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Research article

Analysis of nonlinear ordinary differential equations with the generalized Mittag-Leffler kernel

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Abstract: The Picard iterative approach used in the paper to derive conditions under which nonlinear ordinary differential equations based on the derivative with the Mittag-Leffler kernel admit a unique solution. Using a simple Euler approximation and Heun's approach, we solved this nonlinear equation numerically. Some examples of a nonlinear linear differential equation were considered to present the existence and uniqueness of their solutions as well as their numerical solutions. A chaotic model was also considered to show the extension of this in the case of nonlinear systems.

Keywords: nonlinear equation; Picard iteration; Atangana-Baleanu derivative; Euler and Heun's approaches

1. Introduction

No viable analytical technique is found in the literature that can be utilized to solve nonlinear differential equations with the generalized-Mittag-Leffler kernel based fractional derivative [1–3]. This is caused by both the complexity of the Mittag-Leffler kernel and the nonlinearity of the problem. Because of this, scientists who study this topic typically rely on numerical schemes to find numerical answers to these equations [4–14]. Nonetheless, at least one wants to be certain that these equations admit unique solutions, and the existence and uniqueness theorems are used for this purpose. The existence and uniqueness theorem for initial value issues of ordinary differential equations implies the prerequisite for the existence of a solution to a linear or non-linear initial value problem and guarantees the uniqueness of the discovered solution. Researchers have been working on methods to ensure that nonlinear ordinary differential equations based on the Atangana-Baleanu derivative, a unique solution for the better part of the last ten years [2–5,11–14]. Most of these techniques are, in fact, adaptations of those that have been proposed for nonlinear differential equations with classical derivatives [6]. Let us not forget that Picard proposed one of the earliest methods [10,15–18]. An explicit iteration is then introduced after converting an ordinary differential equation into an integral equation, and in some cases, the Lipschitz condition can be used to ensure that this iteration will eventually converge to a singular solution. The nonlinear equations related to the Atangana-Baleanu fractional derivative will be the subject of our application of this technique.

2. Existence and uniqueness of IVP with Atangana-Baleanu

Nonlinear ordinary differential equations with the Caputo and the Atangana-Baleanu have been recognized as important mathematical differential equations to modeling processes with non-local behaviors. However, due to complexity of these equations, exact solutions are not always easy to obtain using current analytical techniques. Researchers, therefore, rely on numerical approaches to derive numerical solution of these equations. However, before deriving a numerical solution, it is mathematically important to show that the equation has a unique solution under some conditions. One of the approaches to achieve this is to establish a sequence that converges towards a given solution, with some important theorem well-established in literature, and we can conclude that the equation linked to that sequence has a unique solution. In this section, we shall consider the following IVP:

$$\begin{cases} {}^{ABC}_{t_0}D_t^{\alpha} y(t) = f(t, y(t)) \quad t > 0, \\ y(t_0) = y_0. \end{cases} \tag{1}$$

The above is transformed into:

$$\begin{cases} y(t) = y(t_0) + (1-\alpha) f\big(t, y(t)\big) + \frac{\alpha}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(t, y(\tau)) d\tau, \\ y(t_0) = y(0). \end{cases} \tag{2}$$

We consider the Picard approach to obtain:

$$\begin{cases} y_n(t) = y(t_0) + (1-\alpha) f\big(t, y_{n-1}(t)\big) + \frac{\alpha}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau, y_{n-1}(\tau)) d\tau, \\ y(t_0) = y(0). \end{cases} \tag{3}$$

where $\overline{D} \subseteq \mathbb{R} \times \mathbb{R}$ is a closed rectangle with $(t_0, y(t_0)) \in \overline{D}$.

We assume that $f: [t_0 - a, t_0 + a] \times \overline{B}(a, r) \to \mathbb{R}$ is continuous and bounded by δz. Then the ordinary differential equation with the Atangana-Baleanu derivative has a solution.

Proof: We show that each $y_n(t)$ is well-defined. Without loss of generality, we show this only for $t \in [t_0, t_0 + \bar{c}]$.

$$|y_n(t) - y(t_0)| \leq (1-\alpha)\left|f\left(t, y_{n-1}(t)\right)\right| + \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}\left|f\left(t, y_n(t)\right)\right| d\tau, \tag{4}$$

$$\leq (1-\alpha)\Omega + \frac{\alpha}{\Gamma(\alpha)}\Omega\int_{t_0}^{t}(t-\tau)^{\alpha-1}\, d\tau, \leq (1-\alpha)\Omega + \frac{\alpha}{\Gamma(\alpha)}\Omega\frac{(t-t_0)^{\alpha}}{\alpha}$$

$$\leq (1-\alpha)\Omega + \frac{\Omega}{\Gamma(\alpha)}a^{\alpha}, \leq \left(1-\alpha+\frac{a^{\alpha}}{\Gamma(\alpha)}\right)\Omega < \left(1+\frac{a^{\alpha}}{\Gamma(\alpha)}\right)\Omega < r.$$

Such that:

$$a < \left(\frac{r}{\Omega}\Gamma(\alpha)\right)^{\frac{1}{\alpha}}.$$

We show that $\left(y_n(t)\right)$ is uniformly bounded.

$$|y_n(t)| = |y(t_0)| + (1-\alpha)\left|f\left(t, y_{n-1}(t)\right)\right| \tag{5}$$

$$+\frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t} f\left(\tau, y_{n-1}(\tau)\right)(t-\tau)^{\alpha-1}d\tau,$$

$$\leq |y(t_0)| + (1-\alpha)\left|f\left(t, y_{n-1}(t)\right)\right|$$

$$+\frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t}\left|f\left(\tau, y_{n-1}(\tau)\right)(t-\tau)^{\alpha-1}d\tau\right| \leq |y(t_0)| + (1-\alpha)\Omega$$

$$+\frac{\alpha}{\Gamma(\alpha)}\Omega\int_{t_0}^{t}(t-\tau)^{\alpha-1}d\tau \leq |y(t_0)| + (1-\alpha)\Omega + \frac{\alpha}{\Gamma(\alpha)}\Omega\frac{a^{\alpha}}{\alpha},$$

$$\leq |y(t_0)| + \Omega\left(1+\frac{a^{\alpha}}{\Gamma(\alpha)}\right).$$

We shall show that $\left(y_n(t)\right)$ is uniformly equicontinuous.

$t_1, t_2 \in \overline{D}$, such that $t_1 > t_2$.

$$|y_n(t_1) - y_n(t_2)| \tag{6}$$

$$= \Bigg|(1-\alpha)\{f(t_1, y_{n-1}(t_1)) - f(t_2, y_{n-1}(t_2))\}$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1} f(\tau, y_{n-1}(\tau))d\tau$$

$$- \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-\tau)^{\alpha-1} f(\tau, y_{n-1}(\tau))d\tau \,\Big| \le (1-\alpha)|f(t_1, y_{n-1}(t_1))$$

$$- f(t_2, y_{n-1}(t_2))|$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\Bigg|\Bigg\{+\int_{t_0}^{t_2}(t_1-\tau)^{\alpha-1} f(\tau, y_{n-1}(\tau))d\tau$$

$$+ \int_{t_2}^{t_1}(t_1-\tau)^{\alpha-1} f(\tau, y_{n-1}(\tau))d\tau - \int_{t_0}^{t_2}(t_2-\tau)^{\alpha-1} f(\tau, y_{n-1}(\tau))d\tau\Bigg\}\Bigg|$$

$$\le (1-\alpha)\mathcal{L}|y_{n-1}(t_1) - y_{n-1}(t_2)|$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t_2}|f(\tau, y_{n-1}(\tau))|\{(t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1}\}d\tau$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\int_{t_2}^{t_1}(t_1-\tau)^{\alpha-1}|f(\tau, y_{n-1}(\tau))|\,d\tau,$$

$$\le (1-\alpha)\mathcal{L}|y_{n-1}(t_1) - y_{n-1}(t_2)|$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\Omega\left|-\frac{(t_1-t_2)^\alpha}{\alpha} + \frac{(t_1-t_0)^\alpha}{\alpha} - \frac{(t_2-t_0)^\alpha}{\alpha}\right| + \frac{\alpha}{\Gamma(\alpha)}\Omega\frac{(t_1-t_2)^\alpha}{\alpha}.$$

Note that the function $\frac{t^\alpha}{\alpha}$ is differentiable, therefore, there exists $l \in [t_1 - t_0, t_2 - t_0]$ such that by the Mean Value Theorem,

$$\frac{(t_1-t_0)^\alpha}{\alpha} - \frac{(t_2-t_0)^\alpha}{\alpha} = l^{\alpha-1}(t_1 - t_0 - (t_2 - t_0)), \tag{7}$$

$$= l^{\alpha-1}(t_1 - t_2).$$

Therefore:

$$|y_n(t_1) - y_n(t_2)| \tag{8}$$

$$\le (1-\alpha)\mathcal{L}|y_{n-1}(t_1) - y_{n-1}(t_2)| + \frac{\alpha\Omega}{\Gamma(\alpha)} l^{\alpha-1}(t_1-t_2)(t_1-t_0)$$

$$+ \frac{\Omega}{\Gamma(\alpha)}(t_1-t_2)^\alpha.$$

However,

$$
\begin{aligned}
&|y_1(t_1) - y_1(t_2)| \qquad (9)\\
&\le (1-\alpha)\big|f\big(t_1, y_0(t_1)\big)\\
&- f\big(t_2, y_0(t_2)\big)\big| \frac{\alpha}{\Gamma(\alpha)}\bigg|\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1} f\big(\tau, y_0(\tau)\big)d\tau\\
&- \int_{t_0}^{t_2}(t_2-\tau)^{\alpha-1} f\big(\tau, y_0(\tau)\big)d\tau \le (1-\alpha)\mathcal{L}|y_0(t_1)-y_0(t_2)|\\
&+ \frac{\alpha}{\Gamma(\alpha)}\bigg|\int_{t_0}^{t_2}(t_1-\tau)^{\alpha-1} f\big(\tau, y_0(\tau)\big)d\tau\\
&+ \int_{t_2}^{t_1}(t_1-\tau)^{\alpha-1} f\big(\tau, y_0(\tau)\big)d\tau - \int_{t_0}^{t_2}(t_2-\tau)^{\alpha-1} f\big(\tau, y_0(\tau)\big)d\tau\bigg|\\
&\le (1-\alpha)\mathcal{L}|y_0(t_1)-y_0(t_2)|\\
&+ \frac{\alpha}{\Gamma(\alpha)}\Omega\left\{l^{\alpha-1}(t_1-t_2)(t_1-t_0) + \frac{\alpha}{\Gamma(\alpha)}\Omega\frac{(t_1-t_2)^{\alpha}}{\alpha}\right\}.
\end{aligned}
$$

But $y_0(t)$ is constant, therefore:

$$
\begin{aligned}
|y_1(t_1) - y_1(t_2)| &\le \frac{\alpha}{\Gamma(\alpha)}\Omega(t_1-t_2)(t_1-t_0)l^{\alpha-1} + \frac{\alpha\Omega(t_1-t_2)^{\alpha}}{\alpha\Gamma(\alpha)} \qquad (10)\\
&\le \frac{\Omega\alpha}{\Gamma(\alpha)}al^{\alpha-1}(t_1-t_2) + \frac{\Omega(t_1-t_2)^{\alpha}}{\Gamma(\alpha)} < \frac{\alpha\Omega a l^{\alpha-1}}{\Gamma(\alpha)}\delta + \frac{\Omega\delta^{\alpha}}{\Gamma(\alpha)}.
\end{aligned}
$$

Assuming that $\forall n \ge 1\ \big(y_{n-1}(t)\big)$ is equicontinuous.

$$|y_1(t_1) - y_1(t_2)| \tag{11}$$

$$= \Bigg|(1-\alpha)f\big(t_1, y_0(t_1)\big) - (1-\alpha)f\big(t_2, y(t_2)\big)$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t_1} f\big(\tau, y_0(\tau)\big)\,(t_1-\tau)^{\alpha-1}d\tau$$

$$- \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t_2} f\big(\tau, y_0(\tau)\big)\,(t_2-\tau)^{\alpha-1}d\tau\Bigg|$$

$$\leq (1-\alpha)\big|f\big(t_1, y_0(t_1)\big) - f\big(t_2, y_0(t_2)\big)\big|$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\left|\int_{t_0}^{t_2} f\big(\tau, y_0(\tau)\big)\,(t_1-\tau)^{\alpha-1}d\tau - \int_{t_0}^{t_2} f\big(\tau, y_0(\tau)\big)\,(t_2-\tau)^{\alpha-1}d\tau\right|$$

$$+ \left|\frac{\alpha}{\Gamma(\alpha)}\int_{t_2}^{t_1} f\big(\tau, y_0(\tau)\big)\,(t_1-\tau)^{\alpha-1}d\tau\right|$$

$$\leq (1-\alpha)\mathcal{L}|y_0(t_1) - y_0(t_2)|$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\int_{t_1}^{t_2} \big|f\big(\tau, y_0(\tau)\big)\big|\{(t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1}\}d\tau$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\Omega\int_{t_2}^{t_1} (t_1-\tau)^{\alpha-1}d\tau$$

$$\leq (1-\alpha)\mathcal{L}|y_0(t_1) - y_0(t_2)| + \frac{\alpha}{\Gamma(\alpha)}\Omega\left\{\frac{(t_1-t_2)^{\alpha}}{\alpha}\right\}$$

$$+ \frac{\alpha}{\Gamma(\alpha)}\Omega\left\{\frac{(t_1-t_0)^{\alpha}}{\alpha} - \frac{(t_1-t_2)^{\alpha}}{\alpha} - \frac{(t_2-t_0)^{\alpha}}{\alpha}\right\}$$

$$\leq (1-\alpha)\mathcal{L}|y_0(t_1) - y_0(t_2)| + \frac{\Omega}{\Gamma(\alpha)}(t_1-t_2)^{\alpha}$$

$$+ (1-\alpha)\mathcal{L}|y_0(t_1) - y_0(t_2)| + \frac{\Omega}{\Gamma(\alpha)}\{(t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha}\}.$$

Noting that $y_0(t)$ is a constant, therefore, $y_0(t_1) - y_0(t_2) = 0$, also $(t-t_0)^{\alpha}$ is differentiable, therefore, by the Mean Value Theorem, there exists a $l \in [t_1 - t_0, t_2 - t_0]$ such that $\alpha(l-t_0)^{\alpha}(t_1-t_2) = (t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha}$.

Then,

$$|y_1(t_1) - y_1(t_2)| \leq \frac{\alpha}{\Gamma(\alpha)}\Omega(l-t_0)^{\alpha}(t_1-t_2) < \frac{\alpha\Omega l^{\alpha}(t_1-t_2)}{\Gamma(\alpha)}. \tag{12}$$

$$
\begin{aligned}
&|y_2(t_2) - y_2(t_2)| \qquad (13)\\
&\leq (1-\alpha)\left|f\big(t_1, y_1(t_1)\big) - f\big(t_2, y_1(t_2)\big)\right|\\
&+ \frac{\alpha}{\Gamma(\alpha)}\left|\int_{t_0}^{t_2} f\big(\tau, y_1(\tau)\big)\,(t_1-\tau)^{\alpha-1}d\tau - \int_{t_0}^{t_2} f\big(\tau, y_1(\tau)\big)\,(t_2-\tau)^{\alpha-1}d\tau\right.\\
&\left.+ \int_{t_1}^{t_2} f\big(\tau, y_1(\tau)\big)\,(t_1-\tau)^{\alpha-1}d\tau\right|\\
&\leq \mathcal{L}(1-\alpha)|y_1(t_1) - y_1(t_2)| + \frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}\\
&\leq \mathcal{L}(1-\alpha)\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)} \leq \{1+\mathcal{L}(1-\alpha)\}\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}\\
&< (1+\mathcal{L})\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}.
\end{aligned}
$$

$$
\begin{aligned}
&|y_3(t_1) - y_3(t_2)| \qquad (14)\\
&\leq (1-\alpha)\mathcal{L}|y_2(t_1) - y_2(t_2)|\\
&+ \frac{\alpha}{\Gamma(\alpha)}\left|\left\{\int_{t_0}^{t_2} f\big(\tau, y_2(\tau)\big)\,(t_1-\tau)^{\alpha-1}d\tau - \int_{t_0}^{t_2} f\big(\tau, y_2(\tau)\big)\,(t_2-\tau)^{\alpha-1}d\tau\right.\right.\\
&\left.\left.+ \int_{t_1}^{t_2} f\big(\tau, y_2(\tau)\big)\,(t_1-\tau)^{\alpha-1}d\tau\right\}\right|\\
&\leq (1-\alpha)\mathcal{L}(1+\mathcal{L})\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)} + \frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}\\
&\leq \frac{\mathcal{L}(1+\mathcal{L})}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}(t_1-t_2) + \frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}\\
&\leq \big(1+(1+\mathcal{L})\big)\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)} \leq (1+\mathcal{L})^2\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}.
\end{aligned}
$$

$$
\begin{aligned}
|y_4(t_1) - y_4(t_2)| &\leq (1-\alpha)\mathcal{L}|y_3(t_1) - y_3(t_2)| + \frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)} \qquad (15)\\
&\leq \mathcal{L}(1+\mathcal{L})^2\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)} + \frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}\\
&\leq \big(1+\mathcal{L}(1+\mathcal{L})^2\big)\frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)} \leq \frac{(1+\mathcal{L})^3}{\Gamma(\alpha)}l^{\alpha-1}\alpha(t_1-t_2).
\end{aligned}
$$

Therefore:

$$
|y_{n-1}(t_1) - y_{n-1}(t_2)| \leq \frac{(1+\mathcal{L})^{n-2}}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}(t_1-t_2). \qquad (16)
$$

Thus,

$$|y_n(t) - y_n(t)| \le (1-\alpha)\mathcal{L}|y_{n-1}(t_1) - y_{n-1}(t_2)| + \frac{\alpha\Omega l^{\alpha-1}(t_1 - t_2)}{\Gamma(\alpha)} \tag{17}$$
$$\le \frac{(1-\alpha)\mathcal{L}(1+\mathcal{L})^{n-2}}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}(t_1-t_2) + \frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}$$
$$< \frac{\mathcal{L}(1+\mathcal{L})^{n-2}}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}(t_1-t_2) + \frac{\alpha\Omega l^{\alpha-1}(t_1-t_2)}{\Gamma(\alpha)}$$
$$< \frac{(1+\mathcal{L}(1+\mathcal{L})^{n-2})}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}(t_1-t_2) < \frac{(1+\mathcal{L})^{n-1}}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}(t_1-t_0).$$

Therefore:

$\forall\varepsilon > 0$ such that,

$$|y_n(t_1) - y_n(t_2)| < \varepsilon,$$

$$\frac{(1+\mathcal{L})^{n-1}}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}(t_1-t_0) < \frac{(1+\mathcal{L})^{n-1}}{\Gamma(\alpha)}\alpha\Omega l^{\alpha-1}\delta < \varepsilon.$$

We need,

$$\delta < \frac{\Gamma(\alpha)\varepsilon}{(1+\mathcal{L})^{n-1}l^{\alpha-1}\alpha\Omega}.$$

Therefore, $\left(y_n(t)\right)$ is equicontinuous.

By the Arzelà-Ascoli theorem, there exist $\left(y_{n_l}(t)\right)$ of $\left(y_n(t)\right)$ that converges uniformly to a solution says $\bar{y}(n)$. Let $y_1(t)$ and $y_2(t)$ be solutions of the IVP with the ABC derivative, therefore:

$$|y_1(t) - y_2(t)| \tag{18}$$
$$\le (1-\alpha)\left|f\left(t, y_1(t)\right) - f\left(t, y_2(t)\right)\right|1$$
$$+ \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}\left|f\left(\tau, y_1(\tau)\right) - f\left(\tau, y_2(\tau)\right)\right|d\tau$$
$$\le (1-\alpha)\mathcal{L}|y_1(t) - y_2(t)| + \frac{\alpha\mathcal{L}}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}|y_1(\tau) - y_2(\tau)|\,d\tau,$$

$$|y_1(t) - y_2(t)|\{1-(1-\alpha)\mathcal{L}\} \le \frac{\alpha\mathcal{L}}{\Gamma(\alpha)}\int_{t_0}^{t}|y_1(\tau) - y_2(\tau)|(t-\tau)^{\alpha-1}d\tau, \tag{19}$$

$$|y_1(t) - y_2(t)| \le \frac{\alpha\mathcal{L}}{\Gamma(\alpha)}\{1-(1-\alpha)\mathcal{L}\}\int_{t_0}^{t}|y_1(\tau) - y_2(\tau)|(t-\tau)^{\alpha-1}d\tau, \tag{20}$$

$$1-(1-\alpha)\mathcal{L} > 0 \Rightarrow 1 < (1-\alpha)\mathcal{L} \Rightarrow \frac{1}{1-\alpha} < L,$$

$\alpha \neq 1$.

$$|y_1(t) - y_2(t)| \leq \frac{\alpha \mathcal{L}}{\Gamma(\alpha)}\{1-(1-\alpha)\mathcal{L}\}\int_{t_0}^{t}|y_1(\tau)-y_2(\tau)|(t-\tau)^{\alpha-1}d\tau. \tag{21}$$

By the Gronwall's inequality,

$$\begin{aligned} &\leq 0\, exp\left\{\int_{t_0}^{t}(t-\tau)^{\alpha-1}d\tau\right\}, \\ &\leq 0\, exp\left[\frac{(t-t_0)^{\alpha}}{\alpha}\right], \\ &= 0. \end{aligned} \tag{22}$$

Therefore:

$$y_1(t) = y_2(t).$$

3. Linear growth and Lipschitz conditions

Assuming that the function f satisfies the Linear growth condition that is to say,

$$|f(t,u)|^2 < K(1+|u|^2).$$

Then, the following inequality can be derived under the condition that, $2\alpha - 1 > 0$

$$\begin{aligned} |y_n(t)|^2 \leq\ & (1-\alpha)^2\left|f\left(\tau, y_{n-1}(\tau)\right)\right|^2 \\ & + \frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\left|\int_{t_0}^{t}(t-\tau)^{\alpha-1}f\left(\left(\tau, y_{n-1}(\tau)\right)d\tau\right)\right|\left|f\left(\left(\tau, y_{n-1}(\tau)\right)d\tau\right)\right| \\ & + \left|\frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau, y(\tau)d\tau)\right|^2 \\ & \leq (1-\alpha)^2K(1+|y_{n-1}|^2) \\ & + \frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(t-\tau)^{2\alpha-2}\,d\tau\right)^{\frac{1}{2}}\left(\int_{t_0}^{t}\left|f\left(\tau, y_{n-1}(\tau)\right)\right|^2 d\tau\right)^{\frac{1}{2}} \\ & + \left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\int_{t_0}^{t}(t-\tau)^{2\alpha-2}\,d\tau\int_{t_0}^{t}\left|f\left(\tau, y_{n-1}(\tau)\right)\right|^2 d\tau \\ & \leq (1-\alpha)^2K(1+|y_{n-1}|^2) \\ & + \frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\left(\frac{(t-t_0)^{2\alpha-1}}{(2\alpha-1)}\right)^{\frac{1}{2}}\left(\int_{t_0}^{t}K(1+|y_{n-1}|^2)\right)^{\frac{1}{2}} \\ & + \left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\left(\frac{(t-t_0)^{2\alpha-1}}{(2\alpha-1)}\right)\int_{t_0}^{t}K(1+|y_{n-1}|^2)d\tau. \end{aligned} \tag{23}$$

Therefore, $\forall n \geq 1$, $\forall t \in [a,b]$, we have that,

$$|y_n(t)|^2 < (1-\alpha)^2 K(1+|y_{n-1}|^2) \tag{24}$$
$$+\frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\frac{(t-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left(\int_{t_0}^{t} K(1+|y_{n-1}|^2)^{\frac{1}{2}}\,d\tau\right)^{\frac{1}{2}}$$
$$+\left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\left(\frac{(t-t_0)^{2\alpha-1}}{(2\alpha-1)}\right)K\int_{t_0}^{t}(1+|y_{n-1}|^2)d\tau.$$

Therefore, for $n=1$, we have:

$$|y_1(t)|^2 < (1-\alpha)^2 K(1+|y_0|^2)+\frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\frac{(t-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left(\int_{t_0}^{t} K(1+|y_0|^2)^{\frac{1}{2}}\,d\tau\right)^{\frac{1}{2}} \tag{25}$$
$$+\left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\left(\frac{(t-t_0)^{2\alpha-1}}{(2\alpha-1)}\right)K\int_{t_0}^{t}(1+|y_0|^2)d\tau.$$

We note that $\forall t\in[a,b]$, $y_0(t)=y(t_0)$ which is a constant, thus,

$$|y_1(t)|^2 < (1-\alpha)^2 K(1+|y_0|^2) \tag{26}$$
$$+\frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\frac{(t-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left(K(1+|y(t_0)|^2)\right)^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}}$$
$$+\left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\frac{(t-t_0)^{2\alpha}}{(2\alpha-1)}K(1+|y(t_0)|^2).$$

In particular,

$$|y_1(t)|^2 < (1-\alpha)^2 K(1+|y_0|^2) \tag{27}$$
$$+\frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\frac{(b-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left(K(1+|y(t_0)|^2)\right)^{\frac{1}{2}}(b-t_0)^{\frac{1}{2}}$$
$$+\left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\frac{(b-t_0)^{2\alpha}}{2\alpha-1}.$$

Therefore, $\forall t\in[a,b]$.

$$|y_1(t)|^2 < \overline{K}.$$

Where

$$\overline{K} = (1-\alpha)^2 K(1+|t_0|^2)+\frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\frac{(b-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left(K(1+|y(t_0)|^2)\right)^{\frac{1}{2}}(b-t_0)^{\frac{1}{2}}+\left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\frac{(b-t_0)^{2\alpha}}{2\alpha-1}.$$

$$\forall t\in[a,b].$$

$$|y_2(t)|^2 < (1-\alpha)^2 K(1+|y_1(t)|^2) \tag{28}$$
$$+\frac{2\alpha(1-\alpha)}{\Gamma(\alpha)}\frac{(t-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left(K(1+|y_1(t)|^2)\right)^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}}$$
$$+\left(\frac{\alpha}{\Gamma(\alpha)}\right)^2\frac{(t-t_0)^{2\alpha}}{2\alpha-1}K(1+|y_1(t_0)|^2),$$

$$|y_2(t)|^2 < (1-\alpha)^2 K(1+\overline{K}^2) + \frac{2\alpha(1-\alpha)}{\Gamma(\alpha)} \frac{(t-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} \left(K(1+\overline{K}^2)\right)^{\frac{1}{2}} (t-t_0)^{\frac{1}{2}} + \left(\frac{\alpha}{\Gamma(\alpha)}\right)^2 \frac{(t-t_0)^{2\alpha}}{(2\alpha-1)} K(1+\overline{K}^2) < \overline{K_1}. \tag{29}$$

Thus, we assume that $\forall t \in [a,b] \forall n \geq 1$.

$$\left(y_{n-1}(t)\right)^2 < \overline{K_n}$$

Then,

$$|\overline{y_n}(t)|^2 < (1-\alpha)^2 K(1+y_{n-1}(t)^2) + \frac{2\alpha(1-\alpha)}{\Gamma(\alpha)} \frac{(b-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} \left(K(1+|y_{n-1}|^2)\right)^{\frac{1}{2}} (b-t_0)^{\frac{1}{2}} + \left(\frac{\alpha}{\Gamma(\alpha)}\right)^2 \frac{(b-t_0)^{2\alpha}}{2\alpha-1} K(1+|y_{n-1}|^2). \tag{30}$$

By inductive hypothesis, we have that that $\forall t \in [a,b] \forall n \geq 1$.

$$|y_{n-1}|^2 < \overline{K_{n-1}},$$

Therefore,

$$|y_n(t)|^2 < (1-\alpha)^2 K\left(1+\overline{K_{n-1}}^2\right) + \frac{2\alpha(1-\alpha)}{\Gamma(\alpha)} \frac{(b-t_0)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} K\left(1+\overline{K_{n-1}}^2\right)(b-t_0)^{\frac{1}{2}} + \left(\frac{\alpha}{\Gamma(\alpha)}\right)^2 (b-t_0)^{2\alpha} K\left(1+\overline{K_{n-1}}^2\right)^{\overline{K_{n-1}}^2} \frac{(b-t_0)^{2\alpha}}{2\alpha-1} < \overline{K_n}. \tag{31}$$

$\forall n \geq 1$, we take,

$$K^m = \max_{1\leq l\leq n}\{\overline{K}_l\},$$

$$\forall t \in [a,b] \forall n \geq 1 |y_n(t)|^2 < K^m.$$

We can now proceed with the numerical solution since such equations are usually nonlinear.

4. Numerical solution and application

In this section, we aim to provide a numerical solution to the general IVP where the derivative is that of ABC.

$$\begin{cases} {}^{ABC}_{t_0}D^{\alpha}_{t} y(t) = f\left(t, y(t)\right), \ \ t > 0 \\ y(t_0) = y_0 \ \ if \ \ \ t = t_0. \end{cases} \tag{32}$$

Applying the AB integral on both sides, yields:

$$y(t) = \ y(t_0) + (1-\alpha) f\left(t, y(t)\right) + \frac{\alpha}{\Gamma(\alpha)} \int_{t_0}^{t} f\left(\tau, y(\tau)\right)(t-\tau)^{\alpha-1} d\tau. \tag{33}$$

We consider the above when $t = t_{n+1}$.

$$y(t_{n+1}) = y(t_0) + (1-\alpha)f\big(t_{n+1}, y(t_{n+1})\big) + \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t_{n+1}} f\big(\tau, y(\tau)\big)(t_{n+1}-\tau)^{\alpha-1}d\tau, \quad (34)$$

$$y(t_{n+1}) = y(t_0) + (1-\alpha)f\big(t_{n+1}, y(t_{n+1})\big) + \frac{\alpha}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} f\big(\tau, y(\tau)\big)(t_{n+1}-\tau)^{\alpha-1}d\tau, \quad (35)$$

$$y_{n+1} = y_0 + (1-\alpha)f(t_{n+1}, \bar{y}_{n+1}) + \frac{h^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{n} f\big(t_{j+1}, y_{j+1}\big)\{(n-j+1)^\alpha - (n-j)^\alpha\},$$

$$y_{n+1} = y_0 + (1-\alpha)f(t_{n+1}, \bar{y}_{n+1}) + \frac{h^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{n-1} f\big(t_{j+1}, y_{j+1}\big)\{(n-j+1)^\alpha - (n-j)^\alpha\} + \frac{h^\alpha}{\Gamma(\alpha)} f(t_{n+1}, \bar{y}_{n+1}),$$

$$\bar{y}_{n+1} = y_0 + (1-\alpha)f(t_n, y_n) + \frac{h^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{n} f\big(t_j, y_j\big)\{(n-j+1)^\alpha - (n-j)^\alpha\}.$$

The above scheme will be used to solve some ordinary nonlinear differential equations and a system of chaotic problem in the next section. The method used above is the simple Euler approximation of the nonlinear functions $f_z(x,y,z,t)$, $f_y(x,y,z,t)$ and $f_x(x,y,z,t)$ within the interval $[t_j, t_{j+1}]$. However, instead of ending at the point $\mathrm{t_n}$, we decided to end at $\mathrm{t_{n+1}}$, this makes the scheme become implicit. To solve this problem, we introduced the corrector factor $\overline{\mathrm{X}}_{\mathrm{n+1}} = (\bar{\mathrm{x}}_{\mathrm{n+1}}, \bar{\mathrm{y}}_{\mathrm{n+1}}, \bar{\mathrm{z}}_{\mathrm{n+1}})$ these components are obtained via the simple Euler approximation.

5. Illustrative examples

In this first example, we will look at the Solow-Swan model [11,12]. We will mention that the model aimed at augmenting with human capital predicts that if poor countries have similar savings rates for both physical capital and human capital as a share of output, the poor countries' income levels will tend to catch up with or converge towards the income levels of rich countries, a process known as conditional convergence [11,12]. However, savings rates vary greatly between countries. Saving rates for human capital are anticipated to differ because of cultural and ideological factors in each country, given the significant finance restrictions for investment in education [11,12]. The model will be expanded to include the situation of fractional differential equations. However, before applying the presented numerical scheme, we shall first show that the model satisfies conditions under which a unique solution exists. We shall also show that the system satisfies the conditions under which associated Picard iteration converges.

$$\begin{cases} {}^{ABR}_{t_0}D_t^{\alpha} y(t) = f\big(t, y(t)\big), \ \ t > 0 \\ \quad y(t_0) = y_0 \ \ if \ \ \ t = t_0. \end{cases} \tag{36}$$

We shall verify the property of $f\big(t, y(t)\big)$.

$$\begin{aligned} \big|f\big(t, y(t)\big)\big| &= |ay^b(t) - (\lambda_1 + \lambda_2 + \lambda_3)y(t)|, \\ &\le a|y^b(t)| + |\lambda_1 + \lambda_2 + \lambda_3||y(t)|. \\ &\le a|y(t)|^b + |\lambda_1 + \lambda_2 + \lambda_3||y(t)|. \end{aligned} \tag{37}$$

If we assume that $y(t)$ has maximum point, then,

$$\begin{aligned} \big|f\big(t, y(t)\big)\big| &\le a \max_{t\in[t_0,T]} |y(t)|^b + (\lambda_1 + \lambda_2 + \lambda_3)\ |y(t_0)|, \\ &\le a\|y\|_\infty^b + (\lambda_1 + \lambda_2 + \lambda_3\|y\|_\infty), \\ &\le \max\{\|y\|_\infty^b, \|y\|_\infty\}\,(a + 1\lambda_1 + \lambda_2 + \lambda_3) \le \bar{M} < \infty. \end{aligned} \tag{38}$$

Under the condition that $y(t)$ is bounded, then $f\big(t, y(t)\big)$ is bounded too.

We shall now apply the numerical scheme on the equation.

$$f\big(t, y(t)\big) = ay^b(t) - (\lambda_1 + \lambda_2 + \lambda_3)y(t) \tag{39}$$

$$\begin{aligned} y_{n+1} = y(t_0) + (1-\alpha)[a\bar{y}_{n+1}^b \\ - (\lambda_1 + \lambda_2 + \lambda_3)\bar{y}_{n+1}] \frac{h^\alpha}{\Gamma(\alpha)} \sum_{j=2}^{n-1} \big[ay_{j+1}^b \\ - (\lambda_1 + \lambda_2 + \lambda_3)y_{j+1}\big]\{(n-j+1)^\alpha - (n-j)^\alpha\} + \frac{h^\alpha}{\Gamma(\alpha)}\,[a\bar{y}_{n+1}^b \\ - (\lambda_1 + \lambda_2 + \lambda_3)\bar{y}_{n+1}], \end{aligned} \tag{40}$$

where

$$\begin{aligned} \bar{y}_{n+1} = y(t_0) + (1-\alpha)[ay_n^b - (\lambda_1 + \lambda_2 + \lambda_3)y_n] \\ + \frac{h^\alpha}{\Gamma(\alpha)} \sum_{j=0}^{n} \big[ay_j^b - (\lambda_1 + \lambda_2 + \lambda_3)y_j\big]\,\{(n-j+1)^\alpha - (n-j)^\alpha\}. \end{aligned} \tag{41}$$

Numerical simulations will be presented next. However, we shall apply this to a system of nonlinear equations. We consider the following nonlinear system:

$$\begin{cases} {}^{ABC}_{t_0}D_t^{\alpha} x(t) = ay(t) + b\big(\alpha + \beta y^2(t)\big)z(t) \\ {}^{ABC}_{t_0}D_t^{\alpha} y(t) = z(t) \\ {}^{ABC}_{t_0}D_t^{\alpha} z(t) = x(t) - \big(y(t) - z(t) - B\sin h(dx)\big) \end{cases} \tag{42}$$

With initial conditions,

$$(2, 0.1, 1), \qquad a = b = c = 2 \tag{43}$$

We have that,

$$\begin{cases} {}^{ABC}_{t_0}D_t^{\alpha}x(t) = f_x(x,y,z,t), \\ {}^{ABC}_{t_0}D_t^{\alpha}y(t) = f_y(x,y,z,t), \\ {}^{ABC}_{t_0}D_t^{\alpha}z(t) = f_z(x,y,z,t). \end{cases} \tag{44}$$

Indeed, by posing some conditions on x, y, z we can show that f_x, f_y and f_z are bounded and satisfies the Lipschitz condition, therefore, with the Picard iteration, we can conclude the system has a system of unique solutions. The numerical solutions are given as:

$$\begin{aligned} x_{n+1} = x_0 &+ (1-\alpha)f_x(\bar{x}_{n+1}, \bar{y}_{n+1}, \bar{z}_{n+1}, t_{n+1}) \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{n-1} f(t_j, x_j, y_j, z_j)\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\} \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha)} f_x(\bar{x}_{n+1}, \bar{y}_{n+1}, \bar{z}_{n+1}, t_{n+1}). \end{aligned} \tag{45}$$

Where the predictor components are presented as follows:

$$\begin{aligned} \bar{x}_{n+1} = x_0 &+ (1-\alpha)f_x(x_n, y_n, z_n, t_n) \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{n} f_x(t_j, x_j, y_j, z_j)\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\}. \end{aligned} \tag{46}$$

$$\begin{aligned} y_{n+1} = y_0 &+ (1-\alpha)f_y(\bar{x}_{n+1}, \bar{y}_{n+1}, \bar{z}_{n+1}, t_{n+1}) \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{n-1} f_y(t_j, x_j, y_j, z_j)\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\} \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha)} f_y(\bar{x}_{n+1}, \bar{y}_{n+1}, \bar{z}_{n+1}, t_{n+1}). \end{aligned}$$

$$\bar{z}_{n+1} = y_0 + (1-\alpha)f_y(x_n, y_n, z_n, t_n) + \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{n} f_z(t_j, x_j, y_j, z_j)\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\},$$

$$\begin{aligned} z_{n+1} = y_0 &+ (1-\alpha)f_z(\bar{x}_{n+1}, \bar{y}_{n+1}, \bar{z}_{n+1}, t_{n+1}) \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{n-1} f_z(t_j, x_j, y_j, z_j)\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\} \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha)} f_z(\bar{x}_{n+1}, \bar{y}_{n+1}, \bar{z}_{n+1}, t_{n+1}), \end{aligned}$$

$$\bar{z}_{n+1} = z_0 + (1-\alpha)f_z(x_n, y_n, z_n, t_n) + \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{n} f_z(t_j, x_j, y_j, z_j)\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\}.$$

We present here the numerical simulations of the above model for different values of fractional order and initial conditions. For these simulations, we have considered the step size to be 0.01, the initial conditions are given as $x(0) = -2$; $y(0) = 0.1$; $z(0) = 1$; $h = 0.01$. The numerical solutions are depicted in Figures 1–3.

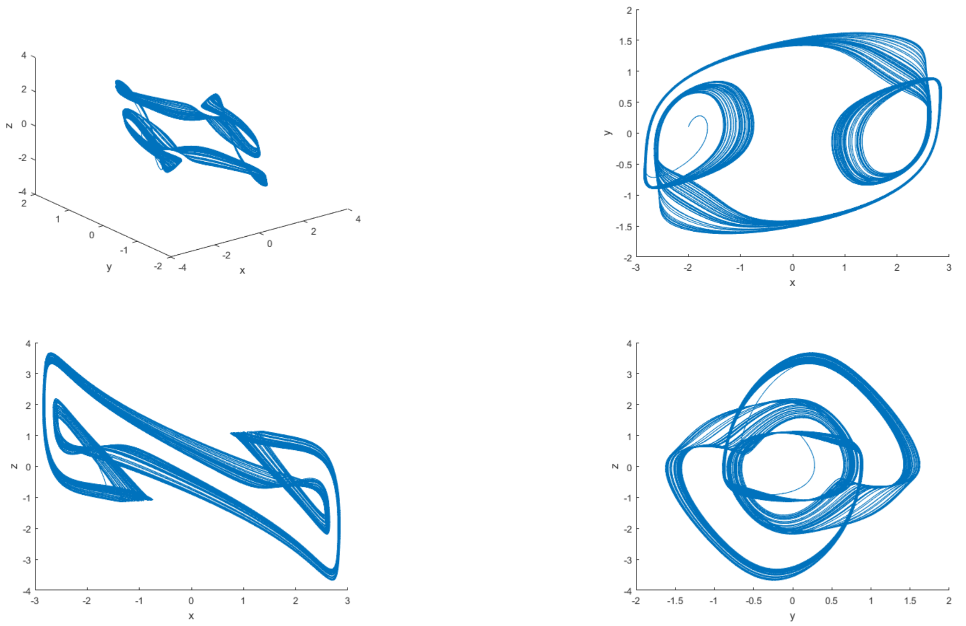

Figure 1. Numerical simulation for the fractional order 0.9 and final time 500.

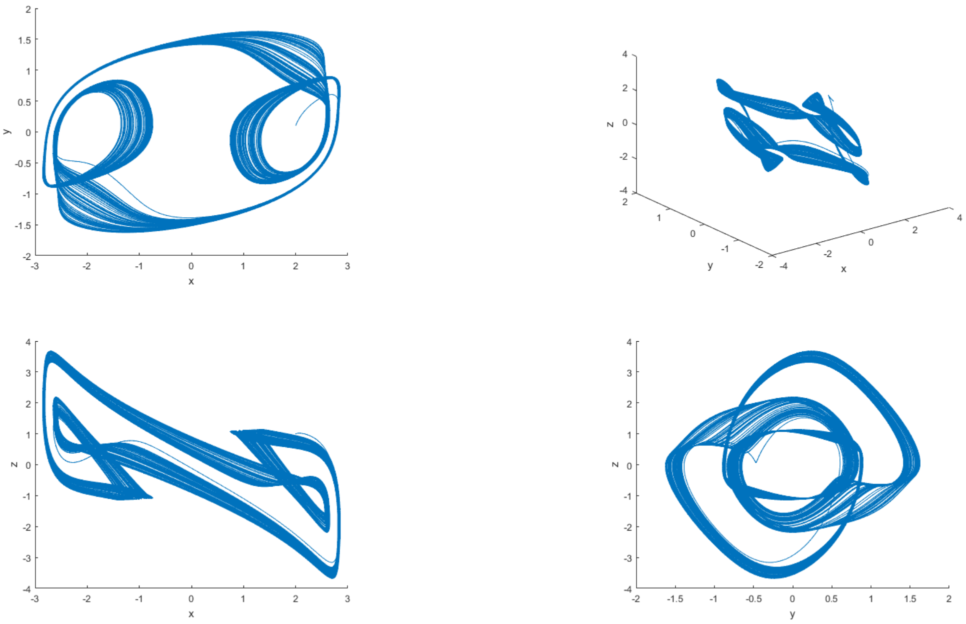

Figure 2. Numerical simulation for the fractional order 1 and final time 500.

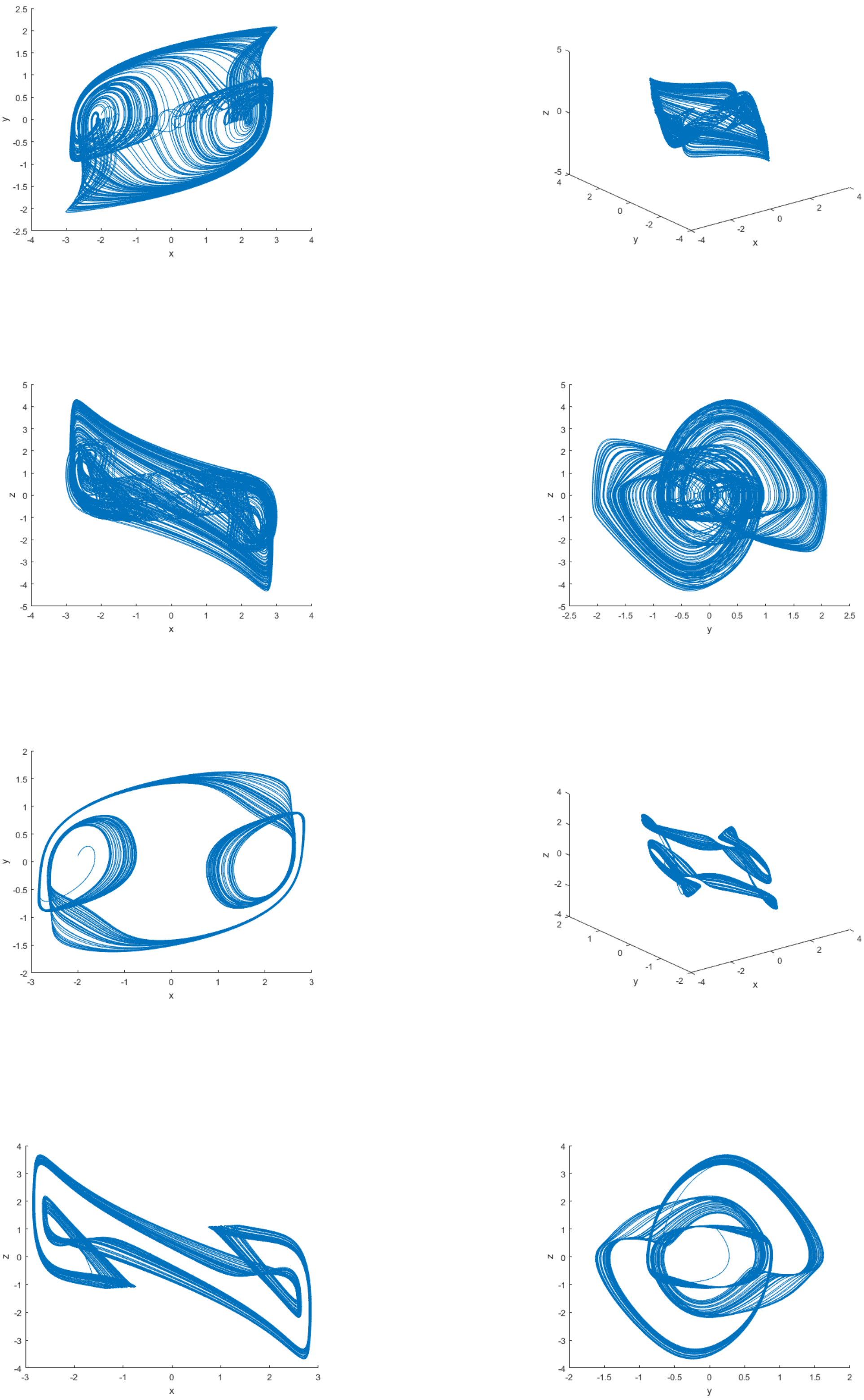

Figure 3. Numerical simulation for the fractional orders1 and 0.99 respectively and final time 100 when the initial conditions are (–2, 0.1, 1).

6. Conclusions

It has been acknowledged that the derivative based on the generalized Mittag-Leffler kernel is an operator with significant features that show up in several real-world applications. An important family of nonlinear ordinary differential equations that are utilized to model processes with crossover from fading to power law patterns have been established because of this derivative. The circumstances under which IVP with this derivative admit a unique solution were determined in this study using an iteration technique. To numerically solve these equations with various examples, we used the predictor corrector method and a straightforward Euler approximation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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