



Research article

A sufficient maximum principle for backward stochastic systems with mixed delays

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Abstract: In this paper, we study the problem of optimal control of backward stochastic differential equations with three delays (discrete delay, moving-average delay and noisy memory). We establish the sufficient optimality condition for the stochastic system. We introduce two kinds of time-advanced stochastic differential equations as the adjoint equations, which involve the partial derivatives of the function f and its Malliavin derivatives. We also show that these two kinds of adjoint equations are equivalent. Finally, as applications, we discuss a linear-quadratic backward stochastic system and give an explicit optimal control. In particular, the stochastic differential equations with time delay are simulated by means of discretization techniques, and the effect of time delay on the optimal control result is explained.

Keywords: stochastic optimal control; stochastic differential equation with time delay; noisy memory; Malliavin derivative

1. Introduction

The forward stochastic differential equation characterizes the dynamic changes of state processes with given initial state trajectories. A backward stochastic differential equation (BSDE) can be seen as an Itô stochastic differential equation in which the terminal rather than the initial condition is given. BSDE will admit a pair of adapted process under some conditions on the coefficient and the terminal value of the BSDE. A stochastic differential equation coupled with a BSDE that has both initial and terminal conditions are known as forward-backward stochastic differential equations. In contrast to the development of forward stochastic differential equations, the study of backward stochastic differential equations (BSDEs) have been started in the last three decades. Pardoux and Peng [1] first proposed BSDEs. Since then, BSDEs have been systematically studied due to their wide applications in social

fields such as applied mathematics, engineering and finance [2–7]. A BSDE can be viewed as an Itô's stochastic differential equation, where the endpoints are given except for the initial conditions. It is worth pointing out that the solution of BSDE includes two processes, which is the essential difference from forward stochastic differential equation. Due to the wide-range of applications and the interesting properties and structure of BSDE, it is valuable to study the optimal control problem of such equations.

There is a lot of research focused on stochastic optimal control problems. The stochastic optimal control for BSDE was studied by Dokuchaer and Zhou [8]. Subsequently, Lim and Zhou [9] solved the linear-quadratic control problem for BSDE, and two alternative expressions for optimal control are obtained. Huang et al. [10] obtained a stochastic maximum principle for partial information control problem of backward stochastic systems. Zhen et al. [11] discussed the optimal control for fully coupled forward-backward stochastic control systems. They obtained the necessary and sufficient stochastic maximum principles. [12] focused on nonzero sum differential games for BSDEs. They established necessary and sufficient conditions for Nash equilibrium points (see [13] and [14] for the game problems of BSDE). For the control problems of BSDEs, see also [15–20].

In the above problems, the BSDEs depend only on the value of the current time. Control systems with time-delay have existed for several decades, which are widely applied in financial areas. For example, investment portfolio management, pricing, hedging problems and automobile traffic model [21–23]. It will be more realistic for an investor to better strategize his investment when considering a delay factor. [24] introduced the time-advanced stochastic differential equations. [25] extended for the first time the applicability of the optimal homotopy asymptotic method to find the algorithm of approximate analytic solution of delay differential equations. However, the control system contained only discrete delay. [26] generalized the results to partial information. Apart from the discrete delays, it is more reasonable and practical to include moving average delays, which indicate that the current situation depends on a past period of history in the form of an integral. As an example of an optimal consumption problem, at time t , let $x(t)$ and $u(t)$ be wealth and consumption, respectively. It is reasonable to assume that the incremental wealth is a combination of the present value $x(t)$ plus some sliding average of previous value $\int_{-\delta}^0 e^{ts} x(t+s) ds$ and negative consumption amount $u(t)$. Therefore, it is meaningful to incorporate moving-average delays, which indicates that the current situation of the system depends on a past period of history in integral form. Moreover, in [27], the authors introduced another delay, the noisy memory or Brownian delay, which is an Itô's integral over a past time periods. Indeed, noisy memories arise naturally in economics and finance, e.g., in modeling stochastic volatility (see, e.g., [28]). [29] obtained necessary and sufficient conditions for mean-field BSDEs with noisy memory over an infinite time span using Malliavin calculus technique. They also applied the theoretical results to pension fund problem. Zhang [31] established the sufficient maximum principle for optimal control problems with three delays and introduced a unified adjoint equation.

In this paper, we consider a BSDE with three delays: The discrete delay, the moving-average delay and the noisy memory. Our work differs from the above in several ways.

1) Compared with [24] and [29], three kinds of delays are incorporated in the optimal control problem: the discrete delay and the moving-average delay and noisy memory, resulting in two kinds of adjoint equations, which will increase the difficulty of analyzing the maximum condition. Thus, our optimal control system can describe more stochastic delay phenomena and cover more backward stochastic delay optimization control problems.

2) We give one kind of a 2-dimensional time-advanced stochastic differential equation as our adjoint

equation in a straightforward form. More importantly, we show that the two kinds of adjoint equations are equivalent. It is also emphasized that only the solution $P_1(t)$ of (3.2) is needed in the theory of the sufficient maximum principle, while $P_2(t)$ is an auxiliary process.

3) As we know from [30], the four-step scheme is used to investigate the adapted solutions for forward-backward stochastic differential equations. Unfortunately, we study the optimal control problem for the state governed by BSDE with delay, that is to say, the state is past-dependence, so the system is not Markovian, and we cannot derive a certain system to get the explicit solution of the state Y and Z . In order to obtain the effect of delay, we divide the interval according to the delay, and then continue to subdivide each subinterval to get the numerical solution using Euler method or improved Euler method.

This paper is organized as follows. Section 2 presents the problem statement and some preliminaries. In Section 3, we introduce two kinds of adjoint equations, obtain the sufficient maximum principle in Section 4. In Section 5, we apply the theoretical results to linear-quadratic systems and derive explicit expressions for the optimal control. In addition, we study an optimal consumption problem. Finally, we conclude the paper with concluding remarks.

2. Preliminary results

Assume that T is a finite time horizon, λ is a constant, and δ is a positive constant time delay such that $\delta < T$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a complete filtered probability space on which one-dimensional standard Brownian motion is defined. Moreover, it is also assumed that $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is a natural filtration generated by $W(t)$ and $\mathcal{F}_T = \mathcal{F}$. Let $\mathbb{E}[\cdot]$ be the expectation under P . Denote $\mathbb{E}'[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ for each $t \in [0, T]$.

Throughout this paper, we denote by $L^2(\mathcal{F}_t; \mathbb{R})$ the space of \mathbb{R} -valued \mathcal{F}_t -measurable random variables η satisfying $\mathbb{E}|\eta|^2 < +\infty$, by $L^2_{\mathcal{F}}(r, s; \mathbb{R})$ the space of \mathbb{R} -valued \mathcal{F}_t -adapted process $\psi(\cdot)$ satisfying $\mathbb{E} \int_r^s |\psi(t)|^2 dt < +\infty$. Let $\phi(\cdot) : [-\delta, 0] \rightarrow \mathbb{R}$ be given deterministic functions such that $\sup_{-\delta \leq t \leq 0} |\phi(t)|^2 < +\infty$. Let U be a nonempty convex control subset of \mathbb{R} . We introduce the admissible control set $\mathcal{U} = \{u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) | u(t) \in U, t \in [0, T]\}$.

Consider the following BSDE with delay:

$$\begin{cases} -dy(t) = f(t, \mathbf{y}(t), z(t), u(t))dt - z(t)dW(t), \\ y(T) = \xi, \quad y(t) = \phi(t), \quad t \in [-\delta, 0), \end{cases} \quad (2.1)$$

where $\mathbf{y}(t) := (y(t), y_1(t), y_2(t), y_3(t))$, $y_1(t) := \int_{t-\delta}^t e^{\lambda(s-t)} y(s) ds$, $y_2(t) := \int_{t-\delta}^t y(s) dW(s)$ and $y_3(t) := y(t - \delta)$. $y_1(t), y_2(t), y_3(t)$ are called the moving-average delayed state, the noisy memory process and the discrete delayed state, respectively. Here $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function in $(t, y, y_1, y_2, y_3, z, u)$, $\xi \in L^2(\mathcal{F}_T, \mathbb{R})$, $\phi(t)$ is a deterministic function.

The following assumption will be inforce.

(H1) f is continuously differentiable in (y, y_1, y_2, y_3, z, u) . Moreover, the partial derivatives $f_y, f_{y_1}, f_{y_2}, f_{y_3}, f_z, f_u$ of f are uniformly bounded.

We give the corresponding functional

$$J(u(t)) = \mathbb{E} \left\{ \int_0^T l(t, \mathbf{y}(t), z(t), u(t)) dt + g(y(0)) \right\}, \quad (2.2)$$

where l and g satisfy the following condition:

(H2) The functions l and g are differentiable in (y, y_1, y_2, y_3, z, u) and y , respectively, and all the derivatives are bounded.

The following is a stochastic optimal control problem.

Problem (BSO) Find an admissible control that maximizes $J(u(\cdot))$ over $u \in \mathcal{U}$.

Any admissible control that solves **Problem (BSO)** is called an optimal control. Assume that $\bar{u}(\cdot) \in \mathcal{U}$ is an admissible control. For convenience, we use the following notations throughout the paper. $f(t, \bar{u}) = f(t, \bar{y}, z, \bar{u})$, $f(t, u) = f(t, y, z, u)$, $l(t, \bar{u}) = f(t, y, z, \bar{u})$, $l(t, u) = l(t, y, z, u)$, and for $k = y, y_1, y_2, y_3, z, u$, we use $f_k(t) = \frac{\partial f}{\partial k}(\bar{y}, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{z}, \bar{u})$ and for $j = y, z, u$, we use $l_j(t) = \frac{\partial l}{\partial j}(\bar{y}, \bar{z}, \bar{u})$.

Remark 2.1. In contrast to [29], the present optimal control problem can also include moving-average delay and noisy memory. Thus, the model can cover more stochastic phenomena and stochastic optimization problems.

3. The adjoint equations

The adjoint equation plays a key role in the theory of maximum principle. In this section, we introduce some adjoint equations. Then, we decompose the two-dimensional adjoint equation (3.1) into two one-dimensional systems of equations and give the equivalent adjoint equation (3.6), which is called a unified one since it connects the two kinds of adjoint equations (3.2), (3.3) and (3.6).

3.1. A primer of Malliavin derivative

A general reference to Malliavin derivative D_t for Lévy processes is the book (Di Nunno et al. [32]). See also Nualart [33] and Di Nunno et al. [34].

For all $F \in L^2(\mathcal{F}_T, P)$, we have $D_t F \in (\mathbb{S})^*$ and $(t, \omega) \mapsto E[D_t F | \mathcal{F}_t]$ belongs to $L^2(\mathcal{F}_T, \lambda \times P)$, where $(\mathbb{S})^* \supseteq L^2(P)$ denotes the Hida space of stochastic distributions and λ denotes the Lebesgue measure on $[0, T]$. Moreover, the following generalized Clark-Ocone theorem holds:

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t).$$

As also noted in Dahl et al. [27], we can also apply the Clark-Ocone theorem to show that:

Proposition 3.1. (The generalized duality formula). Let $F \in L^2(\mathcal{F}_T, P)$ and let $\varphi(t) \in L^2(\lambda \times P)$ be adapted. Then

$$\mathbb{E}\left[F \int_0^T \varphi(t) dB(t)\right] = \mathbb{E}\left[\int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] \varphi(t) dt\right].$$

Proof. By Clark-Ocone theorem and the Itô isometry we have

$$\begin{aligned} \mathbb{E}\left[F \int_0^T \varphi(t) dB(t)\right] &= \mathbb{E}\left[\left(\mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t)\right) \left(\int_0^T \varphi(t) dW(t)\right)\right] \\ &= \mathbb{E}\left[\left(\int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t)\right) \left(\int_0^T \varphi(t) dW(t)\right)\right] \\ &= \mathbb{E}\left[\int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] \varphi(t) dt\right]. \end{aligned}$$

3.2. The first version of the adjoint equations

In this section, we will give two adjoint equations and prove that they are equivalent.

The adjoint equations for **Problem (BSO)** is given by

$$\begin{cases} dP(t) = \left\{ A_1(t)P(t) + \mathbb{E}'[A_2(s)P(s)]|_{s=t+\delta}\chi_{[0,T-\delta]}(t) + \mathbb{E}' \int_t^{t+\delta} D_t(B_1(s)P(s))ds \right. \\ \quad \left. + \tilde{b}_1(t) \right\} dt + (B_2(t)P(t) + \tilde{b}_2(t))dW(t), \\ P(0) = \zeta, \quad P(t) = 0, t \in (T, T + \delta], \end{cases} \quad (3.1)$$

where

$$A_1(t) = \begin{pmatrix} f_y(t) & 1 \\ f_{y_1}(t) & \lambda \end{pmatrix}, \quad A_2(s) = \begin{pmatrix} f_{y_3}(s) & -e^{-\lambda\delta} \\ 0 & 0 \end{pmatrix}, \quad B_1(s) = \begin{pmatrix} f_{y_2}(s) & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_2(t) = \begin{pmatrix} f_z(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{b}_1(t) = \begin{pmatrix} -l_y(t) \\ 0 \end{pmatrix}, \quad \tilde{b}_2(t) = \begin{pmatrix} -l_z(t) \\ q_2(t) \end{pmatrix}, \quad \zeta = \begin{pmatrix} -g_y(\bar{y}(0)) \\ 0 \end{pmatrix}.$$

Here and in what follows, χ denotes the indicator function.

3.3. The decomposition of the adjoint equation

In this section, we decompose the two-dimensional adjoint equation (3.1) into two one-dimensional systems of equations.

Define the solution $P(\cdot)$ of the adjoint equation (3.1) by

$$P(t) = \begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix}.$$

Then by (3.1), it is easy to check that

$$\begin{cases} dP_1(t) = \{f_y(t)P_1(t) + \mathbb{E}'[f_{y_3}(t + \delta)P_1(t + \delta)]\chi_{[0,T-\delta]}(t) - l_y(t) \\ \quad + \int_t^{t+\delta} \mathbb{E}[D_t f_{y_2}(s)P_1(s)|\mathcal{F}_t]ds + P_2(t) - e^{-\lambda\delta}\mathbb{E}'[P_2(t + \delta)]\chi_{[0,T-\delta]}(t)\} dt \\ \quad + \{f_z(t)P_1(t) - l_z(t)\}dW(t) \\ P_1(0) = -g_y(\bar{y}(0)), \end{cases} \quad (3.2)$$

$$\begin{cases} dP_2(t) = \{f_{y_1}(t)P_1(t) + \lambda P_2(t)\}dt - q_2(t)dW(t) \\ P_2(0) = 0, \end{cases} \quad (3.3)$$

Remark 3.2. Note that, (3.2) and (3.3) is a system of coupled equations, where $P_1(\cdot)$ depends on $P_2(\cdot)$. We will show that $P_1(\cdot)$ satisfies an independent time-advanced BSDE involving Malliavin derivatives (see (3.6)).

3.4. The second version of the adjoint equations

Note that, it is difficult to derive the adjoint equations $P_1(\cdot)$ and $P_2(\cdot)$ by focusing on the coupled equations (3.2) and (3.3). We will give the equivalent adjoint equation to show that $P_1(\cdot)$ satisfies an independent time-advanced BSDE involving Malliavin derivatives (see (3.6)). We call it a unified one since it connects the two kinds of adjoint equations (3.2), (3.3) and (3.6).

Lemma 3.3. *Under (H1), it holds that*

$$\begin{aligned} P_2(t) - e^{-\lambda\delta} \mathbb{E}^t[P_2(t + \delta)]\chi_{[0, T-\delta]}(t) \\ = \int_t^{(t+\delta) \wedge (T-\delta)} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds. \end{aligned} \quad (3.4)$$

Proof. Given (3.3), applying Itô's formula to $e^{-\lambda s}P_2(s)$, $0 \leq s \leq t$ yields

$$P_2(t) = - \int_0^t \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds. \quad (3.5)$$

Set $\Sigma(t) = P_2(t) - e^{-\lambda\delta} \mathbb{E}^t[P_2(t + \delta)]\chi_{[0, T-\delta]}(t)$. Then by (3.5), we have

$$\begin{aligned} \Sigma(t) &= \int_0^t -\mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds \\ &+ \int_0^{t+\delta} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds\chi_{[0, T-\delta]}(t). \end{aligned}$$

(i) We first consider the case: $t + \delta < T - \delta$, $\Sigma(t) = \int_t^{t+\delta} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds$.

(ii) Otherwise, $\Sigma(t) = \int_t^{T-\delta} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds$.

Therefore, we have $\Sigma(t) = \int_t^{(t+\delta) \wedge (T-\delta)} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds$.

Thus, the proof is complete.

Substituting (3.4) into (3.2), we get the following equivalent adjoint equation

$$\left\{ \begin{aligned} dP_1(t) &= \{f_y(t)P_1(t) + \mathbb{E}^t[f_{y_3}(t + \delta)P_1(t + \delta)]\chi_{[0, T-\delta]}(t) - l_y(t) \\ &+ \int_t^{t+\delta} \mathbb{E}[D_t f_{y_2}(s)P_1(s)|\mathcal{F}_t]ds \\ &+ \int_t^{(t+\delta) \wedge (T-\delta)} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds\}dt \\ &+ \{f_z(t)P_1(t) - l_z(t)\}dW(t) \\ P_1(0) &= -g_y(\bar{y}(0)). \end{aligned} \right. \quad (3.6)$$

Remark 3.4. *Three types of delays of the state are incorporated in **Problem (BSO)**. Let us consider some special cases.*

(a) *If the moving-average delayed state $y_1(\cdot)$ vanishes, then it holds that $P_2(t) = 0$, (3.6) degenerates to*

$$\left\{ \begin{aligned} dP_1(t) &= \{f_y(t)P_1(t) + \mathbb{E}^t[f_{y_3}(t + \delta)P_1(t + \delta)]\chi_{[0, T-\delta]}(t) - l_y(t) \\ &+ \int_t^{t+\delta} \mathbb{E}[D_t f_{y_2}(s)P_1(s)|\mathcal{F}_t]ds\}dt \\ &+ \{f_z(t)P_1(t) - l_z(t)\}dW(t) \\ P_1(0) &= -g_y(\bar{y}(0)). \end{aligned} \right.$$

This is the case studied in [35] without mean-field term and on finite time horizon.

(b) If the noisy memory $y_2(\cdot)$ vanishes, then (3.6) degenerates to

$$\begin{cases} dP_1(t) = \{f_y(t)P_1(t) + \mathbb{E}^t[f_{y_3}(t+\delta)P_1(t+\delta)]\chi_{[0, T-\delta]}(t) - l_y(t) \\ \quad + \int_t^{(t+\delta)\wedge(T-\delta)} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds\}dt \\ \quad + \{f_z(t)P_1(t) - l_z(t)\}dW(t) \\ P_1(0) = -g_y(\bar{y}(0)). \end{cases}$$

This is the case studied in [36] without mean-field term and on finite time horizon. Note that, in [36], the discrete delay is not considered.

(c) If the discrete state $y_3(\cdot)$ vanishes, then (3.6) degenerates to

$$\begin{cases} dP_1(t) = \{f_y(t)P_1(t) - l_y(t) + \int_t^{t+\delta} \mathbb{E}[D_t f_{y_2}(s)P_1(s)|\mathcal{F}_t]ds \\ \quad + \int_t^{(t+\delta)\wedge(T-\delta)} \mathbb{E}^t[e^{\lambda(t-s)}(l_{y_1}(s) - P_1(s)f_{y_1}(s))]ds\}dt \\ \quad + \{f_z(t)P_1(t) - l_z(t)\}dW(t) \\ P_1(0) = -g_y(\bar{y}(0)). \end{cases}$$

This case has not been studied before.

(d) If both $y_1(\cdot)$ and $y_2(\cdot)$ vanish, then (3.6) degenerates to

$$\begin{cases} dP_1(t) = \{f_y(t)P_1(t) + \mathbb{E}^t[f_{y_3}(t+\delta)P_1(t+\delta)]\chi_{[0, T-\delta]}(t) - l_y(t)\}dt \\ \quad + \{f_z(t)P_1(t) - l_z(t)\}dW(t) \\ P_1(0) = -g_y(\bar{y}(0)). \end{cases}$$

This is the case studied in [37] with full information.

4. The sufficient maximum principle

Define the Hamiltonian function H by

$$\begin{aligned} H(t, y, y_1, y_2, y_3, z, u, P_1, P_2) &= l(t, \mathbf{y}, z, u) - f(t, y, y_1, y_2, y_3, z, u)P_1 \\ &\quad - (y - \lambda y_1 - e^{-\lambda\delta}y_3)P_2. \end{aligned}$$

In this section, we form sufficient condition of optimality based on some concavity assumptions on H and g .

The following assumption will also be needed.

(H3) The functions $H(t, y, y_1, y_2, y_3, z, u, P_1, P_2)$ and $g(y)$ are concave with respect to the corresponding variables, for $t \in [0, T]$ and given P_1, P_2 .

For simplicity, we use $(y(t), u(t))$ and $(\bar{y}(t), \bar{u}(t))$ to denote the state trajectories of system (2.1) corresponding to $u(t)$ and $\bar{u}(t)$. We denote by $\bar{H}(t) = H(t, \bar{y}, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{z}, \bar{u}, P_1, P_2)$, $\bar{H}(t) = H(t, y, y_1, y_2, y_3, z, u, P_1, P_2)$ and the similar definitions for $\bar{f}(t)$ and $f(t)$. We give the sufficient maximum principle now.

Theorem 4.1. Let (H1) – (H3) hold. If an admissible control $\bar{u}(\cdot)$ satisfies the following condition:

$$(\bar{u} - u)H_u(t) \geq 0,$$

then $\bar{u}(\cdot)$ is an optimal control.

Proof. We write $J(\bar{u}(\cdot)) - J(u(\cdot)) = I_1 + I_2$ with

$$I_1 = \mathbb{E} \int_0^T \left\{ l(t, \bar{\mathbf{y}}(t), \bar{z}(t), \bar{u}(t)) - l(t, \mathbf{y}(t), z(t), u(t)) \right\} dt, \quad (4.1)$$

$$I_2 = \mathbb{E}[g(\bar{\mathbf{y}}(0)) - g(\mathbf{y}(0))].$$

Since g is concave on y , we have

$$I_2 \geq \mathbb{E}[g_y(\bar{\mathbf{y}}(0))(\bar{\mathbf{y}}(0) - \mathbf{y}(0))]. \quad (4.2)$$

Applying Itô's formula to $P_1(t)(\bar{\mathbf{y}}(t) - \mathbf{y}(t))$ and taking expectation, we get

$$\begin{aligned} \mathbb{E}(g_y(\bar{\mathbf{y}}(0))(\bar{\mathbf{y}}(0) - \mathbf{y}(0))) &= \mathbb{E} \int_0^T \left\{ (\bar{\mathbf{y}}(t) - \mathbf{y}(t))(f_y(t)P_1(t) + \mathbb{E}^t[f_{y_3}(t + \delta)P_1(t + \delta)]\chi_{[0, T-\delta]}(t)) \right. \\ &\quad - l_y(t) + \int_t^{t+\delta} \mathbb{E}[D_t f_{y_2}(s)P_1(s)|\mathcal{F}_t] ds + P_2(t) - e^{-\lambda\delta} \mathbb{E}^t[P_2(t + \delta)]\chi_{[0, T-\delta]}(t) \left. \right\} dt \\ &\quad + \mathbb{E} \int_0^T \left\{ P_1(t)(-f(t, \bar{\mathbf{y}}(t), \bar{z}(t), \bar{u}(t)) - f(t, \mathbf{y}(t), z(t), u(t))) \right\} dt \\ &\quad + \mathbb{E} \int_0^T \left\{ (\bar{z}(t) - z(t))(f_z(t)P_1(t) - l_z(t)) \right\} dt. \end{aligned} \quad (4.3)$$

By the definition H , it holds that

$$\begin{aligned} I_1 &= \mathbb{E} \int_0^T \left\{ \bar{H}(t) - H(t) + (\bar{f}(t) - f(t))P_1(t) \right. \\ &\quad \left. - (\bar{\mathbf{y}}(t) - \mathbf{y}(t) - \lambda(\bar{y}_1(t) - y_1(t)) - e^{-\lambda\delta}(\bar{y}_3(t) - y_3(t)))P_2(t) \right\} dt. \end{aligned} \quad (4.4)$$

Noting that

$$dy_1(t) = [y(t) - \lambda y_1(t) - e^{-\lambda\delta} y_3(t)], \quad t \in [0, T].$$

Using Itô's formula to $P_2(t)(\bar{y}_1(t) - y_1(t))$ and integrating from 0 to T , and noticing $P_2(0) = 0$ and $\bar{y}_1(T) - y_1(T) = 0$, we get

$$\begin{aligned} &\mathbb{E}[P_2(T)(\bar{y}_1(T) - y_1(T))] - \mathbb{E}[P_2(0)(\bar{y}_1(0) - y_1(0))] \\ &= \mathbb{E} \int_0^T \left\{ -l_{y_1}(t) + P_1(t)f_{y_1}(t) + \lambda P_2(t)(\bar{y}_1(t) - y_1(t)) \right\} dt \\ &\quad + \mathbb{E} \int_0^T \left\{ P_2(t)(\bar{\mathbf{y}}(t) - \mathbf{y}(t) - \lambda(\bar{y}_1(t) - y_1(t)) - e^{-\lambda\delta}(\bar{y}_3(t) - y_3(t))) \right\} dt \\ &= 0. \end{aligned} \quad (4.5)$$

Inserting (4.5) into (4.4), we have

$$I_1 = \mathbb{E} \int_0^T \left\{ \bar{H}(t) - H(t) + (\bar{f}(t) - f(t))P_1(t) + (-l_{y_1}(t) + P_1(t)f_{y_1}(t) + \lambda P_2(t))(\bar{y}_1(t) - y_1(t)) \right\} dt. \quad (4.6)$$

Since H is concave, and from (4.2), (4.3) and (4.6), we can obtain

$$\begin{aligned} J(\bar{u}) - J(u) &\geq \mathbb{E} \int_0^T \left\{ -f_{y_3}(t)(\bar{y}_3(t) - y_3(t))P_1(t) \right. \\ &\quad + \mathbb{E}^t[f_{y_3}(t + \delta)P_1(t + \delta)]\chi_{[0, T-\delta]}(t)(\bar{y}(t) - y(t)) \left. \right\} dt \\ &\quad + \mathbb{E} \int_0^T \left\{ -e^{-\lambda\delta} \mathbb{E}^t[P_2(t + \delta)]\chi_{[0, T-\delta]}(t)(\bar{y}(t) - y(t)) + e^{-\lambda\delta}(\bar{y}_3(t) - y_3(t))P_2(t) \right\} dt \\ &\quad + \mathbb{E} \int_0^T \left\{ -f_{y_2}(t)(\bar{y}_2(t) - y_2(t))P_1(t) + \int_t^{t+\delta} \mathbb{E}[D_s f_{y_2}(s)P_1(s)|\mathcal{F}_t] ds (\bar{y}(t) - y(t)) \right. \\ &\quad \left. + l_u(t)(\bar{u}(t) - u(t)) - f_u(t)(\bar{u}(t) - u(t))P_1(t) \right\} dt \\ &= J_1 + J_2 + J_3 + J_5. \end{aligned}$$

Based on the definition of y_3 and through transformation of variables, we have

$$\begin{aligned} &\mathbb{E} \int_0^T f_{y_3}(t)(\bar{y}_3(t) - y_3(t))P_1(t) dt \\ &= \mathbb{E} \int_0^T f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt \\ &= \mathbb{E} \int_{-\delta}^{T-\delta} f_{y_3}(t + \delta)(\bar{y}(t) - y(t))P_1(t + \delta) dt. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathbb{E} \int_0^T \mathbb{E}^t[f_{y_3}(t + \delta)P_1(t + \delta)]\chi_{[0, T-\delta]}(t)(\bar{y}(t) - y(t)) dt \\ &= \mathbb{E} \int_{\delta}^{T+\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt. \end{aligned}$$

Therefore,

$$\begin{aligned} J_1 &= -\mathbb{E} \int_0^T f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt + \mathbb{E} \int_{\delta}^{T+\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt \\ &= -\mathbb{E} \int_0^{\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt - \mathbb{E} \int_{\delta}^T f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt \\ &\quad + \mathbb{E} \int_{\delta}^{T+\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt \\ &= -\mathbb{E} \int_0^{\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt + \mathbb{E} \int_T^{T+\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt. \end{aligned}$$

By (2.1), we have $\bar{y}(t - \delta) - y(t - \delta) = 0$ when $t \in [0, \delta]$. So $\int_0^{\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt = 0$. By (3.1), $P_1(t) = 0$, $t \in (T, T + \delta]$, we get $\mathbb{E} \int_T^{T+\delta} f_{y_3}(t)(\bar{y}(t - \delta) - y(t - \delta))P_1(t) dt = 0$. Then, we obtain $J_1 = 0$.

$$J_3 = \mathbb{E} \int_0^T \left\{ -f_{y_2}(t)(\bar{y}_2(t) - y_2(t))P_1(t) + \int_t^{t+\delta} \mathbb{E}[D_s f_{y_2}(s)P_1(s)|\mathcal{F}_t] ds (\bar{y}(t) - y(t)) \right\} dt.$$

Using Fubini's theorem and the definition of y_2 , as well as the duality formula for Malliavin derivatives, we get

$$\begin{aligned} & \mathbb{E} \int_0^T f_{y_2}(t)(\bar{y}_2(t) - y_2(t))P_1(t)dt \\ &= \mathbb{E} \int_0^T f_{y_2}(t) \left(\int_{t-\delta}^t \bar{y}(s) - y(s) dW(s) \right) P_1(t) dt \\ &= \int_0^T \mathbb{E} \left[\int_{t-\delta}^t \mathbb{E}[D_s(f_{y_2}(t)P_1(t)) | \mathcal{F}_s] (\bar{y}(s) - y(s)) ds \right] dt \\ &= \mathbb{E} \int_0^T \int_s^{s+\delta} \mathbb{E}[D_s(f_{y_2}(t)P_1(t)) | \mathcal{F}_s] (\bar{y}(s) - y(s)) dt ds. \end{aligned}$$

So, $J_3 = 0$. In the similar ways, we can show that $J_2 = 0$. This implies

$$\begin{aligned} J(\bar{u}) - J(u) &\geq \int_0^T \mathbb{E}[H_u(t)(\bar{u}(t) - u(t))] dt \\ &\geq 0. \end{aligned}$$

From this, it can be seen that $\bar{u}(t)$ is an optimal control. We conclude.

5. Applications

5.1. A linear-quadratic problem

In this section, applying the results obtained in the previous section, we derive an explicit optimal control of the linear-quadratic optimal control with delayed BSDE.

We give the following linear BSDE:

$$\begin{cases} -dy(t) = \{a(t)y(t) + b(t)y_3(t) + c(t)y_1(t) + d(t)y_2(t) + e(t)u(t)\}dt \\ \quad - z(t)dW(t), \\ y(T) = \xi, \quad y(t) = \varphi(t), \quad t \in [-\delta, 0) \end{cases}$$

and the performance functional

$$J(u(t)) = -\frac{1}{2} \mathbb{E} \left[\int_0^T a_0(t)y^2(t) + b_0(t)z^2(t) + r(t)u^2(t) \right] dt - \frac{1}{2} \mathbb{E}[y^2(0)],$$

where $a(t), b(t), c(t), d(t), e(t)$ are deterministic functions, with $y_1(t) := \int_{t-\delta}^t e^{\lambda(s-t)} y(s) ds$, $y_2(t) := \int_{t-\delta}^t y(s) dW(s)$ and $y_3(t) := y(t - \delta)$.

We give the following assumption:

(H4) $a_0(t)$ and $b_0(t)$ are non-negative bounded deterministic functions. Moreover, $r(t), r^{-1}(t)$ are deterministic positive bounded functions.

The Hamiltonian function is

$$H(t, y, y_1, y_2, y_3, z, u, P_1, P_2) = -\frac{1}{2} \left(a_0(t)y^2(t) + b_0(t)z^2(t) + r(t)u^2(t) \right)$$

$$\begin{aligned} & - \left(a(t)y(t) + b(t)y_3(t) + c(t)y_1(t) + d(t)y_2(t) + e(t)u(t) \right) P_1(t) \\ & - (y(t) - \lambda y_1(t) - e^{-\lambda\delta} y_3(t)) P_2(t) \end{aligned}$$

and the adjoint equation satisfies

$$\begin{cases} dP(t) = \{A_1(t)P(t) + \mathbb{E}^t[A_2(s)P(s)]|_{s=t+\delta}\chi_{[0,T-\delta]}(t) + \mathbb{E}^t \int_t^{t+\delta} D_t(B_1(s)P(s))dt \\ \quad + \tilde{b}_1(t)\}dt + (B_2(t)P(t) + \tilde{b}_2(t))dW(t), \\ P(0) = \zeta, \end{cases}$$

where

$$\begin{aligned} A_1(t) &= \begin{pmatrix} a(t) & 1 \\ c(t) & \lambda \end{pmatrix}, \quad A_2(s) = \begin{pmatrix} b(s) & -e^{-\lambda\delta} \\ 0 & 0 \end{pmatrix}, \quad B_1(s) = \begin{pmatrix} d(s) & 0 \\ 0 & 0 \end{pmatrix}, \\ B_2(t) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{b}_1(t) = \begin{pmatrix} a_0(t) \\ 0 \end{pmatrix}, \quad \tilde{b}_2(t) = \begin{pmatrix} b_0(t) \\ q_2(t) \end{pmatrix}, \quad \zeta = \begin{pmatrix} \mathbb{E}[y(0)] \\ 0 \end{pmatrix}. \end{aligned}$$

Our objective is to maximize the performance functional over \mathcal{U} .

According to Theorem 4.1, we can derive the following result.

Theorem 5.1. *Under (H4), if the admissible control $\bar{u}(t)$ satisfies*

$$\bar{u}(t) = -r^{-1}(t)e(t)P_1(t), \quad (5.1)$$

then it is an optimal control.

Proof. Let $\bar{u}(t)$ be a control given by (5.1), $\bar{y}(t)$ and $\bar{z}(t)$ be the corresponding state equation. By (5.1), $(\bar{y}(t), \bar{z}(t), \bar{u}(t))$ satisfies the optimality condition in Theorem 4.1, which means $\mathbb{E}[H_u(t)(\bar{u}(t) - u(t))] \geq 0$. From (H4), it is easy to check that all other conditions in Theorem 5.1 are satisfied, so \bar{u} is optimal.

5.2. Optimal investment problem with delay

We consider a financial market with two investment possibilities. One is a risk-free asset with price $S_0(t)$ and the other is a risky asset with price $S_1(t)$. They satisfies:

$$\begin{cases} dS_0(t) = \mu(t)S_0(t)dt, \\ S_0(0) = 0, \end{cases}$$

where $\mu(t)$ is a bonded deterministic function.

$$\begin{cases} dS_1(t) = S_1(t)(b(t)dt + \sigma(t)dW(t)), \\ S_1(0) = 0, \end{cases} \quad (5.2)$$

where $b(t), \sigma(t)$ are bounded deterministic functions.

Suppose someone invests in bonds and stocks. They want to achieve their wealth goal ξ at time T . Moreover, the investor wants to maximize his utility function J .

We denote by $\pi(t)$ the proportion of fund that the investor invests in the stock. $c(t)$ is the instantaneous consumption rate of investor. In real life, usually there are different levels of consumption. For simplicity, there is only one level. For example, $c(t) \in \mathbb{U} = [0, a_1]$, that is we only consider the ultra-low consumption. Here a_1 is a constant such that $a_1 > (\frac{\mathbb{E}[e^{\gamma t} P_1(t)]}{L})^{-\frac{1}{R}}$. $y(t)$ is the wealth process of the investor at time t and $k(y(t) - y(t - \delta)) - \int_{t-\delta}^t y(s) dW(s) - \int_{t-\delta}^t e^{\lambda(s-t)} y(s) ds$ is the surplus premium.

We propose a wealth process with delay, which is caused by the instantaneous capital inflow or outflow from the investor's current wealth.

Now the investor's wealth process is governed by the following BSDE:

$$\begin{cases} -dy(t) = [(\mu - k)y(t) + ky(t - \delta) + k \int_{t-\delta}^t y(s) dW(s) \\ \quad + k_1 \int_{t-\delta}^t e^{-\lambda(s-t)} y(s) ds - c(t) + \lambda z(t)] dt - z(t) dW(t), \\ y(T) = \xi, \quad y(t) = 0, \quad t \in [-\delta, 0). \end{cases} \quad (5.3)$$

We give the corresponding cost functional:

$$J(c(t)) = \mathbb{E} \left\{ \int_0^T L e^{-\gamma t} \frac{(c(t))^{1-R}}{1-R} dt - M y(0) \right\}, \quad (5.4)$$

where $L > 0, K > 0, M > 0, \gamma > 0$, and $0 < R < 1$.

Here is the Hamiltonian function

$$\begin{aligned} H(t, y, y_1, y_2, y_3, z, c, P_1, P_2) \\ = L e^{-\gamma t} \frac{(c(t))^{1-R}}{1-R} - \left((\mu - k)y(t) + ky_3(t) + ky_2(t) + k_1 y_1(t) \right. \\ \left. - c(t) + \lambda z(t) \right) P_1(t) - (y(t) - \lambda y_1(t) - e^{-\lambda \delta} y_2(t)) P_2(t), \end{aligned}$$

where $y_1(t) := \int_{t-\delta}^t e^{\lambda(s-t)} y(s) ds$, $y_2(t) := \int_{t-\delta}^t y(s) dW(s)$ and $y_3(t) := y(t - \delta)$.

The corresponding adjoint equation satisfies the following:

$$\begin{cases} dP(t) = \{A_1(t)P(t) + \mathbb{E}^t[A_2(s)P(s)]|_{s=t+\delta} \chi_{[0, T-\delta]}(t) \\ \quad + \mathbb{E}^t \int_t^{t+\delta} D_t(B_1(s)P(s)) ds + \tilde{b}_1(t) dt\} + (B_2(t)P(t) + \tilde{b}_2(t)) dW(t), \\ P(0) = \zeta, \end{cases}$$

where

$$A_1(t) = \begin{pmatrix} \mu - k & 1 \\ -k_1 & \lambda \end{pmatrix}, \quad A_2(s) = \begin{pmatrix} -k & -e^{-\lambda \delta} \\ 0 & 0 \end{pmatrix}, \quad B_1(s) = \begin{pmatrix} -k & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_2(t) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{b}_1(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \tilde{b}_2(t) = \begin{pmatrix} 0 \\ q_2(t) \end{pmatrix}, \quad \zeta = \begin{pmatrix} M \\ 0 \end{pmatrix}.$$

Proposition 5.2. Suppose that (5.2), (5.3) and the adjoint process $P(t)$ satisfy the conditions in Theorem 4.1. Then the optimal consumption $c(t)$ is given by $(\frac{\mathbb{E}[e^{\gamma t} P_1(t)]}{L})^{-\frac{1}{R}}$.

Finally, let's look at a numerical example. Take $\delta_1 = 0.1, \delta_2 = 0.15, \delta_3 = 0.2, \delta_4 = 0.23, \delta_5 = 0.26, \lambda = 0.4, \mu = 0.6, k = 0, k_1 = 1, \gamma = 0.3, M = 1, R = 0.5, T = 3$ and $L = 3$.

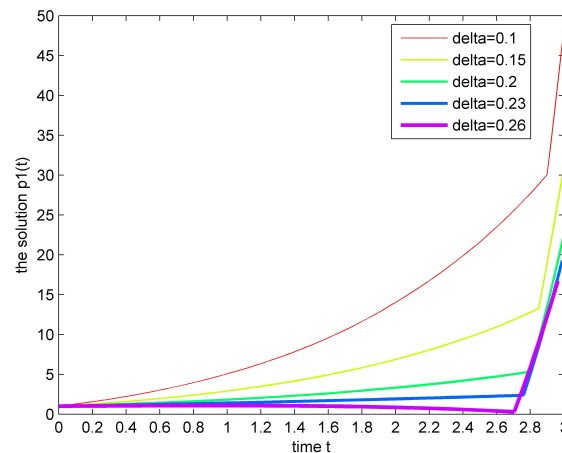


Figure 1. The solution $P_1(t)$ with different time delays.

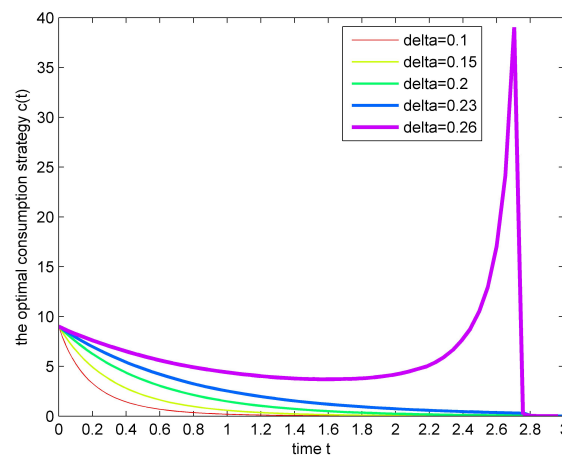


Figure 2. The optimal consumption $c(t)$ with different time-delays.

In order to obtain the effect of delay, we can divide the time horizon according to the delay, and then continue to subdivide each subinterval to get the numerical solution using the Euler method or improved Euler method. The corresponding parameter values are: $\delta_1 = 0.1, \delta_2 = 0.15, \delta_3 = 0.2, \delta_4 = 0.23, \delta_5 = 0.26$. Figure 1 plots the solution $P_1(t)$ for different time delays. We observe that the solution is very sensitive to δ , i.e., the smaller δ is, the larger $P_1(t)$ is.

Figure 2 shows the optimal consumption $c(t)$ for different time delays. We find that the smaller δ is, the lower the consumption rate is. Furthermore, it can be seen from Figure 2 that no matter what value δ , the optimal consumption strategy is close to almost the same number. This is because, for convenience, we consider only one level of consumption: Ultra-low consumption.

For a more comprehensive analysis of the factors affecting the optimal control, we will show how the terminal condition $y(T) = \xi$ affects the optimal control result. Here $k = k_1 = 0, \mu = 1$. The state process satisfies the following BSDE

$$\begin{cases} -dy(t) = (y(t) - c(t) + z(t))dt - z(t)dW(t), \\ y(T) = \xi. \end{cases}$$

The cost function is given by

$$J(c(t)) = \mathbb{E}\left\{ \int_0^T L e^{-\gamma t} \frac{(c(t))^{1-R}}{1-R} dt - y^2(0) \right\}.$$

and the adjoint equation (3.2) is reduced to the following form

$$\begin{cases} dP_1(t) = P_1(t)dt, \\ P_1(0) = 2\bar{y}(0). \end{cases}$$

By Proposition 5.2, the optimal consumption $c(t)$ is given by $(\frac{\mathbb{E}[e^{\gamma t} P_1(t)]}{L})^{-\frac{1}{R}}$.

The following is the numerical result. Figure 3 plots the optimal consumption $c(t)$ with different terminal conditions. The parameter values used in the calculations are: $\gamma = 0.3, R = 0.5, T = 3, L = 3, \xi_1 = 1, \xi_2 = 3, \xi_3 = 5$. We observe that the smaller terminal condition will lead to a smaller consumption.

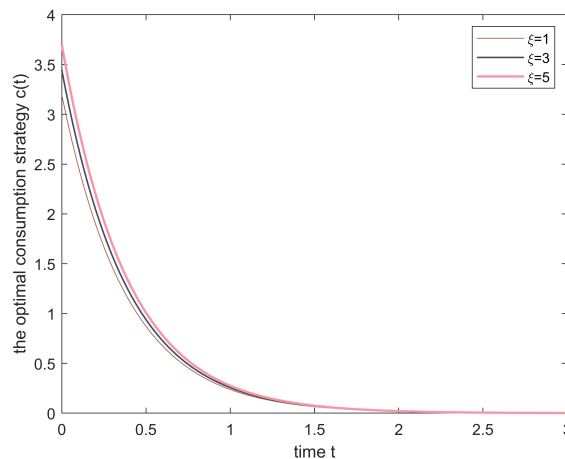


Figure 3. The optimal consumption $c(t)$ with different terminal conditions.

6. Conclusions

Motivated by some interesting financial and economic phenomena, we study the optimal control problem for BSDE with three delays. Some new-type adjoint equations are introduced for the first time in the literature. More especially, we obtain a sufficient condition for BSDE with three delays. The major features of this work include the introduction of a unified adjoint equation as well as the

new type of the adjoint equations. The theoretical results established here are applied to solve a linear-quadratic optimal control problem and a financial problem. Then, a linear-quadratic optimal control and an optimal investment strategy are derived, respectively. There are several interesting issues that deserve further study.

One could consider more complicated optimal control problems, such as problems with mean-field term and partial information rather than full information.

On the one hand, the mean-field stochastic differential equation (SDE) was introduced to study physical systems with a large number of interacting particles. The mean-field models have many applications in economics and finance. Stochastic maximum principle for the optimal control system of mean-field type has become a popular topic. When considering mean-field terms, some new mean-field stochastic differential equations will be introduced as the adjoint equations and the maximum principle is generalized to the mean-field framework.

On the other hand, in general, controllers can get only partial information in most cases. For example, we consider a cash flow process for an insurance firm, where due to the discreteness of account information, it is impossible for the firm to fully observe the cash flow. Unlike the system with full information, we will give a noisy process Y for partially observed optimal control problem. In addition, an extra adjoint equation will be introduced to obtain stochastic maximum principle, which does not appear in the full information case. The optimal control will contain the filtering of adjoint process which will increase the difficulty of analyzing the maximum condition and improve the skills of pure mathematical calculation. We plan to discuss these problems in the future.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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