Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(2): 2874–2889.
DOI: 10.3934/mbe.2023135
Received: 08 September 2022
Revised: 06 November 2022
Accepted: 17 November 2022
Published: 01 December 2022

Research article

Complex dynamics and Bogdanov-Takens bifurcations in a retarded van der Pol-Duffing oscillator with positional delayed feedback

Mohammad Sajid[1,*], Sahabuddin Sarwardi[2], Ahmed S. Almohaimeed[3] and Sajjad Hossain[2]

[1] Department of Mechanical Engineering, College of Engineering, Qassim University, Buraydah 51452, Saudi Arabia

[2] Department of Mathematics and Statistics, Aliah University, IIA/27, New Town, Kolkata 700160 India

[3] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: msajd@qu.edu.sa; Tel: +966163013761.

Abstract: In this article, we will investigate a retarded van der Pol-Duffing oscillator with multiple delays. At first, we will find conditions for which Bogdanov-Takens (B-T) bifurcation occurs around the trivial equilibrium of the proposed system. The center manifold theory has been used to extract second order normal form of the B-T bifurcation. After that, we derived third order normal form. We also provide a few bifurcation diagrams, including those for the Hopf, double limit cycle, homoclinic, saddle-node, and Bogdanov-Takens bifurcation. In order to meet the theoretical requirements, extensive numerical simulations have been presented in the conclusion.

Keywords: retarded van der Pol-Duffing oscillator; damping; Bogdanov-Takens bifurcation; chaos; center manifold; delayed feedback; numerical simulation

1. Introduction

It is well known that the dynamics of harmonic oscillators has long been studied for numerous applications in electrodynamics, engineering, electronics, neurology and biological systems. Time-delayed feedback control has been used to a variety of disciplines of inquiry, including chaos communication, electronic systems, and engineering. It is known that the amplitude of oscillations always differs for the positional delayed feedback. The purpose of the present study is to investigate the global dynamics of a retarded van der-Pol Duffing oscillator with retarded damping and delayed position feedback, which is a new idea in the study of harmonic oscillators. Due to the numerous uses of the harmonic oscillator, including in physics and many other disciplines, it has been discussed for a very long period. If the frictional force is proportional to the velocity, the harmonic oscillator is referred to as a damped oscillator. In the case of damped oscillators, the vibration amplitude decreases with time. Damping is important in realistic oscillatory systems, which can be positive or negative.

We get delay differential equation by introducing time delay into the ordinary differential equation. Clearly it is more realistic. Following ground-breaking discovery done by Pyragas [1], time-delayed feedback control has been applied to numerous areas of study, including chaos communication, engineering and electronic systems. From the study of Atay [2], we know that positional delayed feedback change the amplitude of oscillations. We can describe the damped harmonic oscillator with delayed feedback by the following functional differential equation:

$$\ddot{x}(t) + \epsilon(1 - x^2)\dot{x}(t) + ax(t) + bx^3(t) = g(x(t - \eta)), \tag{1.1}$$

where $x(t) \in \mathbf{R}$ denote the displacement from the point of oscillation, $a > 0$ denote the stiffness of the spring element, g is a C^r $(r \geq 3)$ smooth function which narrates the delayed feedback, ϵ is the damping coefficient, η denote the time delay, and b is the Duffing coefficient of order three. Equation (1.1) has been analyzed by many authors (cf. Jiang and Song [3], Campbell et al. [4], Song et al. [5], Song and Xu [6], Cao et al. [7], Campbell and Yuan [8]).

So far, there have been numerous research findings on Hopf bifurcation, Hopf-zero bifurcation and triple-zero bifurcation of several oscillators studied by Cao and Yuan [9], Qiao et al. [10]. But, there are limited number of research articles that described the B-T bifurcation of the harmonic oscillators with multiple delay and retarded damping. Inspired from the discussions in [11–14] time delay is incorporated in Eq (1.1), which then extended to the following retarded van der Pol-Duffing equation with positional delayed feedback:

$$\ddot{x}(t) + \epsilon(1 - x^2(t))\dot{x}(t - \tau) + ax(t) + bx^3(t) = g(x(t - \eta)). \tag{1.2}$$

It is simple to verify that the system's origin Eq (1.2) is a double-zero singularity subjected to some parametric conditions. The primary target of this study is to discuss the B-T bifurcation and extract the related normal form of the dynamical Eq (1.2). We know that B-T singularity is a critical point at which the Jacobian matrix has double zero eigenvalue and geometric multiplicity one. Also, we know that the order of the ordinary differential equation is at least two if it contains a B-T singularity, which is not true for the delay differential equation.

The system of delay differential equations can be converted to a two-dimensional ordinary differential equation centred on the B-T singular point, using the center manifold reduction and normal form theory. In fact, a variety of dynamical systems, including chemical reaction and neural network models, can accurately depict the B-T bifurcation under certain critical circumstances. This paper will investigate the B-T bifurcation of a retarded delay differential equation with damping effect.

The remaining part of this article is outlined as follows: The parametric restrictions under which the origin is a B-T singularity are stated in Section 2. Section 3 provides a detailed description of the B-T singularity as well as the second and third order normal forms corresponding bifurcation. In Section 4, we carried out numerical simulations to verify the analytical findings made in this study. In Section 5, the article concludes with a brief conclusion.

2. Bogdanov-Takens bifurcation

The Bogdanov-Takens bifurcation is an unusual and fascinating bifurcation of continuous dynamical systems, in which the Jacobian matrix has a double-zero eigenvalue. It's a bit difficult to explain what's going on. For this type of bifurcation, there is a two dimensional parametric space, wherein there exist a curve of saddle-node bifurcations; a curve of Hopf bifurcations; and a curve of homoclinic bifurcations. The intersection of these three curves is know as Bogdanov-Takens bifurcation point. One of the ways that limit cycles are created or annihilated is in the context of a Bogdanov-Takens bifurcation. These global bifurcations are unlike anything we've yet seen in the local bifurcations like Hopf, Saddle-node, Transcritical and Pitchfork bifurcations. Considering the aforementioned discussion, the retarded van der Pol-Duffing oscillator with two delays can represented mathematically with the following retarded delay differential system:

$$\ddot{x}(t) + \epsilon(1 - x^2(t))\dot{x}(t-\tau) + ax(t) + bx^3(t) = g(x(t-\eta)), \tag{2.1}$$

where τ and η represents the retarded delay on damping and positional feedback respectively. Let $x(t) = x_1(t)$ and $\dot{x}_1(t) = x_2(t)$. Then Eq (2.1) becomes

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -ax_1(t) - bx_1^3(t) - \epsilon(1 - x_1^2(t))x_2(t-\tau) + g(x_1(t-\eta)). \end{cases} \tag{2.2}$$

We will study this Eq (2.1) theoretically and numerically for both the focus ($\epsilon > 0$) and saddle ($b < 0$) cases.

Let us consider $a > 0,\ g(0) = 0$, and $g'(0) = d$. Thus, the characteristic equation of the Jacobian matrix of the Eq (2.2) around $(0, 0)$ is given by

$$E(\lambda) = \lambda^2 + \epsilon\lambda e^{-\lambda\tau} + a - de^{-\lambda\eta} = 0. \tag{2.3}$$

Lemma 1. *Let $2 + \epsilon(\eta - 2\tau) > 0,\ d = a$ and $\epsilon = -a\eta$. Then the Eq (2.3) has a double-zero root, namely $\lambda = 0$.*

Proof. Here, $E(0) = a - d$. $E(0) = 0$ as $d = a$. Clearly, $E^{'}(\lambda) = 2\lambda + \epsilon e^{-\lambda\tau} - \epsilon\tau\lambda e^{-\lambda\tau} + d\eta e^{-\lambda\eta}$, consequently $E^{'}(0) = 0$ as $\epsilon = -a\eta$. Also, we have $E''(0) = 2 + \epsilon(\eta - 2\tau)$. Thus $E''(0) \neq 0$, since $2 + \epsilon(\eta - 2\tau) > 0$. Therefore, we conclude that 0 is a characteristic root of multiplicity two.

Lemma 2. *If $2 + \epsilon(\eta - 2\tau) > 0,\ d = a$ and $\epsilon = -a\eta$, and $a \in (0, M)$, then all roots of Eq (2.3) except $\lambda = 0$, have non-zero real parts, i.e., origin of the Eq (2.2) is B-T singularity, where $M = \{\min a_j :\ a_j = \frac{\sigma_j^2}{1 - \eta\sigma_j \sin\tau\sigma_j - \cos\eta\sigma_j} > 0,\ 1 \leq j \leq m,$ and σ_j are the roots of the equation $\sigma^2 - (\epsilon^2 + 2a) + \frac{2a\epsilon\sin(\tau-\eta)\sigma}{\sigma} = 0\}$.*

Proof. The proof is carried out in the same technique adopted by Sarwardi et al. [15].

3. Normal forms of Bogdanov-Takens bifurcation of the Eq (2.2)

From Lemma 2, we see that the Eq (2.2) experiences B-T bifurcation around the origin, if $\epsilon = -a\eta$ and $d = a \in (0, M)$ hold. Hence, we assume d and ϵ as bifurcation parameters. Therefore, we may incorporate two small parameters ν_1 and ν_2 in the vicinity of origin and then discuss perturbation of the Eq (2.2). Expanding the Eq (2.2) according to Taylor's theorem, we have

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -ax_1(t) - (\epsilon + \nu_2)x_2(t-\tau) + (a+\nu_1)x_1(t-\eta) + \cdots . \end{cases} \tag{3.1}$$

After simplification, on the phase space C, we may write the Eq (3.1) as the following retarded functional differential equation :

$$\dot{X}(t) = L(\nu)X_t + H(X_t, \nu), \tag{3.2}$$

where C is the Banach space of all continuous functions from $[-\tau, 0]$ to $\mathbf{R}^2$ with supremum norm $|\xi| = \sup\limits_{\theta\in[-\tau,0]} |\xi(\theta)|$, $X_t \in C$ is defined by $X_t(\theta) = X(t+\theta)$ for $\theta \in [-\tau, 0]$, and $L(\nu) : C \to \mathbf{R}^2$ is a parameterized family of bounded linear operators defined as follows:

$$\begin{aligned} L(\nu)(\xi) &= L_0(\xi) + L_1(\nu)(\xi) \\ &= \begin{pmatrix} \xi_2(0) \\ -a\xi_1(0) - \epsilon\xi_2(-\tau) + a\xi_1(-\eta) \end{pmatrix} + \begin{pmatrix} 0 \\ \nu_1\xi_1(-\eta) - \nu_2\xi_2(-\tau) \end{pmatrix}; \end{aligned} \tag{3.3}$$

and $H : C \times W \to \mathbf{R}^2$ is a C^k $(k \geq 2)$ function satisfying $H(0,0),\ DH(0,0) = 0$ with

$$H(\xi, \nu) = \frac{g''(0)}{2}\begin{pmatrix} 0 \\ \xi_1^2(-\eta) \end{pmatrix} + \begin{pmatrix} 0 \\ -b\xi_1^3(0) + \epsilon\xi_1^2(0)\xi_2(-\tau) + \frac{g'''(0)\xi_1^3(-\eta)}{3!} \end{pmatrix} + \cdots . \tag{3.4}$$

Now, we will linearize the Eq (3.2) around $(X_t, \nu) = (0, 0)$ and obtained the followings:

$$\dot{X}(t) = L(0)X_t, \tag{3.5}$$

where $L(0)(\xi) = A\xi(0) + B_1\xi(-\tau) + B_2\xi(-\eta)$,

$$A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

The linear operator L_1 may be defined in the similar fashion.

Now, we will use Riesz representation theorem on the bounded linear operator L_0 and obtain a 2×2 matrix $\rho(.)$ defined on $[-\tau, 0]$ whose entries are functions of bounded variation, such that

$$L_0\xi = \int_{-\tau}^{0} [d\rho(\theta)]\xi(\theta). \tag{3.6}$$

From [16] and [17], we define the adjoint inner product on $C^* \times C$ as follows:

$$\langle \phi, \psi \rangle = \phi(0)\psi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \phi(\xi - \theta)[d\rho(\theta)]\psi(\xi)d\xi,\ \psi \in C,\ \phi \in C^*, \tag{3.7}$$

where $C^* = C([0, \tau], \mathbf{R}^{2*})$ be the adjoint space of C. Let $A_0 : C \to C$ be the infinitesimal generator, defined by $A_0(\psi) = \dot{\psi}$. Assuming $\Lambda_0 = \{0, 0\}$, the set of eigenvalues with zero real part of A_0, we define the invariant space of A_0 associated with Λ_0, and the dual of it by P and P^* respectively. Now, we will decompose C by the help of adjoint theory for a functional differential equation as follows:

$$C = P \oplus Q, \text{ where } Q = \{\psi \in C : \langle \phi, \psi \rangle = 0, \ \forall \phi \in P^*\}. \tag{3.8}$$

Now, we define the bases for P and P^* by Ψ and Φ respectively. We are assuming Ψ and Φ as follows: $\Psi = (\psi_1(\theta), \psi_2(\theta))$ for $-\tau \leq \theta \leq 0$, and $\Phi = (\phi_1(s), \phi_2(s))^T$ for $0 \leq s \leq \tau$. Then $\langle \Phi, \Psi \rangle = I_2$, and $\dot{\Psi} = \Psi \bar{B}$, $-\dot{\Phi} = \bar{B}\Phi$, where $\bar{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. After using Lemma 3.1 of [18], we get

$$\Psi(\theta) = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \ -\tau \leq \theta \leq 0$$

and

$$\Phi(0) = \begin{pmatrix} \frac{2\epsilon^2(\eta^2-3\tau^2)}{3[2+\epsilon(\eta-2\tau)]^2} + \frac{2(1-\epsilon\tau)}{2+\epsilon(\eta-2\tau)} & \frac{2\epsilon(\eta^2-3\tau^2)}{3[2+\epsilon(\eta-2\tau)]^2} \\ \frac{2\epsilon}{2+\epsilon(\eta-2\tau)} & \frac{2}{2+\epsilon(\eta-2\tau)} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \text{(say)}.$$

In the next part of this section, B-T bifurcation of the Eq (3.2) will be discussed. For that we will extend the phase space C by Banach space $BC = \{\psi : [-\tau, 0] \to \mathbf{R}^2 : \psi \text{ is continuous on } [-\tau, 0), \text{ with a possible jump discontinuity at zero}\}$. Any element ψ of BC may be assumed as $\psi = \chi + Y_0 e$ and norm of the same is given by $||\psi|| = ||\chi + Y_0 c|| = ||\chi||_C + |e|$, where Y_0 is a 2×2 matrix valued function defined as follows

$$Y_0(\theta) = \begin{cases} I_2 & \text{if } \theta = 0, \\ O & \text{if } -\tau \leq \theta < 0. \end{cases} \tag{3.9}$$

Now, in BC, the Eq (3.2) may be transformed to the following form:

$$\frac{d}{dt} w = \bar{A} w + X_0 G(w, \nu), \tag{3.10}$$

where $G(w, \nu) = (L(\nu) - L_0)w + H(w, \nu) = L_1(\nu)w + H(w, \nu), \ \nu \in W$ and $\bar{A} : C^1 \to BC$ is the extension of infinitesimal generator A_0, it is defined by

$$\bar{A}\chi = \dot{\chi} + X_0[L_0\chi - \dot{\chi}(0)] = \begin{cases} \dot{\chi}, & -\tau \leq \theta < 0 \\ L_0\chi, & \theta = 0. \end{cases} \tag{3.11}$$

Now, we will use the result of [19] and letting $w = \psi(\theta)z(t) + y$ to decompose the Eq (3.10) as follows:

$$\begin{cases} \dot{z} = \bar{B}z + \Phi(0)G(\Psi z + y, \mu), \\ \frac{d}{dt} y = A_{Q^1} y + (I - \pi)Y_0 G(\Psi z + y, \nu), \end{cases} \tag{3.12}$$

where $\pi : BC \to P$ is the projection operator and $A_{Q^1} : Q \cap C^1 \to \ker \pi$ is the restriction of $\bar{A}$. Using Taylor series expansion for $G(\Psi z + y, \nu)$ at $(z, y, \nu) = (0, 0, 0)$, above equation becomes

$$\begin{cases} \dot{z} = \bar{B}z + \Phi(0) \sum_{j \geq 2} \frac{1}{j!} G_j(\Psi z + y, \mu) = \bar{B}z + \sum_{j \geq 2} \frac{1}{j!} g_j^1(z, y, \nu), \\ \frac{d}{dt} y = A_{Q^1} y + (I - \pi)Y_0 \sum_{j \geq 2} \frac{1}{j!} G_j(\Psi z + y, \nu) = A_{Q^1} y + \sum_{j \geq 2} \frac{1}{j!} g_j^2(z, y, \nu); \end{cases} \tag{3.13}$$

where

$$G_2(\Psi z + y, \nu) = \begin{pmatrix} 0 \\ 2\nu_1[z_1 - \eta z_2 + y_1(-\eta)] - 2\nu_2[z_2 + y_2(-\tau)] + g''(0)[z_1 - \eta z_2 + y_1(-\eta)]^2 \end{pmatrix},$$

$$G_3(\Psi z + y, \nu) = \begin{pmatrix} 0 \\ -6b[z_1 + y_1(0)]^3 + 6\epsilon[z_1 + y_1(0)]^2[z_2 + y_2(-\tau)] + g'''(0)[z_1 - \eta z_2 + y_1(-\eta)]^3 \end{pmatrix}.$$

Now, we define the operator N_j^1 on $V_j^4(\mathbf{R}^2)$, the vector space of homogeneous polynomials in $z_1,\ z_2,\ \nu_1,\ \nu_2$ of degree j having coefficients in $\mathbf{R}^2$, by

$$N_j^1 \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial p_1}{\partial z_1} z_2 - p_2 \\ \frac{\partial p_2}{\partial z_1} z_2 \end{pmatrix}. \tag{3.14}$$

Using [18] and [20], we may decompose $V_j^4(\mathbf{R}^2)$ as follows:

$$V_j^4(\mathbf{R}^2) = I_m(N_j^1) \oplus I_m(N_j^1)^c. \tag{3.15}$$

Now, we consider a mapping from $V_j^4(\mathbf{R}^2)$ to $I_m(N_j^1)$ $(j = 2, 3)$ denoted by $P_{I,j}^1$ which satisfies

$$q - r \in I_m(N_j^1)^c \text{ if } P_{I,j}^1(q) = r.$$

It is clearly visible that

$$Pr_{I_m(N_j^1)^c} q = \begin{cases} q, & \text{if } q \in I_m(N_j^1)^c \\ 0 & \text{if } q \in I_m(N_j^1). \end{cases} \tag{3.16}$$

Using [19, 21], the normal form of Eq (2.2) on the center manifold associated to the space P can be taken as

$$\dot{z} = \bar{B}z + \sum_{j\geq 2} \frac{1}{j!} f_j^1(z, 0, \nu), \tag{3.17}$$

where $f_2^1(z, 0, \nu) = Pr_{I_m(M_2^1)^c} g_2^1(z, 0, \nu)$

$$= \begin{pmatrix} 0 \\ 2\phi_{22}\nu_1 z_1 + 2[(\phi_{12} - \eta\phi_{22})\nu_1 - \phi_{22}\nu_2]z_2 + g''(0)\phi_{22}z_1^2 + 2g''(0)(\phi_{12} - \eta\phi_{22})z_1 z_2 \end{pmatrix}.$$

The Eq (2.2) may be converted into the following normal form on the center manifold:

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \lambda_1 z_1 + \lambda_2 z_2 + c_2 z_1^2 + d_2 z_1 z_2 + \text{h.o.t.}, \end{cases} \tag{3.18}$$

where $\lambda_1 = \phi_{22}\nu_1,\ \lambda_2 = (\phi_{12} - \eta\phi_{22})\nu_1 - \phi_{22}\nu_2,\ c_2 = \frac{g''(0)\phi_{22}}{2},\ d_2 = g''(0)(\phi_{12} - \eta\phi_{22})$.

We can have a small a such that $\phi_{12} - \eta\phi_{22} < 0$ as $0 < a < M$. Without loss of generality, we are assuming that $\phi_{12} - \eta\phi_{22} < 0$. As per our assumption $2 + \epsilon(\eta - 2\tau) > 0$ it is clear that $\phi_{22} > 0$. Hence, the sign of $g''(0)$ determines the sign of c_2 and d_2.

Case I: If $g''(0) > 0$, then $c_2 > 0$ and $d_2 < 0$. After re-scaling the time and change the coordinates in the following manner:

$$t = -\frac{d_2}{c_2}\bar{t},\ z_1 = \frac{c_2}{d_2^2}\bar{z}_1,\ z_2 = -\frac{c_2^2}{d_2^3}\bar{z}_2.$$

The Eq (3.18) transforms to the following form on the center manifold

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \alpha_1 z_1 + \alpha_2 z_2 + z_1^2 - z_1 z_2 + \text{h.o.t.}, \end{cases} \tag{3.19}$$

where $\alpha_1 = \frac{4(\phi_{12}-\eta\phi_{22})^2}{\phi_{22}}\nu_1$ and $\alpha_2 = -\frac{2(\phi_{12}-\eta\phi_{22})^2}{\phi_{22}}\nu_1 + 2(\phi_{12} - \eta\phi_{22})\nu_2$ (dropping bars).

The following summarizes the bifurcation curves related to the perturbation parameters $\nu_1,\ \nu_2$ [3, 6, 8, 20]:

(i) If $\nu_1 = 0$, then transcritical bifurcation occurs.
(ii) If $\nu_1 < 0$ and $3[2+\epsilon(\eta-2\tau)]\nu_2 + [6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]\nu_1 = 0$, then Hopf bifurcation from the zero equilibrium point (H_0) occurs.
(iii) If $\nu_1 > 0$ and $3[2+\epsilon(\eta-2\tau)]\nu_2 - [6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]\nu_1 = 0$, then Hopf bifurcation from the non-zero equilibrium point (H_1) occurs.
(iv) If $\nu_1 > 0$ and $21[2+\epsilon(\eta-2\tau)]\nu_2 - 5[6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]\nu_1 = 0$, then Homoclinic bifurcation from the zero equilibrium point (H_C^0) occurs.
(v) If $\nu_1 < 0$ and $3[2+\epsilon(\eta-2\tau)]\nu_2 + [6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]\nu_1 = 0$, then Homoclinic bifurcation from the non-zero equilibrium point (H_C^1) occurs.

Case II: If $g''(0) < 0$ then $c_2 < 0$ and $d_2 > 0$. After re-scaling the time and change the coordinates in the following manner:

$$t = -\frac{d_2}{c_2}\bar{t},\ z_1 = -\frac{c_2}{d_2^2}\bar{z}_1,\ z_2 = \frac{c_2^2}{d_2^3}\bar{z}_2. \tag{3.20}$$

The Eq (3.18) transforms to the following form on the center manifold (dropping the bars)

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \alpha_1 z_1 + \alpha_2 z_2 - z_1^2 + z_1 z_2 + \text{h.o.t.}, \end{cases} \tag{3.21}$$

where $\alpha_1,\ \alpha_2$; and the corresponding bifurcation curves are the same as of **Case I**.

Case III: If $g''(0) = 0$, then $c_2 = d_2 = 0$. We must compute higher order normal forms in order to talk about the dynamics near the B-T singularity. From [19] and [20], we have

$$f_3^1(z,0,\nu) = Pr_{I_m(N_3^1)^c}\bar{g}_3^1(z,0,\nu), \tag{3.22}$$

where $\bar{g}_3^1(z,0,\nu) = g_3^1(z,0,\nu) + \frac{3}{2}[(D_z g_2^1)(z,0,\nu)U_2^1(z,\nu) - (D_z U_2^1)(z,\nu)f_2^1(z,0,\nu) + (D_y g_2^1)(z,0,\nu)U_2^2(z,\nu)]$. After computation, we have

$$g_3^1(z,0,\nu) = \begin{pmatrix} \phi_{12}[6(-bz_1^3 + \epsilon z_1^2 z_2) + g'''(0)(z_1 - \eta z_2)^3] \\ \phi_{22}[6(-bz_1^3 + \epsilon z_1^2 z_2) + g'''(0)(z_1 - \eta z_2)^3] \end{pmatrix}, \tag{3.23}$$

where U_2^1 and U_2^2 are defined in [20].

To compute $f_3^1(z,0,\nu)$, we have used similar method as in our previous work [15].

Using the same process as in [20], we have the followings:

$$Pr_{I_m(N_3^1)^c} g_3^1(z,0,\nu) = \phi_{22}[-6b + g'''(0)]\begin{pmatrix} 0 \\ z_1^3 \end{pmatrix} + 3[\phi_{12}(-6b + g'''(0)) - \eta\phi_{22}g'''(0)]\begin{pmatrix} 0 \\ z_1^2 z_2 \end{pmatrix},$$

$$Pr_{I_m(N_3^1)^c} D_z U_2^1(z,0,\nu)U_2^1(z,\nu) = 4\phi_{12}\phi_{22}\begin{pmatrix} 0 \\ \nu_1\nu_2 z_1 \end{pmatrix} + 4\eta\phi_{12}\phi_{22}\begin{pmatrix} 0 \\ \nu_1^2 z_1 \end{pmatrix} + 4\phi_{12}(\phi_{12} - 2\eta\phi_{22})\begin{pmatrix} 0 \\ \nu_1\nu_2 z_2 \end{pmatrix}$$
$$+4\eta\phi_{12}(\phi_{12} - \eta\phi_{22})\begin{pmatrix} 0 \\ \nu_1^2 z_2 \end{pmatrix} - 4\phi_{12}\phi_{22}\begin{pmatrix} 0 \\ \nu_2^2 z_2 \end{pmatrix},$$

$$Pr_{I_m(N_3^1)^c} D_z U_2^1(z,\nu)f_2^1(z,0,\nu) = 4\phi_{12}\phi_{22}\begin{pmatrix} 0 \\ \nu_1\nu_2 z_2 \end{pmatrix} + 4\eta\phi_{12}\phi_{22}\begin{pmatrix} 0 \\ \nu_1^2 z_1 \end{pmatrix} + 4\phi_{12}(\phi_{12} - 2\eta\phi_{22})\begin{pmatrix} 0 \\ \nu_1\nu_2 z_2 \end{pmatrix}$$
$$-4\phi_{12}\phi_{22}\begin{pmatrix} 0 \\ \nu_2^2 z_2 \end{pmatrix} + 4\eta\phi_{12}(\phi_{12} - \eta\phi_{22})\begin{pmatrix} 0 \\ \nu_1^2 z_2 \end{pmatrix},$$

$$Pr_{I_m(N_3^1)^c}(D_y g_2^1(z,0,\nu))U_2^2(z,\nu) = 2[\phi_{12}l_{21010}^1(-\eta) + \phi_{22}l_{20110}^1(-\eta)]\begin{pmatrix} 0 \\ \nu_1^2 z_2 \end{pmatrix} + 2\phi_{22}l_{21010}^1(-\eta)\begin{pmatrix} 0 \\ \nu_1^2 z_1 \end{pmatrix}$$
$$-2\phi_{22}l_{21010}^2(-\tau)\begin{pmatrix} 0 \\ \nu_1\nu_2 z_1 \end{pmatrix} - 2\phi_{22}l_{20101}^2(-\tau)\begin{pmatrix} 0 \\ \nu_2^2 z_2 \end{pmatrix}$$
$$+2[\phi_{22}l_{20101}^1(-\eta) - \phi_{22}l_{20110}^2(-\tau) - \phi_{12}l_{21010}^2(-\tau)]\begin{pmatrix} 0 \\ \nu_1\nu_2 z_2 \end{pmatrix};$$

where $l_2^i(\theta)(z_1,z_2,\nu_1,\nu_2) = l_{22000}^i(\theta)z_1^2 + l_{20200}^i(\theta)z_2^2 + l_{20020}^i(\theta)\nu_1^2 + l_{20002}^i(\theta)\nu_2^2 + l_{21100}^i(\theta)z_1z_2 + l_{21010}^i(\theta)\nu_1 z_1 + l_{21001}^i(\theta)\nu_2 z_1 + l_{20110}^i(\theta)\nu_1 z_2 + l_{20101}^i(\theta)\nu_2 z_2 + l_{20011}^i(\theta)\nu_1\nu_2$ for $i = 1, 2$.

From above equations, we have

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \beta_1 z_1 + \beta_2 z_2 + c_3 z_1^3 + d_3 z_1^2 z_2 + \text{h.o.t.}, \end{cases} \tag{3.24}$$

where $\beta_1 = \phi_{22}\nu_1 + \frac{1}{2}\phi_{22}l_{21010}^1(-\eta)\nu_1^2 - \frac{1}{2}\phi_{22}l_{21010}^2(-\tau)\nu_1\nu_2$, $\beta_2 = (\phi_{12} - \eta\phi_{22})\nu_1 - \phi_{22}\nu_2 + \frac{1}{2}[\phi_{22}l_{20110}^1(-\eta) + \phi_{12}l_{21010}^1(-\eta)]\nu_1^2 + \frac{1}{2}[\phi_{22}l_{20101}^1(-\eta) - \phi_{22}l_{20110}^2(-\tau) - \phi_{12}l_{21010}^2(-\tau)]\nu_1\nu_2 - 3\phi_{22}l_{20101}^2(-\tau)\nu_2^2$, $c_3 = \frac{\phi_{22}}{6}[-6b + g'''(0)]$, $d_3 = \frac{1}{2}[\phi_{12}(-6b + g'''(0)) - \eta\phi_{22}g'''(0)]$.

Utilizing the following time re-scaling and co-ordinate shifting :

$$\bar{t} = -\frac{|c_3|}{d_3}t,\ \gamma_1 = \frac{d_3}{\sqrt{|c_3|}}z_1,\ \gamma_2 = -\frac{d_3^2}{c_3\sqrt{|c_3|}}z_3, \tag{3.25}$$

the Eq (3.24) converted to the following

$$\begin{cases} \dot{\zeta}_1 = \zeta_2, \\ \dot{\zeta}_2 = \gamma_1\zeta_1 + \gamma_2\zeta_2 + s\zeta_1^3 - \zeta_1^2\zeta_2 + \text{h.o.t.}, \end{cases} \tag{3.26}$$

where $\gamma_1 = (\frac{d_3}{c_3})^2\beta_1,\ \gamma_2 = -\frac{d_3}{|c_3|}\beta_2,\ s = sgn(c_3)$.

The sign of s determines the bifurcations of the Eq (3.26) [22,23].

For s = 1, the bifurcations for the small perturbed parameters (ν_1, ν_2) are summarised as follows [3,8,20]:

(i) On the curve $S = \{(\nu_1, \nu_2) : \nu_1 = 0, \ \nu_2 \in \mathbf{R}\}$, the Eq (3.26) possesses pitchfork bifurcation.
(ii) On the curve $H = \{(\nu_1, \nu_2) : \nu_2 = -\frac{6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)}{3[2+\epsilon(\eta-2\tau)]}\nu_1 + O(\nu_1^2), \ \nu_1 < 0\}$, the Eq (3.26) possesses Hopf bifurcation at the zero equilibrium point.
(iii) On the curve $L = \{(\nu_1, \nu_2) : \nu_2 = -\frac{2[6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]}{15[2+\epsilon(\eta-2\tau)]}\nu_1 + O(\nu_1^2), \ \nu_1 < 0\}$, the Eq (3.26) experiences Heteroclinic bifurcation at the zero equilibrium point.

For $s = -1$, the bifurcation curves are concluded as follows [4, 5, 8]:

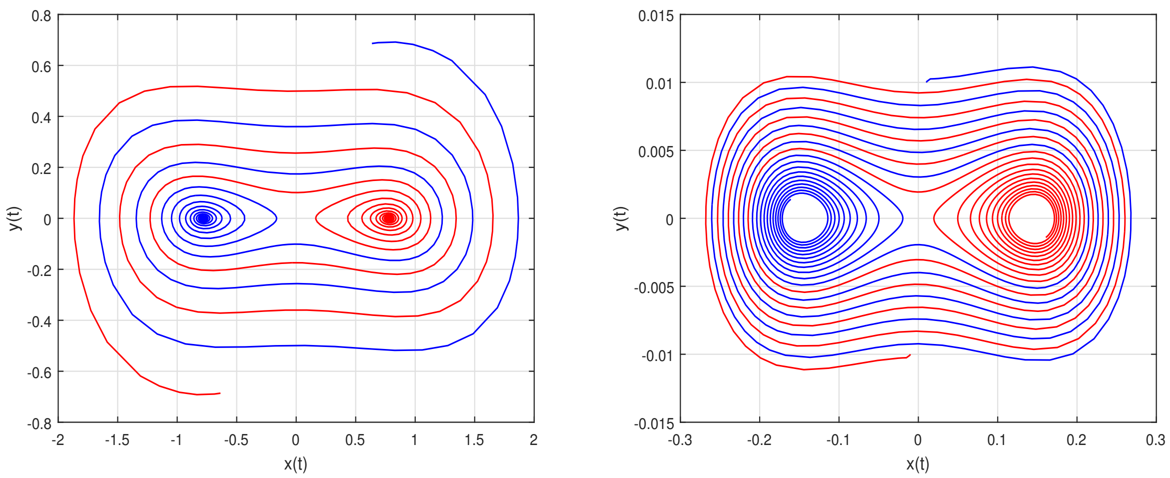
(i) On the curve $S = \{(\nu_1, \nu_2) : \nu_1 = 0, \ \nu_2 \in \mathbf{R}\}$, the Eq (3.26) experiences Pitchfork bifurcation.
(ii) On the curve $H_1 = \{(\nu_1, \nu_2) : \nu_2 = -\frac{6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)}{3[2+\epsilon(\eta-2\tau)]}\nu_1 + O(\nu_1^2), \ \nu_1 < 0\}$, the Eq (3.26) experiences Hopf bifurcation at the zero equilibrium point.
(iii) On the curve $H_2 = \{(\nu_1, \nu_2) : \nu_2 = -\frac{4[6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]}{3[2+\epsilon(\eta-2\tau)]}\nu_1 + O(\nu_1^2), \ \nu_1 > 0\}$, the Eq (3.26) experiences Hopf bifurcation at the non-zero equilibrium point.
(iv) On the curve $T = \{(\nu_1, \nu_2) : \nu_2 = -\frac{17[6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]}{15[2+\epsilon(\eta-2\tau)]}\nu_1 + O(\nu_1^2), \ \nu_1 > 0\}$, the Eq (3.26) experiences Homoclinic bifurcation.
(v) On the curve $H_d = \{(\nu_1, \nu_2) : \nu_2 = -\frac{3.256[6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)]}{3[2+\epsilon(\eta-2\tau)]}\nu_1 + O(\nu_1^2), \ \nu_1 > 0\}$, the Eq (3.26) experiences fold bifurcation of the limit cycle.

4. Numerical results

We have only so far focused on the Eq (2.2)'s dynamical behaviour in terms of theoretical observations. This section discusses some numerical instances to support the validity of our theoretical conclusions and uncover other standout characteristics of the specified system. To show the phase diagrams, Mathematica 7.0, Maple 18, and MATLAB-R2015a were utilized as the computer programmes. The system parameter values are taken from the study done by Siewe et al. [24] with slightest deviation. Let us consider the positional delayed feedback function $g(x(t-\eta)) = \frac{\sin x(t-\eta)}{5}$, we have $g(0) = 0$, $g'(0) = 0.2$, $g''(0) = 0.0$ and $g'''(0) = -0.2 < 0$. For the purpose of the simulation, we have selected the following set of parameters: $a = d = 0.2$, $b = 0.05$, $\epsilon = -0.1$, $\tau = 0.5$, $\eta = 0.5$, for which the condition of double zero eigenvalues (i) $\epsilon = -a\tau_2$ and (ii) $[2 + \epsilon(\eta - 2\tau)] = 2.05 > 0$ are fulfilled. Moreover, it is calculated that the value of $-\frac{6\eta+\epsilon(3\tau^2+2\eta^2-6\tau\eta)}{3[2+\epsilon(\eta-2\tau)]} = -0.4918699187$.

Following numerical studies display an agreement with our analytical findings: The phase portraits and the solution trajectory of Eq (2.2) with the critical B-T bifurcation parameter value $(d, \epsilon) = (0.2, -0.1)$ is demonstrated by Figure 1. Figure 1(a) corresponds to unperturbed Bogdanov-Takens bifurcation parameter $(d, \epsilon) = (0.2, -0.1)$ and Figure 1(b) is the phase diagram with a small perturbation $(\nu_1, \nu_2) = (-0.002, 0.00098)$. If we choose the initial point at $(0.06364, 0.06861)$, the system produce slowly converging diagram around the origin (see Figure 2).

We now take into account a minor alteration to the bifurcation parameters by letting $(d, \epsilon) = (0.2 + \nu_1, -0.1 + \nu_2)$. If we consider the perturbation parameters $(\nu_1, \nu_2) = (-0.01, 0.0)$, there exists a stable limit cycle around the trivial equilibrium of Eq (2.2) (see intersection of blue and red solution curves of Figure 3). From the bifurcation diagram (cf. Figure 4), one can be concluded that the the coefficient of damping ϵ has a crucial impact on regulating the oscillatory nature of Eq (2.2).



(a) (b)

Figure 1. (a) Phase portraits of Eq (2.2) with unperturbed pair of parameters $(d, \epsilon) = (0.2, -0.1)$, with $f(x(t-\eta)) = 0.2 \sin x(t-\eta))$ corresponding to the parameter set : $b = 0.05$, $\epsilon = -0.1$, $\tau = 0.5$, $\eta = 0.5$. (b) Phase diagram corresponding to the perturbation $(\nu_1, \nu_2) = (-0.002, 0.00098)$ to the bifurcation parameters (d, ϵ).

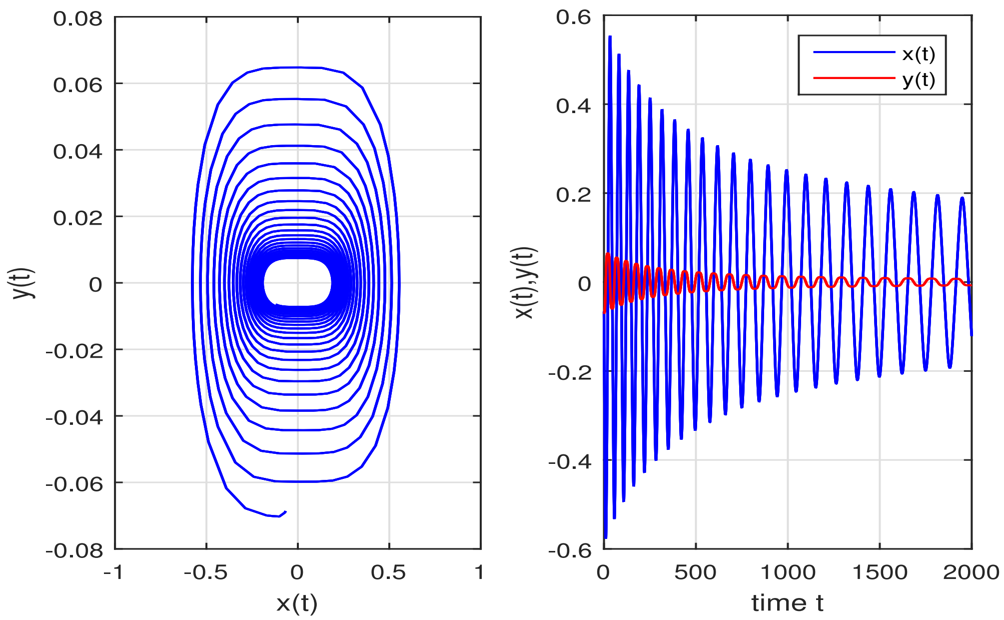


Figure 2. For $(\nu_1, \nu_2) = (0.00, 0.00)$, and $a = 0.2$, $b = 0.05$, $\epsilon = -0.1$, $d = 0.2$, $\tau = 0.5$, $\eta = 0.5$. The left panel shows the phase diagram around the trivial equilibrium $(0, 0)$. The right panel shows time series solution of the Eq (2.2).

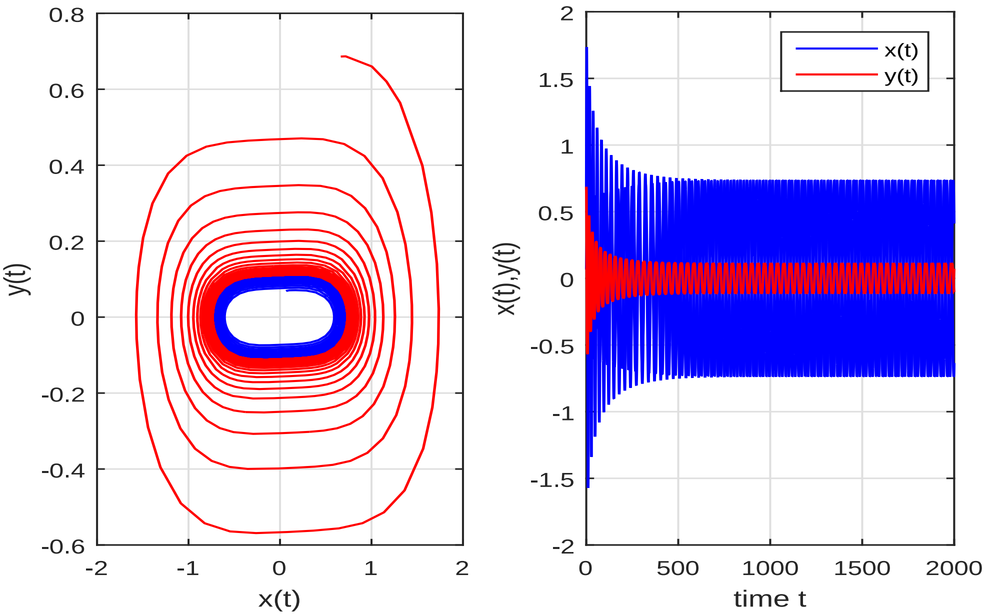


Figure 3. For $(v_1, v_2) = (-0.0, -0.01)$, and $a = 0.2$, $b = 0.05$, $\epsilon = -0.1$, $d = 0.2$, $\tau = 0.5$, $\eta = 0.5$. The left panel demonstrate the existence of stable limit cycle the Eq (2.2) around the equilibrium point $(0, 0)$. The right panel shows the corresponding time series solution.

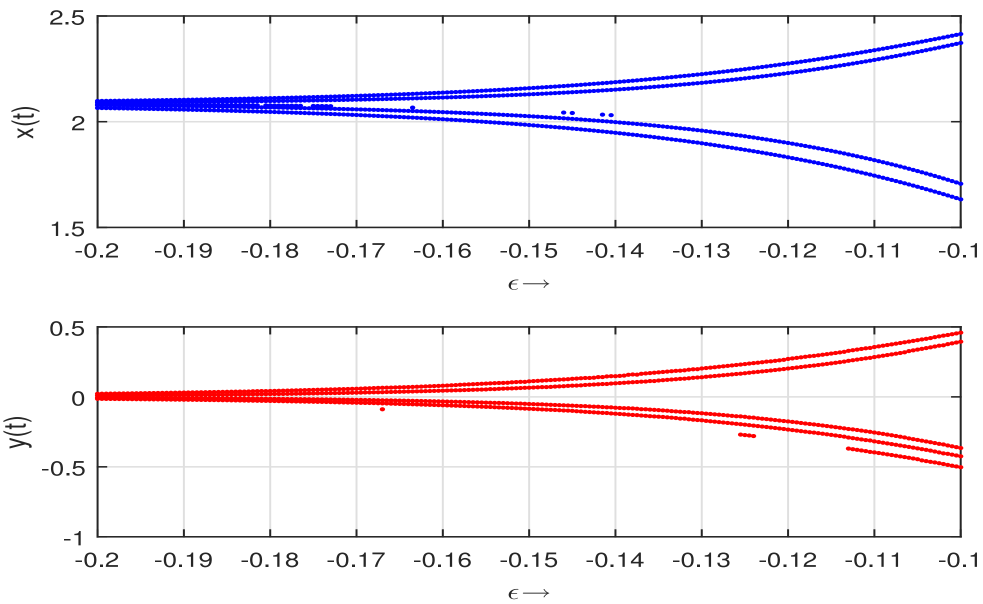
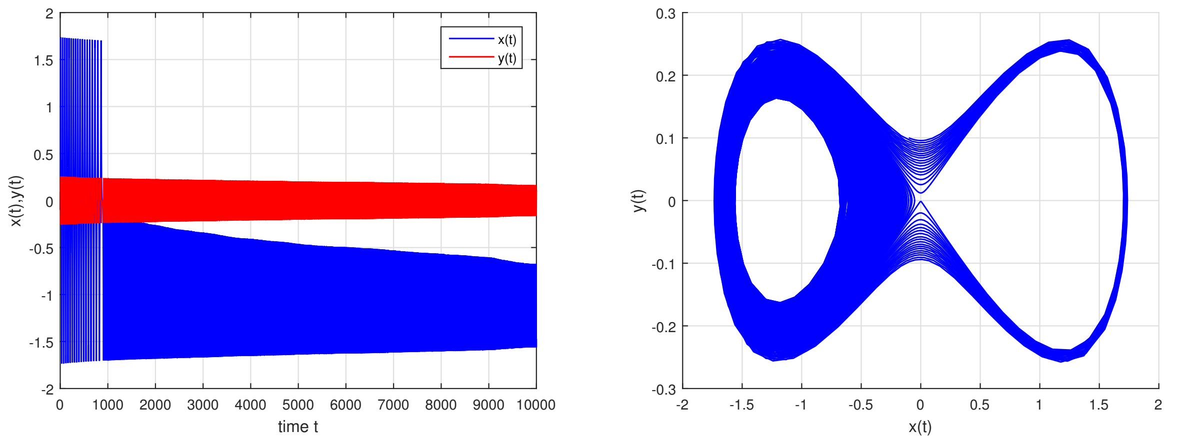
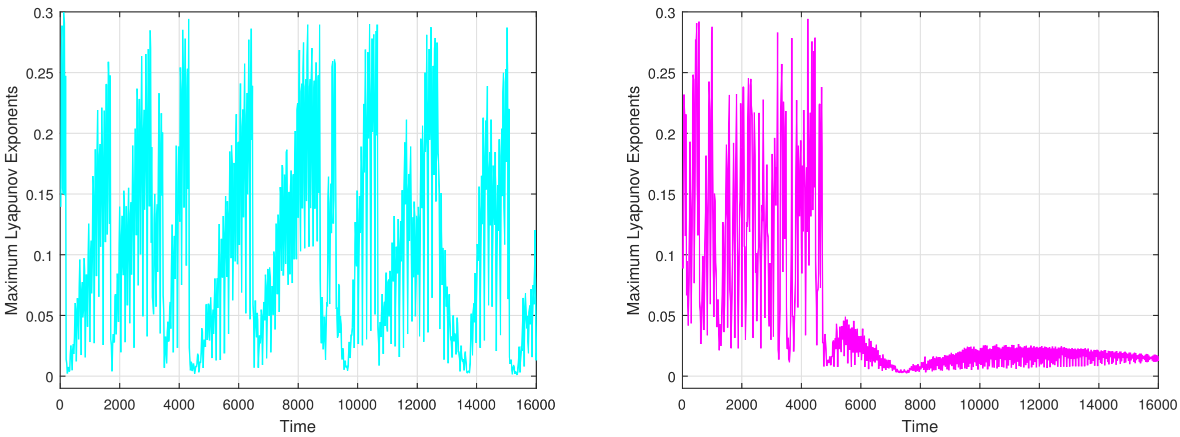


Figure 4. Top panel indicating the Hopf bifurcation diagram for the displacement vector $x(t) = x_1(t)$ of the Eq (2.2) with respect to the damping coefficient "ϵ". Bottom panel indicating the same for the velocity vector $\dot{x} = x_2(t)$.



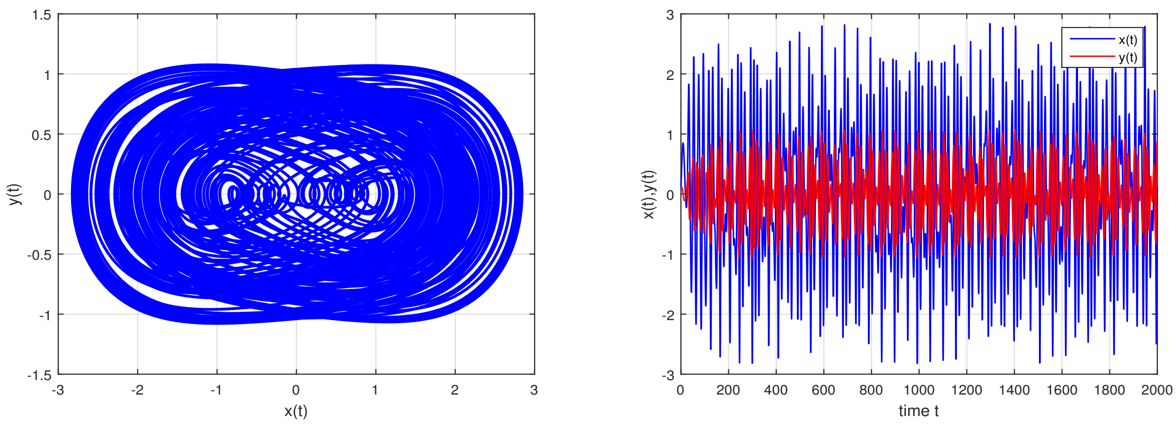
(a) (b)

Figure 5. (a) Time series solution for the positional feedback function $f(x(t-\eta)) = \sin x(t-\eta)$ for the parametric set : $a = 0.02$, $b = 0.04$, $\epsilon = 0.002$, $\tau = 0.5$, $\eta = 0.0005$ with starting position $(-0.1, 0, 1)$. (b) Phase diagram for the same set of parameters, used in Figure 5(a).



(a) (b)

Figure 6. (a) Largest Lyapunov exponent for the positional feedback function $f(x(t-\eta)) = \sin x(t-\eta)$ for the parametric set : $a = 0.02$, $b = 0.04$, $\epsilon = 0.002$, $\tau = 0.5$, $\eta = 0.0005$ with no perturbation $(\nu_1, \nu_2) = (0.0, 0.0)$. (b) Largest Lyapunov exponent for the positional feedback function $f(x(t-\eta)) = \frac{\exp(x(t-\eta))-1}{\exp(x(t-\eta))+2}$ for the same set of parameters, used in Figure 6(a).



(a) (b)

Figure 7. (a) Chaotic time series solution plot of the Eq (2.2) for $f(x(t-\eta)) = \sin x(t-\eta)$ for the parametric set : $a = 0.02$, $b = 0.04$, $\epsilon = 0.002$, $\tau = 0.5$, $\eta = 20.5$ with no perturbation $(\nu_1, \nu_2) = (0.0, 0.0)$. (b) Phase diagram of chaotic solution for the same set of parameters, used in Figure 7(a).

Next, we consider the parameter set $a = 0.02$, $b = 0.04$, $\epsilon = 0.002$, $\tau = 0.5$, $\eta = 0.0005$ and the feedback function $g(x) = \sin x$. A small perturbation $(\nu_1, \nu_2) = (-0.1, 0.1)$ to (d, ϵ) produce two homoclinic loop around the saddle point $(0, 0)$. In Figure 5, there exists a left homoclinic loop, where the trajectory started with the initial point $(-0.1, 0.1)$. The right homoclinic loop is also formed if the trajectory started with the initial point $(0.1, 0.1)$ (Figure is not reported here).

In Figure 6(a),(b), the largest Lyapunov exponents are depicted for the delayed feedback functions $\sin(x(t-\eta))$ and $\frac{e^{x(t-\eta)}-1}{e^{x(t-\eta)}+2}$ respectively for η and other parameters remain the same as in Figure 5.

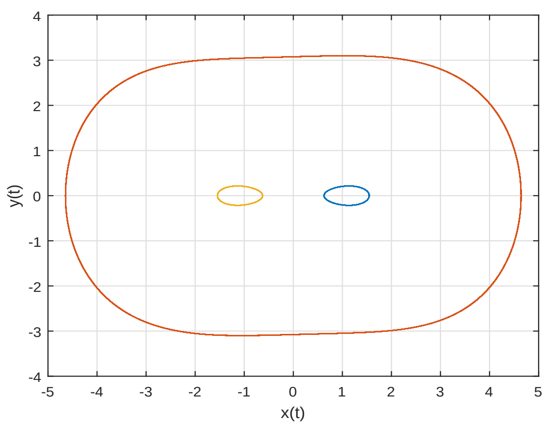


Figure 8. End tail of the system plot Eq (2.2), for different starting positions $(0.1, 0.1)$, $(-0.1, 0.1)$ and $(1.5, 1.5)$ for the period solution with cyan, yellow and magenta colors respectively. The parameter values are given in Figure 7.

Figure 7 describes the Eq (2.2)'s chaotic dynamics. Figure 8 illustrates the existence of two small limit cycles covered by a larger limit cycle.

5. Conclusions

In this present study, we have investigated the normal form for Bogdanov-Takens bifurcation experienced by the proposed retarded van der Pol-Duffing equation with multiple delay having possible degeneration. For the present system, we are aimed to study the dynamics with parametric positional delayed feedback in relation with quasi-periodic solution near and quasi-periodic attractor leading to chaotic solution. Analytically as well as numerically, we have found that the delayed Eq (2.2) experiences a B-T bifurcation if the conditions $\epsilon = -a\eta$ and $[2 + \epsilon(\eta - 2\tau)] > 0$ are satisfied simultaneosly. Utilizing the center manifold and normal form theories, we derived the canonical forms of Bogdanov-Takens singularity. Moreover, the phase and bifurcation diagrams for different positional delayed feedback functions are shown. It is shown that that the the coefficient of damping plays an important function to regulate dynamics of the Eq (2.2) (cf. Figure 4). One of the most crucial mechanisms for examining the chaotic behaviour of dynamical systems is the Lyapunov exponent. The positivity of the computed values of Lyapunov exponents provide a definite conclusion of chaotic dynamics. However, Lyapunov exponents are also notoriously difficult to estimate reliable from experimental data. One should be cognizant of additional evidence proving Sensitive Dependence on Initial Conditions (SDIC) before asserting that any system is chaotic. Growing awareness of this challenge will undoubtedly continue to encourage the creation of newer and more reliable algorithms for finding the Largest Lyapunov Exponents. From the Figure 6(a),(b), it is seen that the Largest Lyapunov exponents are at the positive level for the positional feedback function $\sin(x(t-\eta))$ and $\frac{e^{x(t-\eta)}-1}{e^{x(t-\eta)}+2}$ respectively, when η is increased to 20.5 (other parameters remain the same as in Figure 5). For the both cases, similar chaotic dynamics have been observed and shown in Figure 7. Figure 8 shows three close limit cycles encircling the trivial equilibrium point and other two symmetrically situated axial equilibria corresponding to very neighboring starting position, which exists only for the end of the total time span of evolution of the trajectories. We employ the MatLab version of the approach developed by Wolf et al. [25] to estimate the largest Lyapunov exponent for the current study. Multiple delays, as is well known, make it difficult and tiresome to analyze the distribution of the eigenvalues and cause the system to exhibit considerably deeper dynamical characteristics. There are hardly any unfolding results about the double Hopf bifurcation, quasi-periodic attractors and coexisting attractors enveloping. The engineers can use these results as a guide for selecting the delay settings to obtain the desired dynamical effects. Therefore, there are less number of results about the quasi-periodic solution leading to chaos due to multiple delays for dynamical systems on oscillators. We'll work about these realities in the next study.

Acknowledgements

The authors gratefully acknowledge Qassim University, represented by the Deanship of Scientific Research, on the financial support for this research under the number 10367-cos-2020-1-3-I during the academic year 1442 AH / 2020 AD.

Conflict of interest

The authors declare no conflicts of interest.

References

1. K. Pyragas, Continuous control of chaos by self-controlling feedback, *Phys. Lett. A.*, **170** (1992), 421–428. https://doi.org/10.1016/0375-9601(92)90745-8
2. F. M. Atay, van der Pol's oscillator under delayed feedback, *J. Sound Vib.*, **218** (1998), 333–339. https://doi.org/10.1006/jsvi.1998.1843
3. J. Jiang, Y. Song, Bogdanov-Takens bifurcation in an oscillator with negative damping and delayed position feedback, *Appl. Math. Modell.*, **37** (2013), 8091–8105. https://doi.org/10.1016/j.apm.2013.03.034
4. S. A. Campbell, J. Blair, T. Ohira, J. Milton, Limit cycles, tori, and complex dynamics in a second-order differential equation with delayed negative feedback, *J. Dyn. Differ. Equations*, **7** (1995), 213–236. https://doi.org/10.1007/BF02218819
5. Y. Song, T. Zhang, M. O. Tad, Stability and multiple bifurcations of a damped harmonic oscillator with delayed feedback near zero eigenvalue singularity, *Chaos*, **18** (2008), 043113. https://doi.org/10.1063/1.3013195
6. Z. Song, J. Xu, Codimension-two bursting analysis in the delayed neural system with external stimulations, *Nonlinear Dyn.*, **67** (2012), 309–328. https://doi.org/10.1007/s11071-011-9979-4
7. J. Cao, R. Yuan, H. Jiang, J. Song, Hopf bifurcation and multiple periodic solutions in a damped harmonic oscillator with delayed feedback, *J. Comput. Appl. Math.*, **263** (2014), 14–24. https://doi.org/10.1016/j.cam.2013.11.015
8. S. A. Campbell, Y. Yuan, Zero singularities of codimension two and three in delay differential equations, *Nonlinearity*, **21** (2008), 2671–2691. https://dx.doi.org/10.1088/0951-7715/21/11/010
9. J. Cao, R. Yuan, Multiple bifurcations in a harmonic oscillator with delayed feedback, *Neurocomputing*, **122** (2013), 172–180. https://doi.org/10.1016/j.neucom.2013.06.033
10. Z. Q. Qiao, X. B. Liu, D. M. Zhu, Bifurcation in delay differential systems with triple-zero singularity, *Chin. Ann. Math. A.*, **31** (2010), 59–70.
11. Z. Song, J. Xu, Stability switches and Bogdanov-Takens bifurcation in an inertial two-neuron coupling system with multiple delays, *Sci. China Technol. Sci.*, **57** (2014), 893–904. https://doi.org/10.1007/s11431-014-5536-y
12. J. Wang, W. Jiang, Bogdanov-Takens singularity in the comprehensive national power model with delays, *J. Appl. Anal. Comput.*, **3** (2013), 81–94. https://doi.org/10.11948/2013007
13. J. Wang, X. Liu, J. Liang, Bogdanov-Takens bifurcation in an oscillator with positive damping and multiple delays, *Nonlinear Dyn.*, **87** (2017), 255–269. https://doi.org/10.1007/s11071-016-3040-6
14. Y. Song, J. Jiang, Steady-state, Hopf and steady-state-Hopf bifurcations in delay differential equations with applications to a damped harmonic oscillator with delay feedback, *Int. J. Bifurcation Chaos*, **22** (2012), 1250286. https://doi.org/10.1142/S0218127412502860
15. S. Sarwardi, S. Hossain, M. Sajid, A. S. Almohaimeed, Analysis of the Bogdanov-Takens bifurcation in a retarded oscillator with negative damping and double delay, *AIMS Math.*, **7** (2022), 19770–19793. https://doi.org/10.3934/math.20221084

16. J. K. Hale, *Theory of Functional Differential Equations*, Springer, New York, 1977. https://doi.org/10.1007/978-1-4612-9892-2
17. J. K. Hale, S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7
18. Y. Xu, M. Huang, Homoclinic orbits and Hopf bifurcations in delay differential systems with T-B singularity, *J. Differ. Equations*, **244** (2008), 582–598. https://doi.org/10.1016/j.jde.2007.09.003
19. T. Faria, L. T. Magalhaes, Normal forms for retarded functional differential equations and applications to Bogdanov-Takens singularity, *J. Differ. Equations*, **122** (1995), 201–224. https://doi.org/10.1006/jdeq.1995.1145
20. W. Jiang, Y. Yuan, Bogdanov-Takens singularity in van der Pol's oscillator with delayed feedback, *Physica D*, **227** (2007), 149–161. https://doi.org/10.1016/j.physd.2007.01.003
21. T. Faria, L. T. Magalhaes, Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation, *J. Differ. Equations*, **122** (1995), 181–200. https://doi.org/10.1006/jdeq.1995.1144
22. X. He, C. Li, Y. Shu, Bogdanov-Takens bifurcation in a single inertial neuron model with delay, *Neurocomputing*, **89** (2012), 193–201. https://doi.org/10.1016/j.neucom.2012.02.019
23. T. Dong, X. Liao, Bogdanov-Takens bifurcation in a tri-neuron BAM neural network model with multiple delays, *Nonlinear Dyn.*, **71** (2013), 583–595. https://doi.org/10.1007/s11071-012-0683-9
24. M. S. Siewe, C. Tchawoua, P. Woafo, Melnikov chaos in aperiodically driven Rayleigh-Duffing oscillator, *Mech. Res. Commun.*, **37** (2010), 363–368. https://doi.org/10.1016/j.mechrescom.2010.04.001
25. A. Wolf, J. B. Swift, H. L. Swinney, J. A. Vastano, Determining Lyapunov exponents from a time series, *Physica D*, **16** (1985), 285–317. https://doi.org/10.1016/0167-2789(85)90011-9