Observer-based adaptive fuzzy output feedback control for functional constraint systems with dead-zone input

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Abstract: This paper develops an adaptive output feedback control for a class of functional constraint systems with unmeasurable states and unknown dead zone input. The constraint is a series of functions closely linked to state variables and time, which is not achieved in current research results and is more general in practical systems. Furthermore, a fuzzy approximator based adaptive backstepping algorithm is designed and an adaptive state observer with time-varying functional constraints (TFC) is constructed to estimate the unmeasurable states of the control system. Relying on the relevant knowledge of dead zone slopes, the issue of non-smooth dead-zone input is successfully solved. The time-varying integral barrier Lyapunov functions (iBLFs) are employed to guarantee that the states of the system remain within the constraint interval. By Lyapunov stability theory, the adopted control approach can ensure the stability of the system. Finally, the feasibility of the considered method is confirmed via a simulation experiment.

Keywords: fuzzy state observer; time-varying functional constraints (TFC); dead-zone input; adaptive fuzzy control; backstepping algorithm

1. Introduction

Over the past few decades, a lot of attention has been raised to handle the stability of nonlinear control systems [1, 2]. It is worth noting that adaptive control is favored by many scholars because of its ability to update adaptive parameters online. For dealing with the unknown nonlinear characteristics, fuzzy logic systems (FLSs) [3,4] and neural networks (NNs) [5–7] are widely employed. To list a few, utilizing FLSs control approach, a state feedback adaptive fuzzy method is presented in [8]. The work [9] develops an adaptive fuzzy control scheme to overcome the actuator faults of stochastic nonlinear systems. Moreover, a fixed time tracking control is investigated in [10], where an adaptive fuzzy controller is devised via backstepping technique. In [11], a Lyapunov stability strategy is addressed based on event-triggered mechanism. Considering fault-tolerant control problem, a suitable neural
controller combining with backstepping method is proposed in [12], which ensures the stability of the system in finite-time. Nevertheless, the mentioned adaptive control schemes don’t take the constraint problem into consideration.

As the main factor affecting system performance, constraint problem always appears in most practical systems. Hence, it is a challenging task to construct a suitable controller to maintain the stability of such systems. The barrier Lyapunov functions (BLFs) and backstepping algorithm are selected to stop the signal of the system from exceeding the constraint compact set in [13–16]. According to fuzzy approximate approach and BLFs, the output constraints related to constants are developed in [17,18]. In addition, full-state constant constraints are achieved in [19–21], where all signals are not transgressed the constraint boundary. Under the frame of NNs, a neural network control scheme with external disturbances and uncertainties is introduced in [22]. In particular, Zhao and Song [23] develop a unique approach (nonlinear state-dependent function) to achieve asymmetric state constraint, which completely removes the feasibility conditions that current BLFs exist. Subsequently, the time-varying constraints have attracted scholar’s attention because of its generally. In [24], a neural approach is presented to prevent arms to move to the desired position. Furthermore, Liu et al. [25] address a backstepping feedback control strategy with uncertain parameters, preventing the constraint boundaries from being violated and achieving full state constraints. Differently, a unified barrier function (UBF) with time-varying state constraints is established in [26], where novel coordinate transformations are introduced into the backstepping technique. Remarkably, only a small number of scholars have devoted themselves to the study of complex functional constraints. As far as we know, this breakthrough is only completed in [27]. However, the aforementioned results are realized under the assumption that the system is in good working condition.

In practical systems, the non-smooth input characteristics such as hysteresis, dead zone, saturation signal, etc. are always inevitable, which can lead to system instability. It is emphasized that dead zone regarded as a significant input nonlinearity continually occurs in actual systems. Therefore, the performance of the system will also be greatly affected when dead-zone inputs exist in the system, which should not be ignored. To ensure tracking performance, an adaptive compensation algorithm subject to dead-zone characteristics is proposed in [28]. Considering continuous-time nonlinear dynamic systems, Wang et al. [29] employ an adaptive control scheme by relying on the method of establishing dead zone model. An adaptive asymptotic control is analyzed in [30] where the unknown dead-zone and event trigger input are considered simultaneously. For nonlinear discrete-time systems, a fuzzy approximation combining with backstepping algorithm is constructed in [31]. Especially, not only the above-mentioned nonlinear systems, but also the dead zone input has been introduced into the constraint control systems. To just name a few, a full-state constraint tracking control approach based adaptive backstepping technique is addressed in [32]. The stability of feedback control systems subjected to dead-zone is outlined in [33], while barrier Lyapunov-Krasovskii functional (LKF) is introduced to overcome time-delay terms. It is noteworthy that these dead-zone inputs are investigated under the condition of state constraints, ignoring the problem of immeasurable states.

In addition to the state measurable systems of the above-mentioned researches, there are still a number of states that cannot be directly obtained in many practical systems, which encourages scholars to construct state observers to estimate the unmeasurable states. In [34], a sliding-mode observer is addressed to cope with unmeasurable states of stochastic polynomial systems. According to the approximation of FLSs, various state observer control approaches are achieved in [35–37] via employing
backstepping algorithm. Subsequently, the control strategy has been further developed to stabilize other nonlinear systems, such as discrete-time fuzzy systems [38] and input delay systems [39]. Yoo [40] proposes an output-feedback control scheme considering fault detection and accommodation, where a neural state observer is constructed. Under the framework of constraint control systems, a neural-based output constraints control [41] and an adaptive fuzzy observer with time-varying full state constraints (TFSC) [42] are developed. By relying on BLF, a fuzzy tracking control strategy about backlash-like hysteresis and TFSC is established in [43]. Liu et al. [44] present a constraint control of multi-input-multi-output systems, where the problem of unmeasurable states is well solved. Despite remarkable achievements have been made in nonlinear constrained control systems, the situation of unmeasurable states in functional constrained systems need to be further studied.

Inspired by aforementioned approaches, this paper addresses an output feedback control scheme with functional constraints and dead-zone input, where a state observer is constructed to estimate the unmeasurable states. The major contributions are summarized as follows.

1. The time-varying functional constraints (TFC) are considered by adopting integral BLF. In particular, this paper specifically investigates the impact on system performance when the state variables and time exist simultaneously in the constraint boundary.

2. Most studies tend to develop state measurable systems, but neglect the situation of state unmeasurable. In order to handle this issue, an adaptive fuzzy state observer combining with backstepping technique is presented in this paper. Currently, the output feedback control with functional constraints has not been developed.

3. As a significant input nonlinearity affecting the stability of the system, dead-zone input is successfully solved in the controller design. Finally, an observer based adaptive backstepping algorithm with TFC and dead zone input is achieved in this paper.

The remainder of this paper is organized as follows. Some basic knowledge and system descriptions are elaborated in Section 2. In Section 3, a fuzzy state observer is constructed. The process of controller construction is provided in Section 4. Section 5 gives the simulation results. At last, Section 6 concludes the work of this paper.

2. System description and preliminaries

2.1. System descriptions

Take the following strict feedback nonlinear systems into consideration:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + x_2, \\
\dot{x}_2 &= f_2(x_2) + x_3, \\
\vdots \\
\dot{x}_i &= f_i(X_i) + x_{i+1}, \\
\dot{x}_n &= f_n(X) + u, \\
y &= x_1
\end{align*}
\]  

(2.1)

where \( X_i = [x_1, x_2, ..., x_i]^T \) denotes immeasurable state vectors with \( i \geq 2 \), \( X = [x_1, x_2, ..., x_n]^T \) denotes the state variables, and \( y \in \mathbb{R} \) represents the system output. \( f_i(X_i) \) stands for unknown nonlinear smooth functions. In addition, choose the known functional constraints \( \xi_i(X_{i-1}, t), (i = 1, 2, ..., n) \) with
\[ x_0 = y_r, \] so that the states in this paper are constrained in predefined compact sets \( \Delta_x = \{ x_i | x_i(t) < \xi_c(x_{i-1}, t), \forall t \geq 0 \} \), where \( \xi_c(x_{i-1}, t) \) is a designable function. \( u \in \mathbb{R} \) denotes the input of the dead-zone, which is described as:

\[
u(t) = D(\nu(t)) = \begin{cases} m_r(\nu(t) - k_r), & \text{if } \nu(t) \geq k_r \\ 0, & \text{if } -k_l < \nu(t) < k_r \\ m_l(\nu(t) + k_l), & \text{if } \nu(t) \leq -k_l \end{cases}
\] (2.2)

where \( \nu(t) \) denotes the input of the dead zone, \( m_r, m_l \) represent right and left slopes, \( m_r = m_l = m. k_r, k_l > 0 \) are the break points. The dead zone Eq (2.2) can be expressed as:

\[
D(\nu(t)) = m \nu(t) + k(t) \tag{2.3}
\]

where

\[
k(t) = \begin{cases} -m k_r, & \nu(t) \geq k_r \\ -m \nu(t), & -k_l < \nu(t) < k_r \\ m k_l, & \nu(t) \geq k_r \end{cases}
\]

with \( \bar{k} = \max\{m k_r, m k_l\} \) is the upper bounded of \( |k(t)| \).

Transforming system Eq (2.1) into the following state space form:

\[
\begin{cases}
\dot{X} = AX + \eta y + \sum_{i=1}^{n} B_i f_i(X_i) + \beta u \\
y = CX
\end{cases}
\] (2.4)

where

\[
A = \begin{bmatrix}
-\eta_1 & 1 & \cdots & \cdots & 0 \\
-\eta_2 & 0 & \ddots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\eta_{n-1} & 0 & \cdots & \cdots & 1 \\
-\eta_n & 0 & \cdots & \cdots & 0
\end{bmatrix}_{n \times n}, \quad \beta = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}_{1 \times n}
\]

\[
\eta = [\eta_1, \eta_2, ..., \eta_n]^T, \quad B_i = [0 \ \cdots \ 1 \ \cdots \ 0]^T, \quad C = [1 \ \cdots \ 0 \ \cdots \ 0]_{1 \times n}, \quad \text{and vector } \eta \text{ is selected such that } A \text{ denotes a strict Hurwitz matrix. Thus, given a matrix } Q = Q^T > 0, \text{ there exists a matrix } P = P^T > 0 \text{ satisfying:}
\]

\[
A^T P + PA = -2Q \tag{2.5}
\]

**Remark 1.** A large number of achievements investigated nonlinear constraint systems whose boundary was a constant [19–23] or a time-varying function [24–26]. Differently, this paper takes functional constraints relying on state variables and time into account, which has not achieved in current research. In addition, the states of this system are unmeasurable, leading us to construct a fuzzy observer to estimate the former. The non-smooth input dead-zone is also considered in this paper, which is a challenging task to design a reasonable controller.
Control objective: The control objective is to develop an output feedback control strategy to achieve the following points: a) the output of this system can follow desired signal \( y_r(t) \) and the constructed fuzzy state observer can estimate the unmeasurable states commendably; b) the functional constraints are never violated; c) all signals in the closed-loop system remain within bounds.

Assumption 1 [25]: There exist unknown constants \( \mathcal{Y}_i \) and \( \mathcal{Y}_q \) \( (i = 1, ..., n, \ q = 1, ..., n) \) satisfying \( |\xi_{c_i}(X_{i-1},t)| \leq \mathcal{Y}_i \) and \( |\xi_{c_i}^{(q)}(X_{i-1},t)| \leq \mathcal{Y}_q \), where \( \xi_{c_i}^{(q)}(X_{i-1},t) \) denotes the \( q \)-th order derivative of \( \xi_{c_i}(X_{i-1},t) \), \( \forall t \geq 0 \).

Assumption 2 [18]: For any functional constraints \( \xi_{c_i}(X_{i-1},t) > 0 \), there exist positive constants \( C_0 \) and \( C_i \) such that the desired signal \( y_r(t) \) and its \( i \)-th order derivative \( y_r^{(i)}(t) \) satisfy \( |y_r(t)| \leq C_0 < \xi_{c_i}(y_r(t)) \) and \( |y_r^{(i)}(t)| \leq C_i \).

Remark 2. To make this paper more rigorous, we introduced Assumptions 1 and 2. Assumption 1 indicates that the selected boundary function and its \( q \)-th derivative are bounded. Obviously, it is more meaningful to construct an appropriate controller to maintain the states in a closed set. Assumption 2 guarantees the boundedness of the desired signal \( y_r(t) \), which facilitates the theorem proving. Similar assumptions have also been introduced in existing researches [18,25].

Lemma 1 [14,15]: For \( |x_i(t)| < \xi_{c_i}(X_{i-1},t) \), \( i = 1, ..., n \), the function \( V_{z_i} \) satisfies the following inequality:

\[
V_{z_i} \leq \varepsilon^2 \xi_{c_i}^2(X_{i-1},t) \left( \xi_{c_i}^2(X_{i-1},t) - x_i^2 \right).
\]

2.2. FLSs

A fuzzy approximator is constructed to estimate uncertain nonlinear functions which exists in the function-constrained systems with unmeasurable states. The detailed characteristics are as follows.

Lemma 2: An unknown continuous function \( f(x) \) defined on a compact set \( \Delta \) satisfies the following inequality:

\[
\sup_{x \in \Delta} \left| f(x) - \theta^T \phi(x) \right| \leq \varepsilon
\]  

(2.6)

In this paper, the unknown continuous functions are described as:

\[
f_i(X_i | \theta_i) = \theta_i^T \psi_i(X_i)
\]  

(2.7)

\[
\hat{f}_i(\hat{X}_i | \theta_i) = \theta_i^T \psi_i(\hat{X}_i)
\]  

(2.8)

where \( \hat{X}_i = [\hat{x}_1, \hat{x}_2, ..., \hat{x}_n]^T \) stands for the estimation of \( X_i = [x_1, x_2, ..., x_n]^T \).

Define

\[
\delta_i = f_i(X_i) - \hat{f}_i(\hat{X}_i | \theta_i)
\]  

(2.9)

\[
\zeta_i = f_i(X_i) - \hat{f}_i(\hat{X}_i | \theta_i), \quad i = 1, ..., n
\]  

(2.10)

where \( \delta_i \) denotes the fuzzy minimum approximation error, \( \zeta_i \) is the approximation error, and \( \theta_i^* \) denotes the optional parameter vector. Moreover, there exist positive constants \( \bar{\delta}_i \) and \( \bar{\zeta}_i \), which satisfy \( |\delta_i| \leq \bar{\delta}_i, |\zeta_i| \leq \bar{\zeta}_i, (i = 1, ..., n) \).

According to above analysis, select \( \sigma_i = \delta_i - \zeta_i \), we can obtain \( |\sigma_i| \leq \bar{\sigma}_i \), with constant \( \bar{\sigma}_i > 0 \).

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3. Observer-based adaptive controller construction and stability analysis

To handle the unmeasurable state problem, an adaptive observer is developed by combining backstepping technique in this section. And the stability of this system is analyzed at the end.

Constructing the following fuzzy state observer:

\[
\begin{aligned}
\dot{\hat{X}} &= AX + \eta y + \sum_{i=1}^{n} B_i \tilde{f}_i(\hat{X}|\theta_i) + \beta u \\
\hat{y} &= C\hat{X}
\end{aligned}
\]  

(3.1)

Let \( \bar{X} = X - \hat{X} = [\bar{x}_1, \bar{x}_2, ..., \bar{x}_n]^T \) be the observer errors, based on Eqs (2.4) and (3.1), one obtains

\[
\dot{\bar{X}} = A\bar{X} + \sum_{i=1}^{n} B_i \left[ f_i(X_i) - \tilde{f}_i(\hat{X}|\theta_i) \right] \\
= A\bar{X} + \zeta
\]  

(3.2)

where \( \zeta = [\zeta_1, \zeta_2, ..., \zeta_n]^T \).

Take the following Lyapunov function candidate into account:

\[
V_0^X = \frac{1}{2} \bar{X}^T P \bar{X}
\]  

(3.3)

The time derivative of \( V_0^X \) along Eq (3.2) is given as

\[
\dot{V}_0^X = \frac{1}{2} \bar{X}^T P \dot{\bar{X}} + \frac{1}{2} \bar{X}^T P \bar{X}
\]  

(3.4)

Substituting Eq (2.5) into Eq (3.4) and combining Eq (3.2), one acquires

\[
\dot{V}_0^X = \frac{1}{2} \bar{X}^T (A^T P + PA) \bar{X} + \bar{X}^T P \xi \\
= -\bar{X}^T Q \bar{X} + \bar{X}^T P \xi \\
\leq -\lambda_{\min}(Q) \bar{X}^2 + \bar{X}^T P \xi
\]  

(3.5)

Utilizing the Young’s inequality, we have

\[
\bar{X}^T P \xi \leq \frac{1}{2} \bar{X}^2 + \frac{1}{2} \| P \xi \|^2
\]  

(3.6)

Then, the following inequality holds

\[
\dot{V}_0^X \leq -(\lambda_{\min}(Q) - \frac{1}{2}) \bar{X}^2 + \frac{1}{2} \| P \xi \|^2
\]  

(3.7)

To realize the control objectives, the following coordinate transformation are given:

\[
\begin{aligned}
z_1 &= x_1 - y_r \\
z_i &= \hat{x}_i - \alpha_{i-1}, \ i = 2, ..., n
\end{aligned}
\]  

(3.8) (3.9)
where \( z_i \) represents the tracking error, and \( \alpha_{i-1} \) denotes a virtual controller with \( \alpha_0 = y_r \).

Selecting the following Lyapunov function candidate:

\[
V_{z_i} = \int_0^{z_i} \frac{\gamma \xi_{c_i}^2(\hat{\xi}_{i-1}, t) - (y + \alpha_{i-1})^2}{\xi_c^2(\hat{\xi}_{i-1}, t)} \, dy
\]  

(3.10)

where \( V_{z_i} \) is positive definite and continuously differentiable. The state vectors \( x_i \) and \( \hat{X}_i \), \( (i \geq 2) \) are confined to \( |x_i| < \xi_{c_i}(y_r, t) \) and \( |\hat{X}_i(t)| < \xi_{c_i}(\hat{\xi}_{i-1}, t) \), respectively.

Define \( \gamma = \varepsilon z_i \), one gets

\[
V_{z_i} = \varepsilon z_i ^2 \int_0^{z_i} \frac{\xi_{c_i}^2(\hat{\xi}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2}{\xi_c^2(\hat{\xi}_{i-1}, t)} \, d\varepsilon \geq \frac{1}{2} \varepsilon z_i ^2
\]  

(3.11)

which is applied in stability analysis.

Step 1: Based on Eq (3.8), one acquires

\[
\dot{z}_1 = \tilde{\theta}_1^T \psi_1(\hat{X}_1) + \theta_1^T \psi_1(\hat{X}_1) + \varepsilon \alpha_1 + \tilde{z}_2 + \alpha_1 + \tilde{x}_2 - \dot{y}_r
\]  

(3.12)

in which \( \theta_1 \) represents the estimation of \( \theta_i \), \( \tilde{\theta}_1 \) denotes the estimation error, and \( \tilde{\theta}_1 = \theta_i - \theta_1 \).

According to Eq (3.12), \( \dot{V}_{z_i} \) is obtained as

\[
\dot{V}_{z_i} = \frac{z_1 \xi_{c_i}^2(y_r, t)}{\xi_{c_i}^2(y_r, t) - (z_1 + y_r)^2} \dot{z}_1 + \dot{y}_r \int_0^{z_i} \frac{\partial}{\partial y} \frac{\partial}{\partial \xi_{c_i}(y_r, t)} \frac{\xi_{c_i}^2(y_r, t)}{\xi_{c_i}^2(y_r, t) - (y + y_r)^2} \, dy
\]

\[
+ \xi_{c_i}(y_r, t) \int_0^{z_i} \frac{\partial}{\partial y} \frac{\xi_{c_i}^2(y_r, t)}{\xi_{c_i}^2(y_r, t) - (y + y_r)^2} \, dy
\]  

(3.13)

where

\[
\int_0^{z_i} \frac{\partial}{\partial y} \frac{\xi_{c_i}^2(y_r, t)}{\xi_{c_i}^2(y_r, t) - (y + y_r)^2} \, dy
\]

\[
= z_1 \left( \frac{\xi_{c_i}^2(y_r, t) - x_i^2}{\xi_{c_i}^2(y_r, t) - x_i^2} + \frac{\partial}{\partial y} \ln \frac{\xi_{c_i}^2(y_r, t) - x_i^2}{\xi_{c_i}^2(y_r, t) - y_r^2} \right) M_1(\xi_{c_i}, y_r, z_1) - N_1(\xi_{c_i}, y_r, z_1)
\]

with

\[
M_1(\xi_{c_i}, y_r, z_1) = - \frac{(z_1 + y_r) E_{c_i}(y_r, t)}{\xi_{c_i}^2(y_r, t) - x_i^2} + \frac{1}{2z_1} \left( 2E_{z_i} + y_r \right) \xi_{c_i}(y_r, t) \left( \xi_{c_i}(y_r, t) - x_i^2 \right) \, \, dE
\]

\[
= - \frac{(z_1 + y_r) E_{c_i}(y_r, t)}{\xi_{c_i}^2(y_r, t) - x_i^2} - \xi_{c_i}(y_r, t) \ln \frac{\xi_{c_i}^2(y_r, t) - x_i^2}{\xi_{c_i}^2(y_r, t) - y_r^2} \, \, dE
\]

\[
+ \frac{y_r}{2z_1} \ln \frac{(\xi_{c_i}(y_r, t) - x_i)(\xi_{c_i}(y_r, t) + y_r)}{(\xi_{c_i}(y_r, t) - y_r)(\xi_{c_i}(y_r, t) + x_i)}
\]

\[
N_1(\xi_{c_i}, y_r, z_1) = \frac{\xi_{c_i}^2(y_r, t)}{\xi_{c_i}^2(y_r, t) - (\varepsilon z_i + y_r)^2} \, \, dE
\]

\[
= \frac{\xi_{c_i}(y_r, t)}{2z_1} \ln \frac{(\xi_{c_i}(y_r, t) + x_i)(\xi_{c_i}(y_r, t) + y_r)}{(\xi_{c_i}(y_r, t) + y_r)(\xi_{c_i}(y_r, t) + x_i)}.
\]
The following part of Eq (3.13) is expressed as
\[
\begin{align*}
\int_0^{z_1} \frac{\partial}{\partial \xi_{c_1}(y, t)} & \frac{\gamma \xi_{c_1}^2(y, t)}{\xi_{c_1}^2(y, t) - (y + r)^2} dy \\
= & \int_0^{z_1} -\gamma(y + r) d \frac{\xi_{c_1}(y, t)}{\xi_{c_1}^2(y, t) - (y + r)^2} \\
= & z_1 \left(- \frac{z_1 \xi_{c_1}(y, t)}{\xi_{c_1}(y, t) - x_1^2} + P_1(\xi_{c_1}, y, z_1)\right)
\end{align*}
\] (3.14)

where
\[
P_1(\xi_{c_1}, y, z_1) = - \frac{y \xi_{c_1}(y, t)}{\xi_{c_1}^2(y, t) - x_1^2} + \int_0^1 \frac{(2 \varepsilon_1 + y) \xi_{c_1}(y, t)}{\xi_{c_1}^2(y, t) - (\varepsilon_1 + y)^2} d \varepsilon
\]
\[
= - \frac{y \xi_{c_1}(y, t)}{\xi_{c_1}^2(y, t) - x_1^2} - \frac{\xi_{c_1}(y, t)}{z_1} \ln \left(\frac{\xi_{c_1}^2(y, t) - x_1^2}{\xi_{c_1}^2(y, t) - y_1^2}\right) \\
+ \frac{y_r}{2z_1} \ln \left(\frac{(\xi_{c_1}(y, t) - x_1)(\xi_{c_1}(y, t) + y_r)}{(\xi_{c_1}(y, t) + x_1)(\xi_{c_1}(y, t) - y_r)}\right)
\]

Remark 3. For convenience of description, this paper rewrites \(M_1(\xi_{c_1}, y, z_1), N_1(\xi_{c_1}, y, z_1), \) and \(P_1(\xi_{c_1}, y, z_1)\) as \(M_1, N_1, P_1,\) respectively. Applying L'Hôpital's rule, we get \(\lim_{z_1 \to 0} M_1 = \lim_{z_1 \to 0} P_1 = 0,\) \(\lim_{z_1 \to 0} N_1 = \xi_{c_1}^2(y, t)/(\xi_{c_1}^2(y, t) - y_1^2).\) Assumption 2 supposes that \(y_r\) is bounded satisfying \(|y_r(t)| \leq C_0\), so the boundedness of \(N_1\) is guaranteed when \(z_1 \to 0\). These rules are also established in below steps.

Choose a Lyapunov function as
\[
V_1 = V_0^X + V_{z_1} + \frac{1}{2 \rho_1} \bar{\dot{\gamma}_1} \dot{\gamma}_1
\] (3.15)

where \(\rho_1\) is a designable parameter.

Then, the derivative of \(V_1\) becomes
\[
\begin{align*}
\dot{V}_1 = & \dot{V}_0^X + \frac{z_1 \xi_{c_1}^2(y, t)}{\xi_{c_1}^2(y, t) - x_1^2} \left(\omega_1 + z_2 + \alpha_1 + \dot{x}_2 - \dot{y}_r\right) \\
+ & \frac{z_1 \xi_{c_1}^2(y, t)}{\xi_{c_1}^2(y, t) - x_1^2} \left(\dot{\theta}_1^T \psi_1(\hat{X}_1) + \dot{\theta}_1^T \psi_1(\hat{X}_1) + \dot{y}_r\right) \\
+ & \frac{\partial \xi_{c_1}(y, t)}{\partial y_r} z_1 M_1 \dot{y}_r + z_1 P_1 \dot{\xi}_{c_1}(y, t) \\
- & z_1 N_1 \dot{y}_r - \frac{z_1 \xi_{c_1}^2(y, t)}{\xi_{c_1}^2(y, t) - x_1^2} \dot{\xi}_{c_1}(y, t) - \frac{1}{\rho_1} \ddot{\gamma}_1 \dot{\gamma}_1
\end{align*}
\] (3.16)

where
\[
\dot{\xi}_{c_1}(y, t) = \frac{\partial \xi_{c_1}(y, t)}{\partial y_r} \dot{y}_r + \frac{\partial \xi_{c_1}(y, t)}{\partial t}.
\]

The first virtual controller \(\alpha_1\) and adaption law \(\dot{\gamma}_1\) are constructed as
\[ \alpha_1 = -\iota_1 \zeta_1 - \tilde{c}_1 \zeta_1 - \theta_1^T \psi_1(\tilde{X}_1) - \frac{\partial \xi_c y_1(y_r, t)}{\xi_c^2(y_r, t) - x_1^2} = \frac{\partial \xi_c y_1(y_r, t)}{\theta_1 M_1 \gamma_r} \]
\[ + \theta_1 N_1 \tilde{y}_r - \frac{\partial \xi_c y_1(y_r, t)}{\theta_1 P_1 \gamma_r} - \frac{\partial \xi_c y_1(y_r, t)}{\theta_1 P_1} \]
\[ \dot{\theta}_1 = \frac{\rho_1 \iota_1 \xi_c^2(y_r, t)}{\xi_c^2(y_r, t) - x_1^2} \psi_1(\tilde{X}_1) - \beta_1 \theta_1 \]
(3.17)

where \( \iota_1 > 0, \beta_1 > 0 \) are designable parameters, and \( \theta_1 = (\xi_c^2(y_r, t) - x_1^2) / \xi_c^2(y_r, t) \). \( \tilde{c}_1 \) is a time-varying function described as \( \tilde{c}_1 = \left( \xi_c(y_r, t) \right)^2 + \alpha_1 \) with \( \alpha_1 > 0 \).

Utilizing Young’s inequality, one yields
\[ \frac{z_1 \xi_c^2(y_r, t)}{\xi_c^2(y_r, t) - x_1^2} \left( \sigma_1 + \tilde{x}_2 \right) \leq \left( \frac{z_1 \xi_c^2(y_r, t)}{\xi_c^2(y_r, t) - x_1^2} \right)^2 + \frac{1}{2} \left( \tilde{x}_2 \right)^2 + \frac{1}{2} \tilde{\sigma}_1^2 \]
(3.19)

Substituting Eqs (3.7), (3.17), (3.18) and (3.19) into Eq (3.16) gets
\[ V_1 \leq \frac{z_1 \xi_c^2(y_r, t)}{\xi_c^2(y_r, t) - x_1^2} \left( \sigma_1 + \tilde{x}_2 \right) - \frac{z_1 \xi_c^2(y_r, t)}{\xi_c^2(y_r, t) - x_1^2} + \frac{\beta_1}{\rho_1} \tilde{y}_1^T \tilde{\theta}_1 \]
\[ + \frac{z_1 \xi_c^2(y_r, t)}{\xi_c^2(y_r, t) - x_1^2} + \frac{1}{2} \tilde{\sigma}_1^2 + \frac{1}{2} \| P \xi \|^2 \]
(3.20)

Step ii \((2 \leq i \leq n - 1)\): In view of Eq (3.9), \( \tilde{z}_i \) is calculated as
\[ \tilde{z}_i = \tilde{\dot{x}}_i - \tilde{\alpha}_{i-1} \]
\[ = \eta_i \tilde{x}_1 + \tilde{z}_{i+1} + \alpha_i + \tilde{\psi}_i(\tilde{X}_i) + \tilde{\psi}_i(\tilde{X}_i) + \sigma_1 - \tilde{\alpha}_{i-1} \]
(3.21)

where
\[ \tilde{\alpha}_{i-1} = \sum_{m=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \tilde{x}_m} \left( \tilde{x}_{m+1} - \eta_m(\tilde{x}_1 - y) + \tilde{\psi}_m(\tilde{X}_m) \right) \]
\[ + \frac{\partial \alpha_{i-1}}{\partial \tilde{x}_1} \left( \tilde{x}_2 + \tilde{x}_2 + \tilde{\psi}_1(\tilde{X}_1) + \tilde{\zeta}_1 \right) + \sum_{m=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_r^{(m-1)}} y_r^{(m)} \]
\[ + \sum_{m=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \tilde{y}_m} \tilde{y}_m + \sum_{m=1}^{i-1} \sum_{j=0}^{i-m} \frac{\partial \alpha_{i-1}}{\partial \xi_{c_m}^{(j)}} (\tilde{X}_{m-1}, t) (\tilde{X}_{m-1}, t) \]
(3.22)

Choose the following Lyapunov function
\[ V_i = V_{i-1} + \int_0^\infty \frac{\gamma \xi_c^2(\tilde{X}_{i-1}, t)}{\xi_c^2(\tilde{X}_{i-1}, t) - (y + \alpha_{i-1})^2} d\gamma + \frac{1}{2 \rho_i} \tilde{\psi}_i^T \tilde{\psi}_i \]
(3.23)

The time derivative of \( V_i \) is
\[ \dot{V}_i = \dot{V}_{i-1} + \frac{z_i \xi_i^2(\hat{X}_{i-1}, t)}{\xi_i^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \frac{\hat{x}_i}{\gamma_i} + \alpha_{i-1} \int_0^\infty \frac{\partial}{\partial \alpha_i} \frac{\gamma_i \xi_i^2(\hat{X}_{i-1}, t)}{\xi_i^2(\hat{X}_{i-1}, t) - (\gamma + \alpha_i)^2} \, dy - \frac{1}{\rho_i} \dot{\theta}_i^T \dot{\theta}_i \] (3.24)

Substituting Eq (3.21) into Eq (3.24), one yields

\[ \dot{V}_i = \dot{V}_{i-1} + \frac{z_i \xi_i^2(\hat{X}_{i-1}, t)}{\xi_i^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \frac{\hat{x}_i}{\gamma_i} + \dot{\theta}_i^T \psi_i(\hat{X}_i) + \alpha_i + \frac{\partial \xi_i}{\partial \alpha_i} \frac{\dot{z}_i}{\xi_i^2}(\hat{X}_{i-1}, t) \frac{\dot{X}_i}{\alpha_i} \frac{z_i \xi_i^2(\hat{X}_{i-1}, t)}{\xi_i^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \] (3.25)

where \( M_i, N_i \) and \( P_i \) is similar to step 1, and the detailed calculations of them are provided in the Appendix (a). Besides, \( \dot{\xi}_i(\hat{X}_{i-1}, t) \) is expressed as

\[ \dot{\xi}_i(\hat{X}_{i-1}, t) = \sum_{m=1}^{i-1} \frac{\partial \xi_i(\hat{X}_{i-1}, t)}{\partial \hat{X}_m} \dot{\hat{X}}_m + \frac{\partial \xi_i(\hat{X}_{i-1}, t)}{\partial t} \] (3.26)

Further, Eq (3.25) is rewritten in the following form

\[ \dot{V}_i = \dot{V}_{i-1} + \frac{z_i \xi_i^2(\hat{X}_{i-1}, t)}{\xi_i^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \frac{\hat{x}_i}{\gamma_i} + \dot{\theta}_i^T \psi_i(\hat{X}_i) + \frac{\partial \xi_i}{\partial \alpha_i} \frac{z_i \xi_i^2(\hat{X}_{i-1}, t)}{\xi_i^2(\hat{X}_{i-1}, t) - \hat{x}_i^2} \] (3.27)

Construct the intermediate virtual controller \( \alpha_i \) and adaption law \( \dot{\theta}_i \) as

\[ \alpha_i = -u_i z_i - \hat{c}_i z_i - \eta_i \hat{x}_i - \dot{\theta}_i^T \psi_i(\hat{X}_i) - \frac{\partial \xi_i(\hat{X}_{i-1}, t)}{\partial \alpha_i} \theta_i M_i \dot{\alpha}_i \] (3.28)
where \( \nu_i > 0, \beta_i > 0 \) are designable parameters, and \( \theta_i = \left( \dot{\epsilon}_{c_i}^2(\hat{X}_{i-1}, t) - \ddot{c}_i \right)^2 / \epsilon_{c_i}^2(\hat{X}_{i-1}, t) - \ddot{c}_i^2 \). \( \ddot{c}_i \) is a time-varying function described as \( \ddot{c}_i = \left( \dot{c}_i(\hat{X}_{i-1}, t) / \dot{c}_i(\hat{X}_{i-1}, t)^2 + o_i \right)^2 \) with \( o_i > 0 \).

According to Young’s inequality, one has

\[
\frac{z_i \dot{\epsilon}_{c_i}^2(\hat{X}_{i-1}, t)}{\dot{\epsilon}_{c_i}^2(\hat{X}_{i-1}, t) - \ddot{c}_i^2} \leq \frac{1}{2} \left( \frac{z_i \dot{\epsilon}_{c_i}^2(\hat{X}_{i-1}, t)}{\dot{\epsilon}_{c_i}^2(\hat{X}_{i-1}, t) - \ddot{c}_i^2} \right)^2 + \frac{1}{2} \tilde{\sigma}_i^2
\]

Thus, we have

\[
V_i \leq - (\lambda_{\min}(Q) - 1) \| \dot{X} \|^2 - \sum_{m=1}^{i} \lambda_m \frac{z_m \dot{\epsilon}_{c_m}^2(\hat{X}_{m-1}, t)}{\dot{\epsilon}_{c_m}^2(\hat{X}_{m-1}, t) - \ddot{c}_m^2} \sum_{m=1}^{i} \beta_m \tilde{g}_m \tilde{g}_m^T + \sum_{m=1}^{i} \rho_m \tilde{g}_m^T \tilde{g}_m
\]

\[
+ \frac{\dot{\epsilon}_{c_i}^2(\hat{X}_{i-1}, t)}{\dot{\epsilon}_{c_i}^2(\hat{X}_{i-1}, t) - \ddot{c}_i^2} z_i \ddot{c}_{i+1} + \sum_{m=1}^{i} \frac{1}{2} \tilde{\sigma}_m^2 + \frac{1}{2} \| \dot{P} \zeta \|^2
\]

**Step n:** Form Eq (3.9), \( \ddot{z}_n \) is calculated as

\[
\ddot{z}_n = \dot{z}_n - \dot{\alpha}_{n-1}
\]

\[
= \eta_n \ddot{x}_1 + m \nu(t) + \dot{\kappa}(t) + \ddot{\theta}_n^T \psi_n(\hat{X}_n) + \dot{\theta}_n^T \psi_n(\hat{X}_n) + \sigma_n - \dot{\alpha}_{n-1}
\]

where

\[
\dot{\alpha}_{n-1} = \sum_{m=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \dot{x}_m} (\dot{x}_{m+1} - \eta_m(\dot{x}_1 - y) + \ddot{\theta}_m^T \psi_m(\hat{X}_m))
\]

\[
+ \frac{\partial \alpha_{n-1}}{\partial \dot{x}_1} (\ddot{x}_2 + \ddot{x}_2 + \ddot{\theta}_1^T \psi(\hat{X}_1) + \ddot{c}_i) + \sum_{m=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \dot{\theta}_m} \dot{\theta}_m
\]

\[
+ \sum_{m=1}^{n} \frac{\partial \alpha_{n-1}}{\partial \dot{\theta}_m} \dot{\theta}_m + \sum_{m=1}^{n-1} \sum_{j=0}^{n-m} \frac{\partial \alpha_{n-1}}{\partial \dot{c}_m} (\ddot{c}_m + \xi_m(\hat{X}_{m-1}, t))
\]

Choose the following Lyapunov function

\[
V_n = V_{n-1} + \int_{0}^{n} \frac{\gamma \dot{\epsilon}_{c_m}^2(\hat{X}_{m-1}, t)}{\dot{\epsilon}_{c_m}^2(\hat{X}_{m-1}, t) - (y + \alpha_{n-1})^2} d\gamma + \frac{1}{2 \rho_n} \tilde{\sigma}_n^2
\]

The time derivative of \( V_n \) is

\[
\dot{V}_n = \dot{V}_{n-1} + \frac{z_n \dot{\epsilon}_{c_n}^2(\hat{X}_{n-1}, t)}{\dot{\epsilon}_{c_n}^2(\hat{X}_{n-1}, t) - \ddot{c}_n^2} \ddot{z}_n + \alpha_n \int_{0}^{\gamma_n} \frac{\partial}{\partial \dot{c}_m} (\ddot{c}_m + \xi_m(\hat{X}_{m-1}, t)) \dot{c}_m^2(\hat{X}_{m-1}, t) - (y + \alpha_{n-1})^2 d\gamma
\]

\[
+ \dot{\xi}_c(\hat{X}_{n-1}, t) \int_{0}^{\gamma_n} \frac{\partial}{\partial \dot{c}_m} (\ddot{c}_m + \xi_m(\hat{X}_{m-1}, t)) \dot{c}_m^2(\hat{X}_{m-1}, t) - (y + \alpha_{n-1})^2 d\gamma - \frac{1}{\rho_n} \dot{\dot{\theta}}_n \dot{\theta}_n
\]
Replacing Eq (3.35) by Eq (3.32) results in
\[
\dot{V}_n = \dot{V}_{n-1} + \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} (\eta_n \tilde{x}_1 + m v(t) + k(t) + \vartheta_n^T \psi_n (\hat{X}_n) + \omega_n) \\
+ \frac{\partial \hat{\xi}_{c_n} (\hat{X}_{n-1}, t)}{\partial \alpha_{n-1}} z_n M_{n} \alpha_{n-1} - z_n N_{n} \alpha_{n-1} + z_n P_{n} \hat{\xi}_{c_n} (\hat{X}_{n-1}, t) \\
- \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} \hat{\xi}_{c_n} (\hat{X}_{n-1}, t) \frac{1}{\rho_n} \left( \frac{\rho_n z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} \psi_n (\hat{X}_n) - \beta_n \vartheta_n \right)
\]  
(3.36)
where \( \rho_n, \beta_n \) are positive constants, the definition of \( M_{n}, N_{n} \) and \( P_{n} \) will be explained in the Appendix (b). Besides, \( \hat{\xi}_{c_n} (\hat{X}_{n-1}, t) \) is expressed as
\[
\hat{\xi}_{c_n} (\hat{X}_{n-1}, t) = \sum_{m=1}^{n-1} \frac{\partial \hat{\xi}_{c_n} (\hat{X}_{n-1}, t)}{\partial \hat{X}_m} \dot{\hat{X}}_m + \frac{\partial \hat{\xi}_{c_n} (\hat{X}_{n-1}, t)}{\partial t}
\]  
(3.37)
The real controller \( \nu(t) \) and adaption law \( \vartheta_n \) are given by
\[
\nu(t) = \frac{1}{m} \left[ -l_n \hat{\xi}_{c_n} - \hat{\xi}_{c_n} \tilde{n} - \eta_n \tilde{x}_1 - \vartheta_n^T \psi_n (\hat{X}_n) - \frac{\partial \hat{\xi}_{c_n} (\hat{X}_{n-1}, t)}{\partial \alpha_{n-1}} \theta_{n} M_{n} \alpha_{n-1} \right. \\
+ \theta_{n} N_{n} \alpha_{n-1} - \theta_{n} P_{n} \left( \sum_{m=1}^{n-1} \frac{\partial \hat{\xi}_{c_n} (\hat{X}_{n-1}, t)}{\partial \hat{X}_m} \dot{\hat{X}}_m \right) - \theta_{n} P_{n} \frac{\partial \hat{\xi}_{c_n} (\hat{X}_{n-1}, t)}{\partial t} \\
- \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} - \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-2}, t)}{\xi_{c_n}^2 (\hat{X}_{n-2}, t)} \left( \xi_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2 \right) - \left[ \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-2}, t)}{\xi_{c_n}^2 (\hat{X}_{n-2}, t)} - \hat{\xi}_{c_n}^2 \right] \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} - \hat{\xi}_{c_n}^2 \right)
\]  
(3.38)
where \( \nu_n \) is desirable parameter and \( \theta_{n} = \left( \frac{\xi_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} \right) \). \( \hat{c}_n \) is a time-varying function described as \( \hat{c}_n = \left( \frac{\hat{\xi}_{c_n} (\hat{X}_{n-1}, t)}{\xi_{c_n} (\hat{X}_{n-1}, t)} \right)^2 + o_n \) with \( o_n > 0 \).

Based on Young’s inequality, it has
\[
\frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} - \hat{\xi}_{c_n}^2 \leq \frac{1}{2} \left( \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2 \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2} \right)^2 + \frac{1}{2} \hat{\xi}_{c_n}^2 \hat{\xi}_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2 \right)
\]  
(3.40)
\[
\frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t)} - \hat{\xi}_{c_n}^2 \leq \frac{1}{2} \left( \frac{z_n \xi_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2 \xi_{c_n}^2 (\hat{X}_{n-1}, t)}{\xi_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2} \right)^2 + \frac{1}{2} \hat{\xi}_{c_n}^2 \hat{\xi}_{c_n}^2 (\hat{X}_{n-1}, t) - \hat{\xi}_{c_n}^2 \right)
\]  
(3.41)
Finally, we obtain
\[
\dot{V}_n \leq - (\lambda_{\text{max}}(Q) - 1) \left\| \dot{X}_n \right\|^2 - \sum_{m=1}^{n} \left( l_m \xi_{c_m}^2 (\hat{X}_{m-1}, t) - \hat{\xi}_{c_m}^2 \right) + \sum_{m=1}^{n} \beta_m \vartheta_n^T \vartheta_n
\]  
(3.42)

**Remark 4.** It is worth emphasized that a direct constraint is adopted on the states of this system according to the constraint Eq (3.10). But, through [19] and [24], the boundedness of the virtual
controller must be obtained firstly, then from error transformation \(z_i = x_i - \alpha_{i-1}\), the boundedness of states can be known. The integral BLF-approach introduced in this paper successfully overcomes the aforementioned conservation conditions.

**Theorem 1.** Under the condition of functional constraints \(|\hat{x}_i| < \xi_i(\hat{X}_{i-1}, t)\), consider strict-feedback control systems Eq 2.1, actual fuzzy controller Eq (3.38), and adaptation laws Eqs (3.18), (3.29) and (3.39), if the initial condition satisfies \(x(0) \in \Delta_x = \{\hat{x}_i|\hat{x}_i(0)| < \xi_i(\hat{X}_{i-1}(0), 0)\}\), the following properties can be guaranteed:

- the constructed fuzzy state observer can estimate the unmeasurable states commendably;
- the functional constraints are never violated;
- the boundedness of all the closed-loop signals are ensured.

**Proof.** Applying Young’s inequality, the third term \(\sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \tilde{\vartheta}_m^T \vartheta_m\) of Eq (3.42) is transformed into the following form:

\[
\sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \tilde{\vartheta}_m^T \vartheta_m \leq -\frac{1}{2} \sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \tilde{\vartheta}_m^T \tilde{\vartheta}_m + \frac{1}{2} \sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \vartheta_m^T \vartheta_m
\]  
(3.43)

Substituting Eq (3.43) into Eq (3.42), one acquires

\[
\dot{V}_n \leq -\chi V_n + \Lambda
\]  
(3.44)

where

\[
\chi = \frac{2(\lambda_{\min}(Q) - 1)}{\lambda_{\min}(P)}, \ 2\xi_m, \ 2\beta_m, \ m = 1, ..., n,
\]

and

\[
\Lambda = \sum_{m=1}^{n} \frac{1}{2} \tilde{\vartheta}_m^2 + \frac{1}{2} \|P\xi\|^2 + \sum_{m=1}^{n} \frac{1}{2} \tilde{\vartheta}_m^2 + \frac{1}{2} \sum_{m=1}^{n} \frac{\beta_m}{\rho_m} \vartheta_m^T \vartheta_m^*.
\]

Multiplying Eq (3.44) by \(e^{\pi t}\), one obtains

\[
d(e^{\pi t}V_n)/dt \leq e^{\pi t} \Lambda
\]  
(3.45)

Integrating Eq (3.45) over \([0, t]\) yields

\[
V_n(t) \leq (V_n(0) - \Lambda/\chi)e^{-\pi t} + \Lambda/\chi
\]  
(3.46)

Employing Eqs (3.11), (3.15) and (3.46), the following inequalities are acquired

\[
z_i^2 \leq 2 [(V_n(0) - \Lambda/\chi)e^{-\pi t} + \Lambda/\chi]
\]  
(3.47)

\[
\|	ilde{\vartheta}\|^2 \leq 2 [(V_n(0) - \Lambda/\chi)e^{-\pi t} + \Lambda/\chi]
\]  
(3.48)

According to Eqs (3.42) and (3.44), one has

\[
\bar{X}^T P \bar{X} \leq 2V_n(0)e^{-\pi t} + 2\Lambda/\chi
\]  
(3.49)
Further, we can obtain the following form
\[
\|\tilde{X}\| \leq \sqrt{\lambda_{\min}(P^{-1})(2V_0(0)e^{-\pi t} + 2\Delta/\chi)}.
\] (3.50)

Based on the inequality Eq (3.47), the boundedness of $z_i$ is obtained. From Assumption 1 and 2, it can be known that $|y_i(t)| \leq C_0 < \xi_{c_i}(y_i, t)$. Thus, we conclude that the boundedness of $x_i$ is guaranteed. Since $\tilde{x}_i = x_i - \hat{x}_i$, it’s obvious that $\tilde{x}_i$ and $\hat{x}_i$ are within limits. Therefore, it can be proved that $x_i$ is bounded. According to $z_i = \dot{x}_i - \alpha_{i-1}$ and inequality Eq (3.47), the boundedness of $\alpha_{i-1}$ is implied. Depending on the inequality Eq (3.48), the boundedness of $\vartheta_i$ is ensured. In the same way, it can be deduced that the controller $v(t)$ is bounded.

Depending on the above analyses, we can draw a conclusion that all the signals of this system are within bounds.

4. Simulation example

To further verify the feasibility of this scheme, the following simple pendulum system is considered:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{m_1 g L \sin(x_1)}{2M} - \frac{D x_2}{M} + u \\
y &= x_1
\end{align*}
\] (4.1)

where $m_1 = 1$, $g = 9.8$, $L = 1.5$, $M = 0.5$, $D = 0.4$. The state variables $x_1$ and $x_2$ are constrained by $-\xi_{c_1}(y_i, t) < x_1 < \xi_{c_1}(y_i, t)$ and $-\xi_{c_2}(x_1, t) < x_2 < \xi_{c_2}(x_1, t)$. In this paper, the constraint boundaries are selected as $\xi_{c_1}(y_i, t) = 0.2 \sin(2t) + 0.8e^{-0.1y_1} + 1$ and $\xi_{c_2}(x_1, t) = 5 \sin(0.3x_1) + 0.1e^{-0.5t} + 4$. In addition, $u$ denotes the dead-zone input which is described by:
\[
u = D(v(t)) = \begin{cases} 
0.5(v(t) - 2.5) & v(t) \geq 2.5 \\
0 & -1.5 < v(t) < 2.5 \\
0.5(v(t) + 1.5) & v(t) \leq -1.5
\end{cases}
\] (4.2)

The state observer is constructed as:
\[
\begin{align*}
\dot{\hat{x}}_1 &= \dot{\hat{x}}_2 + \eta_1 \tilde{x}_1 + \theta_1^T \psi_1(\hat{x}_1) \\
\dot{\hat{x}}_2 &= \eta_2 \tilde{x}_1 + \theta_2^T \psi_2(\hat{x}_2) + u
\end{align*}
\] (4.3)

where the initial values are chosen as $x_1(0) = 0.6$, $x_2(0) = 0.2$, $\tilde{x}_1(0) = 0.6$ and $\tilde{x}_2(0) = 0.2$.

Select the following fuzzy membership functions:
\[
\mu_{F_1}(\hat{x}_1) = \exp\left((-\hat{x}_1 + 12 - 0.5i)^2\right),
\]
\[
\mu_{F_2}(\hat{x}_1, \hat{x}_2) = \exp\left((-\hat{x}_1 + 12 - 0.5i)^2\right) \times \exp\left((-0.5\hat{x}_2 + 12 - 0.6i)^2\right),
\]

where $i = 1, 2, ..., 8$.

The fuzzy basis functions are defined as:
\[
\psi_{1j}(\hat{x}_1) = \frac{\mu_{F_1}(\hat{x}_1)}{\sum_{j=1}^{8} \mu_{F_1}(\hat{x}_1)}, \quad \psi_{2j}(\hat{x}_1, \hat{x}_2) = \frac{\mu_{F_1}(\hat{x}_1)\mu_{F_2}(\hat{x}_2)}{\sum_{j=1}^{8} \mu_{F_1}(\hat{x}_1)\mu_{F_2}(\hat{x}_2)}.
\]
Figure 1. Trajectories of $x_1$ and $y_r$.  

Figure 2. Trajectories of $x_1$, $\hat{x}_1$ and estimation error.  

where $j = 1, 2, ..., 8$.

In this simulation, the tracking signal is designed as $y_r = 0.5 \cos(4t)$, $\dot{y}_r = -2 \sin(4t)$. The initial values of the adaptation laws are $\theta_1(0) = [0.2, 0.2, 0.2, 0.1, 0.1, 0.1, 0.1]^T$, $\theta_2(0) =$
Figure 3. Trajectories of $x_2$, $\hat{x}_2$ and estimation error.

Figure 4. Trajectories of real controller $\nu$ and dead-zone input $u$.

$[0.2, 0.2, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1]^T$. The relevant parameters in this paper are chosen as $\rho_1 = 0.5$, $\rho_2 = 0.2$, $\beta_1 = 5$, $\beta_2 = 16$, $t_1 = 30$, $t_2 = 25$, $a_1 = a_2 = 0.2$, $\eta_1 = 5$ and $\eta_2 = 155$.

According to the mentioned above parameter values, the corresponding simulation results are
demonstrated in Figures 1–5. The change curves of state $x_1$ and reference signal $y_r(t)$ are described in Figure 1, where two curves can approximate overlap and the tracking effect is good. In Figures 2 and 3, the constructed observer is able to estimate the system states $x_1$ and $x_2$ well, and their estimation errors are relatively small. Figures 1–3 indicates that system states are strictly restricted in the predetermined ranges, and the TFSC are achieved. The trajectories of actual controller $v$ and dead-zone input $u$ are plotted in Figure 4, and their curves tend to be stable. Finally, Figure 5 diagrams the boundedness of adaptation parameters $\vartheta_1$ and $\vartheta_2$.

5. Conclusions

A state observer and a fuzzy controller for a class of functional constraint systems subject to unknown dead-zone have been constructed in this paper. The former is applied to estimate unmeasurable states, while the latter is established to approximate uncertain nonlinear functions. Relying on backstepping algorithm and iBLFs, the full state TFC are accomplished and the issue of non-smooth dead-zone input is successfully handled. The simulation diagrams further indicate that the developed control scheme is feasible. In the future, how to choose an appropriate barrier function to settle the application of constraint control in practical systems is a key problem.

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Conflict of interest

The authors declare that there is no conflict of interest.

References


Appendix A

In this part, the specific steps of Eq (3.24) are demonstrated as follows. From Eq (3.24), one has

$$
\int_{0}^{z_i} \frac{\partial}{\partial \alpha_{i-1}} \left( \frac{\gamma \xi_c^2(\hat{x}_{i-1}, t)}{\xi_c^2(\hat{x}_{i-1}, t) - (\gamma + \alpha_{i-1})^2} \right) d\gamma
= z_i \left( \frac{\xi_c^2(\hat{x}_{i-1}, t)}{\xi_c^2(\hat{x}_{i-1}, t) - \ddot{x}_i^2} + \frac{\partial \xi_c}{\partial \alpha_{i-1}} M_i(\xi_c, \alpha_{i-1}, z_i) - N_i(\xi_c, \alpha_{i-1}, z_i) \right)
$$

where

$$
M_i(\xi_c, \alpha_{i-1}, z_i) = -\frac{(z_i + \alpha_{i-1}) \xi_c(\hat{x}_{i-1}, t)}{\xi_c^2(\hat{x}_{i-1}, t) - \ddot{x}_i^2} + \int_{0}^{1} \frac{(2 \varepsilon z_i + \alpha_{i-1}) \xi_c(\hat{x}_{i-1}, t)}{\xi_c^2(\hat{x}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2} d\varepsilon
= -\frac{(z_i + \alpha_{i-1}) \xi_c(\hat{x}_{i-1}, t)}{\xi_c^2(\hat{x}_{i-1}, t) - \ddot{x}_i^2} - \frac{\xi_c(\hat{x}_{i-1}, t)}{z_i} \ln \frac{\xi_c^2(\hat{x}_{i-1}, t) - \ddot{x}_i^2}{\xi_c^2(\hat{x}_{i-1}, t) - \alpha_{i-1}^2} + \frac{\alpha_{i-1}}{2 \varepsilon z_i} \left( \frac{\xi_c(\hat{x}_{i-1}, t) - \ddot{x}_i}{\xi_c(\hat{x}_{i-1}, t) + \alpha_{i-1}} \right) \left( \xi_c(\hat{x}_{i-1}, t) + \ddot{x}_i \right)
$$

and

$$
N_i(\xi_c, \alpha_{i-1}, z_i) = \int_{0}^{1} \frac{\xi_c^2(\hat{x}_{i-1}, t)}{\xi_c^2(\hat{x}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2} d\varepsilon
= \frac{\xi_c(\hat{x}_{i-1}, t)}{2 \varepsilon z_i} \ln \left( \frac{\xi_c(\hat{x}_{i-1}, t) + \ddot{x}_i}{\xi_c(\hat{x}_{i-1}, t) + \alpha_{i-1}} \right) \left( \xi_c(\hat{x}_{i-1}, t) - \alpha_{i-1} \right)
$$

The following part of Eq (3.24) is expressed as
\begin{align*}
&\int_0^{z_i} \frac{\partial}{\partial \xi_i} (\mathcal{X}_{i-1}, t) \frac{\gamma \xi_i^2 (\mathcal{X}_{i-1}, t)}{\xi_i (\mathcal{X}_{i-1}, t)} - (\gamma + \alpha_{i-1})^2 d\gamma \\
&= \int_0^{z_i} -\gamma (\gamma + \alpha_{i-1}) d \frac{\xi_i (\mathcal{X}_{i-1}, t)}{\xi_i^2 (\mathcal{X}_{i-1}, t) - (\gamma + \alpha_{i-1})^2} \\
&= z_i \left( -\frac{z_i \xi_i (\mathcal{X}_{i-1}, t)}{\xi_i^2 (\mathcal{X}_{i-1}, t) - \xi_i^2} + P_i (\xi_i, \alpha_{i-1}, z_i) \right)
\end{align*}

where

\begin{align*}
P_i (\xi_i, \alpha_{i-1}, z_i) &= -\alpha_{i-1} \xi_i (\mathcal{X}_{i-1}, t) + \int_0^1 \frac{(2 \varepsilon z_i + \alpha_{i-1}) \xi_i (\mathcal{X}_{i-1}, t) - (\varepsilon z_i + \alpha_{i-1})^2 d \varepsilon}{\xi_i^2 (\mathcal{X}_{i-1}, t) - (\alpha_{i-1})^2} \\
&= -\alpha_{i-1} \xi_i (\mathcal{X}_{i-1}, t) - \frac{\xi_i (\mathcal{X}_{i-1}, t) - \xi_i^2}{\xi_i^2 (\mathcal{X}_{i-1}, t) - \xi_i^2} z_i \ln \left( \frac{\xi_i^2 (\mathcal{X}_{i-1}, t) - \alpha_{i-1}}{\xi_i^2 (\mathcal{X}_{i-1}, t) - (\alpha_{i-1})^2} \right) \\
&+ \frac{\alpha_{i-1}}{2 z_i} \ln \left( \frac{\xi_i (\mathcal{X}_{i-1}, t) - \xi_i}{\xi_i (\mathcal{X}_{i-1}, t) - (\alpha_{i-1})} \right) \left( \frac{\xi_i (\mathcal{X}_{i-1}, t) + \xi_i}{\xi_i (\mathcal{X}_{i-1}, t) + (\alpha_{i-1})} \right)
\end{align*}

Appendix B

Partial calculations of the final step will be described in the following contents. According to Eq (3.35), one has

\begin{align*}
&\int_0^{z_n} \frac{\partial}{\partial \alpha_{n-1}} \frac{\gamma \xi_n^2 (\mathcal{X}_{n-1}, t)}{\xi_n (\mathcal{X}_{n-1}, t)} - (\gamma + \alpha_{n-1})^2 d\gamma \\
&= z_n \left( -\frac{z_n \xi_n (\mathcal{X}_{n-1}, t)}{\xi_n^2 (\mathcal{X}_{n-1}, t) - \xi_n^2} + \frac{\partial \xi_n (\mathcal{X}_{n-1}, t)}{\partial \alpha_{n-1}} M_n (\xi_n, \alpha_{n-1}, z_n) - N_n (\xi_n, \alpha_{n-1}, z_n) \right)
\end{align*}

where

\begin{align*}
M_n (\xi_n, \alpha_{n-1}, z_n) &= -\frac{(z_n + \alpha_{n-1}) \xi_n (\mathcal{X}_{n-1}, t) - \xi_n^2}{\xi_n^2 (\mathcal{X}_{n-1}, t) - \xi_n^2} + \int_0^1 \frac{(2 \varepsilon z_n + \alpha_{n-1}) \xi_n (\mathcal{X}_{n-1}, t) - (\varepsilon z_n + \alpha_{n-1})^2 d \varepsilon}{\xi_n^2 (\mathcal{X}_{n-1}, t) - (\alpha_{n-1})^2} \\
&= -\frac{(z_n + \alpha_{n-1}) \xi_n (\mathcal{X}_{n-1}, t) - \xi_n^2}{\xi_n^2 (\mathcal{X}_{n-1}, t) - \xi_n^2} \ln \left( \frac{\xi_n^2 (\mathcal{X}_{n-1}, t) - \alpha_{n-1}}{\xi_n^2 (\mathcal{X}_{n-1}, t) - (\alpha_{n-1})^2} \right) \\
&+ \frac{\alpha_{n-1}}{2 z_n} \ln \left( \frac{\xi_n (\mathcal{X}_{n-1}, t) - \xi_n}{\xi_n (\mathcal{X}_{n-1}, t) - (\alpha_{n-1})} \right) \left( \frac{\xi_n (\mathcal{X}_{n-1}, t) + \xi_n}{\xi_n (\mathcal{X}_{n-1}, t) + (\alpha_{n-1})} \right)
\end{align*}

and

\begin{align*}
N_n (\xi_n, \alpha_{n-1}, z_n) &= \int_0^1 \frac{\xi_n^2 (\mathcal{X}_{n-1}, t)}{\xi_n^2 (\mathcal{X}_{n-1}, t) - (\varepsilon z_n + \alpha_{n-1})^2} d \varepsilon \\
&= \frac{\xi_n (\mathcal{X}_{n-1}, t)}{2 z_n} \ln \left( \frac{\xi_n (\mathcal{X}_{n-1}, t) + \xi_n}{\xi_n (\mathcal{X}_{n-1}, t) + (\alpha_{n-1})} \right) \left( \frac{\xi_n (\mathcal{X}_{n-1}, t) - \alpha_{n-1}}{\xi_n (\mathcal{X}_{n-1}, t) - (\alpha_{n-1})} \right)
\end{align*}

The following part of Eq (3.35) is changed as
\[
\int_0^{\infty} \frac{d}{\partial \xi_n (\hat{X}_{n-1}, t)} \left( \gamma \xi_n^2 (\hat{X}_{n-1}, t) \right) dy = \int_0^{\infty} -\gamma (\gamma + \alpha_{n-1}) d \frac{\xi_n (\hat{X}_{n-1}, t)}{\xi_n^2 (\hat{X}_{n-1}, t) - (\gamma + \alpha_{n-1})^2} \\
= \int_0^{\infty} -\gamma (\gamma + \alpha_{n-1}) d \frac{\xi_n (\hat{X}_{n-1}, t)}{\xi_n^2 (\hat{X}_{n-1}, t) - (\gamma + \alpha_{n-1})^2} \\
= \xi_n \left( -\frac{\gamma (\gamma + \alpha_{n-1})}{\xi_n^2 (\hat{X}_{n-1}, t) - \xi_n^2} + P_n (\xi_n, \alpha_{n-1}, \xi_n) \right)
\]
where

\[
P_n (\xi_n, \alpha_{n-1}, \xi_n) = -\frac{\alpha_{n-1} \xi_n (\hat{X}_{n-1}, t)}{\xi_n^2 (\hat{X}_{n-1}, t) - \xi_n^2} + \int_0^1 \frac{(\gamma + \alpha_{n-1}) \xi_n (\hat{X}_{n-1}, t)}{\xi_n^2 (\hat{X}_{n-1}, t) - (\gamma + \alpha_{n-1})^2} \ d \xi
\]

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