U(0)}{n} + \ln 3 + \frac{1}{q} + \frac{1}{p3^p}\right\}$,

$$0 \leq U(t) \leq 0, t \geq t_2.$$

That is $\lim\limits_{t\to t_2} |e_i(t)| = 0, \lim\limits_{t\to t_2} |r_j(t)| = 0, |e_i(t)| = 0, |r_i(t)| = 0, t \geq t_2$. In other words,

$$\lim_{t\to t_2} |u_i(t) - x_i(t)| = 0, |u_i(t) - x_i(t)| = 0, t \geq t_2;$$

$$\lim_{t\to t_2} |v_i(t) - y_i(t)| = 0, |v_i(t) - y_i(t)| = 0, t \geq t_2.$$

The proof of Theorem 3.2 is finished.

Theorem 3.3. Presume that (H_1) holds. Then the drive system (2.2) and the response system (2.3) are finite-time synchronized under the controllers (3.4) in a finite time t_3, where $t_3 = \max\{\frac{1}{3n} + \frac{2-3\ln 2}{9V(0)}, \frac{1}{2}\}$, if the following conditions hold:

(H_6)

$$C_i = (k_4 - \lambda_i) + |b - \lambda_i||\lambda_i| + M_1 + c\sum_{j=1}^{n} a_{ji} + \frac{M_2}{1-\sigma} < 0;$$

(H_7)

$$D_i = l_4 + 1 + \lambda_i - b < -\sqrt{l_5^2 + l_6^2}.$$

Proof. We establish the same Lyapunov function as follows:

$$V(t) = V_1(t) + V_2(t),$$

where, $V_1(t) = \sum\limits_{i=1}^{n} |e_i(t)| + \sum\limits_{i=1}^{n} |r_i(t)|, V_2(t) = \sum\limits_{i=1}^{n} \frac{M_2}{1-\sigma} \int_{t-\sigma(t)}^{t} |e_i(s)| ds$.

From the proof of theorem 3.1, we get

$$\begin{aligned} & V'(t) = V_1'(t) + V_2'(t) \\ \leq \quad & \sum_{i=1}^{n} \Big\{ \Big[(k_4 - \lambda_i) + |b - \lambda_i||\lambda_i| + M_1 + c\sum_{j=1}^{n} a_{ji} + \frac{M_2}{1-\sigma} \Big] |e_i(t)| \\ & + (l_4 + 1 + \lambda_i - b)|r_i(t)| - k_3|e_i(t)|^3 + l_6 \sin(|r_i(t)|) + l_5 \cos(|r_i(t)|) + \psi(t) \Big\} \end{aligned}$$

$$= \sum_{i=1}^{n}\Big[C_i|e_i(t)| - k_3|e_i(t)|^3\Big] + \sum_{i=1}^{n}\Big[D_i|r_i(t)| + l_6\sin(|r_i(t)|) + l_5\cos(|r_i(t)|)\Big] + n\psi(t). \quad (3.26)$$

Let

$$\begin{aligned} K_3(|e_i(t)|) &= C_i|e_i(t)| - k_3|e_i(t)|^3, \\ G_3(|r_i(t)|) &= D_i|r_i(t)| + l_6\sin(|r_i(t)|) + l_5\cos(|r_i(t)|). \end{aligned}$$

Then we have $\frac{d[K_3(|e_i(t)|)]}{dt} = C_i - 3k_3|e_i(t)|^2$. From $\frac{d[K_3(|e_i(t)|)]}{dt} = C_i - 3k_3|e_i(t)|^2 = 0$, it yields $|e_i(t)| = \sqrt{\frac{C_i}{3k_3}}$. Thus, when $|e_i(t)| < \sqrt{\frac{C_i}{3k_3}}$, $K_3'(|e_i(t)|) > 0$; when $|e_i(t)| > \sqrt{\frac{C_i}{3k_3}}$, $K_3'(|e_i(t)|) < 0$. Hence,

$$K_3(|e_i(t)|) \le \max\{K_3(|e_i(t)|)\} = K_3\Big(\sqrt{\frac{C_i}{3k_3}}\Big) = \frac{2C_i}{3}\sqrt{\frac{C_i}{3k_3}} < 0. \quad (3.27)$$

From the definition of $G_3(|r_j(t)|)$, we attain

$$\begin{aligned} G_3'(|r_i(t)|) &= D_i + l_6\cos(|r_i(t)|) - l_5\sin(|r_i(t)|) \\ &= D_i + \sqrt{l_5^2 + l_6^2}\Big[\frac{l_6}{\sqrt{l_5^2 + l_6^2}}\cos(|r_i(t)|) - \frac{l_5}{\sqrt{l_5^2 + l_6^2}}\sin(|r_i(t)|)\Big] \\ &= D_i + \sqrt{l_5^2 + l_6^2}\cos(|r_i(t)| + \beta) \\ &\le D_i + \sqrt{l_5^2 + l_6^2} \le 0, \end{aligned}$$

where, $\cos\beta = \frac{l_6}{\sqrt{l_5^2+l_6^2}}$, $\sin\beta = \frac{l_5}{\sqrt{l_5^2+l_6^2}}$. Thus $G_3'(|r_i(t)|) < 0$. Therefore, $G_3(|r_i(t)|)$ is decreasing. So

$$G_3(|r_i(t)|) < G_3(0) = l_5 < 0. \quad (3.28)$$

Substituting (3.27) and (3.28) into (3.26), it yields

$$V'(t) < \sum_{i=1}^{n}\Big[K_3(|e_i(t)|) + G_3(|r_i(t)|)\Big] + n\psi(t) < n\psi(t). \quad (3.29)$$

Integrating (3.29) over $[0, t]$ gives

$$\begin{aligned} V(t) &< V(0) + n\Big(2 - \frac{8}{2t+3} - \ln(2t+1) + \ln 2\Big) \\ &\quad + n\left(\frac{8}{3} - 2 - \ln 2 - 3V(0)t\right), \quad (t > \frac{1}{2}) \\ &= n\Big(\frac{4t-2}{2t+3} - \ln(2t+1) + \ln 2\Big) + (\frac{2}{3} - \ln 2)n + (1 - 3nt)V(0). \end{aligned} \quad (3.30)$$

According to Lemma 2.3, we acquire the following result:

$$\frac{4t-2}{2t+3} - \ln(2t+1) + \ln 2 < 0, t > \frac{1}{2}. \tag{3.31}$$

It is obvious that when $t > \frac{1}{3n} + \frac{2-3\ln 2}{9V(0)}$,

$$(\frac{2}{3} - \ln 2)n + (1-3nt)V(0) < 0. \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.30), it yields that when $t > t_3 = \max\{\frac{1}{3n} + \frac{2-3\ln 2}{9V(0)}, \frac{1}{2}\}$,

$$0 \le V(t) \le 0, t > t_2.$$

That is $\lim_{t\to t_3} |e_i(t)| = 0, \lim_{t\to t_3} |r_i(t)| = 0, |e_i(t)| = 0, |r_i(t)| = 0, t > t_3$. Accordingly,

$$\lim_{t\to t_3} |u_i(t) - x_i(t)| = 0, |u_i(t) - x_i(t)| = 0, t > t_3;$$

$$\lim_{t\to t_3} |v_i(t) - y_i(t)| = 0, |v_i(t) - y_i(t)| = 0, t > t_3.$$

The proof of Theorem 3.3 is finished.

Remark 2. The method in finite-time synchronization used in the paper is completely different from those in [59, 61–66].

Remark 3. Without applying the finite-time stability theorems used in [36–45, 52, 53, 60], integral inequalities used in [61–66] and LMI method, by using three novel inequalities in Lemmas 2.1–2.3, we obtain three criteria to assure the finite-time synchronization for the discussed drive-response complex dynamical networks. The advantage of using three inequalities is to obtain the finite-time T without using the complicated finite-time stability theorems.

4. Numerical examples

In this part, to present our results intuitively, we provide three numerical examples.

Example 4.1. We take into consideration the system (2.2), the response system (2.3) and the error system (3.1) with the controllers (3.2) for $i = 1, 2$, where $b = 15, \lambda_1 = 2, \lambda_2 = 3, c = 1, a_{11} = a_{22} = 0, a_{12} = a_{21} = 2; k_1 = 10 > 0, k_2 = -15, l_1 = 2, l_2 = -35 < 0, 2l_1 + l_2 = -31 < 0, l_3 = 3, \phi(t) = \frac{2}{2t+1} - e^{t+\frac{1}{2}} - 1, p_1(t) = \sin t, p_2(t) = 3\cos t, h(x_i(t), x_i(t-\sigma(t))) = |x_i(t)| + |x_i(t-\sigma(t))|, \sigma(t) = 0.3\sin t$.

Moreover, the initial conditions of drive-response systems are defined as: $x_1(0) = 14.12, x_2(0) = 23.61, y_1(0) = 12.015, y_2(0) = -30.253, u_1(0) = -46.892, u_2(0) = 27.35, v_1(0) = 52.637, v_2(0) = 19.54$. Obviously, the assumptions $(H_2), (H_3)$ is hold:

$$0 < A_1 = (k_2 - \lambda_1) + |b - \lambda_1||\lambda_1| + M_1 + c\sum_{j=1}^{n} a_{j1} + \frac{M_2}{1-\sigma} = 13.4286 < ek_1 = 27.1828,$$

$$0 < A_2 = (k_2 - \lambda_2) + |b - \lambda_2||\lambda_i| + M_1 + c\sum_{j=1}^{n} a_{j2} + \frac{M_2}{1-\sigma} = 22.4286 < ek_1 = 27.1828,$$

$$B_1 = l_3 + 1 + \lambda_1 - b = -9 < -l_2 = 35,$$

$$B_2 = l_3 + 1 + \lambda_2 - b = -8 < -l_2 = 35.$$

Based on the introduction of the inequality in Lemma 2.1, we propose the controller (3.2), which is less conservative than those in the literature [35, 36, 44–49, 51–53]. It is not difficult to verify that the parameters in the above papers do not satisfy the conditions of finite-time synchronization. Therefore, the finite-time synchronization between the drive system (2.2) and response system (2.3) can't be validated with the results in [35–53, 60]. It is easy to know that $(H_1) - (H_3)$ in Theorem 3.1 in our paper are satisfied. Hence, according to Theorem 3.1, the drive system (2.2) and the response system (2.3) are finite-time synchronized under the controllers (3.2).

The trajectories of $x_i(t), u_i(t), y_i(t), v_i(t)$ and errors $e_i(t), r_i(t), i = 1, 2$ are shown with a controller (3.2) in the following Figures 1–3.

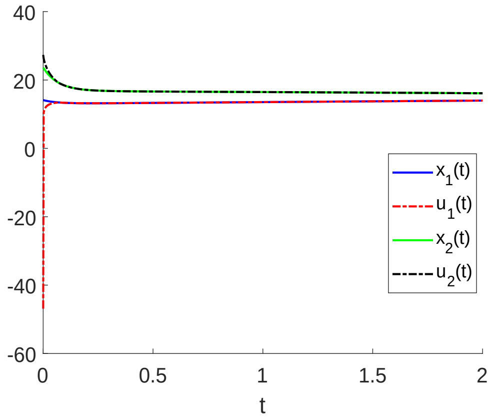

Figure 1. The curves of the $x_1(t), x_2(t), u_1(t), u_2(t)$ in the Example 4.1.

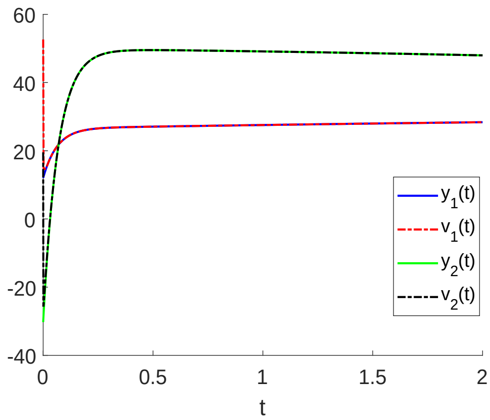

Figure 2. The curves of the $y_1(t), y_2(t), v_1(t), v_2(t)$ in the Example 4.1.

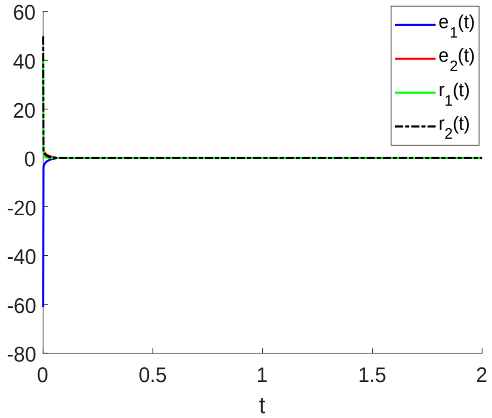

Figure 3. The curves of the $e_1(t), e_2(t), r_1(t), r_2(t)$ in the Example 4.1.

Example 4.2. We focus on the drive system (2.2), the response system (2.3) and the error system (3.1) with the controllers (3.3) for $i = 1, 2$. The value of parameters are chosen as: $b = 7, \lambda_1 = 4.2, \lambda_2 = 5, c = 3, a_{11} = a_{22} = 0, a_{12} = a_{21} = 3.2; w_1 = -5.8 < 0, w_2 = -40 < 0, \theta_1 = 2.2 > 2, \theta_3 = 4 > 0, \theta_2 = -15 < \frac{\theta_3^2}{2R_i}, p = 2; \rho(t) = \frac{3}{3t+1} - t + \frac{1}{3}^{p-1} - 1, p_1(t) = -0.8\cos t, p_2(t) = 1.2\sin t, h(x_i(t), x_i(t-\sigma(t))) = 0.4|x_i(t)| + \sin[x_i(t-\sigma(t))], \sigma(t) = 0.5\sin t$.

Here, we take the initial conditions of systems as follows: $x_1(0) = 2394.12, x_2(0) = -1124.61, y_1(0) = -7421.15, y_2(0) = -1701.25, u_1(0) = -4104.892, u_2(0) = -3562.35, v_1(0) = -3885.637, v_2(0) = -1651.54$. Based on simple calculation, we find that the criteria in Theorem 3.2 is satisfied.

$$0 < L_1 = 2(w_1 - \lambda_1) + |b - \lambda_1||\lambda_1| + 1 + M_1 + \frac{M_2}{1-\sigma} + c\sum_{j=1}^{n} a_{j1} = 3.76 < \min\{4, 4\mathrm{e} - 2\theta_1\};$$

$$0 < L_2 = 2(w_1 - \lambda_2) + |b - \lambda_2||\lambda_2| + 1 + M_1 + \frac{M_2}{1-\sigma} + c\sum_{j=1}^{n} a_{j2} = 0.40 < \min\{4, 4\mathrm{e} - 2\theta_1\};$$

$$R_1 = 2(w_2 + \lambda_1 - b) + 1 + |b - \lambda_1||\lambda_1| + M_1 + M_2 + c\sum_{j=1}^{n} a_{1j} = -61.84 < 0;$$

$$R_2 = 2(w_2 + \lambda_2 - b) + 1 + |b - \lambda_2||\lambda_2| + M_1 + M_2 + c\sum_{j=1}^{n} a_{2j} = -62.00 < 0.$$

The figures of $x_i(t), u_i(t), y_i(t), v_i(t)$ and errors $e_i(t), r_i(t), i = 1, 2$ with controller (3.3) are depicted in Figures 4–6.

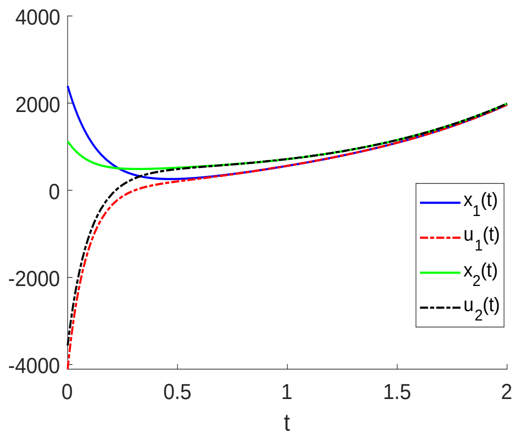

Figure 4. The curves of the $x_1(t), x_2(t), u_1(t), u_2(t)$ in the Example 4.2.

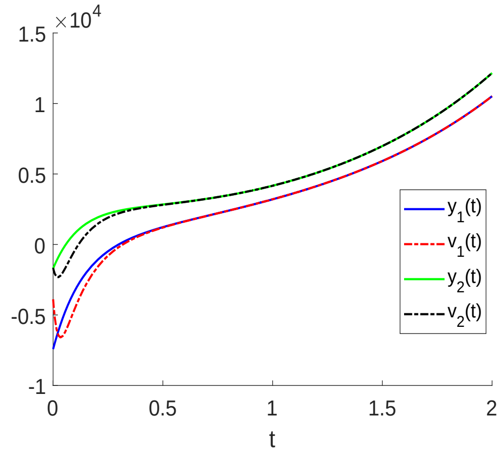

Figure 5. The curves of the $y_1(t), y_2(t), v_1(t), v_2(t)$ in the Example 4.2.

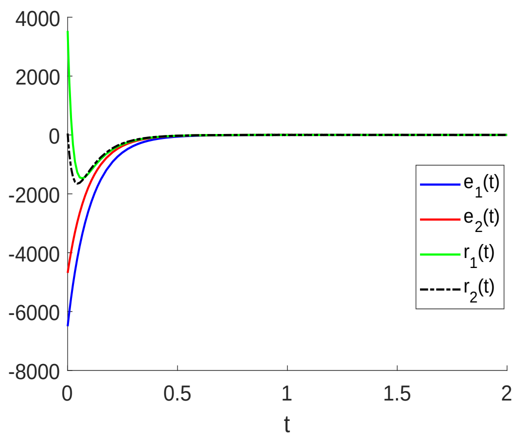

Figure 6. The curves of the $e_1(t), e_2(t), r_1(t), r_2(t)$ in the Example 4.2.

Example 4.3. We take into consideration the system (2.2), the response system (2.3) and the error system (3.1) with the controllers (3.2), the time-varying delay $\sigma(t)$ is set randomly as $0.7 \sin t - 0.4$, The nonlinear restoring force is chosen as $h(x_i(t), x_i(t-\sigma(t))) = 0.2 \sin[x_i(t)] + \sin[x_i(t-\sigma(t))]$, which is satisfied with assumption (H_1). Then the damping constant, coupling strength and rest parameters are represented as: $b = 12, \lambda_1 = 3.8, \lambda_2 = -3, c = 5, a_{11} = a_{22} = 0, a_{12} = a_{21} = 4; k_3 = -32 < 0, k_4 = -75 < 0, l_4 = -4, l_5 = -3.5 < 0, l_6 = 8, \psi(t) = \frac{16}{(2t+3)^2} - \frac{2}{2t+1} - 3V(0), p_1(t) = 5, p_2(t) = -4$.

The initial conditions is defined as: $x_1(0) = 5.9, x_2(0) = -4.63, y_1(0) = -30.48, y_2(0) = -13.4, u_1(0) = 4.5, u_2(0) = -3.13, v_1(0) = -13.57, v_2(0) = -88.95$. By calculation, one has

$$C_1 = (k_4 - \lambda_1) + |b - \lambda_1||\lambda_1| + M_1 + c\sum_{j=1}^{n} a_{j1} + \frac{M_2}{1-\sigma} = -25.44 < 0,$$

$$C_2 = (k_4 - \lambda_2) + |b - \lambda_2||\lambda_2| + M_1 + c\sum_{j=1}^{n} a_{j2} + \frac{M_2}{1-\sigma} = -4.80 < 0,$$

$$D_1 = l_4 + 1 + \lambda_1 - b = -11.20 < -\sqrt{l_5^2 + l_6^2} = -8.7321,$$

$$D_2 = l_4 + 1 + \lambda_2 - b = -18 < -\sqrt{l_5^2 + l_6^2} = -8.7321.$$

The figures of $x_i(t), u_i(t), y_i(t), v_i(t)$ and synchronous error $e_i(t), r_(t)$ are plotted in Figures 7–9.

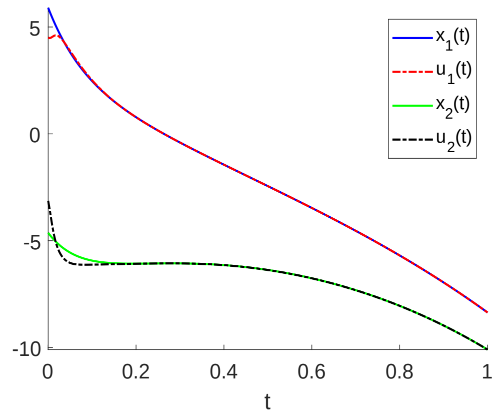

Figure 7. The curves of the $x_1(t), x_2(t), u_1(t), u_2(t)$ in the Example 4.3.

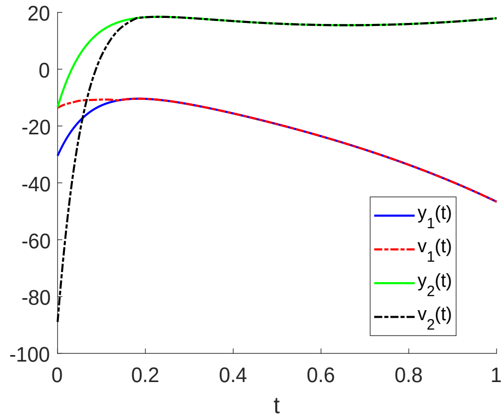

Figure 8. The curves of the $y_1(t), y_2(t), v_1(t), v_2(t)$ in the Example 4.3.

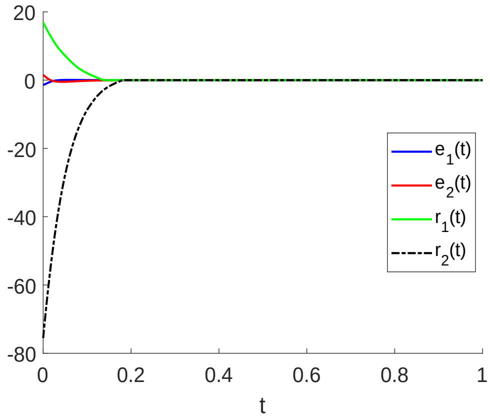

Figure 9. The curves of the $e_1(t), e_2(t), r_1(t), r_2(t)$ in the Example 4.3.

5. Conclusions

In our paper, we are concentrated on the finite-time synchronization of a type of delayed multinonidentical coupled complex dynamical networks. Without using Lyapunov stability theory, graph theory and finite-time stability theorem, by combining the Zero-point theorem approach with novel differential inequality and designing three classes of new controllers. Immediately, we construct three different criteria to guarantee the finite-time synchronization between the drive system and the response system. Our results and method are different from those in existing papers. In the future, we will consider the fixed-time synchronization of delayed multinonidentical coupled complex dynamical networks.

Acknowledgments

We are thankful to the reviewers for their constructive comments which help us to improve the manuscript. This work was supported by Hubei University of Technology Doctoral Research Launch Fund.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. S. Zhu, J. Zhou, X. Yu, J. Lu, Synchronization of complex networks with nondifferentiable time-varying delay, *IEEE Trans. Cybern.*, **52** (2022), 3342–3348. https://doi.org/10.1109/TCYB.2020.3022976
2. X. He, H. Zhang, Exponential synchronization of complex networks via feedback control and periodically intermittent noise, *J. Franklin Inst.*, **359** (2022), 3614–3630. https://doi.org/10.1016/j.jfranklin.2022.03.010

3. S. Zhu, J. Zhou, Q. Zhu, N. Li, J. Lu, Adaptive exponential synchronization of complex networks with nondifferentiable time-varying delay, *IEEE Trans. Neural Networks Learn. Syst.*, 2022. https://doi.org/10.1109/TNNLS.2022.3145843

4. L. Shi, C. Zhang, S. Zhong, Synchronization of singular complex networks with time-varying delay via pinning control and linear feedback control, *Chaos Solitons Fractals*, **145** (2021), 110805. https://doi.org/10.1016/j.chaos.2021.110805

5. X. Liu, Synchronization and control for multiweighted and directed complex networks, *IEEE Trans. Neural Networks Learn. Syst.*, 2021. https://doi.org/10.1109/TNNLS.2021.3110681

6. J. Wang, J. Xia, H. Shen, M. Xing, J. H. Park, H_∞ synchronization for fuzzy markov jump chaotic systems with piecewise-constant transition probabilities subject to PDT switching rule, *IEEE Trans. Fuzzy Syst.*, **29** (2021), 3082–3092. https://doi.org/10.1109/TFUZZ.2020.3012761

7. Y. Wu, B. Shen, C. K. Ahn, W. Li, Intermittent dynamic event-triggered control for synchronization of stochastic complex networks, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **68** (2021), 2639–2650. https://doi.org/10.1109/TCSI.2021.3071034

8. H. Gu, K. Liu, J. Lü, Adaptive PI control for synchronization of complex networks with stochastic coupling and nonlinear dynamics, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **67** (2020), 5268–5280. https://doi.org/10.1109/TCSI.2020.3020146

9. Y. Liu, Z. Wang, L. Ma, Y. Cui, F. E. Alsaadi, Synchronization of directed switched complex networks with stochastic link perturbations and mixed time-delays, *Nonlinear Anal.-Hybrid Syst.*, **27** (2018), 213–224. https://doi.org/10.1016/j.nahs.2017.07.006

10. L. Zhang, X. Yang, C. Xu, J. Feng, Exponential synchronization of complex-valued complex networks with time-varying delays and stochastic perturbations via time-delayed impulsive control, *Appl. Math. Comput.*, **306** (2017), 22–30. https://doi.org/10.1016/j.amc.2017.02.004

11. R. Sakthivel, M. Sathishkumar, B. Kaviarasan, S. M. Anthoni, Synchronization and state estimation for stochastic complex networks with uncertain inner coupling, *Neurocomputing*, **238** (2017), 44–55. https://doi.org/10.1016/j.neucom.2017.01.035

12. A. Hongsri, T. Botmart, W. Weera, P. Junsawang, New delay-dependent synchronization criteria of complex dynamical networks with time-varying coupling delay based on sampled-data control via new integral inequality, *IEEE Access*, **9** (2021), 64958–64971. https://doi.org/10.1109/ACCESS.2021.3076361

13. X. Yi, L. Ren, Z. Zhang, New criteria on global asymptotic synchronization of Duffing-type oscillator system, *Nonlinear Anal. Model. Control*, **25** (2020), 378–399. https://doi.org/10.15388/namc.2020.25.16656

14. D. López-Mancilla, G. López-Cahuich, C. Posadas-Castillo, C. E. Castañeda, J. H. García-López, J. L. Vázquez-Gutiérrez, et al., Synchronization of complex networks of identical and nonidentical chaotic systems via modelmatching control, *PLoS ONE*, **14** (2019), e0216349. https://doi.org/10.1371/journal.pone.0216349

15. K. Sivaranjani, R. Rakkiyappan, J. Cao, A. Alsaedi, Synchronization of nonlinear singularly perturb e d complex networks with uncertain inner coupling via event triggered control, *Appl. Math. Comput.*, **311** (2017), 283–299. https://doi.org/10.1016/j.amc.2017.05.007

16. H. Dai, W. Chen, J. Jia, J. Liu, Z. Zhang, Exponential synchronization of complex dynamical networks with time-varying inner coupling via event-triggered communication, *Neurocomputing*, **245** (2017), 124–132. https://doi.org/10.1016/j.neucom.2017.03.035

17. C. Zhang, X. Wang, C. Wang, Synchronization of complex networks with time-varying inner coupling and outer coupling matrices, *Math. Methods Appl. Sci.*, **40** (2017), 4237–4245. https://doi.org/10.1002/mma.4300

18. R. Sakthivel, R. Sakthivel, F. Alzahrani, P. Selvaraj, S. M. Anthoni, Synchronization of complex dynamical networks with random coupling delay and actuator faults, *ISA Trans.*, **94** (2019), 57–69. https://doi.org/10.1016/j.isatra.2019.03.029

19. Y. Wu, R. Lu, P. Shi, H. Su, Z. Wu, Sampled-data synchronization of complex networks with partial couplings and T-S fuzzy nodes, *IEEE Trans. Fuzzy Syst.*, **26** (2018), 782–793. https://doi.org/10.1109/TFUZZ.2017.2688490

20. X. Yang, X. Li, J. Lu, Z. Cheng, Synchronization of time-delayed complex networks with switching topology via hybrid actuator fault and impulsive effects control, *IEEE Trans. Cybern.*, **50** (2020), 4043–4052. https://doi.org/10.1109/TCYB.2019.2938217

21. Y. Bao, Y. Zhang, Synchronization of complex networks with memristive neural network nodes via impulsive control, in *2019 Chinese Control And Decision Conference (CCDC)*, 2019, 2355–2360. https://doi.org/10.1109/CCDC.2019.8833419

22. X. Yang, J. Lu, D. W. C. Ho, Q. Song, Synchronization of uncertain hybrid switching and impulsive complex networks, *Appl. Math. Model.*, **59** (2018), 379–392. https://doi.org/10.1016/j.apm.2018.01.046

23. X. Yao, Y. Liu, Z. Zhang, W. Wan, Synchronization rather than finite-time synchronization results of fractional-order multi-weighted complex networks, *IEEE Trans. Neural Networks Learn. Syst.*, 2021. https://doi.org/10.1109/TNNLS.2021.3083886

24. V. K. Yadav, V. K. Shukla, S. Das, Exponential synchronization of fractional-order complex chaotic systems and its application, *Chaos Solitons Fractals*, **147** (2021), 110937. https://doi.org/10.1016/j.chaos.2021.110937

25. Y. Yang, C. Hua, J. Yu, H. Jiang, S. Wen, Synchronization of fractional-order spatiotemporal complex networks with boundary communication, *Neurocomputing*, **450** (2021), 197–207. https://doi.org/10.1016/j.neucom.2021.04.008

26. L. Li, X. Liu, M. Tang, S. Zhang, X. Zhang, Asymptotical synchronization analysis of fractional-order complex neural networks with non-delayed and delayed couplings, *Neurocomputing*, **445** (2021), 180–193. https://doi.org/10.1016/j.neucom.2021.03.001

27. Y. Xu, Q. Wang, W. Li, J. Feng, Stability and synchronization of fractional-order delayed multilink complex networks with nonlinear hybrid couplings, *Math. Methods Appl. Sci.*, **44** (2021), 3356–3375. https://doi.org/10.1002/mma.6946

28. H. Li, J. Cao, C. Hu, L. Zhang, Z. Wang, Global synchronization between two fractional-order complex networks with non-delayed and delayed coupling via hybrid impulsive control, *Neurocomputing*, **356** (2019), 31–39. https://doi.org/10.1016/j.neucom.2019.04.059

29. Z. Hu, H. Ren, P. Shi, Synchronization of complex dynamical networks subject to noisy sampling interval and packet loss, *IEEE Trans. Neural Networks Learn. Syst.*, **33** (2022), 3216–3226. https://doi.org/10.1109/TNNLS.2021.3051052

30. J. Feng, L. Zhang, J. Wang, Y. Zhao, The synchronization of complex dynamical networks with discontinuous dynamics and exogenous disturbances, *Asian J. Control*, **23** (2021), 2837–2848. https://doi.org/10.1002/asjc.2414

31. B. Rehák, V. Lynnyk, Synchronization of symmetric complex networks with heterogeneous time delays, in *2019 22nd International Conference on Process Control (PC19)*, 2019, 68–73. https://doi.org/10.1109/PC.2019.8815036

32. A. Kazemy, K. Shojaei, Synchronization of complex dynamical networks with dynamical behavior links, *Asian J. Control*, **22** (2020), 474–485. https://doi.org/10.1002/asjc.1910

33. J. Zhang, J. Sun, Exponential synchronization of complex networks with continuous dynamics and Boolean mechanism, *Neurocomputing*, **307** (2018), 146–152. https://doi.org/10.1016/j.neucom.2018.03.061

34. M. A. Alamin Ahmeda, Y. Liu, W. Zhang, F. E. Alsaadic, Exponential synchronization via pinning adaptive control for complex networks of networks with time delays, *Neurocomputing*, **225** (2017), 198–204. https://doi.org/10.1016/j.neucom.2016.11.022

35. Q. Cui, L. Li, J. Lu, A. Alofi, Finite-time synchronization of complex dynamical networks under delayed impulsive effects, *Appl. Math. Comput.*, **430** (2022), 127290. https://doi.org/10.1016/j.amc.2022.127290

36. H. Zhang, X. Zheng, N. Li, Finite-Time pinning synchronization control for coupled complex networks with time-varying delays, *Discrete Dyn. Nat. Soc.*, **2022** (2022), 7119370. https://doi.org/10.1155/2022/7119370

37. J. Wang, L. Zhao, H. Wu, T. Huang, Finite-time passivity and synchronization of multi-weighted complex dynamical networks under PD control, *IEEE Trans. Neural Networks Learn. Syst.*, 2022. Available from: https://doi.org/10.1109/TNNLS.2022.3175747

38. N. Gunasekaran, M. S. Ali, S. Arik, H. I. Abdul Ghaffar, A. A. Zaki Diab, Finite-time and sampled-data synchronization of complex dynamical networks subject to average dwell-time switching signal, *Neural Networks*, **149** (2022), 137–145. https://doi.org/10.1016/j.neunet.2022.02.013

39. W. Yuan, Y. Ma, Finite-time $\mathcal{H}_\infty$ synchronization for complex dynamical networks with time-varying delays based on adaptive control, *ISA Trans.*, **128** (2021), 109–122. https://doi.org/10.1016/j.isatra.2021.11.018

40. W. Zhang, X. Yang, S. Yang, A. Alsaedi, Finite-time and fixed-time bipartite synchronization of complex networks with signed graphs, *Math. Comput. Simul.*, **188** (2021), 319–329. https://doi.org/10.1016/j.matcom.2021.04.013

41. M. S. Ali, L. Palanisamy, N. Gunasekaran, A. Alsaedi, B. Ahmad, Finite-time exponential synchronization of reaction-diffusion delayed complex-dynamical networks, *Discrete Cont. Dyn.-S*, **14** (2021), 1465–1477. https://doi.org/10.3934/dcdss.2020395

42. X. Li, H. Wub, J. Cao, Synchronization in finite time for variable-order fractional complex dynamic networks with multi-weights and discontinuous nodes based on sliding mode control strategy, *Neural Networks*, **139** (2021), 335–347. https://doi.org/10.1016/j.neunet.2021.03.033
43. J. He, H. Chen, M. Ge, T. Ding, L. Wang, C. Liang, Adaptive finite-time quantized synchronization of complex dynamical networks with quantized time-varying delayed couplings, *Neurocomputing*, **431** (2021), 90–99. https://doi.org/10.1016/j.neucom.2020.12.038
44. Y. Luo, Y. Yao, Z. Cheng, X. Xiao, H. Liu, Event-triggered control for coupled reaction-diffusion complex network systems with finite-time synchronization, *Phys. A*, **562** (2021), 125219. https://doi.org/10.1016/j.physa.2020.125219
45. Y. Ren, H. Jiang, J. Li, B. Lu, Finite-time synchronization of stochastic complex networks with random coupling delay via quantized aperiodically intermittent control, *Neurocomputing*, **420** (2021), 337–348. https://doi.org/10.1016/j.neucom.2020.05.103
46. J. Wang, Z. Qin, H. Wu, T. Huang, Finite-time synchronization and $\mathcal{H}_\infty$ synchronization of multiweighted complex networks with adaptive state couplings, *IEEE Trans. Cybern.*, **50** (2020), 600–612. https://doi.org/10.1109/TCYB.2018.2870133
47. J. Wang, T. Ru, J. Xia, Y. Wei, Z. Wang, Finite-time synchronization for complex dynamic networks with semi-Markov switching topologies: An $\mathcal{H}_\infty$ event-triggered control scheme, *Appl. Math. Comput.*, **356** (2019), 235–251. https://doi.org/10.1016/j.amc.2019.03.037
48. H. Li, J. Cao, H. Jiang, A. Alsaedi, Finite-time synchronization and parameter identification of uncertain fractional-order complex networks, *Phys. A*, **533** (2019), 122027. https://doi.org/10.1016/j.physa.2019.122027
49. X. Liu, D. W. C. Ho, Q. Song, W. Xu, Finite/Fixed-Time Pinning Synchronization of Complex Networks With Stochastic Disturbances, *IEEE Trans. Cybern.*, **49** (2019), 2398–2403. https://doi.org/10.1109/TCYB.2018.2821119
50. W. Jiang, L. Li, Z. Tu, Y. Feng, Semiglobal finite-time synchronization of complex networks with stochastic disturbance via intermittent control, *Int. J. Robust Nonlinear Control*, **29** (2019), 2351–2363. https://doi.org/10.1002/rnc.4496
51. H. Li, J. Cao, H. Jiang, A. Alsaedi, Finite-time synchronization of fractional-order complex networks via hybrid feedback control, *Neurocomputing*, **320** (2018), 69–75. https://doi.org/10.1016/j.neucom.20 18.09.021
52. G. Ji, C. Hu, J. Yu, H. Jiang, Finite-time and fixed-time synchronization of discontinuous complex networks: A unified control framework design, *J. Franklin Inst.*, **355** (2018), 4665–4685. https://doi.org/10.1016/j.jfranklin.2018.04.026
53. D. Zhang, Y. Shen, J. Mei, Finite-time synchronization of multi-layer nonlinear coupled complex networks via intermittent feedback control, *Neurocomputing*, **225** (2017), 129–138. https://doi.org/10.1016/j.neucom.2016.11.005
54. N. Gunasekaran, M. S. Ali, S. Arik, H. I. Abdul Ghaffar, A. A. Zaki Diab, Finite-time and sampled-data synchronization of complex dynamical networks subject to average dwell-time switching signal, *Neural Networks*, **149** (2022), 137–145. https://doi.org/10.1016/j.neunet.2022.02.013

55. N. Gunasekaran, G. Zhai, Q. Yu, Sampled-data synchronization of delayed multi-agent networks and its application to coupled circuit, *Neurocomputing*, **413** (2020), 499–511. https://doi.org/10.1016/j.neucom.2020.05.060

56. M. Thiele, R. Berner, P. A. Tass, E. Schöll, S. Yanchuk, Asymmetric Adaptivity induces Recurrent Synchronization in Complex Networks, preprint, arXiv:2112.08697.

57. N. Li, X. Wu, J. Feng, J. Lü, Fixed-Time Synchronization of Complex Dynamical Networks: A Novel and Economical Mechanism, *IEEE Trans. Cybern.*, **8** (2022), 33072–33079. https://doi.org/10.1109/ACCESS.2020.2970789

58. C. Hu, H. He, H. Jiang, Fixed/Preassigned-time synchronization of complex networks via improving fixed-time stability, *IEEE Trans. Cybern.*, **51** (2021), 2882–2892. https://doi.org/10.1109/TCYB.2020.2977934

59. H. Shen, X. Hu, J. Wang, J. Cao, W. Qian, Non-Fragile H_∞ synchronization for markov jump singularly perturbed coupled neural networks subject to double-layer switching regulation, *IEEE Trans. Neural Networks Learn. Syst.*, 2021. Available from: https://doi.org/10.1109/TNNLS.2021.3107607

60. L. Zhang, J. Zhong, J. Lu, Intermittent control for finite-time synchronization of fractional-order complex networks, *Neural Networks*, **144** (2021), 11–20. https://doi.org/10.1016/j.neunet.2021.08.004

61. Z. Zhang, J. Cao, Novel finite-time synchronization criteria for inertial neural networks with time delays via integral inequality method, *IEEE Trans. Neural Networks Learn. Syst.*, **30** (2019), 1476–1485. https://doi.org/10.1109/TNNLS.2018.2868800

62. Z. Zhang, M. Chen, A. Li, Further study on finite-time synchronization for delayed inertial neural networks via inequality skills, *Neurocomputing*, **373** (2020), 15–23. https://doi.org/10.1016/j.neucom.2019.09.034

63. Z. Zhang, J. Cao, Finite-time synchronization for fuzzy inertial neural networks by maximum-value approach, *IEEE Trans. Fuzzy Syst.*, **30** (2022), 1436–1446. https://doi.org/10.1109/TFUZZ.2021.3059953

64. L. Wang, T. Chen, Finite-time anti-synchronization of neural networks with time-varying delays, *Neurocomputing*, **275** (2018), 1595–1600. https://doi.org/10.1016/j.neucom.2017.09.097

65. Z. Zhang, T. Zheng, S. Yu, Finite-time anti-synchronization of neural networks with time-varying delays via inequality skills, *Neurocomputing*, **356** (2019), 60–68. https://doi.org/10.1016/j.neucom.2019.05.012

66. Z. Zhang, A. Li, S. Yu, Finite-time synchronization for delayed complex-valued neural networks via integrating inequality method, *Neurocomputing*, **318** (2018), 248–260. https://doi.org/10.1016/j.neucom.2018.08.063

AIMS Press