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Research article

Convex radial solutions for Monge-Ampère equations involving the gradient

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Abstract: This paper deals with the existence and multiplicity of convex radial solutions for the Monge-Ampère equation involving the gradient ∇u:

$$\begin{cases} \det(D^2u) = f(|x|, -u, |\nabla u|), x \in B, \\ u|_{\partial B} = 0, \end{cases}$$

where $B := \{x \in \mathbb{R}^N : |x| < 1\}$. The fixed point index theory is employed in the proofs of the main results.

Keywords: Monge-Ampère equation; convex radial solution; Krein-Rutman theorem; fixed point index; a priori estimate

1. Introduction

This paper deals with the existence and multiplicity of convex radial solutions for the Monge-Ampère equation involving the gradient ∇u:

$$\begin{cases} \det(D^2u) = f(|x|, -u, |\nabla u|), x \in B, \\ u|_{\partial B} = 0, \end{cases} \tag{1.1}$$

where $B := \{x \in \mathbb{R}^N : |x| < 1\}$, $|x| := \sqrt{\sum_{i=1}^N x_i^2}$.

The Monge-Ampère equation

$$\det D^2u = f(x, u, Du) \tag{1.2}$$

is fundamental in affine geometry. For example, if

$$f(x, u, Du) := K(x)(1 + |Du|^2)^{(n+2)/n},$$

then Eq (1.2) is called the prescribed Gauss curvature equation. The Monge-Ampère equation also arises in isometric embedding, optimal transportation, reflector shape design, meteorology and fluid mechanics (see [1–3]). As a result, the Monge-Ampère equation is among the most significant of fully nonlinear partial differential equations and has been extensively studied. In particular, the existence of radial solutions of (1.1) has been thoroughly investigated (see [2–15], only to cite a few of them).

In 1977, Brezis and Turner [16] examined a class of elliptic problems of the form

$$\begin{cases} Lu = g(x, u, \mathrm{D}u), x \in \Omega, \\ u = 0, x \in \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^N$ and L is a linear elliptic operator enjoying a maximum principle, $\mathrm{D}u$ is the gradient of u, and g is a nonnegative function. It is worthwhile to point out the function g satisfies a growth condition on u and $\mathrm{D}u$, i.e., $\lim_{u\to+\infty} \frac{g(x,u,p)}{u^{\frac{N+1}{N-1}}} = 0$ uniformly in $x \in \Omega$ and $p \in \mathbb{R}^N$, in comparison with our main results for (1.1) (see Theorems 3.1 and 3.3 in Section 3).

In 1988, Kutev [9] studied the existence of nontrivial convex solutions for the problem

$$\begin{cases} \det(D^2u) = (-u)^p, x \in B_R := \{x \in \mathbb{R}^n : |x| < R\}, \\ u = 0, x \in \partial B_R, \end{cases} \tag{1.4}$$

where $p > 0$ and $p \neq n$. His main results obtained are the three theorems below:

Theorem 1. Let G denote a bounded convex domain in $\mathbb{R}^n$ and $0 < p < n$. Then the problem

$$\begin{cases} \det(D^2u) = (-u)^p, x \in G, \\ u = 0, x \in \partial G \end{cases} \tag{1.5}$$

possesses at most one strictly convex solution $u \in C^2(G) \cap C(\overline{G})$.

Theorem 2. Let $0 < p < n$. Then problem (1.4) possesses a unique strictly convex solution u which is a radially symmetric function and $u \in C^\infty(\overline{B}_R)$.

Theorem 3. Let $p > n$. Then problem (1.4) possesses a unique nontrivial radially symmetric solution u which is a strictly convex function and $u \in C^\infty(\overline{B}_R)$.

In 2004, by means of the fixed point index theory, Wang [12] studied the existence of convex radial solutions of the problem

$$\begin{cases} \det(D^2u) = f(-u), x \in B, \\ u|_{\partial B} = 0. \end{cases} \tag{1.6}$$

His main conditions on f are

1) the superlinear case: $\lim\limits_{v\to0^+} \frac{f(v)}{v^n} = 0$, $\lim\limits_{v\to+\infty} \frac{f(v)}{v^n} = +\infty$,

and

2) the sublinear case: $\lim\limits_{v\to0^+} \frac{f(v)}{v^n} = +\infty$, $\lim\limits_{v\to+\infty} \frac{f(v)}{v^n} = 0$.

In 2006, Hu and Wang [8] studied the existence, multiplicity and nonexistence of strictly convex solutions for the boundary value problem

$$\begin{cases} ((u'(r))^n)' = \lambda n r^{n-1} f(-u(r)), r \in (0, 1), \\ u'(0) = u(r) = 0, \end{cases} \tag{1.7}$$

which is equivalent to (1.6) with $f(-u)$ replaced by $\lambda f(-u)$, λ being a parameter.

In 2009, Wang [13] studied the existence of convex solutions to the Dirichlet problem for the weakly coupled system

$$\begin{cases} \left((u_1'(t))^N\right)' = Nt^{N-1}f(-u_2(t)), \\ \left((u_2'(t))^N\right)' = Nt^{N-1}g(-u_1(t)), \\ u_1'(0) = u_2'(0) = 0, u_1(1) = u_2(1) = 0. \end{cases} \tag{1.8}$$

Dai [4] studied the bifurcation problem

$$\begin{cases} \det(D^2u) = \lambda^N a(x)f(-u), & u \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

In 2020, Feng et al. [17] established an existence criterion of strictly convex solutions for the singular Monge-Ampère equations

$$\begin{cases} \det D^2u = b(x)f(-u) + g(|Du|), \text{ in } \Omega \\ u = 0, \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \det D^2u = b(x)f(-u)(1 + g(|Du|)), \text{ in } \Omega \\ u = 0, \text{on } \partial\Omega \end{cases}$$

where Ω is a convex domain, $b \in C^\infty(\Omega)$ and $g \in C^\infty(0, +\infty)$ being positive and satisfying $g(t) \leqslant c_g t^q$ for some $c_g > 0$ and $0 \leqslant q < n$.

In 2022, Feng [18] analyzed the existence, multiplicity and nonexistence of nontrivial radial convex solutions of the following system coupled by singular Monge-Ampère equations

$$\begin{cases} \det D^2u_1 = \lambda h_1(|x|)f(-u_2), \text{in } \Omega, \\ \det D^2u_2 = \lambda h_2(|x|)f(-u_1), \text{in } \Omega, \\ u_1 = u_2 = 0, \text{on } \partial\Omega, \end{cases}$$

where $\Omega := \{x \in \mathbb{R}^n : |x| < 1\}$.

In 2023, Zhang and Bai [19] studied the following singular Monge-Ampère problems:

$$\begin{cases} \det D^2u = b(x)f(-u) + |Du|^q, \text{ in } \Omega, \\ u = 0, \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \det D^2u = b(x)f(-u)(1 + |Du|^q), \text{ in } \Omega, \\ u = 0, \text{on } \partial\Omega \end{cases}$$

where Ω is a convex domain, $b \in C^\infty(\Omega)$ and $q < n$.

It is interesting to observe that of previous works cited above, except for [16, 17, 19], all nonlinearities under study are not concerned with the gradient or the first-order derivative, in contrast to our one in (1.1) that involves the gradient ∇u. The presence of the gradient makes it indispensable to estimate the contribution of its presence to the associated nonlinear operator A and, that is a difficult task. In order to overcome the difficulty created by the gradient, we use the Nagumo-Berstein type condition [20, 21] to restrict the growth of the gradient at infinity, thereby facilitating the obtention of a priori estimation of the gradient through Jensens's integral inequalities. Additionally, in [17, 19], the dimension n is an unreachable growth ceiling of $|\nabla u|$ in their nonlinearities, compared to our nonlinearities in the present paper (see (H3) in the next section). Thus our methods in the present paper are entirely different from these in the existing literature, for instance, in [8, 10, 12–14, 16–19].

The remainder of the present article is organized as follows. Section two is concerned with some preliminary results. The main results, i.e., Theorems 3.1–3.3, will be stated and shown in Section 3.

2. Preliminary results

Let $t := |x| = \sqrt{\sum_{i=1}^{N} x_i^2}$. Then (1.1) reduces to

$$\begin{cases} ((u'(t))^N)' = Nt^{N-1} f(t, -u, u'), \\ u'(0) = u(1) = 0; \end{cases} \tag{2.1}$$

see [8]. Substituting $v := -u$ into (2.1), we obtain

$$\begin{cases} ((-v'(t))^N)' = Nt^{N-1} f(t, v, -v'), \\ v'(0) = v(1) = 0. \end{cases} \tag{2.2}$$

It is easy to see that every solution u of (2.1), under the very condition $f \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$, must be convex, increasing and nonpositive on $[0, 1]$. Naturally, every solution v of (2.2) must be concave, decreasing and nonnegative on $[0, 1]$. This explains why we will work in a positive cone of $C^1[0, 1]$ whose elements are all decreasing, nonnegative functions.

Let $E := C^1[0, 1]$ be endowed with the norm

$$\|v\|_1 := \max\{\|v\|_0, \|v'\|_0\}, v \in E,$$

where $\|v\|_0$ denotes the maximum of $|v(t)|$ on the interval $[0, 1]$ for $v \in C[0, 1]$. Thus, $(E, \|\cdot\|_1)$ becomes a real Banach space. Furthermore, let P be the set of C^1 functions that are nonnegative and decreasing on $[0, 1]$. It is not difficult to verify that P represents a cone in E. Additionally, in our context, (2.2) and, in turn, (1.1), is equivalent to the nonlinear integral equation

$$v(t) = \int_t^1 \left(\int_0^s N\tau^{N-1} f(\tau, v(\tau), -v'(\tau))\, \mathrm{d}\tau \right)^{1/N} \mathrm{d}s, v \in P.$$

For our forthcoming proofs of the main results, we define the the nonlinear operator A to be

$$(Av)(t) := \int_t^1 \left(\int_0^s N\tau^{N-1} f(\tau, v(\tau), -v'(\tau))\, \mathrm{d}\tau \right)^{1/N} \mathrm{d}s, v \in P. \tag{2.3}$$

If $f \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$, then $A : P \to P$ is completely continuous. Now, the existence of convex radial solutions of (1.1) is tantamount to that of concave fixed points of the nonlinear operator A.

Denote by

$$k(t,s) := \min\{1-t, 1-s\}. \tag{2.4}$$

By Jenesen's integral inequality, we have the basic inequality

$$(Av)(t) \geqslant N^{1/N} \int_0^1 k(t,s) s^{1-1/N} f^{1/N}(s, v(s), -v'(s))\, \mathrm{d}s, v \in P. \tag{2.5}$$

Associated with the righthand of the inequality above is the linear operator B_1, defined by

$$(B_1 v)(s) := N^{1/N} \int_0^1 k(t,s) s^{1-1/N} v(t)\, \mathrm{d}t. \tag{2.6}$$

Clearly, $B_1 : P \to P$ is completely continuous with its spectral radius $r(B_1)$ being positive. The Krein-Rutman theorem [22] asserts that there exists $\varphi \in P \setminus \{0\}$ such that $B_1\varphi = r(B_1)\varphi$, which may be written in the form

$$N^{1/N} \int_0^1 k(t,s) s^{1-1/N} \varphi(t)\, \mathrm{d}t = r(B_1)\varphi(s). \tag{2.7}$$

For convenience, we require in addition

$$\int_0^1 \varphi(t)\, \mathrm{d}t = 1. \tag{2.8}$$

Lemma 2.1. *(see [23]) Let E be a real Banach space and P a cone in E. Suppose that $\Omega \subset E$ is a bounded open set and that $T : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $w_0 \in P \setminus \{0\}$ such that*

$$w - Tw \neq \lambda w_0, \forall \lambda \geqslant 0, w \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 0$, where i indicates the fixed point index.

Lemma 2.2. *(see [23]) Let E be a real Banach space and P a cone in E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $T : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$w - \lambda Tw \neq 0, \forall \lambda \in [0,1], w \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 1$.

3. Existence and multiplicity of negative convex solutions of (1.1)

Below are the conditions posed on the nonlinearity f.

(H1) $f \in C([0,1] \times \mathbb{R}_+^2, \mathbb{R}_+)$.

(H2) One may find two constants $a > (r(B_1))^{-N}$ and $c > 0$ such that

$$f(t,x,y) \geqslant ax^N - c, t \in [0,1], (x,y) \in \mathbb{R}_+^2.$$

(H3) For every $M > 0$ there exists a strictly increasing function $\Phi_M \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t,x,y) \leqslant \Phi_M(y^N), \forall (t,x,y) \in [0,1] \times [0,M] \times \mathbb{R}_+$$

and $\int_{2^{N-1}c_0}^{\infty} \frac{\mathrm{d}\xi}{\Phi_M(\xi)} > 2^{N-1}N$, where $\varphi \in P \setminus \{0\}$ is determined by (2.7) and (2.8), and $c_0 := \left(-\frac{\varphi'(1)c^{1/N}N^{1/N}}{\varphi(0)(a^{1/N}r(B_1)-1)\int_0^1(1-t)\varphi(t)\,\mathrm{d}t}\right)^N$.

(H4) $\limsup\limits_{x\to 0^+, y\to 0^+} \frac{f(t,x,y)}{q(x,y)} < 1$ holds uniformly for $t \in [0,1]$, where

$$q(x,y) := \max\{x^N, y^N\}, x \in \mathbb{R}_+, y \in \mathbb{R}_+. \tag{3.1}$$

(H5) There exist two constants $r > 0$ and $b > (r(B_1))^{-N}$ so that

$$f(t,x,y) \geqslant bx^N, t \in [0,1], x \in [0,r], y \in [0,r].$$

(H6) $\limsup\limits_{x+y\to\infty} \frac{f(t,x,y)}{q(x,y)} < 1$ holds uniformly for $t \in [0,1]$, with $q(x,y)$ being defined by (3.1).

(H7) There exists $\omega > 0$ so that $f(t,x,y) \leqslant f(t,\omega,\omega)$ for all $t \in [0,1], x \in [0,\omega], y \in [0,\omega]$ and $\int_0^1 Ns^{N-1}f(s,\omega,\omega)\,\mathrm{d}s < \omega^N$.

Theorem 3.1. *If (H1)–(H4) hold, then* (1.1) *has at least one convex radial solution.*

Proof. Let

$$\mathscr{M} := \{v \in P : v = Av + \lambda\varphi, \text{for some } \lambda \geqslant 0\},$$

where φ is specified in (2.7) and (2.8). Clearly, if $v \in \mathscr{M}$, then v is decreasing on $[0,1]$, and $v(t) \geqslant (Av)(t), t \in [0,1]$. We shall now prove that $\mathscr{M}$ is bounded. We first establish the a priori bound of $\|v\|_0$ on $\mathscr{M}$. Recall (2.5). If $v \in \mathscr{M}$, then Jensen's inequality and (H2) imply

$$\begin{aligned} v(t) &\geqslant N^{1/N}\int_0^1 k(t,s)s^{1-1/N}f^{1/N}(s,v(s),-v'(s))\,\mathrm{d}s \\ &\geqslant a^{1/N}N^{1/N}\int_0^1 k(t,s)s^{1-1/N}v(s)\,\mathrm{d}s - c^{1/N}N^{1/N}. \end{aligned}$$

Then, by (2.7) and (2.8) we obtain

$$\int_0^1 v(t)\varphi(t)\,\mathrm{d}t \geqslant a^{1/N}r(B_1)\int_0^1 v(t)\varphi(t)\,\mathrm{d}t - c^{1/N}N^{1/N},$$

so that

$$\int_0^1 v(t)\varphi(t)\,\mathrm{d}t \leqslant \frac{c^{1/N}N^{1/N}}{a^{1/N}r(B_1)-1}, \forall v \in \mathscr{M}.$$

Since v is concave and $\|v\|_0 = v(0)$, we obtain that

$$\begin{aligned} \|v\|_0 &\leqslant \frac{\int_0^1 v(t)\varphi(t)\,\mathrm{d}t}{\int_0^1(1-t)\varphi(t)\,\mathrm{d}t} \\ &\leqslant \frac{c^{1/N}N^{1/N}}{(a^{1/N}r(B_1)-1)\int_0^1(1-t)\varphi(t)\,\mathrm{d}t} \\ &:= M_0, \forall v \in \mathscr{M}, \end{aligned} \tag{3.2}$$

which proves the a priori estimate of $\|v\|_0$ on $\mathscr{M}$. Now we are going to establish the a priori estimate of $\|v'\|_0$ on $\mathscr{M}$. By (H3), there exists a strictly increasing function $\Phi_{M_0} \in C(\mathbb{R}_+, \mathbb{R}_+)$ so that

$$f(t, v(t), -v'(t)) \leqslant \Phi_{M_0}((-v')^N(t)), \forall v \in \mathscr{M}, t \in [0, 1].$$

Now, (3.2) implies $\lambda \leqslant \frac{M_0}{\varphi(0)}$ for all $\lambda \in \Lambda$, where

$$\Lambda := \{\lambda \in \mathbb{R}_+ : \text{there is } v \in P \text{ so that } v = Av + \lambda\varphi\}.$$

If $v \in \mathscr{M}$, then

$$v'(t) = -\left(\int_0^t Ns^{N-1} f(s, v(s), -v'(s))\,\mathrm{d}s\right)^{1/N} + \lambda\varphi'(t)$$

for some $\lambda \geqslant 0$, and

$$\begin{aligned}(-v')^N(t) &\leqslant 2^{N-1}\left(\int_0^t Ns^{N-1} f(s, v(s), -v'(s))\,\mathrm{d}s + c_0\right)\\ &\leqslant 2^{N-1}N\int_0^t \Phi_{M_0}((-v')^N(s))\,\mathrm{d}s + 2^{N-1}c_0,\end{aligned}$$

where $c_0 := (-\frac{M_0\varphi'(1)}{\varphi(0)})^N$. Let $w(t) := (-v')^N(t)$. Then $w \in C([0, 1], \mathbb{R}_+)$ and $w(0) = 0$. Moreover,

$$w(t) \leqslant 2^{N-1}N\int_0^t \Phi_{M_0}(w(s))\,\mathrm{d}s + 2^{N-1}c_0, \forall v \in \mathscr{M}.$$

Let $F(t) := \int_0^t \Phi_{M_0}(w(\tau))\,\mathrm{d}\tau$. Then $F(0) = 0$, $w(t) \leqslant 2^{N-1}NF(t) + 2^{N-1}c_0$, and

$$F'(t) = \Phi_{M_0}(w(t)) \leqslant \Phi_{M_0}\left(2^{N-1}NF(t) + 2^{N-1}c_0\right), \forall v \in \mathscr{M}.$$

Therefore

$$\int_{2^{N-1}c_0}^{2^{N-1}NF(1)+2^{N-1}c_0} \frac{\mathrm{d}\xi}{\Phi_{M_0}(\xi)} = \int_0^1 \frac{2^{N-1}NF'(\tau)\,\mathrm{d}\tau}{\Phi_{M_0}(2^{N-1}NF(\tau) + 2^{N-1}c_0)} \leqslant 2^{N-1}N.$$

Now (H3) indicates that there exists $M_1 > 0$ so that $F(1) \leqslant M_1$ for every $v \in \mathscr{M}$. Consequently, one obtains

$$\|(-v')^N\|_0 = \|w\|_0 = w(1) \leqslant 2^{N-1}NM_1 + 2^{N-1}c_0$$

for all $v \in \mathscr{M}$. Let $M := \max\{M_0, (2^{N-1}NM_1 + 2^{N-1}c_0)^{1/N}\} > 0$. Then

$$\|v\|_1 \leqslant M, \forall v \in \mathscr{M}.$$

This shows that $\mathscr{M}$ is bounded. Choosing $R > \max\{M, r\} > 0$, we obtain

$$v \neq Av + \lambda\varphi, \forall v \in \partial B_R \cap P,$$

where $B_R := \{v \in E : \|v\|_1 < R\}$. Then Lemma 2.1 implies

$$i(A, B_R \cap P, P) = 0. \tag{3.3}$$

By (H4), there exist two constants $r > 0$ and $\delta \in (0, 1)$ so that

$$f(t, x, y) \leqslant \delta q(x, y), \forall 0 \leqslant x, y \leqslant r, 0 \leqslant t \leqslant 1.$$

Therefore, for all $v \in \overline{B}_r \cap P$, $t \in [0, 1]$, one sees that

$$\begin{aligned}(Av)^N(t) &\leqslant \int_t^1 \left(\int_0^s N\delta\tau^{N-1} q(v(\tau), -v'(\tau))\, \mathrm{d}\tau\right) \mathrm{d}s \\ &= N\delta \int_0^1 k(t,s) s^{N-1} q(v(s), -v'(s))\, \mathrm{d}s \\ &\leqslant N\delta \|v\|_1^N \tfrac{1-t^{N+1}}{N(N+1)} \\ &\leqslant \delta \|v\|_1^N,\end{aligned}$$

and

$$\begin{aligned}-(Av)'(t) &\leqslant \left(\int_0^t N\delta s^{N-1} q(v(s), -v'(s))\, \mathrm{d}s\right)^{1/N} \\ &\leqslant \left(N\delta \|v\|_1^N \cdot \tfrac{t^N}{N}\right)^{1/N} \\ &\leqslant \delta^{1/N} \|v\|_1.\end{aligned}$$

Now, the preceding two inequalities imply

$$\|Av\|_1 \leqslant \delta^{1/N}\|v\|_1 < \|v\|_1, \forall v \in \overline{B}_r \cap P,$$

and, in turn,

$$v \neq \lambda Av, \forall v \in \partial B_r \cap P, \lambda \in [0, 1].$$

Invoking Lemma 2.2 begets

$$i(A, B_r \cap P, P) = 1.$$

Recalling (3.3), we obtain

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 0 - 1 = -1.$$

Thus, A possesses at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$, which proves that (1.1) possesses at least one convex radial solution. This finishes the proof.

Theorem 3.2. *If (H1), (H5) and (H6) hold, then* (1.1) *possesses at least one convex radial solution.*

Proof. Let $r > 0$ be specified by (H5) and $\varphi \in P \setminus \{0\}$ be given by (2.7) and (2.7). Denote by

$$\mathscr{N} := \{v \in \overline{B}_r : v = Av + \lambda\varphi, \text{for certain } \ \lambda \geqslant 0\},$$

where $r > 0$ is specified by (H5) and $\varphi \in P \setminus \{0\}$ ia given by (2.7) and (2.8). Now we assert that $\mathscr{N} \subset \{0\}$ and indeed, (H5) implies that

$$\begin{aligned}(Av)(t) &\geqslant \int_t^1 \left(\int_0^s Nb\tau^{N-1} v^N(\tau)\, \mathrm{d}\tau\right)^{1/N} \mathrm{d}s \\ &\geqslant N^{1/N} b^{1/N} \int_0^1 k(t,s) s^{1-1/N} v(s)\, \mathrm{d}s\end{aligned}$$

for every $v \in \overline{B}_r \cap P$. If $v \in \mathscr{N}$, then

$$v(t) \geqslant N^{1/N} b^{1/N} \int_0^1 k(t,s) s^{1-1/N} v(s)\, \mathrm{d}s.$$

By (2.7) and (2.8), one obtains

$$\int_0^1 v(t)\varphi(t)\, \mathrm{d}t \geqslant b^{1/N} r(B_1) \int_0^1 v(t)\varphi(t)\, \mathrm{d}t,$$

so that

$$\int_0^1 v(t)\varphi(t)\ \mathrm{d}t = 0, \forall v \in \mathscr{N}.$$

Therefore, we have $v \equiv 0$ and, hence, $\mathscr{N} \subset \{0\}$ as asserted. Finally, one finds

$$v \neq Av + \lambda\varphi, \forall v \in \partial B_r \cap P, \lambda \geqslant 0.$$

Applying Lemma 2.1 begets

$$i(A, B_r \cap P, P) = 0. \tag{3.4}$$

Alternatively, (H6) indicates that there exist two constants $\delta \in (0, 1)$ and $c > 0$ such that

$$f(x, y) \leqslant \delta q(x, y) + c, \forall x \geqslant 0, y \geqslant 0, t \in [0, 1]. \tag{3.5}$$

Denote by

$$\mathscr{S} := \{v \in P : v = \lambda Av, \text{for certain } \lambda \in [0, 1]\}.$$

We are going to prove the boundedness of $\mathscr{S}$. In fact, $v \in \mathscr{S}$ indicates

$$v^N(t) \leqslant (Av)^N(t), (-v')^N(t) \leqslant ((-Av)')^N(t).$$

Hence, for every $v \in \mathscr{S}$, $t \in [0, 1]$, (3.5) implies the inequalities below:

$$\begin{aligned} v^N(t) &\leqslant \int_t^1 \left(\int_0^s N\tau^{N-1}[\delta q(v(\tau), -v'(\tau)) + c]\ \mathrm{d}\tau\right)\ \mathrm{d}s \\ &= \int_0^1 k(t, s)Ns^{N-1}[\delta q(v(s), -v'(s)) + c]\ \mathrm{d}s \\ &\leqslant N(\delta\|v\|_1^N + c)\tfrac{1-t^{N+1}}{N(N+1)} \\ &\leqslant \delta\|v\|_1^N + c \end{aligned}$$

and

$$\begin{aligned} (-v')^N(t) &\leqslant \int_0^t Ns^{N-1}[\delta q(v(s), -v'(s)) + c]\ \mathrm{d}s \\ &\leqslant N(\delta\|v\|_1^N + c)\tfrac{t^N}{N} \\ &\leqslant \delta\|v\|_1^N + c. \end{aligned}$$

Now, the preceding two inequalities allude to

$$\|v\|_1^N \leqslant \delta\|v\|_1^N + c$$

and, hence,

$$\|v\|_1 \leqslant \left(\frac{c}{1-\delta}\right)^{1/N}$$

for all $v \in \mathscr{S}$, which asserts that $\mathscr{S}$ is bounded, as desired. Choosing $R > \max\{\sup\{\|v\|_1 : v \in \mathscr{S}\}, r\} > 0$, one finds

$$v \neq \lambda Av, \forall v \in \partial B_R \cap P, \lambda \in [0, 1].$$

Applying Lemma 2.2 begets

$$i(A, B_R \cap P, P) = 1.$$

This, together with (3.4), concludes that

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Therefore, A possesses at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and (1.1) possesses at least one convex radial solution. This finishes the proof.

Theorem 3.3. *If (H1)–(H3), (H5) and (H7) hold, then* (1.1) *possesses at least two convex radial solutions.*

Proof. The proofs of Theorems 3.1 and 3.2 suggest that (3.3) and (3.4) may be derived from (H1)–(H3) and (H5). Alternatively, (H7) indicates

$$\|(Av)^N\|_0 = (Av)^N(0) \leqslant \int_0^1 N(1-s)s^{N-1}f(s,\omega,\omega)\,\mathrm{d}s < \omega^N$$

and

$$\|\left[(Av)'\right]^N\|_0 = \left[(-Av)'\right]^N(1) \leqslant \int_0^1 Ns^{N-1}f(s,\omega,\omega)\,\mathrm{d}s < \omega^N$$

for all $v \in \overline{B}_\omega \cap P$. Consequently,

$$\|Av\|_1 < \|v\|_1, \forall v \in \partial B_\omega \cap P.$$

This means

$$v \neq \lambda Av, \forall v \in B_\omega \cap P, \lambda \in [0,1].$$

Applying Lemma 2.2 begets

$$i(A, B_\omega \cap P, P) = 1. \tag{3.6}$$

Notice that $R > 0$ in (3.4) may be sufficiently large and $r > 0$ may be sufficiently small. This means that we may assume $R > \omega > r$. Now (3.6), together with (3.3) and (3.4), implies

$$i(A, (B_R \setminus \overline{B}_\omega) \cap P, P) = 0 - 1 = -1,$$

and

$$i(A, (B_\omega \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Consequently, A possesses at least two positive fixed points, one on $(B_R \setminus \overline{B}_\omega) \cap P$ and the other on $(B_\omega \setminus \overline{B}_r) \cap P$. Thus, (1.1) possesses at least two convex radial solutions. This finishes the proof.

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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