Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(2): 2998–3022.
DOI: 10.3934/mbe.2023142
Received: 05 October 2022
Revised: 07 November 2022
Accepted: 12 November 2022
Published: 01 December 2022

Research article

Spatial decay bound and structural stability for the double-diffusion perturbation equations

Yuanfei Li* **and Xuejiao Chen**

School of Data Science, Guangzhou Huashang College, Guangdong 511300, China

* **Correspondence:** Email: liqfd@163.com, A10314063@163.com.

Abstract: In this paper, we study the double-diffusion perturbation equations when the flow is through a porous medium. If the initial conditions satisfy some constraint conditions, the Saint-Venant type spatial decay of solutions for double-diffusion perturbation equations is obtained. Based on the spatial decay bound, the structural stability for the double-diffusion perturbation equations is also established.

Keywords: perturbation equations; spatial decay; Saint-Venant type; structural stability

1. Introduction

The asymptotic behavior or norm of solutions to initial-boundary value problems of various partial differential equations has received attention for many years, and a large number of results have been obtained (see [1–10]). These results can be regarded as a study of the Saint-Venant principle type which has been commonly used in engineering mechanics.

We note that Payne and Song [11] studied the Forchheimer equation

$$\begin{aligned} b|\boldsymbol{u}|\boldsymbol{u} + (1+\gamma T)\boldsymbol{u} + \nabla p - \boldsymbol{g}T &= 0, \\ \nabla\cdot\boldsymbol{u} &= 0, \\ T_t + \boldsymbol{u}\cdot\nabla T - \Delta T &= 0, \\ |\boldsymbol{u}|, |T| = O(1), |u_3|, |\nabla T|, |p| = o(x_3^{-1}), as\ x_3 &\to \infty, \end{aligned}$$

where $\boldsymbol{u} = (u_1, u_2, u_3), T$ and p denote the velocity, temperature and pressure in the semi-infinite pipe, respectively. b and γ are positive constants. The vector $\boldsymbol{g} = (g_1, g_2, g_3)$ represents a gravity field. Using the maximum principle, they first got the maximum value of the temperature and then obtained the exponential decay result of the solutions. Using a similar method, other studies that obtained the desired results can be seen in [12–17].

In this paper, we consider the following cylindrical domain

$$\Omega = \Big\{(x_1, x_2, x_3)\Big|(x_1, x_2) \in D,\ x_3 > 0\Big\},$$

where D is a bounded simply connected region in the (x_1, x_2)-plane with the piecewise smooth boundary ∂D. We let $D(z)$ denote the cross-section of Ω at $x_3 = z$, i.e.,

$$D(z) = \Big\{(x_1, x_2, x_3)\Big|(x_1, x_2) \in D,\ x_3 = z > 0\Big\}.$$

We present the non-dimensional perturbation equations in the form of (see [18, 19])

$$u_i + RTk_i - p_{,i} - Ck_i\varphi = 0,\ in\ \Omega \times (0, t), \tag{1.1}$$
$$u_{i,i} = 0,\ in\ \Omega \times (0, t) \tag{1.2}$$
$$T_t + u_iT_{,i} = u_3 + \Delta T,\ in\ \Omega \times (0, t) \tag{1.3}$$
$$\epsilon_1\varphi_t + Leu_i\varphi_{,i} = u_3 + \Delta\varphi,\ in\ \Omega \times (0, t), \tag{1.4}$$

with the following initial-boundary conditions

$$u_i = 0,\ T = \varphi = 0,\ \ on\ \partial D \times \{x_3 > 0\} \times (0, t), \tag{1.5}$$
$$u_i = f_i(x_1, x_2, t),\ T = h(x_1, x_2, t), \varphi = H(x_1, x_2, t),\ \ on\ D \times (0, t), \tag{1.6}$$
$$T = \varphi = 0,\ \ in\ \Omega \times \{t = 0\}, \tag{1.7}$$
$$|\boldsymbol{u}|, |T|, |\varphi| = O(1), |u_3|, |\nabla T|, |\nabla\varphi|, |p| = o(x_3^{-1}), as\ x_3 \to \infty. \tag{1.8}$$

Here φ is the concentration perturbation, Le is the Lewis number, $\epsilon_1 = \epsilon Le$ (ϵ is the porosity), R and C are the Rayleigh and salt Rayleigh numbers, respectively, and $\boldsymbol{k} = (k_1, k_2, k_3) = (0, 0, 1)$. The prescribed functions $\boldsymbol{f} = (f_1, f_2, f_3)$, h and H are continuously differentiable and f_i satisfies the constraint condition

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0.$$

It can be seen from Eqs (1.3) and (1.4) that we cannot obtain the maximum values of temperature and concentration perturbations as in the previous studies. Therefore, the spatial decay results we derive will not be obtained by using the previous methods. We must adopt a new method to overcome the difficulty of being able to obtain the maximum value of the temperature. To do this, we derive the L^4 norms of T and φ. There is no Laplacian term in Eq (1.1), so we need to use the Soblev inequality to derive a nonlinear differential inequality about the L^2 norms of the the velocity and its gradient.

The second aim is to study the structural stability of solutions of Eqs (1.1)–(1.8). The concept of structural stability was first proposed by Hirsch and Smale [20]. Structural stability of this type involves studying whether a small change in a coefficient in the equations will induce a dramatic change in the solution. There is a large number of studies that investigated have studied the structural stability of various types of partial differential equations. Scott [12] considered a porous medium of Darcy type and obtained the continuous dependence on boundary reaction terms. Considering the simultaneous existence of multiple fluids in a bounded region, Li et al. [21] obtained the structural stability of resonant penetrative convection in a Brinkman-Forchheimer fluid interfacing with a Darcy fluid. Liu et al. [22] assumed that Boussinesq fluid interfaced with a Darcy fluid in a bounded region in $\mathbb{R}^2$, and they obtained the continuous dependence on the interface parameters. For more papers one can refer to [23–30]. Clearly, most of the above articles mainly focused on the structural stability of solutions on bounded regions. The innovation of this work is to extend the study of structural stability in a bounded region to a semi-infinite cylinder. In this work, we still need the prior bounds of the L^4 norm of T and φ, the H^1 norm of u, and the spatial decay estimate of solutions.

We declare that in Eqs (1.1)–(1.8) and in the whole paper, the usual summation convention is employed with repeated Latin subscripts summed from 1 to 3 and repeated Greek alphabet summed from 1 to 2. The comma is used to indicate partial differentiation, e.g., $u_{i,j}u_{i,j} = \Sigma_{i,j=1}^{3}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2$ and $u_{\alpha,\beta}u_{\alpha,\beta} = \Sigma_{\alpha,\beta=1}^{2}\Big(\frac{\partial u_\alpha}{\partial x_\beta}\Big)^2$.

The plan of the paper is as follows. In the next section, we derive the a priori bounds for T, φ and u in the region Ω. The third section is devoted to deriving the spatial decay bound for the solution. In Section 4, we prove the continuous dependence on the coefficients R and C. In the Section 5, a convergence result is proved for Eqs (1.1)–(1.8) when $(R, C) \to (0, 0)$.

2. Preliminaries of the problem

To obtain the main result, we shall make frequent use of the following inequalities.

Lemma 2.1 (see [11,31]) If $v\Big|_{\partial D} = 0$, then

$$\lambda_1 \int_D |v|^2 dA \leq \int_D v_{,\alpha}v_{,\alpha}dA,$$

where λ_1 is the smallest positive eigenvalue of

$$\Delta_2\vartheta + \lambda\vartheta = 0, in\ D,\ \ \vartheta = 0,\ \ on\ \partial D.$$

Lemma 2.2 (see [32,33]) If $v_i \in C^1(D \times (0, \infty))$ and v_i vanishes on ∂D for $x_3 \geq z$ and if v_i vanishes as $x_3 \to \infty$, then

$$\int_z^\infty \int_{D(\xi)} (v_iv_i)^3 dAd\xi d\eta \leq k_1\Big[\int_z^\infty \int_{D(\xi)} v_{i,j}v_{i,j}dAd\xi\Big]^3,$$

where $k_1 = \frac{1}{27}\Big(\frac{3}{4}\Big)^4$.

We can prove the following lemmas.

Lemma 2.3 If $\int_D f_\alpha dA = 0, f_i \in L^2(D)$, then

$$\int_D u_3pdA \leq n_1(t),$$

where $n_1(t)$ is a positive computable function.

Proof. From Eq (1.1), we can have

$$u_\alpha(x_1,x_2,x_3,t)-p_{,\alpha}(x_1,x_2,x_3,t)=0 \tag{2.1}$$

for $\alpha=1,2$. Since $\int_D f_\alpha dA=0$, we integrate Eq (2.1) on D to obtain

$$p|_{\partial D}=\int_D p_{,\alpha}dA=\int_D u_\alpha dA=\int_D f_\alpha dA=0.$$

Therefore, using Lemma 2.1 we obtain

$$\int_D u_3 p dA \le \Big[\int_D f_3^2 dA\int_D p^2 dA\Big]^{\frac{1}{2}} \le \frac{1}{\sqrt{\lambda_1}}\Big[\int_D f_3^2 dA\int_D p_{,\alpha}p_{,\alpha}dA\Big]^{\frac{1}{2}}. \tag{2.2}$$

In light of Eq (2.1), from Eq (2.2) we can obtain

$$\int_D u_3 p dA \le \frac{1}{2\sqrt{\lambda_1}}\int_D f_i f_i dA. \tag{2.3}$$

Choosing $n_1(t)=\frac{1}{2\sqrt{\lambda_1}}\int_D f_i f_i dA$, from Eq (2.3) we can have Lemma 2.3.
Lemma 2.4 Letting

$$\widehat{h}(x_1,x_2,x_3,t)=h(x_1,x_2,t)e^{-\sigma_1 x_3},\ \sigma_1>0 \tag{2.4}$$

and assuming $h\in L^2(\Omega\times(0,t))$, then

$$\begin{aligned}-\int_0^t\int_{D(0)} e^{-\omega\eta}TT_{,3}dAd\eta \le{}& n_2(t)+\varepsilon_1\int_0^t\int_\Omega e^{-\omega\eta}T_{,i}T_{,i}dAd\xi d\eta\\ &+\varepsilon_2 e^{-\omega t}\int_\Omega T^2 dAd\xi+(\varepsilon_3+\frac{1}{4\varepsilon_4}L_M^2)\int_0^t\int_\Omega e^{-\omega\eta}T^2 dAd\xi d\eta\\ &+\varepsilon_4\int_0^t\int_\Omega e^{-\omega\eta}u_iu_i dAd\xi d\eta+\varepsilon_5\int_0^t\int_\Omega e^{-\omega\eta}u_3^2 dAd\xi d\eta,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}-\int_0^t\int_{D(0)} e^{-\omega\eta}T^3T_{,3}dAd\eta \le{}& n_2'(t)+\varepsilon_1'\int_0^t\int_\Omega e^{-\omega\eta}T_{,i}T_{,i}dAd\xi d\eta\\ &+\varepsilon_2' e^{-\omega t}\int_\Omega T^2 dAd\xi+(\varepsilon_3'+\frac{1}{4\varepsilon_4'}L_M^6)\int_0^t\int_\Omega e^{-\omega\eta}T^2 dAd\xi d\eta\\ &+\varepsilon_4'\int_0^t\int_\Omega e^{-\omega\eta}u_iu_i dAd\xi d\eta+\varepsilon_5'\int_0^t\int_\Omega e^{-\omega\eta}u_3^2 dAd\xi d\eta,\end{aligned} \tag{2.6}$$

where $L_M=\sup_{\Omega\times(0,t)}\{|\nabla h(x_1,x_2,x_3,t)|,|\nabla H(x_1,x_2,x_3,t)|\}$, $n_2(t)$ and $n_2'(t)$ are computable functions and $\omega,\varepsilon_i,\varepsilon_i'(i=1,2,...,5)$ are positive constants.

Proof. From the definition of $\widehat{h}$, we can conclude that $\widehat{h}$ has the same initial-boundary conditions as T. Therefore, we have

$$
\begin{aligned}
-\int_0^t\int_{D(0)} e^{-\omega\eta}TT_{,3}dAd\eta &= -\int_0^t\int_{D(0)} e^{-\omega\eta}\widehat{h}T_{,3}dAd\eta = \int_0^t\int_{\Omega} e^{-\omega\eta}(\widehat{h}T_{,i})_{,i}dAd\xi d\eta \\
&= \int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_{,i}T_{,i}dAd\xi d\eta + \int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}(T_\eta + u_iT_{,i} - u_3)dAd\xi d\eta \\
&= \int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_{,i}T_{,i}dAd\xi d\eta + e^{-\omega t}\int_{\Omega}\widehat{h}TdAd\xi \\
&+ \omega\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}TdAd\xi d\eta - \int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_\eta TdAd\xi d\eta \\
&- \int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_{,i}u_iTdAd\xi d\eta - \int_0^t\int_{\Omega}\widehat{h}u_3dAd\xi d\eta \\
&- \int_0^t\int_{\Omega} e^{-\omega\eta}h^2f_3dAd\xi d\eta \\
&\doteq A_1(t) + A_2(t) + A_3(t) + A_4(t) + A_5(t) + A_6(t) \\
&- \int_0^t\int_{\Omega} e^{-\omega\eta}h^2f_3dAd\xi d\eta. \qquad (2.7)
\end{aligned}
$$

By the Hölder inequality and the Young inequality, we obtain

$$A_1(t) \le \varepsilon_1\int_0^t\int_{\Omega} e^{-\omega\eta}T_{,i}T_{,i}dAd\xi d\eta + \frac{1}{4\varepsilon_1}\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_{,i}\widehat{h}_{,i}dAd\xi d\eta, \qquad (2.8)$$

$$A_2(t) \le \varepsilon_2 e^{-\omega t}\int_{\Omega}T^2dAd\xi + \frac{1}{4\varepsilon_2}e^{-\omega t}\int_{\Omega}\widehat{h}^2dAd\xi, \qquad (2.9)$$

$$A_3(t) \le \frac{1}{2}\varepsilon_3\int_0^t\int_{\Omega} e^{-\omega\eta}T^2dAd\xi d\eta + \frac{1}{2\varepsilon_3}\omega^2\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}^2dAd\xi d\eta, \qquad (2.10)$$

$$A_4(t) \le \frac{1}{2}\varepsilon_3\int_0^t\int_{\Omega} e^{-\omega\eta}T^2dAd\xi d\eta + \frac{1}{2\varepsilon_3}\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_\eta^2dAd\xi d\eta, \qquad (2.11)$$

$$A_5(t) \le \varepsilon_4\int_0^t\int_{\Omega} e^{-\omega\eta}u_iu_idAd\xi d\eta + \frac{1}{4\varepsilon_4}L_M^2\int_0^t\int_{\Omega} e^{-\omega\eta}T_{,i}T_{,i}dAd\xi d\eta, \qquad (2.12)$$

$$A_6(t) \le \varepsilon_5\int_0^t\int_{\Omega} e^{-\omega\eta}u_3^2dAd\xi d\eta + \frac{1}{4\varepsilon_5}\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}^2dAd\xi d\eta. \qquad (2.13)$$

Inserting Eqs (2.8)–(2.13) into Eq (2.7) and choosing

$$
\begin{aligned}
n_3(t) &= \frac{1}{4\varepsilon_1}\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_{,i}\widehat{h}_{,i}dAd\xi d\eta + \frac{1}{4\varepsilon_2}e^{-\omega t}\int_{\Omega}\widehat{h}^2dAd\xi \\
&+ \frac{1}{4\varepsilon_3}\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}_\eta^2dAd\xi d\eta + (\frac{1}{2\varepsilon_3}\omega^2 + \frac{1}{4\varepsilon_5})\int_0^t\int_{\Omega} e^{-\omega\eta}\widehat{h}^2dAd\xi d\eta \\
&- \int_0^t\int_{\Omega} e^{-\omega\eta}h^2f_3dAd\xi d\eta, \qquad (2.14)
\end{aligned}
$$

we can obtain Eq (2.5). Using a similar method, we can also obtain Eq (2.6).

If we let

$$\widehat{H}(x_1,x_2,x_3,t)=H(x_1,x_2,t)e^{-\sigma_2 x_3},\sigma_2>0, \tag{2.15}$$

we can obtain the following lemma.

Lemma 2.5 Assuming $H\in L^\infty(\Omega\times(0,t))$, then

$$\begin{aligned}-\int_0^t\int_{D(0)}e^{-\omega\eta}\varphi\varphi_{,3}dAd\eta\le n_3(t)&+(\delta_1+\frac{1}{4\delta_4}L_M^2)\int_0^t\int_\Omega e^{-\omega\eta}\varphi_{,i}\varphi_{,i}dAd\xi d\eta\\&+\delta_2e^{-\omega t}\int_\Omega\varphi^2dAd\xi+\delta_3\int_0^t\int_\Omega e^{-\omega\eta}\varphi^2dAd\xi d\eta\\&+\delta_4\int_0^t\int_\Omega e^{-\omega\eta}u_iu_idAd\xi d\eta+\delta_5\int_0^t\int_\Omega e^{-\omega\eta}u_3^2dAd\xi d\eta,\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}-\int_0^t\int_{D(0)}e^{-\omega\eta}\varphi^3\varphi_{,3}dAd\eta\le n_3'(t)&+(\delta_1'+\frac{1}{4\delta_4'}L_M^2)\int_0^t\int_\Omega e^{-\omega\eta}\varphi_{,i}\varphi_{,i}dAd\xi d\eta\\&+\delta_2'e^{-\omega t}\int_\Omega\varphi^2dAd\xi+\delta_3'\int_0^t\int_\Omega e^{-\omega\eta}\varphi^2dAd\xi d\eta\\&+\delta_4'\int_0^t\int_\Omega e^{-\omega\eta}u_iu_idAd\xi d\eta+\delta_5'\int_0^t\int_\Omega e^{-\omega\eta}u_3^2dAd\xi d\eta,\end{aligned} \tag{2.17}$$

where $n_3(t)$ and $n_3'(t)$ are computable functions and $\delta_i,\delta_i'(i=1,2,...,5)$ are positive constants.

Now, we multiply Eq (1.1) by u_i, and integrate in $[z,\infty)\times D(\xi)$ to have

$$\int_z^\infty\int_{D(\xi)}\Big[u_i+RTk_i-p_{,i}-Ck_i\varphi\Big]u_idAd\xi=0.$$

Therefore, we have

$$\begin{aligned}\int_z^\infty\int_{D(\xi)}u_iu_idAd\xi=-R&\int_z^\infty\int_{D(\xi)}Tu_3dAd\xi\\&+C\int_z^\infty\int_{D(\xi)}\varphi u_3dAd\xi-\int_{D(z)}pu_3dA.\end{aligned} \tag{2.18}$$

Using the Schwarz inequality, we have

$$\begin{aligned}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}u_iu_idAd\xi d\eta\le&-R\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}Tu_3dAd\xi d\eta+\frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}u_3^2dAd\xi d\eta\\&+\frac{1}{2}C^2\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}\varphi^2dAd\xi d\eta-\int_0^t\int_{D(z)}e^{-\omega\eta}pu_3dAd\eta,\end{aligned}$$

or

$$\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_iu_i dAd\xi d\eta \le -2R\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}Tu_3 dAd\xi d\eta + C^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\varphi^2 dAd\xi d\eta - \int_0^t\int_{D(z)} e^{-\omega\eta}pu_3 dAd\eta. \quad (2.19)$$

To obtain the bound for $\|\boldsymbol{u}\|^2_{L^2(\Omega\times(0,t))}$, we have to seek bounds for $\|T\|^2_{L^2(\Omega\times(0,t))}$ and $\|\varphi\|^2_{L^2(\Omega\times(0,t))}$. To do this, we multiply Eq (1.3) by $e^{-\omega\eta}T$, and integrate in $[z,\infty)\times D(\xi)\times(0,t)$ to have

$$\begin{aligned}\frac{1}{2}e^{-\omega t}\int_z^\infty\int_{D(\xi)} T^2 dAd\xi &+ \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[\frac{1}{2}\omega T^2 + T_{,i}T_{,i}\Big]dAd\xi d\eta \\ &= \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}Tu_3 dAd\xi d\eta \\ &+ \frac{1}{2}\int_0^t\int_{D(z)} u_3T^2 dAd\eta - \int_0^t\int_{D(z)} e^{-\omega\eta}TT_{,3} dAd\eta. \end{aligned}\quad (2.20)$$

Similarly, we obtain

$$\begin{aligned}\frac{1}{2}e^{-\omega t}\epsilon_1\int_z^\infty\int_{D(\xi)} \varphi^2 dAd\xi &+ \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[\frac{1}{2}\epsilon_1\omega\varphi^2 + \varphi_{,i}\varphi_{,i}\Big]dAd\xi d\eta \\ &\le \frac{1}{4}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3^2 dAd\xi d\eta + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\varphi^2 dAd\xi d\eta \\ &+ \frac{1}{2}Le\int_0^t\int_{D(z)} e^{-\omega\eta}u_3\varphi^2 dAd\eta - \int_0^t\int_{D(z)} e^{-\omega\eta}\varphi\varphi_{,3} dAd\eta. \end{aligned}\quad (2.21)$$

Combining Eqs (2.19)–(2.21) with $z = 0$, using Lemma 2.3 and Eqs (2.5), (2.16) and (1.6) and choosing

$$\varepsilon_1 = \varepsilon_2 = \delta_1 = \frac{1}{2}, \varepsilon_3 = \delta_3 = 1, \varepsilon_4 = \varepsilon_5 = \frac{1}{32R}, \delta_4 = \delta_5 = \frac{1}{16}, \omega > \max\{4 + \frac{1}{\varepsilon_4}L_M^2, \frac{4C^2}{\epsilon_1} + \frac{4}{\epsilon_1} + \frac{1}{\epsilon_1\delta_4}L_M^2\},$$

we can obtain the following lemma.

Lemma 2.6 If $h, H \in L^\infty(\Omega\times(0,t))$, then

$$\begin{aligned} e^{-\omega t}R\int_\Omega T^2 dAd\xi + e^{-\omega t}\epsilon_1\int_\Omega \varphi^2 dAd\xi \\ &+ \int_0^t\int_\Omega e^{-\omega\eta}\Big[u_iu_i + R\omega T^2 + \frac{1}{2}\epsilon_1\omega\varphi^2 + 2RT_{,i}T_{,i} + \varphi_{,i}\varphi_{,i}\Big]dAd\xi d\eta \\ &\le n_4(t), \end{aligned}$$

where

$$n_4(t) = 4\int_0^t e^{-\omega\eta}n_1(\eta)d\eta + 4Rn_32(t) + 2n_3(t)$$

$$+ 2R\int_0^t\int_{D(0)} f_3h^2dAd\eta + Le\int_0^t\int_{D(0)} e^{-\omega\eta}f_3H^2dAd\eta. \tag{2.22}$$

Now using the Schwarz inequality and Lemmas 2.2 and 2.6 in Eq (2.18), we have

$$\begin{aligned}\int_\Omega u_iu_idAd\xi &\le 2R^2\int_\Omega T^2dAd\xi + 2C^2\int_\Omega \varphi^2dAd\xi + n_1(t)\\ &\le 2\max\Big\{R, \frac{C^2}{\epsilon_1}\Big\}e^{\omega t}n_4(t) + n_1(t).\end{aligned} \tag{2.23}$$

To get our main result, we also need the bound for $\|\nabla \boldsymbol{u}\|_{L^2(\Omega\times(0,t))}$. We note that

$$\int_0^\infty\int_{D(\xi)} u_{i,j}u_{i,j}dAd\xi = \int_0^\infty\int_{D(\xi)} (u_{i,j}-u_{j,i})u_{i,j}dAd\xi + \int_0^\infty\int_{D(\xi)} u_{i,j}u_{j,i}dAd\xi. \tag{2.24}$$

Using the divergence theorem and Eqs (1.2) and (1.5), we have

$$-\int_z^\infty\int_\zeta^\infty\int_{D(\xi)} u_{i,j}u_{j,i}dAd\xi d\zeta = \int_z^\infty\int_{D(\zeta)} u_{3,j}u_jdAd\zeta = -\int_{D(z)} u_3^2dA.$$

Therefore, we obtain

$$\int_z^\infty\int_{D(\xi)} u_{i,j}u_{j,i}dAd\xi = -\int_{D(z)} u_3u_{3,3}dA. \tag{2.25}$$

Now letting

$$F_i(x_1,x_2,x_3,t) = f_i(x_1,x_2,t)e^{-\sigma_3x_3},\ \sigma_3>0,$$

we can know that F_i has the same boundary condition as u_i. Using Eq (1.6), from Eq (2.25) we obtain

$$\begin{aligned}\int_0^\infty\int_{D(\xi)} u_{i,j}u_{j,i}dAd\xi d\zeta &= -\int_D F_3u_{3,3}dA = \int_0^\infty\int_{D(\xi)}(F_3u_{i,3})_{,i}dAd\xi\\ &= \int_0^\infty\int_{D(\xi)} F_{3,i}u_{i,3}dAd\xi\\ &\le \frac{1}{4}\int_0^\infty\int_{D(\xi)} u_{i,3}u_{i,3}dAd\xi + \int_0^\infty\int_{D(\xi)} F_{3,i}F_{3,i}dAd\xi.\end{aligned} \tag{2.26}$$

For the first term on the right of Eq (2.24), we can compute

$$\begin{aligned}\int_0^\infty\int_{D(\xi)}(u_{i,j}-u_{j,i})u_{i,j}dAd\xi &= -R\int_0^\infty\int_{D(\xi)}(T_{,j}u_{3,j}-T_{,i}u_{i,3})dAd\xi\\ &\quad + C\int_0^\infty\int_{D(\xi)}(\varphi_{,j}u_{3,j}-\varphi_{,i}u_{i,3})dAd\xi\\ &\le \frac{1}{8}\int_0^\infty\int_{D(\xi)} u_{3,j}u_{3,j}dAd\xi + \frac{1}{8}\int_0^\infty\int_{D(\xi)} u_{i,3}u_{i,3}dAd\xi\\ &\quad + 8R^2\int_0^\infty\int_{D(\xi)} T_{,j}T_{,j}dAd\xi + 8C^2\int_0^\infty\int_{D(\xi)}\varphi_{,j}\varphi_{,j}dAd\xi.\end{aligned} \tag{2.27}$$

Inserting Eqs (2.26) and (2.27) into Eq (2.24), we obtain

$$\int_0^\infty \int_{D(\xi)} u_{i,j}u_{i,j}dAd\xi \le 16R^2 \int_0^\infty \int_{D(\xi)} T_{,j}T_{,j}dAd\xi + 16C^2 \int_0^\infty \int_{D(\xi)} \varphi_{,j}\varphi_{,j}dAd\xi + 2\int_0^\infty \int_{D(\xi)} F_{3,i}F_{3,i}dAd\xi, \tag{2.28}$$

or

$$\begin{aligned}\int_0^t \int_\Omega e^{-\omega\eta}u_{i,j}u_{i,j}dAd\xi d\eta &\le 16R^2 \int_0^t \int_\Omega e^{-\omega\eta}T_{,j}T_{,j}dAd\xi \\ &+ 16C^2 \int_\Omega e^{-\omega\eta}\varphi_{,j}\varphi_{,j}dAd\xi d\eta \\ &+ 2\int_0^t \int_\Omega e^{-\omega\eta}F_{3,i}F_{3,i}dAd\xi d\eta \\ &\le 8\max\{R, 2C^2\}n_4(t) + 2\int_0^t \int_\Omega e^{-\omega\eta}F_{3,i}F_{3,i}dAd\xi d\eta. \end{aligned} \tag{2.29}$$

We summarize the above results as the following lemma.

Lemma 2.7 If $\boldsymbol{f} \in H^1(\Omega \times (0,t))$ and $h, H \in L^\infty(\Omega \times (0,t))$, then

$$\int_\Omega u_iu_idAd\xi \le n_5(t), \int_0^t \int_\Omega e^{-\omega\eta}u_{i,j}u_{i,j}dAd\xi d\eta \le n_6(t),$$

where $n_5(t) = 2\max\{R, \frac{C^2}{\epsilon_1}\}e^{\omega t}n_4(t) + n_1(t)$ and $n_6(t) = 8\max\{R, 2C^2\}n_4(t) + 2\int_0^t \int_\Omega e^{-\omega\eta}F_{3,i}F_{3,i}dAd\xi d\eta$.

For the bounds of $||T||_{L^4(\Omega}$ and $||\varphi||_{L^4(\Omega)}$, we can prove the following lemma.

Lemma 2.8 If $\boldsymbol{f} \in H^1(\Omega \times (0,t))$ and $h, H \in L^\infty(\Omega \times (0,t))$, then

$$\int_\Omega e^{-\omega t}T^4dAd\xi \le n_7(t), \epsilon_1 \int_\Omega e^{-\omega t}\varphi^4dAd\xi \le n_8(t),$$

where $n_7(t)$ and $n_8(t)$ are positive computable functions.

Proof. We compute

$$\begin{aligned}\frac{d}{dt}\Big\{\int_\Omega e^{-\omega t}T^4dAd\xi\Big\} + \omega \int_\Omega e^{-\omega t}T^4dAd\xi &= 4\int_\Omega e^{-\omega t}T^3\Big[\Delta T + u_3 - u_iT_{,i}\Big]dAd\xi \\ &= -12\int_\Omega e^{-\omega t}T^2T_{,i}T_{,i}dAd\xi + 4\int_\Omega e^{-\omega t}T^3u_3dAd\xi \\ &+ \int_{D(0)} e^{-\omega t}h^3f_3dA + 4\int_{D(0)} e^{-\omega t}T^3T_{,3}dA. \end{aligned} \tag{2.30}$$

Using the Hölder inequality and Lemmas 2.2 and 2.6, we obtain

$$
\begin{aligned}
4\int_{\Omega} e^{-\omega t}T^3u_3dAd\xi &\leq 4\Big[\int_{\Omega} e^{-\omega t}T^6dAd\xi\Big]^{\frac{1}{6}}\Big[\int_{\Omega} e^{-\omega t}T^2dAd\xi\Big]^{\frac{1}{2}}\Big[\int_{\Omega} e^{-\omega t}u_3^6dAd\xi\Big]^{\frac{1}{12}}\Big[\int_{\Omega} e^{-\omega t}u_3^2dAd\xi\Big]^{\frac{1}{4}}\\
&\leq \frac{4}{\sqrt{\lambda_1}}\sqrt{n_4(t)}\sqrt[4]{k_1}\Big[\int_{\Omega} e^{-\omega t}T_{,i}T_{,i}dAd\xi\Big]^{\frac{1}{2}}\Big[\int_{\Omega} e^{-\omega t}u_{3,i}u_{3,i}dAd\xi\Big]^{\frac{1}{2}}\\
&\leq \frac{2\sqrt[4]{k_1}}{\sqrt{\lambda_1}}\sqrt{n_4(t)}\Big[\int_{\Omega} e^{-\omega t}T_{,i}T_{,i}dAd\xi+\int_{\Omega} e^{-\omega t}u_{3,i}u_{3,i}dAd\xi\Big]. \qquad (2.31)
\end{aligned}
$$

Inserting Eq (2.31) into Eq (2.30) and integrating Eq (2.30) from 0 to t, we have

$$
\begin{aligned}
&\int_{\Omega} e^{-\omega t}T^4dAd\xi+\omega\int_0^t\int_{\Omega} e^{-\omega\eta}T^4dAd\xi d\eta\\
&\qquad\leq \frac{2\sqrt[4]{k_1}}{\sqrt{\lambda_1}}\sqrt{n_4(t)}\Big[\int_0^t\int_{\Omega} e^{-\omega\eta}T_{,i}T_{,i}dAd\xi d\eta+\int_0^t\int_{\Omega} e^{-\omega\eta}u_{3,i}u_{3,i}dAd\xi d\eta\Big]\\
&\qquad+\int_0^t\int_{D(0)} e^{-\omega\eta}h^3f_3dAd\eta+4\int_0^t\int_{D(0)} e^{-\omega\eta}T^3T_{,3}dAd\eta. \qquad (2.32)
\end{aligned}
$$

Using Lemma 2.7, Eq (2.6) and Lemma 2.6, we have, from Eq (2.32)

$$
\int_{\Omega} e^{-\omega t}T^4dAd\xi+\omega\int_0^t\int_{\Omega} e^{-\omega\eta}T^4dAd\xi d\eta\leq n_7(t), \qquad (2.33)
$$

where

$$
\begin{aligned}
n_7(t)=&\frac{2\sqrt[4]{k_1}}{\sqrt{\lambda_1}}\sqrt{n_4(t)}\Big[\frac{1}{2R}n_4(t)+n_6(t)\Big]\\
&+\frac{1}{2R}\delta_i\varepsilon_1'n_4(t)+\varepsilon_2'n_4(t)+\frac{1}{R\omega}(\varepsilon_3'+\frac{1}{4\varepsilon_4'}L_M^6)n_4(t)\\
&+\varepsilon_4'n_4(t)+\varepsilon_5'n_4(t)+n_2'(t).
\end{aligned}
$$

Similarly, for φ we have

$$
\epsilon_1\int_{\Omega} e^{-\omega t}\varphi^4dAd\xi+\epsilon_1\omega\int_0^t\int_{\Omega} e^{-\omega t}\varphi^4dAd\xi d\eta\leq n_8(t), \qquad (2.34)
$$

where $n_8(t)$ is a positive function which is similar to $n_7(t)$.

3. Spatial decay bound

In this section, we shall derive the spatial decay bounds for the solutions of Eqs (1.1)–(1.8). To do this, we define

$$
\begin{aligned}
F(z,t)=&e^{-\omega t}\int_z^{\infty}\int_{D(\xi)}\Big[2RT^2+\epsilon_1\varphi^2\Big]dAd\xi\\
&+\int_0^t\int_z^{\infty}\int_{D(\xi)} e^{-\omega\eta}\Big[\frac{1}{2}u_iu_i+R\omega T^2+\frac{1}{2}\epsilon_1\omega\varphi^2+2RT_{,i}T_{,i}+2\varphi_{,i}\varphi_{,i}\Big]dAd\xi d\eta. \qquad (3.1)
\end{aligned}
$$

If we choose $z = 0$ in Eq (3.1), using Lemma 2.6, we can conclude that $F(0, t)$ can be bounded by known data.

We define

$$\mathcal{F}(z,t) = \int_z^\infty F(\xi,t)d\xi = e^{-\omega t}\int_z^\infty\int_{D(\xi)}(\xi - z)\Big[2RT^2 + \epsilon_1\varphi^2\Big]dAd\xi + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}u_iu_i + R\omega T^2 + \frac{1}{2}\epsilon_1\omega\varphi^2 + 2RT_{,i}T_{,i} + 2\varphi_{,i}\varphi_{,i}\Big]dAd\xi d\eta. \quad (3.2)$$

Combining Eqs (2.19)–(2.21) and choosing that $\omega \geq \frac{2(2+C^2)}{\epsilon_1}$, we have

$$\begin{aligned}\mathcal{F}(z,t) \leq & -\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}pu_3dAd\xi d\eta + R\int_0^t\int_z^\infty\int_{D(\xi)} u_3T^2dAd\xi d\eta \\ & - 2R\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}TT_{,3}dAd\xi d\eta \\ & + Le\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3\varphi^2dAd\xi d\eta - 2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\varphi\varphi_{,3}dAd\xi d\eta \\ & \doteq I_1(z,t) + I_2(z,t) + I_3(z,t) + I_4(z,t) + I_5(z,t). \end{aligned} \quad (3.3)$$

Using the Hölder inequality, Lemma 2.1 and Young's inequality, we obtain

$$\begin{aligned} I_1(z,t) &\leq \Big[\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}p^2dAd\xi d\eta\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3^2dAd\xi d\eta\Big]^{\frac{1}{2}} \\ &\leq \frac{1}{2\sqrt{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_iu_idAd\xi d\eta, \end{aligned} \quad (3.4)$$

$$\begin{aligned} I_3(z,t) &\leq 2R\Big[\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}T^2dAd\xi d\eta\int_0^t\int_z^\infty\int_{D(\xi)} d\xi e^{-\omega\eta}T_{,3}^2dAd\xi d\eta\Big]^{\frac{1}{2}} \\ &\leq R\frac{1}{\sqrt{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}T_{,i}T_{,i}dAd\xi d\eta, \end{aligned} \quad (3.5)$$

$$I_5(z,t) \leq \frac{1}{\sqrt{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\varphi_{,i}\varphi_{,i}dAd\xi d\eta. \quad (3.6)$$

Using the Hölder inequality, Eq (2.23), Lemma 2.2 and Young's inequality, we obtain

$$\begin{aligned} I_2(z,t) &\leq R\int_0^t\Big[\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3^2dAd\xi\Big]^{\frac{1}{2}}\Big[\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}T^2dAd\xi\Big]^{\frac{1}{4}}\Big[\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}T^6dAd\xi\Big]^{\frac{1}{4}}d\eta \\ &\leq \frac{\sqrt[4]{k_1n_5^2(t)}}{\sqrt[4]{4\omega}}\int_0^t\Big[\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}R\omega T^2dAd\xi\Big]^{\frac{1}{4}}\Big[\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}2RT_{,i}T_{,i}dAd\xi\Big]^{\frac{3}{4}}d\eta \\ &\leq \frac{\sqrt[4]{k_1n_5^2(t)}}{\sqrt[4]{4\omega}}\Big[\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}R\omega T^2dAd\xi d\eta + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}2RT_{,i}T_{,i}dAd\xi d\eta\Big], \end{aligned} \quad (3.7)$$

and

$$I_4(z,t) \leq \frac{\sqrt[4]{2k_1 n_5^2(t)}}{\sqrt[4]{\epsilon_1\omega}} Le\Big[\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\frac{1}{2}\epsilon_1\omega\varphi^2 dAd\xi d\eta + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}2\varphi_{,i}\varphi_{,i}dAd\xi d\eta\Big]. \quad (3.8)$$

Inserting Eqs (3.4)–(3.8) into Eq (3.3) and combining Eq (3.2), we obtain

$$\mathcal{F}(z,t) \leq \frac{1}{a_1(t)}\Big[-\frac{\partial}{\partial z}\mathcal{F}(z,t)\Big], \quad (3.9)$$

where

$$\frac{1}{a_1(t)} = \sqrt{\lambda_1} + \frac{\sqrt[4]{k_1 n_5^2(t)}}{\sqrt[4]{4\omega}}\max\{\frac{\sqrt[4]{2}}{\sqrt{\epsilon_1}}Le, 1\}. \quad (3.10)$$

Integrating Eq (3.9) from 0 to z, we obtain

$$\mathcal{F}(z,t) \leq \mathcal{F}(0,t)e^{-a_1(t)z}. \quad (3.11)$$

Combining Eqs (3.2) and (3.11) we can obtain the following theorem.

Theorem 3.1 If $f \in H^1(\Omega\times(0,t))$ and $h, H \in L^\infty(\Omega\times(0,t))$, then the solutions of Eqs (1.1)–(1.8) decay exponentially as $z\to\infty$. Specifically,

$$\begin{aligned} &e^{-\omega t}\int_z^\infty\int_{D(\xi)}(\xi - z)\big[2RT^2 + \epsilon_1\varphi^2\big]dAd\xi \\ &+ \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}u_iu_i + R\omega T^2 + \frac{1}{2}\epsilon_1\omega\varphi^2 + 2RT_{,i}T_{,i} + 2\varphi_{,i}\varphi_{,i}\Big]dAd\xi d\eta \\ &\leq \mathcal{F}(0,t)e^{-a_1(t)z}, \end{aligned}$$

where $\omega > \max\{\frac{2(2+C^2)}{\epsilon_1}, 4 + \frac{1}{\varepsilon_4}L_M^2, \frac{4C^2}{\epsilon_1} + \frac{4}{\epsilon_1} + \frac{1}{\epsilon_1\delta_4}L_M^2\}$ and $a_1(t)$ has been defined in Eq (3.10).

Remark 3.1 Theorem 1 shows that the solutions of Eqs (1.1)–(1.8) decay exponentially with the space variable. This decay result can be regarded as the Saint-Venant principle type result.

4. Continuous dependence on R and C

We now consider two solutions to Eqs (1.1)–(1.8), namely (u_i, T, φ, p) and $(u_i^*, T^*, \varphi^*, p^*)$, for different coefficients (R, C) and (R^*, C^*) in Eq (1.1), respectively, but they have the same initial-boundary conditions. Letting

$$v_i = u_i - v_i^*, \Sigma = T - T^*, \theta = \varphi - \varphi^*, \pi = p - p^*, r = R - R^*, c = C - C^*,$$

then $(v_i, \Sigma, \theta, \pi)$ satisfy

$$v_i + rTk_i + R^*\Sigma k_i - \pi_{,i} - ck_i\varphi - C^*k_i\theta = 0, \ in\ \Omega\times(0,t), \quad (4.1)$$

$$v_{i,i} = 0, \ in\ \Omega\times(0,t) \quad (4.2)$$

$$\Sigma_t + v_i T_{,i} + u_i^* \Sigma_{,i} = v_3 + \Delta\Sigma, \ in\ \Omega \times (0,t) \tag{4.3}$$

$$\epsilon_1 \theta_t + Lev_i \varphi_{,i} + Leu_i^* \theta_{,i} = v_3 + \Delta\theta, \ in\ \Omega \times (0,t), \tag{4.4}$$

with the following initial-boundary conditions

$$v_i = 0,\ \Sigma = \theta = 0, \quad on\ \partial D \times (0,t), \tag{4.5}$$

$$v_i = 0,\ \Sigma = \theta = 0, \quad on\ D \times (0,t), \tag{4.6}$$

$$\Sigma = \theta = 0, \quad in\ \Omega \times \{t = 0\}, \tag{4.7}$$

$$|\boldsymbol{v}|, |\Sigma|, |\theta| = O(1), |v_3|, |\nabla\Sigma|, |\nabla\theta|, |\pi| = o(x_3^{-1}), as\ x_3 \to \infty. \tag{4.8}$$

To obtain our main result, we prove the following lemmas.

Lemma 4.1 The solutions of Eqs (4.1)–(4.8) satisfy

$$\begin{aligned}\int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} v_{i,j} v_{i,j} dA d\xi d\eta \le & -2 \int_0^t \int_{D(z)} e^{-\omega\eta} v_3 v_{3,3} dA d\eta \\ & + 8 \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} \Big[(R^*)^2 \Sigma_{,i} \Sigma_{,i} + (C^*)^2 \theta_{,i} \theta_{,i} \Big] dA d\xi d\eta \\ & + 8 \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} \Big[r^2 T_{,i} T_{,i} + c^2 \varphi_{,i} \varphi_{,i} \Big] dA d\xi d\eta.\end{aligned}$$

Proof. We start with the identity

$$\begin{aligned}\int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} v_{i,j} v_{i,j} dA d\xi d\eta = & \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} (v_{i,j} - v_{j,i}) v_{i,j} dA d\xi d\eta \\ & + \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} v_{i,j} v_{j,i} dA d\xi d\eta.\end{aligned} \tag{4.9}$$

Using Eq (4.1), we have

$$\begin{aligned}\int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} (v_{i,j} - v_{j,i}) v_{i,j} dA d\xi d\eta = & \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} \Big[v_{i,j} - v_{j,i} \Big] v_{i,j} dA d\xi d\eta \\ = & -r \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} \Big[T_{,j} v_{3,j} - T_{,i} v_{i,3} \Big] dA d\xi d\eta \\ & - R^* \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} \Big[\Sigma_{,j} v_{3,j} - \Sigma_{,i} v_{i,3} \Big] dA d\xi d\eta \\ & + c \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} \Big[\varphi_{,j} v_{3,j} - \varphi_{,i} v_{i,3} \Big] dA d\xi d\eta \\ & + C^* \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta} \Big[\theta_{,j} v_{3,j} - \theta_{,i} v_{i,3} \Big] dA d\xi d\eta.\end{aligned} \tag{4.10}$$

By using the Hölder inequality and Young's inequality, from Eq (4.10) we have

$$\begin{aligned}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}(v_{i,j}-v_{j,i})v_{i,j}dAd\xi d\eta \le{}& \frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_{i,j}v_{i,j}dAd\xi d\eta\\ &+4r^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}T_{,i}T_{,i}dAd\xi d\eta\\ &+4(R^*)^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dAd\xi d\eta\\ &+4c^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\varphi_{,i}\varphi_{,i}dAd\xi d\eta\\ &+4(C^*)^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\theta_{,i}\theta_{,i}dAd\xi d\eta. \qquad (4.11)\end{aligned}$$

Using a similar method to that of Eq (2.25), we can obtain

$$\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_{i,j}v_{j,i}dAd\xi d\eta = -\int_0^t\int_{D(z)} e^{-\omega\eta}v_3v_{3,3}dAd\eta. \qquad (4.12)$$

Inserting Eqs (4.11) and (4.12) into Eq (4.9), we may have Lemma 4.1.

Lemma 4.2 The L^2 norm of $\boldsymbol{v}$ satisfies

$$\begin{aligned}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_iv_idAd\xi d\eta \le{}& -2R^*\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Sigma v_3dAd\xi d\eta\\ &+4\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\left[r^2T^2+c^2\varphi^2\right]dAd\xi d\eta\\ &+(C^*)^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\theta^2dAd\xi d\eta\\ &+\frac{1}{\sqrt{\lambda_1}}\int_0^t\int_{D(z)} e^{-\omega\eta}v_iv_idAd\eta.\end{aligned}$$

Proof. We multiply Eq (4.1) by $e^{-\omega\eta}v_i$ and integrate in $D(z)\times(z,\infty)\times(0,t)$ to have

$$\begin{aligned}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_iv_idAd\xi d\eta ={}& -r\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}Tv_3dAd\xi d\eta - R^*\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Sigma v_3dAd\xi d\eta\\ &+c\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\varphi v_3dAd\xi d\eta + C^*\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\theta v_3dAd\xi d\eta\\ &-\int_0^t\int_{D(z)} e^{-\omega\eta}\pi v_3dAd\eta.\end{aligned}$$

Noting that $\pi_{,\alpha} = v_\alpha$, and by using Lemma 2.1, the Hölder inequality and Young's inequality again, we have

$$\begin{aligned}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_iv_i dAd\xi d\eta \le{}& \frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_3^2 dAd\xi d\eta + 2r^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}T^2 dAd\xi d\eta\\ &- R^*\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Sigma v_3 dAd\xi d\eta + 2c^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\varphi^2 dAd\xi d\eta\\ &+ \frac{1}{2}(C^*)^2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\theta^2 dAd\xi d\eta + \frac{1}{2\sqrt{\lambda_1}}\int_0^t\int_{D(z)} e^{-\omega\eta}v_iv_i dAd\eta.\end{aligned} \tag{4.13}$$

From Eq (4.13) we may have Lemma 4.2.

Next, we seek the bounds for the L^2 norms of Σ and θ. We write the results as the following lemma.

Lemma 4.3 If $f \in L^4(\Omega\times(0,t))$ and $h, H \in L^\infty(\Omega\times(0,t))$, then

$$\begin{aligned}e^{-\omega t}\int_z^\infty\int_{D(\xi)}\Sigma^2 dAd\xi + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[\omega\Sigma^2 + \Sigma_{,i}\Sigma_{,i}\Big]dAd\xi d\eta&\\ \le -2\int_0^t\int_{D(z)} e^{-\omega\eta}\Sigma\Sigma_{,3} dAd\eta + 2\int_0^t\int_{D(z)} e^{-\omega\eta}u_3^*\Sigma^2 dAd\eta&\\ + 2\int_0^t\int_{D(z)} e^{-\omega\eta}v_3T\Sigma dAd\eta + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_3\Sigma dAd\xi d\eta&\\ + \frac{\sqrt[4]{k_2}}{\sqrt[4]{\lambda_1}}\sqrt{n_7(t)e^{\omega t}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_{i,j}v_{i,j} dAd\xi d\eta.&\end{aligned}$$

Proof. We multiply Eq (4.3) by $e^{-\omega\eta}\Sigma$ and integrate in $D(\xi)\times(z,\infty)\times(0,t)$ to have

$$\begin{aligned}\frac{1}{2}e^{-\omega t}\int_z^\infty\int_{D(\xi)}\Sigma^2 dAd\xi + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[\frac{1}{2}\omega\Sigma^2 + \Sigma_{,i}\Sigma_{,i}\Big]dAd\xi d\eta&\\ = -\int_0^t\int_{D(z)} e^{-\omega\eta}\Sigma\Sigma_{,3} dAd\eta + \frac{1}{2}\int_0^t\int_{D(z)} e^{-\omega\eta}u_3^*\Sigma^2 dAd\eta&\\ + \int_0^t\int_{D(z)} e^{-\omega\eta}v_3T\Sigma dAd\eta + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_iT\Sigma_{,i} dAd\xi d\eta&\\ + \int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_3\Sigma dAd\xi d\eta.&\end{aligned} \tag{4.14}$$

By using the Hölder inequality and Lemmas 2.1, 2.2 and 2.8, we have

$$\begin{aligned}\int_z^\infty\int_{D(\xi)} v_iT\Sigma_{,i} dAd\xi \le{}& \Big[\int_z^\infty\int_{D(\xi)}\Sigma_{,i}\Sigma_{,i} dAd\xi\Big]^{\frac{1}{2}}\\ &\cdot\Big[\int_z^\infty\int_{D(\xi)}(v_iv_i)^2 dAd\xi\Big]^{\frac{1}{4}}\Big[\int_z^\infty\int_{D(\xi)} T^4 dAd\xi\Big]^{\frac{1}{4}}\\ \le{}& \frac{1}{2}\int_z^\infty\int_{D(\xi)}\Sigma_{,i}\Sigma_{,i} dAd\xi\end{aligned}$$

$$+\frac{1}{2}\sqrt{n_7(t)}\Big[\int_z^\infty\int_{D(\xi)}(v_iv_i)^3dAd\xi\Big]^{\frac{1}{4}}\Big[\int_z^\infty\int_{D(\xi)}v_iv_idAd\xi\Big]^{\frac{1}{4}}$$
$$\leq\frac{1}{2}\int_z^\infty\int_{D(\xi)}\Sigma_{,i}\Sigma_{,i}dAd\xi+\frac{\sqrt[4]{k_2}}{2\sqrt[4]{\lambda_1}}\sqrt{n_7(t)e^{\omega t}}\int_z^\infty\int_{D(\xi)}v_{i,j}v_{i,j}dAd\xi.$$

Therefore, we have

$$\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_iT\Sigma_{,i}dAd\xi d\eta\leq\frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dAd\xi d\eta$$
$$+\frac{\sqrt[4]{k_2}}{2\sqrt[4]{\lambda_1}}\sqrt{n_7(t)e^{\omega t}}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_{i,j}v_{i,j}dAd\xi d\eta. \tag{4.15}$$

Inserting Eq (4.15) into Eq (4.14), we can obtain Lemma 4.3.
Similar to Lemma 4.3, we can obtain the following lemma.
Lemma 4.4 If $f\in L^4(\Omega\times(0,t))$ and $h,H\in L^\infty(\Omega\times(0,t))$, then

$$e^{-\omega t}\epsilon_1\int_z^\infty\int_{D(\xi)}\theta^2dAd\xi+\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}\Big[\frac{1}{2}\omega\epsilon_1\theta^2+\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta$$
$$\leq-2\int_0^t\int_{D(z)}e^{-\omega\eta}\theta\theta_{,3}dAd\eta+2Le\int_0^t\int_{D(z)}e^{-\omega\eta}u_3^*\theta^2dAd\eta$$
$$+2\int_0^t\int_{D(z)}e^{-\omega\eta}v_3\varphi\theta dAd\eta+\frac{2}{\omega}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_3^2dAd\xi d\eta$$
$$+\frac{\sqrt[4]{k_2}}{\sqrt[4]{\lambda_1}}\sqrt{n_8(t)e^{\omega t}}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_{i,j}v_{i,j}dAd\xi d\eta.$$

In Lemma 4.4, we have used the inequality

$$\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_3\theta dAd\xi d\eta\leq\frac{1}{4}\omega\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}\theta^2dAd\xi d\eta$$
$$+\frac{1}{\omega}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_3^2dAd\xi d\eta. \tag{4.16}$$

Now, we assume that δ_1 and δ_2 are positive constants such that

$$\delta_1\geq16(C^*)^2\delta_2,\delta_2\leq\frac{1}{16R^*},\omega\geq\max\{4\delta_1,\frac{12}{\delta_1}(C^*)^2\}, \tag{4.17}$$

and the boundary conditions satisfy

$$2\delta_1\frac{\sqrt[4]{k_2}}{\sqrt[4]{\lambda_1}}\sqrt{n_8(t)e^{\omega t}}+2R^*\frac{\sqrt[4]{k_2}}{\sqrt[4]{\lambda_1}}\sqrt{n_7(t)e^{\omega t}}\leq\delta_2. \tag{4.18}$$

If we define

$$E(z,t) = e^{-\omega t}\int_z^\infty \int_{D(\xi)} \Big[R^*\Sigma^2 + \delta_1\epsilon_1\theta^2\Big] dAd\xi$$
$$+ \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta}\Big[\frac{1}{2}R^*\omega\Sigma^2 + \frac{1}{2}R^*\Sigma_{,i}\Sigma_{,i} + \frac{1}{4}\delta_1\omega\epsilon_1\theta^2 + \frac{1}{2}\delta_1\theta_{,i}\theta_{,i}\Big] dAd\xi d\eta$$
$$+ \frac{1}{2}\int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta}\Big[v_iv_i + \delta_2 v_{i,j}v_{i,j}\Big] dAd\xi d\eta, \tag{4.19}$$

then combining Lemmas 4.1–4.4, we have

$$E(z,t) \le \frac{1}{\sqrt{\lambda_1}}\delta_2 \int_0^t \int_{D(z)} e^{-\omega\eta} v_3 v_{3,3} dAd\eta + \frac{1}{\sqrt{\lambda_1}}\int_0^t \int_{D(z)} e^{-\omega\eta} v_i v_i dAd\eta$$
$$- 2R^* \int_0^t \int_{D(z)} e^{-\omega\eta}\Sigma\Sigma_{,3} dAd\eta - 2\delta_1 \int_0^t \int_{D(z)} e^{-\omega\eta}\theta\theta_{,3} dAd\eta$$
$$+ 2R^* \int_0^t \int_{D(z)} e^{-\omega\eta} u_3^*\Sigma^2 dAd\eta + 2Le\delta_1 \int_0^t \int_{D(z)} e^{-\omega\eta} u_3^*\theta^2 dAd\eta$$
$$+ 2R^* \int_0^t \int_{D(z)} e^{-\omega\eta} v_3 T\Sigma dAd\eta + 2\delta_1 \int_0^t \int_{D(z)} e^{-\omega\eta} v_3\varphi\theta dAd\eta$$
$$+ 4\int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta}\Big[r^2T^2 + 2r^2T_{,i}T_{,i} + c^2\varphi^2 + 2c^2\varphi_{,i}\varphi_{,i}\Big] dAd\xi d\eta. \tag{4.20}$$

Based on the above lemmas, we can obtain the following theorem.

Theorem 4.1 If $f \in L^4(\Omega\times(0,t))$, $h, H \in L^\infty(\Omega\times(0,t))$ and the inequality given by Eq (4.18) holds, then the solutions of Eqs (1.1)–(1.8) continuously depend on the coefficients R and C, i.e.,

$$(u_i, T, \varphi, p) \to (u_i^*, T^*, \varphi^*, p^*),\ as\ (R, C) \to (R^*, C^*).$$

Specifically, either the inequality

$$e^{-\omega t}\int_z^\infty \int_{D(\xi)} (\xi - z)\Big[R^*\Sigma^2 + \delta_1\epsilon_1\theta^2\Big] dAd\xi$$
$$+ \int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}R^*\omega\Sigma^2 + \frac{1}{2}R^*\Sigma_{,i}\Sigma_{,i} + \frac{1}{4}\delta_1\epsilon_1\omega\theta^2 + \frac{1}{2}\delta_1\theta_{,i}\theta_{,i}\Big] dAd\xi d\eta$$
$$+ \frac{1}{2}\int_0^t \int_z^\infty \int_{D(\xi)} e^{-\omega\eta}(\xi - z)\Big[v_iv_i + \delta_2 v_{i,j}v_{i,j}\Big] dAd\xi d\eta$$
$$\le a_5(t)(r^2 + c^2)e^{-a_2(t)z} + (r^2 + c^2)\frac{a_2(t)a_3(t)}{a_2(t) - a_1(t)}\Big[e^{-a_1(t)z} - e^{-a_2(t)z}\Big]$$

holds, or the inequality

$$
\begin{aligned}
&e^{-\omega t}\int_z^\infty\int_{D(\xi)}(\xi-z)\Big[R^*\Sigma^2+\delta_1\epsilon_1\theta^2\Big]dAd\xi\\
&+\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[\frac{1}{2}R^*\omega\Sigma^2+\frac{1}{2}R^*\Sigma_{,i}\Sigma_{,i}+\frac{1}{4}\delta_1\epsilon_1\omega\theta^2+\frac{1}{2}\delta_1\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta\\
&+\frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[v_iv_i+\delta_2v_{i,j}v_{i,j}\Big]dAd\xi d\eta\\
&\le a_5(t)(r^2+c^2)e^{-a_2(t)z}+(r^2+c^2)a_2(t)a_3(t)ze^{-a_2(t)z}
\end{aligned}
$$

holds, where $a_1(t), a_2(t), a_3(t)$ and $a_5(t)$ are positive computable functions and ω is a sufficiently large positive constant.

Remark 4.1 In particular, the continuous dependence of the pressure p on the coefficients R and C can be obtained by Eq (3.1) easily.

Remark 4.2 Theorem 4.1 shows that small perturbations of the coefficients R and C will not have a huge impact on the solution of Eqs (1.1)–(1.8).

Proof. We define

$$
\begin{aligned}
\mathcal{E}(z,t)=&\int_z^\infty E(\xi,t)d\xi=e^{-\omega t}\int_z^\infty\int_{D(\xi)}(\xi-z)\Big[R^*\Sigma^2+\delta_1\epsilon_1\theta^2\Big]dAd\xi\\
&+\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[\frac{1}{2}R^*\omega\Sigma^2+\frac{1}{2}R^*\Sigma_{,i}\Sigma_{,i}+\frac{1}{4}\delta_1\epsilon_1\omega\theta^2+\frac{1}{2}\delta_1\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta\\
&+\frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[v_iv_i+\delta_2v_{i,j}v_{i,j}\Big]dAd\xi d\eta. \qquad (4.21)
\end{aligned}
$$

Using Theorem 3.1, from Eq (4.20) we have

$$
\begin{aligned}
\mathcal{E}(z,t)\le&\frac{1}{\sqrt{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_iv_idAd\xi d\eta\\
&-2R^*\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}\Sigma\Sigma_{,3}dAd\xi d\eta-2\delta_1\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}\theta\theta_{,3}dAd\xi d\eta\\
&+2R^*\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}u_3^*\Sigma^2dAd\xi d\eta+2Le\delta_1\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}u_3^*\theta^2dAd\xi d\eta\\
&+2R^*\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_3T\Sigma dAd\xi d\eta+2\delta_1\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_3\varphi\theta dAd\xi d\eta\\
&+4(r^2+c^2)\max\{\frac{1}{R},\frac{1}{\epsilon_1},1\}\mathcal{F}(0,t)e^{-a_1(t)z}, \qquad (4.22)
\end{aligned}
$$

where we have used the fact that

$$
\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}v_3v_{3,3}dAd\xi d\eta=-\frac{1}{2}\int_0^t\int_{D(z)}e^{-\omega\eta}v_3^2dAd\eta\le 0.
$$

By using the Hölder inequality and Young's inequality, we have

$$-2R^*\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Sigma\Sigma_{,3}dAd\xi d\eta \le \frac{R^*}{\sqrt{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Sigma_{,i}\Sigma_{,i}dAd\xi d\eta, \tag{4.23}$$

$$-2\delta_1\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\theta\theta_{,3}dAd\xi d\eta \le \frac{\delta_1}{\sqrt{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\theta_{,i}\theta_{,i}dAd\xi d\eta. \tag{4.24}$$

Similar to Eq (3.7), we have

$$\begin{aligned}
&2R^*\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3^*\Sigma^2 dAd\xi d\eta \\
&\quad\le \frac{\sqrt[4]{2k_1n_5^2(t)}}{\sqrt[4]{\omega}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\frac{1}{2}R^*\Big[\omega\Sigma^2+\Sigma_{,i}\Sigma_{,i}\Big]dAd\xi d\eta, \qquad (4.25)\\
&2Le\delta_1\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3^*\theta^2 dAd\xi d\eta \\
&\quad\le \frac{2\sqrt[4]{k_1n_5^2(t)}}{\sqrt[4]{\omega\epsilon_1}}Le\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\frac{1}{4}\delta_1\Big[\epsilon_1\omega\theta^2+2\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta. \qquad (4.26)
\end{aligned}$$

Using the Hölder inequality, Young's inequality, Lemmas 2.1, 2.2 and 2.8 and Eq (2.34), we have

$$\begin{aligned}
&2R^*\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_3T\Sigma dAd\xi d\eta \\
&\le 2R^*\int_0^t e^{-\omega\eta}\Big(\int_z^\infty\int_{D(\xi)} v_3^2dAd\xi\Big)^{\frac{1}{2}}\Big(\int_0^\infty\int_{D(\xi)} T^4dAd\xi\Big)^{\frac{1}{4}}\Big(\int_z^\infty\int_{D(\xi)}\Sigma^4dAd\xi\Big)^{\frac{1}{4}}d\eta \\
&\le 2R^*\sqrt[4]{n_7(t)e^{\omega t}}\int_0^t e^{-\omega\eta}\Big(\int_z^\infty\int_{D(\xi)} v_3^2dAd\xi\Big)^{\frac{1}{2}}\Big(\int_z^\infty\int_{D(\xi)}\Sigma^2dAd\xi\Big)^{\frac{1}{8}}\Big(\int_z^\infty\int_{D(\xi)}\Sigma^6dAd\xi\Big)^{\frac{1}{8}}d\eta \\
&\le \frac{2R^*\sqrt[8]{n_7^2(t)e^{2\omega t}k_1}}{\sqrt[8]{\lambda_1}}\int_0^t e^{-\omega\eta}\Big(\int_z^\infty\int_{D(\xi)} v_3^2dAd\xi\Big)^{\frac{1}{2}}\Big(\int_z^\infty\int_{D(\xi)}\Sigma_{,i}\Sigma_{,i}dAd\xi\Big)^{\frac{1}{2}}d\eta \\
&\le \frac{4\sqrt[8]{(R^*)^4n_7^2(t)e^{2\omega t}k_1}}{\sqrt[8]{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[\frac{1}{2}v_3^2+\frac{1}{4}R^*\Sigma_{,i}\Sigma_{,i}\Big]dAd\xi d\eta, \qquad (4.27)
\end{aligned}$$

and

$$\begin{aligned}
&2\delta_1\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_3\varphi\theta dAd\xi d\eta \\
&\quad\le \frac{2\sqrt[8]{(\delta_1)^4n_8^2(t)e^{2\omega t}k_1}}{\sqrt[8]{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[\frac{1}{2}v_3^2+\frac{1}{2}\delta_1\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta. \qquad (4.28)
\end{aligned}$$

Inserting Eqs (4.23)–(4.28) into Eq (4.22), we obtain

$$\mathcal{E}(z,t)\le\frac{1}{a_2(t)}\Big[-\frac{\partial}{\partial z}\mathcal{E}(z,t)\Big]+a_3(t)(r^2+c^2)e^{-a_1(t)z}, \tag{4.29}$$

where

$$\frac{1}{a_2(t)} = \max\Big\{\frac{R^*}{\sqrt{\lambda_1}}, \frac{\delta_1}{\sqrt{\lambda_1}}, \frac{\sqrt[4]{2k_1n_5^2(t)}}{\sqrt[4]{\omega}}, \frac{2\sqrt[4]{k_1n_5^2(t)}}{\sqrt[4]{\omega\epsilon_1}}Le, \frac{1}{\sqrt{\lambda_1}} + \frac{4\sqrt[8]{(R^*)^4n_7^2(t)e^{2\omega t}k_1}}{\sqrt[8]{\lambda_1}} + \frac{2\sqrt[8]{(\delta_1)^4n_8^2(t)e^{2\omega t}k_1}}{\sqrt[8]{\lambda_1}}\Big\},$$

$$a_3(t) = 4\max\{\frac{1}{R}, \frac{1}{\epsilon_1}, 1\}\mathcal{F}(0,t).$$

Integrating Eq (4.29) from 0 to z, we obtain

$$\mathcal{E}(z,t) \leq \mathcal{E}(0,t)e^{-a_2(t)z} + a_2(t)a_3(t)e^{-a_2(t)z}(r^2+c^2)\int_0^z e^{(a_2(t)-a_1(t))\xi}d\xi. \tag{4.30}$$

If $a_2(t) \neq a_1(t)$, it follows from Eq (4.30) that

$$\mathcal{E}(z,t) \leq \mathcal{E}(0,t)e^{-a_2(t)z} + (r^2+c^2)\frac{a_2(t)a_3(t)}{a_2(t)-a_1(t)}\Big[e^{-a_1(t)z} - e^{-a_2(t)z}\Big]. \tag{4.31}$$

If $a_2(t) = a_1(t)$, it follows from Eq (4.30) that

$$\mathcal{E}(z,t) \leq \mathcal{E}(0,t)e^{-a_2(t)z} + (r^2+c^2)a_2(t)a_3(t)ze^{-a_2(t)z}. \tag{4.32}$$

On the other hand, we choose $z = 0$ in Eq (4.20) and use the boundary conditions Eqs (4.5) and (4.6) to obtain

$$-\frac{\partial}{\partial z}\mathcal{E}(0,t) \leq 4\int_0^t\int_\Omega e^{-\omega\eta}\Big[r^2T^2 + 2r^2T_{,i}T_{,i} + c^2\varphi^2 + 2c^2\varphi_{,i}\varphi_{,i}\Big]dAd\xi d\eta.$$

Using Lemma 2.6, we have

$$-\frac{\partial}{\partial z}\mathcal{E}(0,t) \leq a_4(t)[r^2+c^2], \tag{4.33}$$

where $a_4(t) = 4\max\{\frac{1}{R\omega}, \frac{1}{R}, 1, \frac{2}{\epsilon_1\omega}\}$. Choosing $z = 0$ in Eq (4.29) and then inserting Eq (4.33) into Eq (4.29), we have

$$\mathcal{E}(0,t) \leq a_5(t)(r^2+c^2), \tag{4.34}$$

where $a_5(t) = \frac{a_4(t)}{a_2(t)} + a_3(t)$.

Combining Eqs (4.21), (4.31), (4.32) and (4.34), we can complete the proof of Theorem 4.1.

5. Convergence result on R and C

We now assume that $(u_i^*, T^*, \varphi^*, p^*)$ are the solutions to Eqs (1.1)–(1.8) with $R = C = 0$, but have the same initial-boundary conditions as (u_i, T, φ, p). We also let

$$v_i = u_i - v_i^*, \Sigma = T - T^*, \theta = \varphi - \varphi^*, \pi = p - p^*,$$

then $(v_i, \Sigma, \theta, \pi)$ satisfy

$$v_i + RTk_i - \pi_{,i} - Ck_i\varphi = 0, \ in\ \Omega\times(0,t), \tag{5.1}$$

$$v_{i,i} = 0,\ in\ \Omega\times(0,t) \tag{5.2}$$

$$\Sigma_t + v_iT_{,i} + u_i^*\Sigma_{,i} = v_3 + \Delta\Sigma,\ in\ \Omega\times(0,t) \tag{5.3}$$

$$\epsilon_1\theta_t + Lev_i\varphi_{,i} + Leu_i^*\theta_{,i} = v_3 + \Delta\theta,\ in\ \Omega\times(0,t), \tag{5.4}$$

with the following initial-boundary conditions

$$v_i = 0,\ \Sigma = \theta = 0,\ \ on\ \partial D\times\{x_3 > 0\}\times(0,t), \tag{5.5}$$

$$v_i = 0,\ \Sigma = \theta = 0,\ \ on\ D\times(0,t), \tag{5.6}$$

$$\Sigma = \theta = 0,\ \ in\ \Omega\times\{t = 0\}, \tag{5.7}$$

$$|\boldsymbol{v}|, |\Sigma|, |\theta| = O(1), |v_3|, |\nabla\Sigma|, |\nabla\theta|, |\pi| = o(x_3^{-1}), as\ x_3 \to \infty. \tag{5.8}$$

Similar to Lemmas 4.1 and 4.2, noting that $R^* = C^* = 0$ by recalculation we can obtain

$$\begin{aligned}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_{i,j}v_{i,j}dAd\xi d\eta \le& -2\int_0^t\int_{D(z)} e^{-\omega\eta}v_3v_{3,3}dAd\eta\\ &+2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[R^2T_{,i}T_{,i} + C^2\varphi_{,i}\varphi_{,i}\Big]dAd\xi d\eta,\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_iv_idAd\xi d\eta \le& 2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Big[R^2T^2 + C^2\varphi^2\Big]dAd\xi d\eta\\ &+\frac{1}{\sqrt{\lambda_1}}\int_0^t\int_{D(z)} e^{-\omega\eta}v_iv_idAd\eta.\end{aligned} \tag{5.10}$$

Now we define a new function

$$\begin{aligned}F(z,t) =& e^{-\omega t}\int_z^\infty\int_{D(\xi)}(\xi - z)\Big[\Sigma^2 + \epsilon_1\theta^2\Big]dAd\xi\\ &+\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}\omega\Sigma^2 + \frac{1}{2}\Sigma_{,i}\Sigma_{,i} + \frac{1}{4}\omega\epsilon_1\theta^2 + \frac{1}{2}\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta\\ &+\frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}(\xi - z)\Big[v_iv_i + \delta v_{i,j}v_{i,j}\Big]dAd\xi d\eta.\end{aligned} \tag{5.11}$$

Choosing $\omega > 8$ and $\delta = 2\frac{\sqrt[4]{k_2}}{\sqrt[4]{\lambda_1}}\sqrt{e^{\omega t}}[\sqrt{n_7(t)} + \sqrt{n_8(t)}]$ and combining Lemmas 4.3 and 4.4 and Eqs (5.9) and (5.10), we obtain

$$\begin{aligned}F(z,t) \le& \frac{1}{\sqrt{\lambda_1}}\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_iv_idAd\xi d\eta\\ &-2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\Sigma\Sigma_{,3}dAd\xi d\eta - 2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}\theta\theta_{,3}dAd\xi d\eta\\ &+2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3^*\Sigma^2dAd\xi d\eta + 2Le\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}u_3^*\theta^2dAd\xi d\eta\end{aligned}$$

$$+2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_3T\Sigma dAd\xi d\eta+2\int_0^t\int_z^\infty\int_{D(\xi)} e^{-\omega\eta}v_3\varphi\theta dAd\xi d\eta$$
$$+2(R^2+C^2)\max\{\frac{1}{R},\frac{1}{\epsilon_1},1\}\mathcal{F}(0,t)e^{-a_1(t)z}, \tag{5.12}$$

where we have used Theorem 3.1.

Combining Eqs (4.23)–(4.28), (5.11) and (5.12), we can conclude that

$$F(z,t)\le\frac{1}{a_2(t)}\Big[-\frac{\partial}{\partial z}F(z,t)\Big]+a_3(t)(R^2+C^2)e^{-a_1(t)z}. \tag{5.13}$$

Through the analysis similar to that in Section 3, from Eq (5.13) we can obtain the following theorem.

Theorem 5.1 If $f\in L^4(\Omega\times(0,t))$, $\int_D f_\alpha dA=0$ and $h,H\in L^\infty(\Omega\times(0,t))$, then

$$(u_i,T,\varphi,p)\to(0,0,0,0),\ as\ (R,C)\to(0,0).$$

Specifically, either the inequality

$$e^{-\omega t}\int_z^\infty\int_{D(\xi)}(\xi-z)\Big[\Sigma^2+\epsilon_1\theta^2\Big]dAd\xi$$
$$+\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[\frac{1}{2}\omega\Sigma^2+\frac{1}{2}\Sigma_{,i}\Sigma_{,i}+\frac{1}{4}\omega\epsilon_1\theta^2+\frac{1}{2}\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta$$
$$+\frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[v_iv_i+\delta v_{i,j}v_{i,j}\Big]dAd\xi d\eta$$
$$\le a_5(t)(R^2+C^2)e^{-a_2(t)z}+(R^2+C^2)\frac{a_2(t)a_3(t)}{a_2(t)-a_1(t)}\Big[e^{-a_1(t)z}-e^{-a_2(t)z}\Big]$$

holds, or the inequality

$$e^{-\omega t}\int_z^\infty\int_{D(\xi)}(\xi-z)\Big[\Sigma^2+\epsilon_1\theta^2\Big]dAd\xi$$
$$+\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[\frac{1}{2}\omega\Sigma^2+\frac{1}{2}\Sigma_{,i}\Sigma_{,i}+\frac{1}{4}\omega\epsilon_1\theta^2+\frac{1}{2}\theta_{,i}\theta_{,i}\Big]dAd\xi d\eta$$
$$+\frac{1}{2}\int_0^t\int_z^\infty\int_{D(\xi)}e^{-\omega\eta}(\xi-z)\Big[v_iv_i+\delta v_{i,j}v_{i,j}\Big]dAd\xi d\eta$$
$$\le a_5(t)(R^2+C^2)e^{-a_2(t)z}+(R^2+C^2)a_2(t)a_3(t)ze^{-a_2(t)z}$$

holds.

6. Conclusions

In this paper, we prove the spatial decay estimate and structural stability on the coefficients R and C of the solutions of Eqs (1.1)–(1.8) in a semi-infinite cylinder, where it is assumed that the solution satisfies the homogeneous boundary conditions on the side of the cylinder. This is a generalization of the literatures. However, if the solutions satisfy the nonlinear conditions on the side of the cylinder, the method in this paper will not be fully applicable. We note that Shi and Luo [34] studied the structural stability for the double-diffusion perturbation equations with nonlinear boundary conditions in a bounded region. How to deal with nonlinear boundary conditions is still an open problem in an unbounded domain. We suggest that this problem can be solved in the future by establishing an appropriate "energy function". Using methods similar to those in Li et al. [35–37], we will also study the problem of the function perturbation to Eqs (1.1)–(1.8).

Acknowledgments

The authors express their heartfelt thanks to the editors and referees who have provided some important suggestions. This work was supported by the Key Projects of Universities in Guangdong Province (NATURAL SCIENCE) (2019KZDXM042) and A Research Team Project of Guangzhou Huashang College (2021HSKT01).

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. Y. Li, C. Lin, Spatial decay for solutions to 2-D Boussinesq system with variable thermal diffusivity, *Acta Appl. Math.*, **154** (2017), 111–130. https://doi.org/10.1007/s10440-017-0136-z
2. Y. Liu, H. Qiu, C. Lin, Sptial decay bounds of solutions to the Navier-Stokes equations for transient compressible viscous flow, *J. Korean Math. Soc.*, **48** (2011), 1153–1170. https://doi.org/10.4134/JKMS.2011.48.6.1153
3. J. C. Song, Spatial decay estimates in time-dependent double-diffusive Darcy plane flow, *J. Math. Anal. Appl.*, **267** (2002), 76–88. https://doi.org/10.1006/jmaa.2001.7750
4. R. J. Knops, R. Quintanilla, Spatial decay in transient heat conduction for general elongated regions, *Q. Appl. Math.*, **76** (2018), 611–625. https://doi.org/10.1090/qam/1497
5. R. Quintanilla, Some remarks on the fast spatial growth/decay in exterior regions, *Z. Angew. Math. Phys.*, **70** (2019), 83. https://doi.org/10.1007/s00033-019-1127-x
6. Y. Li, S. Xiao, X. Chen, Spatial alternative and stability of type III Thermoelastic equations, *Appl. Math. Mech.*, **42** (2021), 431–440. https://doi.org/10.21656/1000-0887.410270
7. M. C. Leseduarte, R. Quintanilla, On the decay of solutions for the heat conduction with two temperatures, *Acta Mech.*, **224** (2013), 631–643. https://doi.org/10.1007/s00707-012-0777-y
8. X. Chen, Y. Li, Structural stability on the boundary coefficient of the Thermoelastic equations of type III, *Mathematics*, **10** (2022), 366. https://doi.org/10.3390/math10030366
9. W. Chen, Cauchy problem for thermoelastic plate equations with different damping mechanisms, *Commun. Math. Sci.*, **18** (2020), 429–457. https://doi.org/10.4310/CMS.2020.v18.n2.a7
10. W. Chen, Decay properties and asymptotic profiles for elastic waves with Kelvin-Voigt damping in 2D, *Asymptotic Anal.*, **117** (2020), 113–140. https://doi.org/10.3233/ASY-191548

11. L. E. Payne, J. C. Song, Spatial decay bounds for the Forchheimer equations, *Int. J. Eng. Sci.*, **40** (2002), 943–956. https://doi.org/10.1016/S0020-7225(01)00102-1

12. N. L. Scott, Continuous dependence on boundary reaction terms in a porous medium of Darcy type, *J. Math. Anal. Appl.*, **399** (2013), 667–675. https://doi.org/10.1016/j.jmaa.2012.10.054

13. Y. Liu, Continuous dependence for a thermal convection model with temperaturedependent solubitity, *Appl. Math. Comput.*, **308** (2017), 18–30. https://doi.org/10.1016/j.amc.2017.03.004

14. L. E. Payne, J. C. Song, Spatial decay estimates for the Brinkman and Darcy flows in a semi-infinite cylinder, *Continuum Mech. Thermodyn.*, **9** (1997), 175–190. https://doi.org/10.1007/s001610050064

15. K. A. Ames, L. E. Payne, J. C. Song, Spatial decay in the pipe flow of a viscous fluid interfacing a porous medium, *Math. Models Methods Appl. Sci.*, **118** (2001), 1547–1562. https://doi.org/10.1142/S021820250100146X

16. Y. Liu, S. Xiao, Structural stability for the Brinkman fluid interfacing with a Darcy fluid in an unbounded domain, *Nonlinear Anal. Real World Appl.*, **42** (2018), 308–333. https://doi.org/10.1016/j.nonrwa.2018.01.007

17. Y. Liu, S. Xiao, Y. Lin, Continuous dependence for the Brinkman-Forchheimer fluid interfacing with a Darcy fluid in a bounded domain, *Math. Comput. Simul.*, **150** (2018), 66–82. https://doi.org/10.1016/j.matcom.2018.02.009

18. D. A. Nield, A. V. Kuznetsov, Do isoflux boundary conditions inhibit oscillatory double-diffusive convection, *Transp. Porous Media*, **112** (2016), 609–618. https://doi.org/10.1007/s11242-016-0666-2

19. B. Straughan, Heated and salted below porous convection with generalized temperature and solute boundary conditions, *Transp. Porous Media*, **131** (2020), 617–631. https://doi.org/10.1007/s11242-019-01359-y

20. M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974. https://doi.org/10.1016/s0079-8169(08)x6044-1

21. Y. Li, X. Chen, J. Shi, Structural stability in resonant penetrative convection in a Brinkman-Forchheimer fluid interfacing with a Darcy fluid, *Appl. Math. Optim.*, **84** (2021), 979–999. https://doi.org/10.1007/s00245-021-09791-7

22. Y. Liu, X. Qin, J. Shi, W. Zhi, Structural stability of the Boussinesq fluid interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^2$, *Appl. Math. Comput.*, **411** (2021), 126488. https://doi.org/10.1016/j.amc.2021.126488

23. N. L. Scott, B. Straughan, Continuous dependence on the reaction terms in porous convection with surface reactions, *Q. Appl. Math.*, **71** (2013), 501–508. https://doi.org/10.1090/S0033-569X-2013-01289-X

24. M. Gentile, B. Straughan, Structural stability in resonant penetrative convection in a Forchheimer porous material, *Nonlinear Anal. Real World Appl.*, **14** (2013), 397–401. https://doi.org/10.1016/j.nonrwa.2012.07.003

25. Y. Li, S. Xiao, P. Zeng, The applications of some basic mathematical inequalities on the convergence of the primitive equations of moist atmosphere, *J. Math. Inequal.*, **15** (2021), 293–304. https://doi.org/10.7153/jmi-2021-15-22

26. C. Lin, L. E. Payne, Continuous dependence on the Soret coefficient for double diffusive convection in Darcy flow, *J. Math. Anal. Appl.*, **342** (2008), 311–325. https://doi.org/10.1016/j.jmaa.2007.11.036

27. R. Quintanilla, Convergence and structural stability in thermoelasticity, *Appl. Math. Comput.*, **135** (2003), 287–300. https://doi.org/10.1016/S0096-3003(01)00331-9

28. R. Quintanilla, Structural stability and continuous dependence of solutions in thermoelasticity of type III, *Discrete Contin. Dyn. Syst. B*, **1** (2001), 463–470. https://doi.org/10.3934/dcdsb.2001.1.463

29. B. Straughan, Continuous dependence on the heat source in resonant porous penetrative convection, *Stud. Appl. Math.*, **127** (2011), 302–314. https://doi.org/10.1111/j.1467-9590.2011.00521.x

30. M. Ciarletta, B. Straughan, V. Tibullo, Structural stability for a thermal convection model with temperature-dependent solubility, *Nonlinear Anal. Real World Appl.*, **22** (2015), 34–43. https://doi.org/10.1016/j.nonrwa.2014.07.012

31. C. O. Horgan, L. T. Wheeler, Spatial decay estimates for the Navier–Stokes equations with application to the problem of entry flow, *SIAM J. Math. Anal.*, **35** (1978), 97–116. https://doi.org/10.1137/0135008

32. H. A. Levine, An estimate for the best constant in a Sobolev inequality involving three integral norms, *Ann. Mat. Pura Appl.*, **4** (1980), 181–197. https://doi.org/10.1007/BF01795392

33. Y. Li, C. Lin, Continuous dependence for the nonhomogeneous Brinkman-Forchheimer equations in a semi-infinite pipe, *Appl. Math. Comput.*, **244** (2014), 201–208. https://doi.org/10.1016/j.amc.2014.06.082

34. J. Shi, S. Luo, Convergence results for the double-diffusion perturbation equations, *Symmetry*, **14** (2022), 67. https://doi.org/10.3390/sym14010067

35. H. Li, S. Wang, X. Li, G. Zhao, Perturbation analysis for controllability of logical control networks, *SIAM J. Control Optim.*, **584** (2020), 3632–3657. https://doi.org/10.1137/19M1281332

36. H. Li, X. Yang, S. Wang, Robustness for stability and stabilization of Boolean networks with stochastic function perturbations, *IEEE Trans. Autom. Control*, **66** (2021), 1231–1237. https://doi.org/10.1109/TAC.2020.2997282

37. S. Liang, H. Li, S. Wang, Structural controllability of Boolean control networks with unknown function structure, *Sci. China Inf. Sci.*, **63** (2020), 219203. https://doi.org/10.1007/s11432-018-9770-4