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Research article

Persistence and boundedness in a two-species chemotaxis-competition system with singular sensitivity and indirect signal production

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Abstract: This paper deals with a two-species chemotaxis-competition system involving singular sensitivity and indirect signal production:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi_1\nabla\cdot(\frac{u}{z^k}\nabla z) + \mu_1 u(1-u-a_1 v), & x\in\Omega,\ t>0,\\ v_t = \nabla\cdot(D(v)\nabla v) - \chi_2\nabla\cdot(\frac{v}{z^k}\nabla z) + \mu_2 v(1-v-a_2 u), & x\in\Omega,\ t>0,\\ w_t = \Delta w - w + u + v, & x\in\Omega,\ t>0,\\ z_t = \Delta z - z + w, & x\in\Omega,\ t>0, \end{cases}$$

where $\Omega\subset R^n$ is a convex smooth bounded domain with homogeneous Neumann boundary conditions. The diffusion functions $D(u), D(v)$ are assumed to fulfill $D(u)\ge(u+1)^{\theta_1}$ and $D(v)\ge(v+1)^{\theta_2}$ with $\theta_1,\theta_2>0$, respectively. The parameters are $k\in(0,\frac{1}{2})\cup(\frac{1}{2},1]$, $\chi_i>0, (i=1,2)$. Additionally, μ_i should be large enough positive constants, and a_i should be positive constants which are less than the quantities associated with $|\Omega|$. Through constructing some appropriate Lyapunov functionals, we can find the lower bounds of $\int_\Omega u$ and $\int_\Omega v$. This suggests that any occurrence of extinction, if it happens, will be localized spatially rather than affecting the population as a whole. Moreover, we demonstrate that the solution remains globally bounded if $\min\{\theta_1,\theta_2\}>1-\frac{2}{n+1}$ for $n\ge2$.

Keywords: persistence; boundedness; singular sensitivity; chemotaxis-competition system

1. Introduction

Chemotaxis refers to the process by which cells move directionally along a concentration gradients of chemical stimuli [1]. The classic Keller-Segel model, along with its numerous variations, has undergone extensive investigations and analyses by numerous researchers following the groundbreaking work of Keller and Segel. The mechanisms underlying this model, including aspects such as cell diffusion, chemotaxis sensitivity, and cell growth and death, have been deeply explored.

The pioneering system for single-species, single-stimulus chemotaxis was as follows:

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(u\chi(v)\nabla v) + f(u), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ (u,v)(x,0) = (u_0,v_0)(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $f(u)$ represents logistic sources. If $\chi(v) = \chi$ is a positive constant, then $d_1 = 1, d_2 = 1$, and $f(u) = 0$; this is the most primitive chemotactic model presented by Keller and Segel [2], dating back to 1970. In this case, the solution of the system (1.1), especially in high dimensional space, may exhibit a blow-up phenomenon in either finite or infinite time [3]. However, in the event of the logistic source $f(u) = \mu_0 u - \mu_1 u^2$, Winkler has demonstrated that it possesses globally bounded solutions in high-dimensional systems for a sufficiently large $\mu_1 > 0$ [4]. Additionally, it has a classical solution in three dimensions for any $\mu_1 > 0$, provided that μ_0 is not too large [5]. The findings indicate that the blow-up phenomenon can be effectively mitigated by implementing a suitable logistic source term. Importantly, these results hold true even when considering the conditions where $d_1 > 0, d_2 > 0$ [6]. Additionally, Tao and Winkler established the mass persistence of system (1.1) by constructing an energy function, which explains the persistence of the population as a whole, and the fact that any extinction must occur within a localized spatial region [7]. Apart from that, if $\chi_i(w)$ is a nonlinear chemotaxis sensitivity function given by $\chi_i(v) = \frac{\chi}{v}$, then it indicates that the sensitivity to chemotaxis is inversely proportional to the density of the signal function. Furthermore, the global existence and boundedness of classical solutions have been established in [8], as well as the global existence of weak solutions under various conditions. For more comprehensive information regarding the qualitative dynamics of system (1.1) and its variants, please consult [9–11] and the references therein.

In the aforementioned systems, the signal are directly generated by the cells themselves; however, in realistic situations, signal production may be indirect or multiple by different mechanisms. For instance, Strohm et al. [12] examined the reproductive and accumulation patterns of mountain pine beetles in forest habitats, where flying mountain pine beetles were attracted towards signals secreted by nesting mountain pine beetles, which served as indirect signals. A single-stimulus chemotaxis system with indirect signal production is presented by the following:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\chi(v)\nabla v) + f(u), & x\in\Omega,\ t>0,\\ v_t = \Delta v + h(v,w) & x\in\Omega,\ t>0,\\ w_t = \varepsilon\Delta w - \delta w + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ (u,v,w)(x,0) = (u_0,v_0,w_0)(x), & x\in\Omega. \end{cases} \tag{1.2}$$

Under the conditions $f(u) = ru - \mu u^2, \chi(v) = \chi$ and $\varepsilon = 0$, Hu et al. [13] investigated the boundedness and exponential convergence of solutions. For the case where $\chi(v) = \frac{\chi}{v}$, $\varepsilon = 1$ and $\delta = 1$, Xing et al. [14] proved that the solution of the system (1.2) is globally bounded in two dimensions and converges exponentially to the steady state if $h(v,w) = -v + w$ and $f(u) = 0$. For the case where

$\chi(v) = \frac{\chi}{v^k}$, $f(u) = ru - \mu u^2$ and $\varepsilon = \delta = 1$, [15] obtained the global boundedness of classical solution of the system (1.2).

On the other hand, there is often an interaction between multiple populations and multiple chemicals that simultaneously occur in a particular environment, thus resulting in competition among them. Therefore, we proceed to directly introduce the following chemotaxis system that incorporates two-species single-stimulus competitive kinetics without indirect signal production:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w - \lambda w + b_1 u + b_2 v, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\lambda, a_i, b_i, \mu_i > 0$ for $i = 1, 2$, $\tau = 0$ or 1. For the case where $\chi_i(w) = \chi_i > 0$, if $\tau = 0$, the coexisting equilibrium state for $n \geq 1$ can be found in [16]. Tell and Winkler [17] proved the global asymptotic stability of system (1.3) in high-dimensional scenarios, thus indicating that species groups can coexist under appropriate conditions. Moreover, some blow-up phenomena of system (1.3) can be found in [18, 19]. If $\tau = 1$, then the large-time behavior and global existence of the system have been extensively investigated in numerous studies [20, 21]. For the case where $\chi_i(w) = \frac{\chi_i}{w}$, Mizukami obtained the asymptotic stability of system (1.3) if $a_1, a_2 \in (0, 1)$ under $n \geq 2$. Qiu et al. demonstrated in [22] that under appropriate parameter conditions, after substituting $w_t = \Delta w - (\alpha u + \beta v)w$ into the third equation, the system possesses a unique globally uniform bounded solution. Furthermore, related variations of system (1.3) have been investigated to understand the asymptotic behavior of diffused Lotka-Volterra competition models. For more details, please refer to [23].

Now we are in the position to introduce the two-species chemotaxis-competition system involving indirect signal production:

$$\begin{cases} u_t = d_1\Delta u - \nabla \cdot (u\chi_1(z)\nabla z) + \mu_1 u(1 - u - a_1 v), & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v - \nabla \cdot (v\chi_2(z)\nabla z) + \mu_2 v(1 - v - a_2 u), & x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w - w + u + v, & x \in \Omega,\ t > 0, \\ \tau z_t = \Delta z + h(w, z), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w, z)(x, 0) = (u_0, v_0, w_0, z_0)(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where τ can be either 0 or 1. For the case where $\chi_i(z) = \chi_i$, when $h(w, z) = -wz$ and $\tau = 1$, the boundedness of solution for $\mu_1, \mu_2 > 0$ and $n \leq 2$ is obtained. Furthermore, the asymptotic stabilization of solutions in two dimensions has been shown if $a_1, a_2 \in (0, 1)$ along with $a_1 = 1 > a_2 > 0$ [24]. The boundedness and stabilization of system (1.4) for $h(w, z) = -z + w$ and $\tau \in \{0, 1\}$ were derived in [25]. Tu et al. [26] studied the global boundedness and regularity of the classical solution of the system, for $\tau = 1$, and the third and fourth equations of the system (1.4) are replaced with $w_t = \Delta w - \lambda_1 w + \alpha_{11} u + \alpha_{12} v$, $z_t = \Delta z - \lambda_2 z + \alpha_{21} u + \alpha_{22} v$. For the case where $\chi_i(z) = \frac{\chi_i}{z}$ and $\tau = 1$, if $0 < \max\{\chi_1, \chi_2\} < \frac{2}{n}$, $n \geq 2$ or $\chi_i > 0$, $n = 2$, then system (1.4) exhibits a unique global solution. Furthermore, if $\tau = 0$, then the boundedness of solution is also established in two-dimensional systems

[27]. For the latest research on the well-posedness behavior for two-competing-species two-stimuli chemotaxis models, please refer to the relevant literature [28–30] and their references.

From both mathematical and biological perspectives, it is of great interest to investigate whether populations persist and remain limited in size. To the best of our knowledge, there are few findings on the persistence of mass and boundedness in multi-species and multi-chemicals issues, particularly those involving chemotaxis singular sensitivity functions $\chi_i(z) = \frac{\chi_i}{z^k}$ $(k > 0)$. In addition, the nonlinear diffusion functions $D(u), D(v)$ are expected to prevent blow-up solutions in chemotaxis systems [31, 32]. Consequently, we deal with the following two-species two-chemicals chemotaxis-competition system involving singular sensitivity and indirect signal production:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi_1 \nabla \cdot (\frac{u}{z^k}\nabla z) + \mu_1 u(1 - u - a_1 v), & x \in \Omega,\ t > 0, \\ v_t = \nabla \cdot (D(v)\nabla v) - \chi_2 \nabla \cdot (\frac{v}{z^k}\nabla z) + \mu_2 v(1 - v - a_2 u), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - w + u + v, & x \in \Omega,\ t > 0, \\ z_t = \Delta z - z + w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w, z)(x, 0) = (u_0, v_0, w_0, z_0)(x), & x \in \Omega, \end{cases} \tag{1.5}$$

which is associated with smooth boundary $\partial\Omega$ in a bounded convex domain $\Omega \subset R^n$; the nonlinear diffusion functions $D(u)$ and $D(v)$ satisfy the following:

$$D(u), D(v) \in C^2([0, \infty)), \tag{1.6}$$

$$D(u) \geq (u + 1)^{\theta_1}, \quad \text{for all } u > 0 \tag{1.7}$$

and

$$D(v) \geq (v + 1)^{\theta_2}, \quad \text{for all } v > 0 \tag{1.8}$$

with $\theta_1, \theta_2 > 0$. The initial data (u_0, v_0, w_0, z_0) satisfy the following:

$$0 < u_0 \in C^0(\bar{\Omega}), 0 < v_0 \in C^0(\bar{\Omega}), 0 < w_0 \in W^{1,\infty}(\Omega), 0 < z_0 \in W^{1,\infty}(\Omega). \tag{1.9}$$

In this scenario, u and v represent the population densities of two competing species, while w and z denote the concentrations of chemical substances. Importantly, both biological species from the two competing populations are attracted to the same chemical signal z. It is worth noting that z is secreted by w, which in turn is secreted by u and v.

In the current paper, we aim to delve deeper into the fundamental questions mentioned above. The main results are presented as follows.

Theorem 1.1. *(Persistence) Consider that $\Omega \subset R^n (n \geq 2)$ is a bounded convex domain with a smooth boundary. Suppose that $D(u)$ and $D(v)$ satisfy (1.6), (1.7) and (1.8). Let the parameters $\chi_i > 0$, $0 < a_1 < \frac{|\Omega|}{m_2}, 0 < a_2 < \frac{|\Omega|}{m_1}$, $k \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $\mu_i = \mu_i(\chi_i, k, a_i, \Omega, u_0, v_0, w_0, z_0)(i = 1, 2)$ be large enough. If that the initial data (u_0, v_0, w_0, z_0) satisfy (1.9) and for any choice of constants $C_w, C_z > 0, K > 0$ and $S > 0$ satisfying*

$$\int_\Omega u_0 \leq m_1, \quad \int_\Omega v_0 \leq m_2, \quad \int_\Omega w_0^2 \leq C_w, \quad \int_\Omega z_0^2 \leq C_z,$$

and

$$\int_\Omega \ln u_0 \geq -K, \quad \int_\Omega \ln v_0 \geq -K, \quad \int_\Omega \ln z_0 \geq -S,$$

where $m_1 := \max\{\int_\Omega u_0, |\Omega|\}$, $m_2 := \max\{\int_\Omega v_0, |\Omega|\}$, *then for all* $t \in (0, T_{max})$, *we can find positive constants* $m_u(m_1, m_2, C_w, C_z, K, S, \chi_1, \mu_1, a_1), m_v(m_1, m_2, C_w, C_z, K, S, \chi_2, \mu_2, a_2)$ *such that*

$$\int_\Omega u \geq m_u \quad \text{and} \quad \int_\Omega v \geq m_v.$$

Remark 1.1. In Section 3, we intricately classify our discussion into three separate cases depending on the value range of k, namely $k \in (0, \frac{1}{2})$, $k \in (\frac{1}{2}, 1)$ and $k = 1$. It is worth mentioning that handling the second term on the right-hand side of inequality (3.13), specifically when $k = \frac{1}{2}$, poses a significant challenge in finding a suitable differential inequality related to $\int_\Omega \frac{|\nabla z|^2}{z}$. Consequently, this aspect remains an open problem that requires further investigation.

Theorem 1.2. *(Boundedness) Consider a bounded convex domain* $\Omega \subset R^n (n \geq 2)$ *with a smooth boundary, and assume that the conditions are the same as those within Theorem 1.1. Moreover, it is required that* $\theta > 1 - \frac{2}{n+1}$. *Then, system (1.5) has a unique classical solution that remains globally bounded.*

Remark 1.2. Based on the result in Theorem 1.1, it becomes feasible to establish a lower bound for z. Since $\inf_{x\in\Omega} z(x, t) \geq \delta$ with δ independent of t, the system (1.5) has a globally bounded solution.

2. Preliminaries

Lemma 2.1. *Consider a bounded domain* $\Omega \subset R^n (n \geq 1)$ *with a smooth boundary, and the parameters* $a_i, \mu_i, \chi_i (i = 1, 2)$ *are assumed to be positive. Let the initial data* (u_0, v_0, w_0, z_0) *satisfy condition (1.9). For any* $q > n$, *there exist* $T_{\max} \in (0, \infty]$ *and a unique quadruple* (u, v, w, z) *of nonnegative functions fulfilling the following:*

$$\begin{aligned} &u \in C^0\left(\bar{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{\max})\right), \\ &v \in C^0\left(\bar{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{\max})\right), \\ &w \in C^0\left(\bar{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{\max})\right) \cap L^\infty_{loc}\left([0, T_{\max}); W^{1,q}(\Omega)\right), \\ &z \in C^0\left(\bar{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{\max})\right) \cap L^\infty_{loc}\left([0, T_{\max}); W^{1,q}(\Omega)\right), \end{aligned} \tag{2.1}$$

which classically solve (1.5) in $\Omega \times [0, T_{\max})$.

Additionally, if $T_{\max} < \infty$, *it follows that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)} + \|z(\cdot, t)\|_{W^{1,q}(\Omega)} \to \infty \text{ as } t \nearrow \infty. \tag{2.2}$$

Proof. Based on the parabolic regularity theory and the standard contraction mapping argument described in [34, 35], the local existence of the classical solution to (1.5) can be similarly derived. □

To obtain the upper bound for $\int_\Omega w^2$, it is necessary to utilize the following auxiliary lemma that ensures the boundedness of solutions to a linearly damped ordinary differential equation with an inhomogeneity.

Lemma 2.2. *[36] Let us consider the assumption where f is a nonnegative absolutely continuous function on $[0,\tau)$, and g is a nonnegative function belonging to $C^0[0,\tau)$. They satisfy the following conditions:*

$$f'(t)+\alpha f(t)\le g(t), \quad a.e.\ t\in(0,\tau),$$
$$\int_t^{t+1} g(s)ds\le\beta, \quad t\in[0,\tau-1),$$

where $\alpha>0$ and $\beta>0$. Under these assumptions, we can then conclude that for $0<t<\tau$,

$$f(t)\le\max\{f(0)+\beta,\frac{\beta}{\alpha}+2\beta\}.$$

Subsequently, we present a number of well-established findings regarding the lower bound of $\int_\Omega \ln\psi$ in accordance with $\int_\Omega \frac{|\nabla\psi|^2}{\psi^2}$, where $\psi\in C^1(\bar{\Omega})$ is positive. Additionally, the results provide quantitative information regarding the magnitude of the point set and a variation of the Poincaré inequality. For detailed proofs of these results, we refer to [7].

Lemma 2.3. *[7] Let $\alpha,\beta>0$ and $\varphi\in L^2(\Omega)$ be a nonnegative function satisfying the following:*

$$\int_\Omega\varphi\ge\alpha \ \ and \ \ \int_\Omega\varphi^2\le\beta.$$

Then,

$$|\{x\in\Omega|\varphi(x)\ge\frac{\alpha^2}{2|\Omega|}\}|\ge\frac{\alpha^2}{4\beta}.$$

Lemma 2.4. *[7] There exists a constant $C(\gamma)>0$ for any $\gamma>0$ such that the inequality*

$$\int_\Omega\psi^2\le C(\gamma)\int_\Omega|\nabla\psi|^2$$

holds for every $\psi\in W^{1,2}(\Omega)$ satisfying

$$|\{x\in\Omega|\psi=0\}|\ge\gamma.$$

Lemma 2.5. *[7] Suppose $\psi>0$ belong to $C^1(\bar{\Omega})$ and $|\{x\in\Omega|\psi\ge\xi\}|\ge\gamma$ for every $\xi>0,\gamma>0$. Then,*

$$\int_\Omega\ln\psi\ge|\Omega|\ln\xi-\sqrt{C(\gamma)|\Omega|\cdot\int_\Omega\frac{|\nabla\psi|^2}{\psi^2}}$$

with the $C(\gamma)$ taken from Lemma 2.4.

Next, we employ a generalized form of Gagliardo-Nirenberg inequality [37].

Lemma 2.6. *[37] Consider a bounded domain $\Omega\subset R^n (n\ge1)$ with a smooth boundary. Let $0<r<p<\infty$, and let $\lambda\in(0,1)$ be determined by the following identity:*

$$-\frac{n}{p}=(1-\frac{n}{2})\lambda-\frac{n}{r}(1-\lambda);$$

then, there exists a positive constant C such that

$$\|\phi\|_{L^p(\Omega)}\le C(\|\nabla\phi\|^{\lambda}_{L^2(\Omega)}\|\phi\|^{1-\lambda}_{L^r(\Omega)}+\|\phi\|_{L^r(\Omega)})$$

for all $\phi\in W^{1,2}(\Omega)\cap L^r(\Omega)$.

3. Lower bounds for $\int_\Omega u$ and $\int_\Omega v$

First, we establish the L^1 boundedness of the solution.

Lemma 3.1. *Let $n \geq 1$, for all $t \in (0, T_{max})$; the solution of system (1.5) satisfies the following properties:*

$$\int_\Omega u(\cdot,t) \leq m_1 := \max\{\int_\Omega u_0, |\Omega|\}, \tag{3.1}$$

$$\int_\Omega v(\cdot,t) \leq m_2 := \max\{\int_\Omega v_0, |\Omega|\}, \tag{3.2}$$

$$\int_\Omega w(\cdot,t) \leq m_3 := m_1 + m_2 + \int_\Omega w_0 \tag{3.3}$$

and

$$\int_\Omega z(\cdot,t) \leq m_4 := m_3 + \int_\Omega z_0. \tag{3.4}$$

Proof. By integrating the first equation of (1.5), it follows that

$$\frac{d}{dt}\int_\Omega u = \mu_1 \int_\Omega u - \mu_1 \int_\Omega u^2 - \mu_1 a_1 \int_\Omega uv. \tag{3.5}$$

Invoking the Hölder inequality, we derive the following:

$$\frac{d}{dt}\int_\Omega u \leq \mu_1 \int_\Omega u - \mu_1 \int_\Omega u^2 \leq \mu_1 \int_\Omega u - \frac{\mu_1}{|\Omega|}(\int_\Omega u)^2, \tag{3.6}$$

for all $t \in (0, T_{max})$. Next, by utilizing an ODE comparison argument, we can deduce (3.1). Similarly, employing the same method, we can easily obtain (3.2)–(3.4). □

Subsequently, we present the following result concerning the size of specific time sets where $\int_\Omega u^2$ is large.

Lemma 3.2. *Suppose $t_0 \geq 0$, $L > 0$ and $T \in (0, T_{max} - t_0) > 0$. Then,*

$$|\{t \in (t_0, t_0 + T)| \int_\Omega u^2 > L\}| \leq \frac{\mu_1 m_1 T + m_1}{\mu_1 L} \tag{3.7}$$

and

$$\int_{t_0}^{t_0+T}\int_\Omega u^2 + \int_{t_0}^{t_0+T}\int_\Omega v^2 \leq \frac{\mu_1 m_1 T + m_1}{\mu_1} + \frac{\mu_2 m_2 T + m_2}{\mu_2} \tag{3.8}$$

where m_1, m_2 are given by (3.1) and (3.2), respectively.

Proof. Integrating in time for (3.6), we derive the following:

$$\begin{aligned}\mu_1 \int_{t_0}^{t_0+T}\int_\Omega u^2 &\leq \mu_1 \int_{t_0}^{t_0+T}\int_\Omega u + \int_\Omega u(\cdot,t_0) - \int_\Omega u(\cdot,t_0+T)\\ &\leq \mu_1 \int_{t_0}^{t_0+T}\int_\Omega u + \int_\Omega u(\cdot,t_0)\\ &\leq \mu_1 m_1 T + m_1.\end{aligned} \tag{3.9}$$

Similarly, we can obtain the following:

$$\mu_2 \int_{t_0}^{t_0+T} \int_\Omega v^2 \le \mu_2 m_2 T + m_2.$$

Setting

$$G_1 := \{t \in (t_0, t_0+T) | \int_\Omega u^2 > L\},$$

we have

$$\int_{t_0}^{t_0+T} \int_\Omega u^2 \ge \int_{G_1} \int_\Omega u^2 \ge L|G_1|.$$

In view of (3.9), this readily yields (3.7). □

In the following process of establishing differential inequalities, we will often use the boundedness of $\int_\Omega w^2$ and $\int_\Omega z^2$.

Lemma 3.3. *There exist two constants $C_z, C_w > 0$, for all $t \in (0, T_{max})$; the components w and z of the solution satisfy*

$$\int_\Omega w^2 \le C_w \tag{3.10}$$

and

$$\int_\Omega z^2 \le C_z. \tag{3.11}$$

Proof. Invoking the Poincaré inequality, we have the following:

$$\int_\Omega h^2 \le c_1 \int_\Omega |\nabla h|^2 + \frac{1}{|\Omega|} (\int_\Omega h)^2$$

for all $h \in W^{1,2}(\Omega)$, where $c_1 > 0$. Therefore, by utilizing (3.3), we derive the following:

$$\int_\Omega |\nabla w|^2 \ge \frac{1}{c_1} \int_\Omega w^2 - \frac{1}{c_1|\Omega|} m_3^2. \tag{3.12}$$

By multiplying the third equation of (1.5) by w, combining Young's inequality, and integrating by parts, we have the following:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \int_\Omega w^2 &= -\int_\Omega |\nabla w|^2 + \int_\Omega (-w^2 + wu + wv) \\ &\le -\int_\Omega |\nabla w|^2 + \frac{1}{2} \int_\Omega (u^2 + v^2).\end{aligned}$$

By substituting (3.12) into the inequality mentioned above, we derive the following:

$$\frac{d}{dt} \int_\Omega w^2 + \frac{2}{c_1} \int_\Omega w^2 \le \int_\Omega (u^2 + v^2) + \frac{2m_3^2}{c_1|\Omega|}.$$

The inequality (3.8) yields the following:

$$\int_t^{t+1} \left\{ \int_\Omega u^2(\cdot, s) + \int_\Omega v^2(\cdot, s) + \frac{2m_3^2}{c_1|\Omega|} \right\} \le \frac{\mu_1 m_1 + m_1}{\mu_1} + \frac{\mu_2 m_2 + m_2}{\mu_2} + \frac{2m_3^2}{c_1|\Omega|}.$$

Invoking Lemma 2.2, it follows that

$$\int_\Omega w^2 \le \max\{\int_\Omega w_0^2 + c_2, \frac{c_1 c_2}{2} + 2c_2\} := C_w.$$

for all $t > 0$, where $c_2 > 0$. This concludes the proof of (3.10).

Next, by testing the fourth equation of (1.5) with $2z$ and applying Young's inequality, we can deduce the following:

$$\begin{aligned}\frac{d}{dt}\int_\Omega z^2 &= 2\int_\Omega (z\Delta z - z^2 + zw)\\ &\le -2\int_\Omega |\nabla z|^2 - 2\int_\Omega z^2 + \int_\Omega z^2 + \frac{1}{4}\int_\Omega w^2\\ &= -\int_\Omega z^2 + \frac{C_w}{4}.\end{aligned}$$

Furthermore, we derive the following:

$$\int_\Omega z^2 \le \max\{\int_\Omega z_0^2, \frac{C_w}{4}\} := C_z.$$

□

Lemma 3.4. *For a sufficiently small $\epsilon > 0$, for all $t \in (0, T_{max})$, the solution of (1.5) satisfies the following:*

$$\frac{d}{dt}\int_\Omega \ln u \ge \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{1+4\epsilon}{4}\chi_1^2\int_\Omega \frac{|\nabla z|^2}{z^{2k}} + \mu_1|\Omega| - \mu_1\int_\Omega u - a_1\mu_1 m_2. \tag{3.13}$$

Proof. By multiplying the first equation of (1.5) by $\frac{1}{u}$ and applying integration by parts, we derive the following:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \ln u &= \int_\Omega \frac{\nabla(D(u)\nabla u)}{u} - \chi_1\int_\Omega \frac{\nabla(\frac{u}{z^k}\nabla z)}{u} + \int_\Omega \mu_1(1-u-a_1 v)\\ &= \int_\Omega \frac{D(u)|\nabla u|^2}{u^2} - \chi_1\int_\Omega \frac{\nabla u}{u}\cdot\frac{\nabla z}{z^k} + \mu_1|\Omega| - \mu_1\int_\Omega u - a_1\mu_1\int_\Omega v.\end{aligned} \tag{3.14}$$

Employing Young's inequality, we have the following:

$$\int_\Omega \frac{\nabla u}{u}\cdot\frac{\nabla z}{z^k} \le \frac{1}{(1+4\epsilon)\chi_1}\int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{1+4\epsilon}{4}\chi_1\int_\Omega \frac{|\nabla z|^2}{z^{2k}}.$$

Combining (1.7) and (3.14), we derive the following:

$$\frac{d}{dt}\int_\Omega \ln u \ge \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{1+4\epsilon}{4}\chi_1^2\int_\Omega \frac{|\nabla z|^2}{z^{2k}} + \mu_1|\Omega| - \mu_1\int_\Omega u - a_1\mu_1 m_2.$$

□

Lemma 3.5. *For all $t \in (0, T_{max})$, it follows that*

$$\frac{d}{dt}\int_\Omega z^{2-2k} = -2(1-k)(1-2k)\int_\Omega \frac{|\nabla z|^2}{z^{2k}} - 2(1-k)\int_\Omega z^{2-2k} + 2(1-k)\int_\Omega z^{1-2k}w. \tag{3.15}$$

Proof. By multiplying the fourth equation of (1.5) by z^{1-2k} and integrating by parts, we readily derive equation (3.15). □

Next, we combine Lemmas 3.4 and 3.5, along with the boundedness of $\int_\Omega z^2$ and $\int_\Omega w^2$ to establish the Lyapunov functional, which directly affects the estimation of the integral $\int_\Omega u$ time set and provide the fundamental groundwork for proving the integral's lower bound. Based on (3.15), the range of k influences the estimation of $\frac{d}{dt}\int_\Omega z^{2-2k}$. Therefore, we will examine this issue in three distinct cases: $k \in (0, \frac{1}{2})$, $k \in (\frac{1}{2}, 1)$ and $k = 1$.

Case 1: In this case, we consider the scenario where $k \in (0, \frac{1}{2})$. This allows us to obtain $1 - 2k \in (0, 1)$, $1 - k \in (\frac{1}{2}, 1)$; then,

$$\frac{d}{dt}\int_\Omega z^{2-2k} \le -2(1-k)(1-2k)\int_\Omega \frac{|\nabla z|^2}{z^{2k}} + 2(1-k)\int_\Omega z^{1-2k}w. \tag{3.16}$$

Lemma 3.6. *Let $n \ge 1$, $\chi_1 > 0$, $0 < a_1 < \frac{|\Omega|}{m_2}$, $\mu_1 = \mu_1(\chi_1, k, a_1, \Omega, u_0, v_0, w_0, z_0)$ be a large enough positive constant, $k \in (0, \frac{1}{2})$ and $D(u)$ satisfy (1.6),(1.7). Then, for all $t \in (0, T_{max})$, the solution of (1.5) satisfies the following:*

$$\frac{d}{dt}\left\{\int_\Omega \ln u - B\int_\Omega z^{2-2k}\right\} \ge \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_\Omega u + A|\Omega|, \tag{3.17}$$

where $A, B > 0$.

Proof. From (3.13) and (3.16), we can derive the following:

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega \ln u - \frac{(1+4\epsilon)\chi_1^2}{8(1-k)(1-2k)}\int_\Omega z^{2-2k}\right\} \\
&\ge \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_\Omega u - \frac{(1+4\epsilon)\chi_1^2}{4(1-2k)}\int_\Omega z^{1-2k}w + \mu_1|\Omega| - a_1\mu_1 m_2.
\end{aligned}$$

Invoking Young's inequality, (3.10), (3.11) and considering the condition $k \in (0, \frac{1}{2})$, we derive the following:

$$\begin{aligned}
&\frac{(1+4\epsilon)\chi_1^2}{4(1-2k)}\int_\Omega z^{1-2k}w \\
&\le \frac{(1+4\epsilon)\chi_1^2}{4(1-2k)}\left\{\frac{(1-2k)|\Omega|}{2C_z}\int_\Omega z^2 + \frac{(1+2k)C_z}{2|\Omega|}\int_\Omega w^{\frac{2}{1+2k}}\right\} \\
&\le \frac{(1+4\epsilon)\chi_1^2}{8}|\Omega| + \frac{(1+4\epsilon)\chi_1^2(1+2k)C_z}{8(1-2k)|\Omega|}\left\{\frac{(1-2k)|\Omega|}{(1+2k)C_z}|\Omega| + \frac{2kC_z}{(1+2k)(1-2k)}\int_\Omega w^2\right\} \\
&\le \frac{(1+4\epsilon)\chi_1^2}{4}|\Omega|\left(1 + \frac{\chi_1^2 C_z^2 C_w}{(1-2k)^2|\Omega|^2}\right).
\end{aligned}$$

Combining the above inequality, we can obtain the following:

$$\frac{d}{dt}\left\{\int_\Omega \ln u - \frac{(1+4\epsilon)\chi_1^2}{8(1-k)(1-2k)}\int_\Omega z^{2-2k}\right\}$$
$$\geq \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_\Omega u + \left[\left(1-\frac{a_1 m_2}{|\Omega|}\right)\mu_1 - \frac{(1+4\epsilon)\chi_1^2}{4}|\Omega|\left(1+\frac{\chi_1^2 C_z^2 C_w}{(1-2k)^2|\Omega|^2}\right)\right].$$

Since $\mu_1 > 0$, $0 < a_1 < \frac{|\Omega|}{m_2}$, we fix $\epsilon > 0$ such that it is sufficiently small to satisfy the following:

$$A := \left(1-\frac{a_1 m_2}{|\Omega|}\right)\mu_1 - \frac{(1+4\epsilon)\chi_1^2}{4}|\Omega|\left(1+\frac{\chi_1^2 C_z^2 C_w}{(1-2k)^2|\Omega|^2}\right) > 0.$$

Furthermore, let $B := \frac{(1+4\epsilon)\chi_1^2}{8(1-k)(1-2k)} > 0$, it can readily yield that (3.17) holds. □

Lemma 3.7. *Let $n \geq 1$, $\chi_1 > 0$, $0 < a_1 < \frac{|\Omega|}{m_2}$, $\mu_1 = \mu_1(\chi_1, k, a_1, \Omega, u_0, v_0, w_0, z_0)$ be a large enough positive constant, $k \in (0, \frac{1}{2})$ and $D(u)$ satisfy (1.6),(1.7). Suppose $t_0 \geq 0$, $K_0 \geq 0$ satisfies the following:*

$$\int_\Omega \ln u(\cdot, t_0) \geq -K_0 \tag{3.18}$$

and for any $T \in (0, T_{max} - t_0) > 0$ fulfills the following:

$$T \geq \frac{K_0 + m_1 + BC_1}{\frac{1}{2}A|\Omega|} \tag{3.19}$$

with m_1 and C_1 given by (3.1) and (3.25); we can derive

$$\int_{t_0}^{t_0+T}\int_\Omega u\, dxdt \geq \frac{AT|\Omega|}{2\mu_1}, \tag{3.20}$$

along with

$$\left|\left\{t \in (t_0, t_0+T) | \int_\Omega u(\cdot, t) \geq \varepsilon\right\}\right| \geq \frac{\varepsilon T}{m_1}, \tag{3.21}$$

where ε is defined as

$$\varepsilon := \min\{\frac{A|\Omega|}{4\mu_1}, m_1\}. \tag{3.22}$$

Proof. By integrating (3.16) in time from t_0 to $t_0 + T$, we can derive the following:

$$\int_\Omega \ln u(\cdot, t_0+T) - \int_\Omega \ln u(\cdot, t_0) - B\int_\Omega z^{2-2k}(\cdot, t_0+T) + B\int_\Omega z^{2-2k}(\cdot, t_0)$$
$$\geq \frac{4\epsilon}{1+4\epsilon}\int_{t_0}^{t_0+T}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_{t_0}^{t_0+T}\int_\Omega u + AT|\Omega|.$$

Since u, z are positive and (3.18), we deduce the following:

$$\begin{aligned}\mu_1\int_{t_0}^{t_0+T}\int_\Omega u &\geq \int_\Omega \ln u(\cdot, t_0) - \int_\Omega \ln u(\cdot, t_0+T) + B\int_\Omega z^{2-2k}(\cdot, t_0+T) \\ &\quad - B\int_\Omega z^{2-2k}(\cdot, t_0) + AT|\Omega| \\ &\geq -\int_\Omega \ln u(\cdot, t_0+T) - B\int_\Omega z^{2-2k}(\cdot, t_0) - K_0 + AT|\Omega|.\end{aligned} \tag{3.23}$$

Considering that $\ln\xi < \xi$ for $\xi > 0$, we can conclude that

$$-\int_\Omega \ln u(\cdot, t_0 + T) \geq -\int_\Omega u(\cdot, t_0 + T) \geq -m_1. \tag{3.24}$$

By utilizing Lemma 3.3 and Young's inequality, we can derive the following:

$$\int_\Omega z^{2-2k} \leq \int_\Omega z^2 + |\Omega|(1-k)^{\frac{1-k}{k}} k := C_1, \tag{3.25}$$

where $C_1 > 0$.

Therefore, invoking (3.19), (3.24) and (3.25), the inequality (3.23) can be restated as follows

$$\begin{aligned}\int_{t_0}^{t_0+T}\int_\Omega u &\geq \frac{1}{\mu_1}(A|\Omega|T - K_0 - m_1 - BC_1)\\ &\geq \frac{A|\Omega|T}{2\mu_1},\end{aligned}$$

which yields (3.20). Next, let

$$G_2 := \left\{ t \in (t_0, t_0 + T) | \int_\Omega u(\cdot, t) \geq \varepsilon \right\};$$

it follows that

$$\begin{aligned}\int_{t_0}^{t_0+T}\int_\Omega u &= \int_{G_2}\int_\Omega u + \int_{(t_0,t_0+T)\setminus G_2}\int_\Omega u\\ &\leq m_1|G_2| + \varepsilon T.\end{aligned}$$

Thanks to (3.20), we derive the following:

$$|G_2| \geq \frac{A|\Omega|T}{2\mu_1 m_1} - \frac{\varepsilon T}{m_1}.$$

The proof of Lemma 3.7 has been completed. □

For another application of (3.17), we can get the size of the time set where $\int_\Omega \frac{|\nabla u|^2}{u^2}$ is big enough.

Lemma 3.8. *Let $n \geq 1$, $T \in (0, T_{max} - t_0) > 0$, $\chi_1 > 0$, $0 < a_1 < \frac{|\Omega|}{m_2}$, $\mu_1 = \mu_1(\chi_1, k, a_1, \Omega, u_0, v_0, w_0, z_0)$ be a large enough positive constant, $k \in (0, \frac{1}{2})$ and $D(u)$ satisfy (1.6),(1.7), suppose that (3.13) is true. Then, we have the following:*

$$\left|\left\{ t \in (t_0, t_0 + T) | \int_\Omega \frac{|\nabla u|^2}{u^2} > M \right\}\right| \leq \frac{(1 + 4\epsilon)(m_1 + K_0 + BC_1 + \mu_1 m_1 T)}{4\epsilon M} \tag{3.26}$$

where $M > 0$.

Proof. Integrating (3.16) over $t \in (t_0, t_0 + T)$, we derive the following:

$$\frac{4\epsilon}{1+4\epsilon}\int_{t_0}^{t_0+T}\int_\Omega \frac{|\nabla u|^2}{u^2} \le \int_\Omega \ln u(\cdot, t_0+T) - \int_\Omega \ln u(\cdot, t_0) - B\int_\Omega z^{2-2k}(\cdot, t_0+T) + B\int_\Omega z^{2-2k}(\cdot, t_0) + \mu_1 \int_{t_0}^{t_0+T}\int_\Omega u - AT|\Omega|.$$

Hence, by utilizing (3.18), (3.25) and positive of z, we have the following:

$$\frac{4\epsilon}{1+4\epsilon}\int_{t_0}^{t_0+T}\int_\Omega \frac{|\nabla u|^2}{u^2} \le m_1 + K_0 + BC_1 + \mu_1 m_1 T. \tag{3.27}$$

We define

$$G_3 := \left\{ t \in (t_0, t_0+T) | \int_\Omega \frac{|\nabla u|^2}{u^2} > M \right\};$$

then,

$$\frac{4\epsilon}{1+4\epsilon}\int_{t_0}^{t_0+T}\int_\Omega \frac{|\nabla u|^2}{u^2} \ge \frac{4\epsilon}{1+4\epsilon}\int_{G_3}\int_\Omega \frac{|\nabla u|^2}{u^2} \ge \frac{4\epsilon}{1+4\epsilon} M|G_3|.$$

Combining with (3.27), it yields the following:

$$|G_3| \le \frac{m_1 + K_0 + BC_1 + \mu_1 m_1 T}{\frac{4\epsilon}{1+4\epsilon}M}.$$

□

By applying the aforementioned lemmas, we can combine the time sets concerning $\int_\Omega u$, $\int_\Omega u^2$, and $\int_\Omega \frac{|\nabla u|^2}{u^2}$ to obtain an upper bound on $\int_\Omega \ln u$. The following proof process is based on the method described in [7].

Lemma 3.9. *Let $n \ge 1$, $T \in (0, T_{max} - t_0) > 0$, $\chi_1 > 0$, $0 < a_1 < \frac{|\Omega|}{m_2}$, $\mu_1 = \mu_1(\chi_1, k, a_1, \Omega, u_0, v_0, w_0, z_0)$ be a large enough positive constant, $k \in (0, \frac{1}{2})$ and $D(u)$ satisfy (1.6), (1.7), if (u_0, v_0, w_0, z_0) satisfy initial condition and*

$$\int_\Omega u_0 \le m_1, \quad \int_\Omega w_0^2 \le C_w, \quad \int_\Omega z_0^2 \le C_z, \quad \int_\Omega \ln u_0 \ge -K_0. \tag{3.28}$$

Then, there exist constants K_1 and a sequence $(t_i)_{i\in N} \subset [0, \infty)$ fulfills $t_i \to \infty$ as $i \to \infty$ and $t_i < t_{i+1} < t_i + T$ along with

$$\int_\Omega \ln u(\cdot, t_i) \ge -K_1 \quad \text{for all } i \in N. \tag{3.29}$$

Proof. Let

$$\varepsilon := \min\{\frac{A|\Omega|}{4\mu_1}, m_1\} \tag{3.30}$$

be defined in Lemma 3.7; we choose $L > 0$, $M > 0$ such that

$$\frac{m_1}{L} < \frac{\varepsilon}{4m_1} \tag{3.31}$$

and

$$\frac{(1+4\epsilon)\mu_1 m_1}{\epsilon M} < \frac{\varepsilon}{2m_1}. \tag{3.32}$$

Furthermore, we can choose

$$\gamma := \frac{\varepsilon^2}{4L} \tag{3.33}$$

and

$$\xi := \frac{\varepsilon}{2|\Omega|}. \tag{3.34}$$

To demonstrate that (3.29) can be achieved through a suitably selected sequence $(t_i)_{i\in N} \subset [0,\infty)$ fulfills $t_i \to \infty$ as $i \to \infty$, we define t_i inductively. Initially, let $t_1 := 0$ and for every $i \geq 1$ and with the assumption that $t_1, \cdots, t_i$ possess the following property

$$\int_\Omega \ln u(\cdot, t_k) \geq -K_1 \tag{3.35}$$

for all $k \in \{1, \cdots, i\}$, where

$$K_1 := \max\{K_0, -|\Omega| \ln \xi + \sqrt{C(\gamma)|\Omega|M}\}, \tag{3.36}$$

then one can find $t_{i+1} \in \left(t_i + \frac{\varepsilon T}{4m_1}, t_i + T\right)$ such that (3.35) holds for $j = i + 1$.

Let

$$Q_1 := \left\{t \in (t_i, t_i + T) | \int_\Omega u(\cdot, t) \geq \varepsilon\right\},$$

$$Q_2 := \left\{t \in (t_i, t_i + T) | \int_\Omega u^2(\cdot, t) \leq L\right\},$$

$$Q_3 := \left\{t \in (t_i, t_i + T) | \int_\Omega \frac{|\nabla u(\cdot, t)|^2}{u^2(\cdot, t)} \leq M\right\},$$

we need to ensure that

$$\left|Q_1 \cap Q_2 \cap Q_3 \cap \left(t_i + \frac{\varepsilon T}{4m_1}, t_i + T\right)\right| > 0. \tag{3.37}$$

Prior to proving it, we fix $T > 0$ enough such that it satisfies

$$\frac{m_1}{\mu_1 L} < \frac{\varepsilon T}{4m_1}, \tag{3.38}$$

$$\frac{(1+4\epsilon)(m_1 + K_1 + BC_1)}{\epsilon M} < \frac{\varepsilon T}{2m_1}, \tag{3.39}$$

and

$$T > \frac{K_1 + m_1 + BC_1}{\frac{1}{2}A|\Omega|}. \tag{3.40}$$

Clearly, we can set $K_0 := K_1$ and $t_0 := t_i$. By (3.35) and (3.40), we can infer (3.18) and (3.19). Furthermore, invoking Lemma 3.7, we can directly derive the following:

$$|Q_1| > \frac{\varepsilon T}{m_1}, \tag{3.41}$$

where ε is defined in (3.30). Subsequently, combining (3.28) and (3.31), we can employ Lemma 3.2 with $t_0 := t_i$ to obtain the following:

$$\begin{aligned}|Q_2| &= T - \left|\left\{t \in (t_i, t_i + T) | \int_\Omega u^2(\cdot, t) > L\right\}\right| \\ &\geq T - \frac{\mu_1 T m_1 + m_1}{\mu_1 L} \\ &= (1 - \frac{m_1}{L})T - \frac{m_1}{\mu_1 L} \\ &> (1 - \frac{\varepsilon}{2m_1})T,\end{aligned} \tag{3.42}$$

where $1 - \frac{\varepsilon}{2m_1} > 0$. Notably, due to (3.22), we have $\varepsilon \leq m_1$. With this observation in mind, considering (3.29) (3.32) (3.39), and using Lemma 3.8, we obtain the following:

$$\begin{aligned}|Q_3| &= T - \left|\left\{t \in (t_i, t_i + T) | \int_\Omega \frac{|\nabla u(\cdot, t)|^2}{u^2(\cdot, t)} > M\right\}\right| \\ &\geq T - \frac{(1 + 4\epsilon)\mu_1 T m_1}{4\epsilon M} - \frac{(1 + 4\epsilon)(m_1 + K_0 + BC_1)}{4\epsilon M} \\ &> T - \frac{\varepsilon T}{8m_1} - \frac{\varepsilon T}{8m_1} \\ &> (1 - \frac{\varepsilon}{4m_1})T.\end{aligned}$$

Given (3.41) and (3.42), it yields the following:

$$|Q_1 \cap Q_2| > \frac{\varepsilon T}{2m_1}.$$

Therefore, it follows that

$$\begin{aligned}|Q_1 \cap Q_2 \cap Q_3| &> \frac{\varepsilon T}{2m_1} + (1 - \frac{\varepsilon}{4m_1})T - T \\ &= \frac{\varepsilon T}{4m_1},\end{aligned}$$

which clearly yields (3.37).

Utilizing (3.37) and setting $t_0 = 0$, we can find a $t_1 > t_0 + \frac{\varepsilon T}{4m_1}$ such that $t_1 \in Q_1 \cap Q_2 \cap Q_3$. Subsequently, by Lemmas 2.3 and 2.5 and (3.36), we derive the following:

$$\int_\Omega \ln u(\cdot, t_1) \geq -K_1,$$

where Lemma 2.5 is applied according to the definitions (3.33) and (3.34) of γ and ξ.

Similarly, we can also find $t_2 > t_1 + \frac{\varepsilon T}{4m_1}$ such that

$$\int_\Omega \ln u(\cdot, t_2) \geq -K_1.$$

Therefore, we can find a sequence $\{t_i\}$ satisfying

$$t_i + T > t_{i+1} > t_i + \frac{\varepsilon T}{4m_1}$$

and

$$\int_\Omega \ln u(\cdot, t_i) \geq -K_1,$$

without a loss of the fact $t_i \to \infty$ as $i \to \infty$. □

Case 2: In this case, we consider the scenario where $k \in (\frac{1}{2}, 1)$. This allows us to obtain the following:

$$1 - 2k \in (-1, 0), \quad 1 - k \in (0, \frac{1}{2});$$

then,

$$\frac{d}{dt}\int_\Omega z^{2-2k} \geq 2(1-k)(2k-1)\int_\Omega \frac{|\nabla z|^2}{z^{2k}} - 2(1-k)\int_\Omega z^{2-2k}. \tag{3.43}$$

Lemma 3.10. *Let $n \geq 1$, $\chi_1 > 0$, $0 < a_1 < \frac{|\Omega|}{m_2}$, $\mu_1 = \mu_1(\chi_1, k, a_1, \Omega, u_0, v_0, w_0, z_0)$ be a large enough positive constant, $k \in (\frac{1}{2}, 1)$ and $D(u)$ satisfy (1.6),(1.7). Then, for all $t \in (0, T_{max})$, the solution of (1.5) satisfies the following:*

$$\frac{d}{dt}\left\{\int_\Omega \ln u + D\int_\Omega z^{2-2k}\right\} \geq \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_\Omega u + E|\Omega|, \tag{3.44}$$

where $D, E > 0$.

Proof. We can use Young's inequality to obtain the following:

$$\int_\Omega z^{2-2k} \leq 2(1-k)\int_\Omega z + (2k-1)|\Omega| \leq 2(1-k)m_4 + (2k-1)|\Omega|.$$

By combining (3.13) and (3.43), we can obtain the following:

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega \ln u + \frac{(1+4\epsilon)\chi_1^2}{8(1-k)(2k-1)}\int_\Omega z^{2-2k}\right\} \\
\geq\ & \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_\Omega u + \mu_1|\Omega|\frac{(1+4\epsilon)\chi_1^2}{4}|\Omega| - \frac{2(1+4\epsilon)\chi_1^2(1-k)m_4}{4(2k-1)} - a_1\mu_1 m_2 \\
\geq\ & \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_\Omega u + \left[\left(1 - \frac{a_1 m_2}{|\Omega|}\right)\mu_1 - \frac{(1+4\epsilon)\chi_1^2}{4}\left(1 + \frac{m_4}{2k-1}\right)\right]|\Omega|.
\end{aligned}$$

Since $\mu_1 > 0$ is large enough and $0 < a_1 < \frac{|\Omega|}{m_2}$, we fix $\epsilon > 0$ sufficiently small such that

$$E := \left(1 - \frac{a_1 m_2}{|\Omega|}\right)\mu_1 - \frac{(1+4\epsilon)\chi_1^2}{4}\left(1 + \frac{m_4}{2k-1}\right) > 0.$$

Thus, let $D := \frac{(1+4\epsilon)\chi_1^2}{8(1-k)(2k-1)} > 0$, which entails (3.44). □

By using a similar method in proving Lemmas 3.7–3.9, we can obtain a lower bounded estimate of $\int_\Omega \ln u(\cdot, t_i)$, $t_i \to \infty$ as $i \to \infty$. We have omitted some details here.

Case 3: Finally, we consider the case of $k = 1$, it follow from (3.13) that

$$\frac{d}{dt}\int_\Omega \ln u \geq \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{1+4\epsilon}{4}\chi_1^2\int_\Omega \frac{|\nabla z|^2}{z^2} + \mu_1|\Omega| - \mu_1\int_\Omega u - a_1\mu_1 m_2.$$

Utilizing the fourth equation of (1.5), we can derive the following:

$$\frac{d}{dt}\int_\Omega \ln z = \int_\Omega \frac{|\nabla z|^2}{z^2} - |\Omega| + \frac{w}{z} \geq \int_\Omega \frac{|\nabla z|^2}{z^2} - |\Omega|,$$

for all $t \in (0, T_{max})$. We can combine the above two inequality to obtain the following:

$$\frac{d}{dt}\left\{\int_\Omega \ln u + \frac{1+4\epsilon}{4}\chi_1^2\int_\Omega \ln z\right\} \geq \frac{4\epsilon}{1+4\epsilon}\int_\Omega \frac{|\nabla u|^2}{u^2} - \mu_1\int_\Omega u + F|\Omega|, \tag{3.45}$$

where $F := \mu_1 - \frac{a_1\mu_1 m_2}{|\Omega|} - \frac{1+4\epsilon}{4}\chi_1^2 > 0$. Because we chose a large enough $\mu_1 > 0$ and $0 < a_1 < \frac{|\Omega|}{m_2}$, we can also select a small enough value for $\epsilon > 0$ in order to achieve the following:

$$\mu_1 > \frac{(1+4\epsilon)\chi_1^2|\Omega|}{4(|\Omega| - a_1 m_2)}.$$

Then, we will structure the lower bound for $\int_\Omega \ln z$.

Lemma 3.11. *There exists a constant $S > 0$, for all $t \in (0, T_{max})$; the component z of the solution satisfies the following:*

$$\int_\Omega \ln z \geq -S.$$

Proof. By employing a similar methodology to that presented in Lemma 2.3 of [33], we can easily establish the validity of Lemma 3.11. However, in the interest of brevity, we refrain from providing the details here. □

By applying the method of discussing $k \in (0, \frac{1}{2})$, combining with the reconstructed functional (3.45) and Lemma 3.11, we can also find a lower bound for $\int_\Omega \ln u(\cdot, t_i)$, $t_i \to \infty$ as $i \to \infty$.

Proof of Theorem 1.1 We infer from (3.5) that

$$\frac{d}{dt}\int_\Omega u(\cdot, t) \leq \mu_1\int_\Omega u(\cdot, t).$$

By applying Lemmatas 3.9–3.11, it indicates that

$$\int_\Omega u(\cdot, t) \geq \int_\Omega u(\cdot, t_i)e^{-\mu_1(t_i - t)} \geq \varepsilon e^{-\mu_1(t_i - t)}, \tag{3.46}$$

where $t \in [0, t_i)$ and $i \in N$. Consequently, we directly obtain $\int_\Omega u(\cdot, t) \geq \varepsilon e^{-\mu_1 t_1}$. We observe that the inequality $t_{i+1} < t_i + T$ holds for all $i \geq 1$ and large values of t. Hence, we can denote (3.46) as $\int_\Omega u(\cdot, t) \geq \varepsilon e^{-2\mu_1 T}$ for all $t \in [t_i, t_{i+1})$. Notably, without a loss of generality, $t_i \to \infty$ as $i \to \infty$, this implies that it possesses a lower bound $m_u := \min\{\varepsilon e^{-\mu_1 t_1}, \varepsilon e^{-2\mu_1 T}\}$ for $\int_\Omega u$. Through constructing some other Lyapunov functionals, we can then utilize the same methodology to determine lower the bound m_v for $\int_\Omega v$.

4. Boundedness of solutions

Assuming that $\Omega \subset R^n (n \geq 2)$ is a bounded convex domain with a smooth boundary, for the sake of convenience, we define $\theta := \min\{\theta_1, \theta_2\}$. First, we establish a lower bound for z based on Theorem 1.1; other approach details can be also found in [40].

Lemma 4.1. *Assuming that the conditions in Theorem 1.1 hold and $t \in [0, T_{max})$; then, we can find a constant $\delta > 0$ independent of t such that*

$$\inf_{x\in\Omega} z(x,t) \geq \delta > 0. \tag{4.1}$$

Proof. By taking the positivity of u and applying the comparison principle, we can infer from the fourth equation of system (1.5) that

$$z(x,t) \geq \delta(t) := \inf_{y\in\Omega} z_0(y) e^{-t}. \tag{4.2}$$

Let us fix $\delta_1 = \frac{1}{2}\inf_{x\in\Omega} z_0(x)$; then, there exists a $t_0 > 0$ such that $z(x,t) > \delta_1$ for $t \in [0, t_0]$. To conclude the proof, it is sufficient to show that $t \in [t_0, T_{max})$. By referencing the established result of Lemma 3.1 in [38] and for all nonnegative $\varphi \in C^0(\bar{\Omega})$, we can derive the following:

$$e^{t\Delta}\varphi \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(diam\Omega)^2}{4t}} \int_\Omega \varphi dx. \tag{4.3}$$

By virtue of the variation of constants formula and Theorem 1.1, we derive the following:

$$\begin{aligned} w(x,t) &= e^{t(\Delta-1)} w_0 + \int_0^t e^{(t-s)(\Delta-1)}(u(x,t)+v(x,t))ds \\ &\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-(t-s)-\frac{(diam\Omega)^2}{4(t-s)}} \int_\Omega (u(x,t)+v(x,t))dxds \\ &\geq (m_u+m_v)\int_0^{t_0} \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-s-\frac{(diam\Omega)^2}{4s}} ds \\ &=: c > 0. \end{aligned} \tag{4.4}$$

Reusing the variation of constants formula to z and (4.4), we deduce the following:

$$\begin{aligned} z(x,t) &= e^{t(\Delta-1)} z_0 + \int_0^t e^{(t-s)(\Delta-1)} w(x,t)ds \\ &\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-(t-s)-\frac{(diam\Omega)^2}{4(t-s)}} \int_\Omega w(x,t)dxds \\ &\geq c|\Omega| \int_0^{t_0} \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-s-\frac{(diam\Omega)^2}{4s}} ds \\ &=: \delta_2 > 0. \end{aligned}$$

Now, we can set $\delta := \min\{\delta_1, \delta_2\}$ to complete the proof. □

In order to more conveniently utilize the Gagliardo-Nirenberg inequality in Lemma 4.7, we need to select some parameters in advance. For any $p > 1$ and $q > 1$, we define

$$\alpha_1 = \frac{2(p+1)}{1+\theta}, \tag{4.5}$$

$$\alpha_2 = \frac{2(p+1)(q-1)}{p-1}, \tag{4.6}$$

$$\lambda_i = \frac{q - \frac{q}{\alpha_i}}{q + \frac{1}{n} - \frac{1}{2}} \tag{4.7}$$

and

$$f_i = \frac{\alpha_i}{q}\lambda_i = \frac{\alpha_i - 1}{q + \frac{1}{n} - \frac{1}{2}} \tag{4.8}$$

for $i = 1, 2$.

Lemma 4.2. *Let $n \geq 2$; then, for $\theta > 1 - \frac{2}{n+1}$ and a sufficiently large $p > 1$, there exists a $q > 1$ such that*

$$\lambda_i \in (0,1) \quad \textit{and} \quad f_i < 2, \quad \textit{for } i = 1, 2. \tag{4.9}$$

Proof. First, a simple calculation reveals that the first inequality in (4.9) is equivalent to the following:

$$0 < q - \frac{q}{\alpha_i} < q + \frac{1}{n} - \frac{1}{2}, \quad \text{for } i = 1, 2. \tag{4.10}$$

Regarding the above inequality , we can obtain $\alpha_i > 1$ and $q > \frac{\alpha_i}{2} - \frac{1}{n}$. Additionally, $f_i < 2$ is equivalent to $\alpha_i - 1 < 2(q + \frac{1}{n} - \frac{1}{2})$, which means that $q > \frac{\alpha_i}{2} - \frac{1}{n}$. Therefore, we can conclude that (4.9) holds if $\alpha_i > 1$ and $q > \frac{\alpha_i}{2} - \frac{1}{n}$. Moreover, we need to make sure

$$p > \frac{1+\theta}{2} - 1, \;\; q > \frac{p-1}{2(p+1)} + 1, \;\; \frac{p+1}{1+\theta} - \frac{1}{n} < q < \frac{p+1}{2} + \frac{p-1}{2n}.$$

Thus, the existence of q is dependent on $\frac{p+1}{1+\theta} < \frac{p+1}{2} + \frac{p-1}{2n} + \frac{1}{n}$, which can be easily derived thanks to $\theta > 1 - \frac{2}{n+1}$. Thus, when p is large enough, we can choose a q such that the inequality (4.10) makes sense. As a result, we can conclude that (4.9) is valid. □

Lemma 4.3. *Let $n \geq 2$; there is a constant $C_1 > 0$ such that*

$$\|\nabla z(\cdot, t)\|_{L^1(\Omega)} \leq C_1, \quad \textit{for all } t \in (0, T_{max}). \tag{4.11}$$

Proof. By applying integration by parts and Young's inequality, we can obtain the following result when testing the fourth equation of (1.5) with $-2\Delta z$:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla z|^2 &= -2\int_\Omega (|\Delta z|^2 + |\nabla z|^2 + w\Delta z) \\
&\leq -2\int_\Omega |\Delta z|^2 - 2\int_\Omega |\nabla z|^2 + 2\int_\Omega |\Delta z|^2 + \frac{1}{2}\int_\Omega w^2 \\
&= -2\int_\Omega |\nabla z|^2 + \frac{1}{2}\int_\Omega w^2.
\end{aligned}$$

By applying (3.10) and the ODE comparison principle, we have $\int_\Omega |\nabla z|^2 \le c_1$, where $c_1 > 0$ is a constant. By utilizing Young's inequality once more, we obtain $\int_\Omega |\nabla z| \le \int_\Omega |\nabla z|^2 + \frac{1}{4}|\Omega| \le c_2$, where $c_2 > 0$ is a constant. Therefore, we can easily derive (4.11). □

Lemma 4.4. *Assuming that the conditions stated in Theorem 1.1 hold, we can conclude that for any $p > 1$, for all $t \in (0, T_{max})$, there exist positive constants C_2, C_3 that are independent of t, such that*

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le -\mu_1 \int_\Omega u^{p+1} + C_2 \int_\Omega |\nabla z|^{\frac{2(p+1)}{1+\theta}} + C_2 \tag{4.12}$$

and

$$\frac{d}{dt}\int_\Omega v^p + \int_\Omega v^p \le -\mu_2 \int_\Omega v^{p+1} + C_3 \int_\Omega |\nabla z|^{\frac{2(p+1)}{1+\theta}} + C_3. \tag{4.13}$$

Proof. By multiplying the first equation of (1.5) by u^{p-1} and performing integration by parts, we derive the following:

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= \int_\Omega u^{p-1}\nabla(D(u)\nabla u) - \chi_1 \int_\Omega u^{p-1}\nabla(\frac{u}{z^k}\nabla z) + \mu_1 \int_\Omega u^p(1 - u - a_1 v) \\ &\le -(p-1)\int_\Omega u^{p-2+\theta}|\nabla u|^2 + \frac{\chi_1(p-1)}{\delta^k}\int_\Omega u^{p-1}\nabla u \cdot \nabla z + \mu_1 \int_\Omega u^p - \mu_1 \int_\Omega u^{p+1} - \mu_1 a_1 \int_\Omega u^p v,\end{aligned} \tag{4.14}$$

where δ is from Lemma 4.1. By applying Young's inequality twice, we deduce the following:

$$\int_\Omega u^{p-1}\nabla u \cdot \nabla z \le \frac{\delta^k}{2\chi_1}\int_\Omega u^{p-2+\theta}|\nabla u|^2 + \frac{\mu_1 \delta^k}{2\chi_1 p(p-1)}u^{p+1} + c_3 \int_\Omega |\nabla z|^{\frac{2(p+1)}{1+\theta}}, \tag{4.15}$$

where $c_3 > 0$ is a constant. Invoking the Young's inequality once more, we have the following:

$$(\mu_1 + \frac{1}{p})\int_\Omega u^p \le \frac{(2p-3)\mu_1}{2p}\int_\Omega u^{p+1} + c_4, \tag{4.16}$$

where $c_4 > 0$. Together (4.14)–(4.16), we can derive (4.12). Furthermore, a similar argument used to obtain the inequality in (4.12) implies that (4.13) also holds. □

To obtain the first term on the right side of inequalities (4.12) and (4.13), we introduce a differential inequality about $\int_\Omega w^{p+1}$ for any $p > 1$.

Lemma 4.5. *Assuming that the conditions stated in Theorem 1.1 hold, for all $t \in (0, T_{max})$, the solution of the system* (1.5) *satisfies the following:*

$$\frac{d}{dt}\int_\Omega w^{p+1} + \int_\Omega w^{p+1} \le 2^p \int_\Omega u^{p+1} + 2^p \int_\Omega v^{p+1}. \tag{4.17}$$

Proof. By multiplying the third equation in (1.5) with $(p+1)w^p$ and Young's inequality, we have the following:

$$\begin{aligned}\frac{d}{dt}\int_\Omega w^{p+1} &= -p(p+1)\int_\Omega w^{p-1}|\nabla w|^2 - (p+1)\int_\Omega w^{p+1} + (p+1)\int_\Omega w^p u + (p+1)\int_\Omega w^p v \\ &\le -\int_\Omega w^{p+1} + 2^p \int_\Omega u^{p+1} + 2^p \int_\Omega v^{p+1},\end{aligned}$$

which implies (4.17). □

To hand the second term on the right side of inequalities (4.12) and (4.13), we will outline the properties of the component solution z of system (1.5).

Lemma 4.6. *Assuming that the conditions stated in Theorem 1.1 hold, for all $t \in (0, T_{max})$, for any $q > 1$, there exists $C_4 > 0$ independent of t such that*

$$\frac{d}{dt}\int_\Omega |\nabla z|^{2q} + 2q\int_\Omega |\nabla z|^{2q} \le -\frac{2(q-1)}{q}\int_\Omega |\nabla|\nabla z|^q|^2 + \frac{1}{2}\int_\Omega w^{p+1} + C_4\int_\Omega |\nabla z|^{\frac{2(p+1)(q-1)}{p-1}}. \tag{4.18}$$

Proof. The result is a well-established inequality, which can be found in Lemma 4.2 of [41]. Therefore, we have omitted the proof process, and interested readers are encouraged to refer to the original literature and the references therein for further details. □

By invoking the above four lemmas, we can provide a bound for L^p norm of u for any $p > 1$.

Lemma 4.7. *Let $n \ge 2$. Assuming that the conditions stated in Theorem 1.1 hold, and considering $\theta > 1 - \frac{2}{n+1}$, then for all $t \in (0, T_{max})$ and any $p > 1$, we can find a constant $C_5 > 0$ independent of t such that*

$$\int_\Omega u^p \le C_5 \quad \text{and} \quad \int_\Omega v^p \le C_5. \tag{4.19}$$

Proof. Combing (4.12), (4.13), (4.17) and (4.18), we can derive the following:

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{2^p}{\mu_1}\int_\Omega u^p + \frac{2^p}{\mu_2}\int_\Omega v^p + \int_\Omega w^{p+1} + \int_\Omega |\nabla z|^{2q}\right\}\\
&+ \frac{2^p}{\mu_1}\int_\Omega u^p + \frac{2^p}{\mu_2}\int_\Omega v^p + \frac{1}{2}\int_\Omega w^{p+1} + 2q\int_\Omega |\nabla z|^{2q} + \frac{2(q-1)}{q}\int_\Omega |\nabla|\nabla z|^q|^2 \\
\le& c_5\int_\Omega |\nabla z|^{\alpha_1} + c_6\int_\Omega |\nabla z|^{\alpha_2} + c_7,
\end{aligned} \tag{4.20}$$

where $c_5, c_6, c_7 > 0$ and α_1, α_2 are defined in (4.5) and (4.6). In view of the Gagliardo-Nirenberg inequality, there exists constants $c_8, c_9 > 0$ such that

$$c_5\int_\Omega |\nabla z|^{\alpha_1} = c_5\||\nabla z|^q\|^{\frac{\alpha_1}{q}}_{L^{\frac{\alpha_1}{q}}(\Omega)} \le c_8\|\nabla|\nabla z|^q\|^{\frac{\lambda_1\alpha_1}{q}}_{L^2(\Omega)}\||\nabla z|^q\|^{\frac{\alpha_1}{q}(1-\lambda_1)}_{L^{\frac{1}{q}}(\Omega)} + c_8\||\nabla z|^q\|^{\frac{\alpha_1}{q}}_{L^{\frac{1}{q}}(\Omega)},$$

$$c_6\int_\Omega |\nabla z|^{\alpha_2} = c_6\||\nabla z|^q\|^{\frac{\alpha_2}{q}}_{L^{\frac{\alpha_2}{q}}(\Omega)} \le c_9\|\nabla|\nabla z|^q\|^{\frac{\lambda_2\alpha_2}{q}}_{L^2(\Omega)}\||\nabla z|^q\|^{\frac{\alpha_2}{q}(1-\lambda_2)}_{L^{\frac{1}{q}}(\Omega)} + c_9\||\nabla z|^q\|^{\frac{\alpha_2}{q}}_{L^{\frac{1}{q}}(\Omega)},$$

where $\lambda_i (i = 1, 2)$ are defined in (4.7) and $\lambda_i \in (0, 1)$ from Lemma 4.2. Thus, thanks to $\||\nabla z|^q\|_{L^{\frac{1}{q}}(\Omega)} = \|\nabla z\|^q_{L^1(\Omega)}$ and applying Young's inequality, we can infer from Lemma 4.3 that

$$c_5\int_\Omega |\nabla z|^{\alpha_1} \le \frac{q-1}{q}\int_\Omega |\nabla|\nabla z|^q|^2 + c_{10}, \tag{4.21}$$

$$c_6\int_\Omega |\nabla z|^{\alpha_2} \le \frac{q-1}{q}\int_\Omega |\nabla|\nabla z|^q|^2 + c_{11}, \tag{4.22}$$

where $c_{10}, c_{11} > 0$. Substituting (4.21), (4.22) into (4.20), we obtain $f'(t) + c_{12}f(t) \le c_{13}$, where $f(t) := \frac{2^p}{\mu_1}\int_\Omega u^p + \frac{2^p}{\mu_2}\int_\Omega v^p + \int_\Omega w^{p+1} + \int_\Omega |\nabla z|^{2q}$, $c_{12} := \min\{\frac{1}{2}, 2q\}$, $c_{13} > 0$.

Therefore, employing the standard ODE comparison principle, we can infer that (4.19) is valid. □

Proof of Theorem 1.2 Due to (4.19) and Lemma 4.1 in [34], for all $t \in (0, T_{max})$, we can readily deduce that for all $\sigma > 1$,

$$\|w(\cdot, t)\|_{W^{1,\sigma}(\Omega)} \leq C_6,$$

where $C_6 > 0$ is a constant.

By applying some parabolic regularity and utilizing Lemma 4.3, for all $t \in (0, T_{max})$, we obtain the following result:

$$\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_7,$$

where $C_7 > 0$. Taking advantage of a standard Alikakos-Moser iteration [39] and Lemma 4.6, for all $t \in (0, T_{max})$, there exists a constant $C_8 > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_8.$$

By combining with Lemma 2.1, we can establish the validity of Theorem 1.2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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