Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(11): 19686–19709.
DOI: 10.3934/mbe.2023872
Received: 31 July 2023
Revised: 22 September 2023
Accepted: 19 October 2023
Published: 26 October 2023

Research article

Improved uniform persistence for partially diffusive models of infectious diseases: cases of avian influenza and Ebola virus disease

Ryan Covington[1]**, Samuel Patton**[1]**, Elliott Walker**[1] **and Kazuo Yamazaki**[2,*]

[1] Texas Tech University, Department of Mathematics and Statistics, Lubbock, TX, 79409-1042, USA

[2] University of Nebraska, Lincoln, Department of Mathematics, 203 Avery Hall, PO Box, 880130, Lincoln, NE 68588-0130, USA

* **Correspondence:** Email: kyamazaki2@unl.edu; Tel: +14024733731; Fax: +14024728466.

Abstract: Past works on partially diffusive models of diseases typically rely on a strong assumption regarding the initial data of their infection-related compartments in order to demonstrate uniform persistence in the case that the basic reproduction number $\mathcal{R}_0$ is above 1. Such a model for avian influenza was proposed, and its uniform persistence was proven for the case $\mathcal{R}_0 > 1$ when all of the infected bird population, recovered bird population and virus concentration in water do not initially vanish. Similarly, a work regarding a model of the Ebola virus disease required that the infected human population does not initially vanish to show an analogous result. We introduce a modification on the standard method of proving uniform persistence, extending both of these results by weakening their respective assumptions to requiring that only one (rather than all) infection-related compartment is initially non-vanishing. That is, we show that, given $\mathcal{R}_0 > 1$, if either the infected bird population or the viral concentration are initially nonzero anywhere in the case of avian influenza, or if any of the infected human population, viral concentration or population of deceased individuals who are under care are initially nonzero anywhere in the case of the Ebola virus disease, then their respective models predict uniform persistence. The difficulty which we overcome here is the lack of diffusion, and hence the inability to apply the minimum principle, in the equations of the avian influenza virus concentration in water and of the population of the individuals deceased due to the Ebola virus disease who are still in the process of caring.

Keywords: avian influenza; basic reproduction number; Ebola virus disease; global attractivity; uniform persistence; spatial diffusion

1. Introduction

1.1. Motivation from real-world applications

In the past few decades, there has been a growing interest in modeling real-world phenomena via systems of partial differential equations (PDEs) solved by functions of not only time but also space, rather than ordinary differential equations (ODEs). In the study of mathematical models of infectious diseases, the basic reproduction number $\mathcal{R}_0$ measures the expected number of secondary infections caused by one infectious individual during its infectious period in an otherwise susceptible population. Proposing and analyzing PDE models of infectious diseases have become very popular, led by important works such as [1–3].

For clarity of discussions let us state the following definition precisely.

Definition 1.1. *Consider a system of m-many PDEs solved by* $(u_1, \ldots, u_m)(x, t)$*:*

$$\partial_t u_1(x,t) = D_1 \Delta u_1(x,t) + F_1(x,t,u_1,\ldots,u_m), \quad (1.1a)$$
$$\partial_t u_2(x,t) = D_2 \Delta u_2(x,t) + F_2(x,t,u_1,\ldots,u_m), \quad (1.1b)$$
$$\vdots \quad (1.1c)$$
$$\partial_t u_m(x,t) = D_m \Delta u_m(x,t) + F_m(x,t,u_1,\ldots,u_m), \quad (1.1d)$$

where we denoted $\partial_t := \frac{\partial}{\partial t}$*. We say that this system* (1.1) *is*

1) "fully diffusive" if $D_j > 0$ *for all* $j \in \{1, \ldots, m\}$*,*

2) "partially diffusive" if there exists some $k \in \{1, \ldots, m\}$ *such that* $D_k = 0$*.*

The spatially diffusive terms such as $D_j \Delta u_j$ within such PDE models can help capture the movement of human hosts and other entities such as viruses or bacteria. The diffusivity coefficients D_j's describe the distinct level of mobility for different population groups. It has been documented for some time (see [4, 5]) that common diffusivity coefficients (i.e., $D_1 = \ldots D_m$) are desperately needed so that one can add equations together, cancel out the nonlinear terms and thereby verify the global existence of a unique solution, only after which asymptotic behavior of solutions such as global stability may be discussed. Thus, many models with fully diffusive PDEs have assumed common diffusivity coefficients; we refer to as examples

1) [6, Eq (1.14)] for a dengue fever model,
2) [7, Eqs (2), (4) and (5)] for a malaria model,
3) [8, Eq (1.1)] for a Lyme disease model,
4) [9, hypothesis of Theorem 2.2] and [10, hypothesis of Theorem 2.1] for a cholera model,
5) [11, hypothesis of Theorem 2.1] for a Zika virus disease model.

As examples of models that are not necessarily of infectious diseases but which assumed uniform diffusivity coefficients, we refer to

1) [12, Eq (1.9)] for an internal storage model,

2) [13, Eq (1.5)] for a periodically-pulsed bio-reactor model,

3) [14, Eq (2.1)] for a hydraulic storage zone model,

4) [15, Eq (3.1)] for a zooplankton and harmful algae model.

Proving uniform persistence of disease when $\mathcal{R}_0$ is above 1 under additional conditions on initial data is an important topic that has received much attention (see [16]). The purpose of this work is to address the uniform persistence results in more extreme cases of partially diffusive models so that not all equations consist of diffusion. Primary examples of our concern are an avian influenza (AI) model from [17] in which only the equation of the concentration of the AI virus in water lacked diffusion (see (1.2)), and an Ebola virus disease (EVD) model from [18] in which only the equation of the population of individuals deceased due to the EVD lacked diffusion (see (1.4)). Although we will not study this issue directly in this work, one may also consider the evidence that the coronavirus of 2019 (COVID-19) survives in the environment for several days (see [19]), and thus including a non-diffusive equation of concentration of COVID-19 in the environmental reservoir may seem reasonable (see [20]).

A lack of diffusion in general implies an inability to apply the minimum principle on its own across all equations. This seems to be the main reason why Vaidya et al. [17] obtained uniform persistence of AI when its $\mathcal{R}_0$ is above one under a strong assumption that all of the infected bird population, the recovered bird population and the AI virus concentration are initially non-vanishing, rather than when *any* of the infection-related compartments are initially non-vanishing (see Theorem 1.1). Similarly, Yamazaki in [18] obtained uniform persistence of EVD when its $\mathcal{R}_0$ is above one, and the infected human population is initially non-vanishing, but not when only the population of the human individuals who died due to EVD but are still in the process of caring are initially non-vanishing (see Theorem 1.2).

It would be biologically meaningful if the condition on the initial data may be improved to an assumption that any, rather than all, of the infection-related compartments are initially non-vanishing. For example, one can try to prove the uniform persistence of AI when only the concentration of the AI virus in water is initially non-vanishing or the uniform persistence of the EVD when only the population of the deceased individuals due to EVD still in the process of caring is initially non-vanishing. As we will explain in detail, in the case of AI, in addition to transmission of AI from a bird to another bird, uninfected birds may drink water and ingest the virus through excretion of the viruses by infected birds (see [21–23]). In the case of EVD, we know that burial ceremonies that involve direct contact with the body of the deceased can also contribute to the transmission (see [24]).

Interestingly, such uniform persistence results starting from initial data that only guarantees that at least one of the infection-related compartments is non-vanishing exist for systems of fully diffusive PDEs modeling various infectious diseases. For example, Yamazaki and Wang [10] proved uniform persistence of the cholera virus in the case $\mathcal{R}_0 > 1$ when either infectious population or the concentration of cholera bacteria in water such as rivers is initially non-vanishing; that is, only one of the infection-related compartments had to be initially non-vanishing for such uniform persistence result to hold. A key difference was that in this cholera model, the equation of the cholera bacteria had diffusion. Similarly, Yamazaki in [11] proved uniform persistence of the Zika virus disease when its $\mathcal{R}_0$ is above 1 and any of the exposed human female population, exposed human male population or exposed mosquito population, and hence any of the infection-related compartments, is initially non-vanishing. We emphasize again that all the equations of these infection-related compartments had diffusion. Proofs in [10, 11] crucially relied on the minimum principle (see [25, Theorem 7.1.12]), which was available only because all the equations governing the infection-related compartments had diffusion.

To the best of our knowledge, for this reason, uniform persistence results of partially diffusive models when the corresponding $\mathcal{R}_0$ is above one under a weak assumption that at least one of its infection-related compartments is initially non-vanishing has never been proven. We achieve this goal in the cases of the AI and the EVD models (see Theorems 2.1 and 2.2). Our approach is generalizable to other systems of PDEs too (as we do in [26]). The mathematical novelty consists of discovering the hidden ability of solutions to non-diffusive equations to induce strict positivity everywhere for not only their own compartment, but for all other infection-related solution components via an indirect approach (see Propositions 3.1 and 4.1).

1.2. Main equation

Let us describe the main equations, namely those of AI in (1.2) and EVD in (1.4). In both cases, we consider the spatial domain Ω which is a bounded, open and connected subset of $\mathbb{R}^n$ for $n \in \mathbb{N}$ with a smooth boundary $\partial\Omega$.

As a first main model, we introduce the AI model from [17]. AI viruses are of major concern due to their potential socioeconomic impact and risks to wildlife conservation. Since October 2020, highly pathogenic AI viruses belonging to geese have been responsible for more than 70 million poultry deaths and 100 discrete infections in many wild mesocarnivore species (see [27]). It is known that aquatic birds are the primary reservoir of AI viruses and that AI viruses can be transmitted from bird to bird, but they may also be transmitted through excretion of the AI virus by infected birds followed by ingestion of the virus in the drinking water of uninfected birds (see [21–23]). For this reason, the authors in [17] chose to include an equation $V(x,t)$ for the AI virus concentration in water. For further details of the AI virus and the derivation of the model, we refer to [17, Sections 1 and 2]. In addition to $V(x,t)$, let us denote by $S(x,t)$, $I(x,t)$ and $R(x,t)$ the population density of the bird population that are susceptible, infected and recovered, respectively. We refer to Table 1 for further notations.

Table 1. Definition of parameters in model (1.2a)–(1.2d).

Parameter	Definition
α	Shedding rate of infected bird hosts in their feces
β_1	Direct transmission rate from infectious bird
β_2	Indirect transmission rate concentration of AI virus
d	Natural death rate of birds
D	Diffusivity coefficients of S, I, R
η	Rate of bird hosts immunity loss
γ	Recovery rate of infected bird hosts
λ	Recruitment rate of susceptible bird hosts

Considering the significant mobility of birds in contrast to that of the concentration of the AI virus in water, the following model was proposed in [17, Eqs (1)–(4)]:

$$\partial_t S = D\Delta S + \lambda - (\beta_1 I + \beta_2 V)S - dS + \eta R, \tag{1.2a}$$

$$\partial_t I = D\Delta I + (\beta_1 I + \beta_2 V)S - (\gamma + d)I, \tag{1.2b}$$

$$\partial_t R = D\Delta R + \gamma I - (\eta + d)R, \tag{1.2c}$$

$$\partial_t V = \alpha I - c(x)V, \tag{1.2d}$$

subjected to an initial condition

$$(S, I, R, V)(x, 0) = (\phi_1, \phi_2, \phi_3, \phi_4)(x)$$

and standard Neumann boundary condition. We observe that (1.2) has a disease-free equilibrium (DFE) of

$$(S, I, R, V) = (m_A^*, 0, 0, 0) \text{ where } m_A^* := \frac{\lambda}{d}. \tag{1.3}$$

As a second main model, we introduce the EVD model from [18] (its ODE version was initially studied in [28]). The EVD is transmitted from wild animals to susceptible humans, and infectious humans may spread to other humans as well. As recently as September 2022, the Sudan Ebola virus was confirmed by the Uganda Virus Research Institute in which a total of 164 cases with 77 deaths were discovered (see [29]). A special feature of the EVD model of concern from [18, 28] is the inclusion of an equation of $D(x, t)$, the population of the individuals deceased due to the EVD. This is due to the warm-hearted West African tradition and customs of caring for the deceased individuals by kissing them, washing them and dressing them up, even those who passed away due to the EVD despite the risk of infection. Infections during funeral and subsequent death have been documented in various articles (see [30]). For further details of EVD and the derivation of the model, we refer to [28, Sections 1 and 2] and [18, Sections 1 and 2]. In addition to $D(x, t)$, let us denote $S(x, t), I(x, t), R(x, t)$ and $P(x, t)$ as the human population of susceptible, infected, recovered individuals and the Ebola virus pathogens in the environment, respectively. We refer to Table 2 for further notations.

Table 2. Definition of parameters in model (1.4).

Parameter	Definition
α	Shedding rate of deceased human individuals
$\frac{1}{b}$	Mean caring duration of deceased human individuals
β_1	Effective contact rate of infectious human individuals
β_2	Effective contact rate of deceased human individuals
$D_i, i = 1, 2, 3, 4$	Diffusion coefficients of S, I, R, P respectively
δ	Disease-induced death rate of human individuals
η	Decay rate of Ebola virus in the environment
γ	Recovery rate of infectious human individuals
λ	Effective contact rate of Ebola virus
μ	Natural death rate of human individuals
π	Recruitment rate of susceptible human individuals
ξ	Shedding rate of infectious human individuals

Considering the lack of mobility of the deceased individuals due to the EVD, the following model was proposed in [18, Eqs (1) and (2)]:

$$\partial_t S = D_1 \Delta S + \pi - (\beta_1 I + \beta_2 D + \lambda P)S - \mu S, \tag{1.4a}$$

$$\partial_t I = D_2\Delta I + (\beta_1 I + \beta_2 D + \lambda P)S - (\mu + \delta + \gamma)I, \tag{1.4b}$$
$$\partial_t R = D_3\Delta R + \gamma I - \mu R, \tag{1.4c}$$
$$\partial_t P = D_4\Delta P + \xi I + \alpha D - \eta P, \tag{1.4d}$$
$$\partial_t D = \delta I - bD, \tag{1.4e}$$

subjected to the initial conditions of

$$(S, I, R, P, D)(x, 0) = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)(x)$$

and standard Neumann boundary conditions. We see that (1.4) has a DFE of

$$(S, I, R, P, D) = (m_{\mathrm{E}}^*, 0, 0, 0, 0) \text{ where } m_{\mathrm{E}}^* := \frac{\pi}{\mu} \tag{1.5}$$

Remark 1.1. *We observe that the diffusivity coefficients in* (1.2) *were same while those in* (1.4) *had the freedom to differ. We refer to [4] for technical details.*

Before we proceed further, let us fix the standard notation to represent the solution as u. Because no confusion occurs, by an abuse of notation, we write

$$u := (u_1, u_2, u_3, u_4) := (S, I, R, V) \text{ while } u := (u_1, u_2, u_3, u_4, u_5) := (S, I, R, P, D) \tag{1.6}$$

as solutions to (1.2) and (1.4), respectively. We consider the space of continuous functions with supremum norm; i.e., $\|f\|_{C(D)} := \sup_{x\in D}|f(x)|$ and define

$$X := C(\overline{\Omega}, \mathbb{R}^k) = \prod_{i=1}^{k} X_i \quad \text{where } X_i := C(\overline{\Omega}, \mathbb{R}) \text{ and } \bar{\Omega} := \Omega \cup \partial\Omega, \tag{1.7}$$

which is the space of $\mathbb{R}^k$-valued functions that are continuous in $x \in \overline{\Omega}$ and equipped with the supremum norm of $\|u\|_{C(\overline{\Omega})} := \sum_{i=1}^{k}\|u_i\|_{C(\overline{\Omega})}$, where $k = 4$ for the AI model while $k = 5$ for the EVD model. Furthermore, we define

$$X^+ := C(\overline{\Omega}, \mathbb{R}_+^k) = \prod_{i=1}^{k} X_i^+ \quad \text{where } X_i^+ := \{f \in C(\overline{\Omega}, \mathbb{R}) : f \geq 0\}. \tag{1.8}$$

1.3. Previous works and mathematical challenges

In this section, we describe previous results, and challenges brought by the absence of diffusion in some equations. First, let us recall two important definitions.

Definition 1.2. *Let Y be any metric space, $Y_0 \subset Y$ an open set, $\partial Y_0 := Y \setminus Y_0$ and $\Psi_t : Y \mapsto Y$ be any semiflow on Y.*

1) ([31, p. 6172]; see also [16, Definition 1.3.1]) A continuous function $q : Y \mapsto [0, \infty)$ is called a generalized distance function for Ψ_t if it satisfies $q(\Psi_t(y)) > 0$ for all $t > 0$ whenever

(a) $q(y) = 0$ and $y \in Y_0$,

(b) or $q(y) > 0$,

2) ([16, Definition 1.3.3]) Let q be a generalized distance function for Ψ_t. Then, Ψ_t is said to be uniformly persistent with respect to (w.r.t.) $(Y_0, \partial Y_0, q)$ if there exists $\eta > 0$ such that

$$\liminf_{n\to\infty} q(\Psi_t^n(\phi)) \geq \eta \quad \forall\, \phi \in Y_0. \tag{1.9}$$

Remark 1.2. *It is clear from* (1.9) *that the larger the Y_0 chosen, the stronger uniform persistence result becomes.*

Let us list some of the main results obtained in [17, 18].

List 1.1. *(a) For any $\phi \in X^+$, both* (1.2) *and* (1.4) *admit a unique non-negative solution that remains in X^+, and thus a semiflow $\Phi_t : X^+ \mapsto X^+$ defined by $\Phi_t(\phi) := u(t)$ for all $t > 0$ such that $u(x, 0) = \phi(x)$ (see [17, Theorem 3.3] and [18, Theorem 2.1]).*

(b) The basic reproduction number $\mathcal{R}_0$ was rigorously derived for both systems (1.2) *and* (1.4) *from their corresponding infection-related compartments, namely I and V in* (1.2) *and I, P, and D in* (1.4) *(see [17, Eq (27)] and [18, Eq (34)]).*

(c) When $\mathcal{R}_0 < 1$, the corresponding DFE are globally attractive in X^+ for both (1.2) *and* (1.4), *although this result for* (1.4) *requires $D_1 = D_2 =: \bar{D}$ (see [17, Theorem 3.8 (1)] and [18, Theorem 2.2]).*

In more detail, to deduce the result from List 1.1 (b), [17] considered the eigenvalue problem associated with the infection-related compartments I and V of (1.2), namely

$$\theta\psi_2 = D\Delta\psi_2 + \beta_1 H(x)\psi_2 + \beta_2 H(x)\psi_4 - (\gamma + d)\psi_2, \tag{1.10a}$$

$$\theta\psi_4 = \alpha\psi_2 - c\psi_4, \tag{1.10b}$$

for an arbitrary $H : \bar{\Omega} \mapsto (0, \infty)$ (see [17, Eq (22)]) and proved the following result.

Lemma 1.1. ([17, Lemma 3.4 and Proposition 1]) Let $\mathcal{R}_0$ be the basic reproduction number of the AI system (1.2) and $(m_A^*, 0, 0, 0)$ its DFE from (1.3). Then, for any $H : \bar{\Omega} \mapsto (0, \infty)$, the eigenvalue problem (1.10) has a principal eigenvalue $\theta(H)$ associated with a strictly positive eigenfunction. Moreover, $\mathcal{R}_0 - 1$ and $\theta(m_A^*)$ have the same sign.

Similarly, to deduce the result from List 1.1 (b), [18] considered the eigenvalue problem associated with the infection-related compartments I, P and D of the EVD system (1.4), namely

$$\theta\psi_2 = \overline{D}\Delta\psi_2 + (\beta_1\psi_2 + \beta_2\psi_5 + \lambda\psi_4)H(x) - (\mu + \delta + \gamma)\psi_2, \tag{1.11a}$$

$$\theta\psi_4 = \overline{D}_4\Delta\psi_4 + \xi\psi_2 + \alpha\psi_5 - \eta\psi_4, \tag{1.11b}$$

$$\theta\psi_5 = \delta\psi_2 - b\psi_5, \tag{1.11c}$$

for an arbitrary $H : \bar{\Omega} \mapsto [0, \infty)$ (see [18, Eq (24)]) and proved the following result.

Lemma 1.2. ([18, Propositions 4.3 and 4.4]) Let $\mathcal{R}_0$ be the basic reproduction number of the EVD system (1.4) and $(m_E^*, 0, 0, 0, 0)$ its DFE from (1.5). Then, for any $H : \bar{\Omega} \mapsto (0, \infty)$, the eigenvalue problem (1.11) has a principal eigenvalue $\theta(H)$ associated with a strictly positive eigenfunction. Moreover, $\mathcal{R}_0 - 1$ and $\theta(m_E^*)$ have the same sign.

Now, let us state the main results from [17, Theorem 3.8 (ii)] and [18, Theorem 2.3].

Theorem 1.1. ([17, Theorem 3.8 and its proof]) Define

$$\mathbb{W}_0 := \{\psi \in X^+ : \psi_2 \not\equiv 0, \psi_3 \not\equiv 0, \text{ and } \psi_4 \not\equiv 0\}, \tag{1.12a}$$

$$\partial\mathbb{W}_0 := X^+ \setminus \mathbb{W}_0 = \{\psi \in X^+ : \psi_2 \equiv 0 \text{ or } \psi_3 \equiv 0 \text{ or } \psi_4 \equiv 0\}, \tag{1.12b}$$

and

$$p : X^+ \mapsto [0,\infty) \text{ by } p(\psi) := \min\{\min_{x\in\bar{\Omega}} \psi_2(x), \min_{x\in\bar{\Omega}} \psi_3(x), \min_{x\in\bar{\Omega}} \psi_4(x)\}. \tag{1.13}$$

If $\mathcal{R}_0$ defined by [17, Eq (27)] satisfies $\mathcal{R}_0 > 1$, then the AI system (1.2) admits at least one positive steady state $\hat{u}$ and there exists $\sigma > 0$ such that for any $\phi \in X^+$ where $\phi_i \not\equiv 0$ for all $i \in \{2,3,4\}$,

$$\liminf_{t\to\infty} u_i(x,t,\phi) \geq \sigma \quad \forall\, i \in \{1,2,3,4\}, \forall\, x \in \bar{\Omega}, \tag{1.14}$$

where $u_i(x,0,\phi) = \phi_i(x)$; i.e., uniform persistence w.r.t. $(\mathbb{W}_0, \partial\mathbb{W}_0, p)$ holds for $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ defined in (1.12).

Theorem 1.2. ([18, Theorem 2.3 and its proof]) Define

$$\mathbb{W}_0 := \{\psi \in X^+ : \psi_2 \not\equiv 0\}, \tag{1.15a}$$

$$\partial\mathbb{W}_0 := X^+ \setminus \mathbb{W}_0 = \{\psi \in X^+ : \psi_2 \equiv 0\}, \tag{1.15b}$$

and

$$p : X^+ \mapsto [0,\infty) \text{ by } p(\psi) := \min\{\min_{x\in\bar{\Omega}} \psi_2(x), \min_{x\in\bar{\Omega}} \psi_3(x), \min_{x\in\bar{\Omega}} \phi_4(x), \min_{x\in\bar{\Omega}} \phi_5(x)\}. \tag{1.16}$$

If $D_1 = D_2 =: \bar{D}$ and $\mathcal{R}_0$ defined by [18, Eq (34)] satisfies $\mathcal{R}_0 > 1$, then the EVD system (1.4) admits at least one positive steady state $\hat{u}$ and there exists $\sigma > 0$ such that for any $\phi \in X^+$ where $\phi_2 \not\equiv 0$,

$$\liminf_{t\to\infty} u_i(x,t,\phi) \geq \sigma \quad \forall\, i \in \{1,2,3,4,5\}, \forall\, x \in \bar{\Omega}, \tag{1.17}$$

where $u_i(x,0,\phi) = \phi_i(x)$; i.e., uniform persistence w.r.t. $(\mathbb{W}_0, \partial\mathbb{W}_0, p)$ holds for $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ defined in (1.15).

As we pointed out already, the infection-related compartments of the AI system (1.2) are I and V, while those of the EVD system (1.4) are I, P and D. Further, for the fully diffusive systems of PDEs such as those for cholera [10] and the Zika virus disease [11], uniform persistence of the disease when the basic reproduction number was above 1 was successfully proven as long as any one of the infection-related compartments is initially non-vanishing. Therefore, it is a natural question to ask if we can improve $\mathbb{W}_0$ and consequently $\partial\mathbb{W}_0$ in (1.12) to

$$\mathbb{W}_0 := \{\psi \in X^+ : \psi_2 \not\equiv 0 \text{ or } \psi_4 \not\equiv 0\}, \tag{1.18a}$$

$$\partial\mathbb{W}_0 := X^+ \setminus \mathbb{W}_0 = \{\psi \in X^+ : \psi_2 \equiv 0 \text{ and } \psi_4 \equiv 0\}, \tag{1.18b}$$

and improve $\mathbb{W}_0$ in (1.15) to

$$\mathbb{W}_0 := \{\psi \in X^+ : \psi_2 \not\equiv 0 \text{ or } \psi_4 \not\equiv 0 \text{ or } \psi_5 \not\equiv 0\}, \tag{1.19a}$$

$$\partial\mathbb{W}_0 := X^+ \setminus \mathbb{W}_0 = \{\psi \in X^+ : \psi_2 \equiv 0, \psi_4 \equiv 0, \text{ and } \psi_5 \equiv 0\}; \tag{1.19b}$$

these are clear improvements based on Remark 1.2. In order to understand the difficulty of this aim, let us briefly review the proofs of Theorems 1.1 and 1.2. It turns out that both Theorems 1.1 and 1.2 rely crucially on the following consequence of the minimum principle.

Lemma 1.3. ([17, Lemma 3.7]) Define m_A^* by (1.3). Suppose that $u(x,t,\phi)$ is a solution to (1.2) such that $u(\cdot,0,\phi)=\phi(\cdot)\in X^+$.

(1) For any $i\in\{2,3\}$, if there exists some $t_0^i\geq 0$ such that $u_i(\cdot,t_0^i,\phi)\not\equiv 0$, then $u_i(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and all $t>t_0^i$.

(2) For any $\phi\in X^+$, $u_1(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and all $t>0$, and

$$\liminf_{t\to\infty} u_1(\cdot,t,\phi)\geq \frac{\lambda}{\Upsilon_A}, \tag{1.20}$$

where

$$\tilde{c}:=\min_{x\in\bar{\Omega}} c(x) \quad \text{and} \quad \Upsilon_A := 2m_{\mathrm{A}}^*\beta_1+4\frac{\alpha m_{\mathrm{A}}^*}{\tilde{c}}\beta_2+d. \tag{1.21}$$

(3) If there exists some $t_0^2\geq 0$ such that $u_2(\cdot,t_0^2,\phi)\not\equiv 0$, then $u_4(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and all $t>t_0^2$.

In particular, Lemma 1.3 (3) follows from solving for u_4 of (1.2) directly as

$$u_4(x,t)=e^{-c(x)(t-t_0^2)}u_4(x,t_0^2)+\alpha e^{-c(x)t}\int_{t_0^2}^t u_2(x,s)e^{c(x)s}ds \quad \forall\, x\in\Omega, t\geq t_0^2$$

and relying on Lemma 1.3 (1).

Lemma 1.4. ([18, Proposition 4.7]) Define m_E^* by (1.5). Suppose that $D_1=D_2=:\bar{D}$ and $u(x,t,\phi)$ is a solution to (1.4) such that $u(\cdot,0,\phi)=\phi(\cdot)\in X^+$.

(1) For any $i\in\{2,3,4\}$, if there exists some $t_0^i\geq 0$ such that $u_i(\cdot,t_0^i,\phi)\not\equiv 0$, then $u_i(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and all $t>t_0^i$.

(2) For any $\phi\in X^+$, $u_1(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and all $t>0$, and

$$\liminf_{t\to\infty} u_1(\cdot,t,\phi)\geq \frac{\pi}{\Upsilon_E}, \tag{1.22}$$

where

$$\Upsilon_E := \beta_1 2m_{\mathrm{E}}^*+\beta_2\left(\frac{4m_{\mathrm{E}}^*\delta}{b}\right)+\lambda\left(\frac{\xi 4m_{\mathrm{E}}^*+[\frac{\alpha 8m_{\mathrm{E}}^*\delta}{b}]}{\eta}\right)+\mu. \tag{1.23}$$

(3) If there exists some $t_0^2\geq 0$ such that $u_2(\cdot,t_0^2,\phi)\not\equiv 0$, then $u_5(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and all $t>t_0^2$.

Proof of Lemma 1.4. The only improvement that we included here is in part (2); in contrast, [18, Proposition 4.7 (2)] stated that there exists $T>0$ such that $u_1(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and $t>T$ rather than $u_1(x,t,\phi)>0$ for all $x\in\bar{\Omega}$ and $t>0$. As we see, it is also stated in Lemma 1.3 (2) (from [17, Lemma 3.7 (iii)]) but its proof was omitted. We give a quick proof here. Suppose that there exists $(x^*,t^*)\in\Omega\times(0,\infty)$ at which $u_1(x^*,t^*,\phi)=0$. By non-negativity of the solution, this is a local minimum, and thus by Lemma A.1 we have $\Delta S(x^*,t^*)\geq 0$ and thus (1.4a) gives us $\partial_t S(x^*,t^*)\geq\pi>0$, which leads to a contradiction because, for some $\epsilon\in(0,t^*)$ sufficiently small, $S(x^*,t^*-\epsilon)<0$. Therefore, $S(x,t)>0$ for all $x\in\Omega$ and all $t>0$. It follows immediately from this that $S(x,t)>0$ for all $x\in\bar{\Omega}$ and all $t>0$ as claimed (see [32, Theorem 7.4.1]).

Remark 1.3. *We make a key observation from both Lemmas 1.3 and 1.4 that an assumption of the existence of* $t_0^4 \geq 0$ *such that* $u_4(\cdot, t_0^4, \phi) \not\equiv 0$ *in* (1.2) *or an assumption of the existence of* $t_0^5 \geq 0$ *such that* $u_5(\cdot, t_0^5, \phi) \not\equiv 0$ *in* (1.4) *does not immediately give us any positivity on any solution component, not even itself. This is completely reasonable; the proofs of Lemma 1.3 (1) and Lemma 1.4 (1) consist of a standard application of the minimum principle (see [25, Theorem 7.1.12]) and it is false in general that a solution to a non-diffusive equation can attain strict positivity everywhere for all* $t > t_0$ *based only on the assumption that it does not vanish at time* t_0.

Let us point out specifically two occasions where the difficulty would arise if we replace the $\mathbb{W}_0$ in (1.12a) by that in (1.18a) for the AI model (1.2) and similarly $\mathbb{W}_0$ in (1.15a) by that in (1.19a) for the EVD model (1.4).

First, in the case of the AI model (1.2), in an effort to prove that the DFE is a uniform weak repeller for $\mathbb{W}_0$, they claim on [17, p. 2839]

> let $\tilde{\psi} := (\tilde{\psi}_2, \tilde{\psi}_4)$ be the strict positive eigenfunction corresponding to $[\theta(m_{\mathrm{A}}^* - \delta_0) > 0]$. Since $u_i(x, t, \phi_0) > 0, \forall x \in \bar{\Omega}, t > 0, i = 2, 4$, there exists $\epsilon_0 > 0$ such that $[(u_2(x, t_1, \phi_0), u_4(x, t_1, \phi_0)) \geq \epsilon_0 e^{\theta(m_{\mathrm{A}}^* - \delta_0)t_1} \tilde{\psi}]$.

Here, if $\phi \in \mathbb{W}_0$ with $\mathbb{W}_0$ from (1.18a), then it is possible that the only available assumption is that $\phi_4 \not\equiv 0$. That alone and Lemma 1.3 cannot guarantee us that both $u_2(x, t, \phi)$ and $u_4(x, t, \phi)$ are strictly positive on $\bar{\Omega}$ for all $t > 0$; in fact, it cannot even guarantee the strictly positivity of either one. However, the strict positivity on $\bar{\Omega}$ for all $t > 0$ is needed to achieve the claim $(u_2(x, t_1, \phi_0), u_4(x, t_1, \phi_0)) \geq \epsilon_0 e^{\theta(m_{\mathrm{A}}^* - \delta_0)t_1} \tilde{\psi}$ because $\tilde{\psi}$ is strictly positive on $\bar{\Omega}$ due to Lemma 1.1.

In the case of the EVD model (1.4), an analogous situation is slightly improved as follows (see [18, p. 24]).

> Now, by hypothesis, we have $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in \mathbb{W}_0$ so that $\phi_2(\cdot) \not\equiv 0$. By [Lemma 1.4 (3)] this implies that $u_2(x, t, \phi) > 0, u_5(x, t, \phi) > 0$ for all $x \in \bar{\Omega}, t > 0$. Suppose that for any $t_0 > 0$, we have $u_4(\cdot, t_0, \phi) \equiv 0$ on $\bar{\Omega}$. But, then by [(1.4d)], we deduce $\partial_t P > 0$ for all $(x, t) \in \Omega \times \{t_0\}$, which is a contradiction to [the non-negativity of the solution.] Therefore, we must have $u_4(\cdot, t_0, \phi) \not\equiv 0$ for all $t_0 > 0$. By [Lemma 1.4 (1)], this implies $u_4(x, t, \phi) > 0$ for all $x \in \bar{\Omega}, t > 0$ by arbitrariness of $t_0 > 0$. Thus, we see that in particular $(u_2, u_4, u_5)(x, t_1, \phi) \gg 0$ so that we may find $\epsilon > 0$ sufficiently small such that $(I, P, D)(x, t_1, \phi) \geq \epsilon \tilde{\psi}(x)$.

Although this argument had a nice outcome that only $\phi_2 \not\equiv 0$ led to not only $u_2, u_5 > 0$ due to Lemma 1.4 (3) but also $u_4 > 0$, an analogous proof in the case that $\mathbb{W}_0$ in (1.15a) is replaced by that in (1.19a) seems hopeless. In particular, if all we know is that $\phi_5 \not\equiv 0$, then Lemma 1.4 alone is not sufficient to guarantee the positivity of all of u_2, u_4 and u_5 on $\bar{\Omega}$ and all $t > 0$; in fact, it does not guarantee the strict positivity of any one of them. Yet, strict positivity of all of them is needed to achieve the desired $(I, P, D)(x, t_1, \phi) \geq \epsilon \tilde{\psi}(x)$ considering that $\tilde{\psi}$ is strictly positive due to Lemma 1.2.

Next, the following difficulty to be described next is even more dire than the first. In the case of the AI model (1.2), the function p defined in (1.13) is claimed to be a generalized distance function for the semiflow Φ_t for (1.2) (recall Definition 1.2).

> By [Lemma 1.3], it follows that ... p has the property that if $p(\phi) > 0$ or $\phi \in \mathbb{W}_0$ with $p(\phi) = 0$, then $p(\Phi_t \phi) > 0, \forall t > 0$

from [17, p. 2839]. Suppose we replace $\mathbb{W}_0$ in (1.12a) by that in (1.18a). Then, in the case that $\phi \in \mathbb{W}_0$ with $p(\phi) = 0$, the only assumption we can make may be that $\phi_4 \not\equiv 0$, from which Lemma 1.3 alone cannot guarantee that P from (1.13) satisfies

$$p(\Phi_t\phi) = \min\{\min_{x\in\bar{\Omega}} u_2(x,t), \min_{x\in\bar{\Omega}} u_3(x,t), \min_{x\in\bar{\Omega}} u_4(x,t)\} > 0 \quad \forall\, t > 0.$$

In the case of the EVD model (1.4), an analogous situation is again slightly improved as follows (see [18, p. 24]).

> The hypothesis that $\phi \in \mathbb{W}_0$ implies $\phi_2(\cdot) \not\equiv 0$. By the identical argument in the proof of [the DFE being a weak repeller], it follows that $u_2(x,t,\phi) > 0$ which implies $u_5(x,t,\phi) > 0$ by [Lemma 1.4] which leads to that $u_4(x,t,\phi) > 0$ for all $x \in \overline{\Omega}, t > 0$ as well. Moreover, suppose that for any $t_0 > 0, u_3(\cdot,t_0,\phi) \equiv 0$. Then, by [(1.4c)], we obtain $\partial_t R|_{t=t_0} > 0, R(\cdot,t_0) \equiv 0$ for all $x \in \Omega$ and hence contradiction to [the non-negativity of the solution]. Therefore, for all $t_0 > 0, u_3(\cdot,t_0,\phi) \not\equiv 0$. By [Lemma 1.4], this implies $R(x,t) > 0$ for all $x \in \overline{\Omega}, t > 0$. Hence, $p(\Phi_t(\phi)) > 0$ for all $t > 0$.

The extra argument presented here allowed the only assumption of $\phi_2 \not\equiv 0$ to verify that p from (1.16) satisfies

$$p(\Phi_t\phi) = \min\{\min_{x\in\bar{\Omega}} u_2(x,t), \min_{x\in\bar{\Omega}} u_3(x,t), \min_{x\in\bar{\Omega}} u_4(x,t), \min_{x\in\bar{\Omega}} u_5(x,t)\} > 0 \quad \forall\, t > 0;$$

thus, it is a generalized distance function. Nevertheless, extending this argument to $\mathbb{W}_0$ from (1.19a) seems difficult; in particular, starting from only an assumption of $\phi_5 \not\equiv 0$, it is not clear if we can guarantee that all of u_2, u_3, u_4 and u_5 are strictly positive on $\bar{\Omega}$ for all $t > 0$. Again, Lemma 1.4 does not immediately guarantee that even one of them is strictly positive, even u_5 itself.

2. Statement of main results and new ideas to overcome difficulty

2.1. Statement of main results

In this section we present our main results: Theorems 2.1 and 2.2. In sharp contrast to Remark 1.3, starting from the non-vanishing assumption at any $t_0 \geq 0$ of the solution component of the non-diffusive equation, strict positivity of all infection-related compartments for all $x \in \bar{\Omega}$ and all $t > t_0$ can be shown (see Propositions 3.1 and 4.1). This ability of a solution to a non-diffusive equation such that it does not vanish at time t_0 to induce strict positivity on $\bar{\Omega}$ for all $t > 0$ on not only itself but even other infection-related components is surprising and it can be derived by taking advantage of the structure of the infectious disease models. Therefore, our approach is general and may prove to be suitable for many other models, such as the COVID-19 model in [26]. Consequently, Theorems 2.1 and 2.2 improve Theorems 1.1 and 1.2 on systems (1.2) and (1.4) by replacing $\mathbb{W}_0$ with those of (1.18a) and (1.19a), respectively.

Theorem 2.1. *Define* $\mathbb{W}_0$ *and* $\partial\mathbb{W}_0$ *by* (1.18)*, and p by* (1.13)*. If* $\mathcal{R}_0$ *defined by [17, Eq (27)] satisfies* $\mathcal{R}_0 > 1$*, then* (1.2) *admits at least one positive steady state* $\hat{u}$ *and there exists* $\sigma > 0$ *such that for any* $\phi \in X^+$ *such that* $\phi_2 \not\equiv 0$ *or* $\phi_4 \not\equiv 0$*,* (1.14) *is satisfied so that uniform persistence w.r.t.* $(\mathbb{W}_0, \partial\mathbb{W}_0, p)$ *holds for* $\mathbb{W}_0$ *and* $\partial\mathbb{W}_0$ *defined in* (1.18)*.*

Theorem 2.2. *Let $D_1 = D_2 =: \bar{D}$ in* (1.4)*. Define $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ by* (1.19)*, and p by* (1.16)*. If $\mathcal{R}_0$ defined by [18, Eq (34)] satisfies $\mathcal{R}_0 > 1$, then* (1.4) *admits at least one positive steady state $\hat{u}$ and there exists $\sigma > 0$ such that for any $\phi \in X^+$ such that $\phi_2 \not\equiv 0$ or $\phi_4 \not\equiv 0$ or $\phi_5 \not\equiv 0$,* (1.17) *is satisfied so that uniform persistence w.r.t.* $(\mathbb{W}_0, \partial\mathbb{W}_0, p)$ *holds for $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ defined in* (1.19).

In comparison of Theorems 2.1 and 2.2 with Theorems 1.1 and 1.2, there is no direct improvement concerning the existence of a positive steady state $\hat{u}$. Nevertheless, its proof relies on the latter's uniform persistence results and therefore definitions of $\mathbb{W}_0$; thus, we included them for completeness.

Before we proceed, we emphasize the novelties of this work and make comments.

(1) Obviously, the $\mathbb{W}_0$ in (1.12a) is strictly embedded in the $\mathbb{W}_0$ from (1.18a); the same statement can be made concerning (1.15a) and (1.19a). Thus, both Theorems 2.1 and 2.2 are direct improvements of Theorems 1.1 and 1.2 according to Remark 1.2.

(2) Theorems 2.1 and 2.2 are not only mathematically but also biologically meaningful. In words, Theorem 2.1 states that, in the case $\mathcal{R}_0 > 1$, if initially there are any concentration of AI virus in water, then the disease persists. Similarly, Theorem 2.2 states that, in the case $\mathcal{R}_0 > 1$, if initially there is any deceased individuals due to EVD still in the process of caring (recall $\frac{1}{b}$ from Table 2), then the disease persists.

The risks of infections from concentration of AI viruses in water to susceptible birds, as well as from the deceased individuals due to EVD to susceptible individuals during the burial ceremonies have been documented in the past, e.g., [21–23] in the case of the AI viruses and [24,28,30] in the case of the EVD. Theorems 2.1 and 2.2 are results with rigorous proofs that support the severity of such risks by predicting uniform persistence of the diseases even in the initial absence of other infection-related compartments.

(3) Besides the actual results stated in Theorems 2.1 and 2.2, their proofs offer interesting features to researchers on PDEs in general. As we pointed out in Remark 1.3, it is false that a solution to an equation without diffusion has a property similar to Lemma 1.3 (1) or Lemma 1.4 (1). However, in the case of the AI model, u_4 non-vanishing at $t_0 \geq 0$ can, together with the fact that $u_1(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$ due to Lemma 1.3 (2), show that $u_2(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$, and this in turn actually shows that $u_4(x,t) > 0$ by Lemma 1.3 (3) (see Proposition 3.1 (1)f). Similarly in the case of EVD model, u_5 non-vanishing at $t_0 \not\equiv 0$ can, together with the fact that $u_1(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$ due to Lemma 1.4 (2), show that $u_2(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$, and this, in turn shows that $u_5(x,t) > 0$ by Lemma 1.4 (3) (see Proposition 4.1 (1)).

In subsequent sections, we prove Theorems 2.1 and 2.2 in order; due to their similarities, we will elaborate in detail for the proof of Theorem 2.1, while we will give main ideas only in the proof of Theorem 2.2. Throughout the rest of the work, for both $(\mathbb{W}_0, \partial\mathbb{W}_0)$ defined by (1.18) or (1.19) and Φ_t, the solution semiflow for (1.2) or (1.4) that is guaranteed to exist by List 1.1, we define

$$M_\partial := \{\phi \in \partial\mathbb{W}_0 : \Phi_t(\phi) \in \partial\mathbb{W}_0 \ \forall\, t \geq 0\} \tag{2.1}$$

and $\omega(\phi)$ to be the omega limit set of the orbit $O^+(\phi) := \{\Phi_t(\phi) : t \geq 0\}$. Moreover, we denote the Kuratowski measure of non-compactness by κ (see Definition A.1 for precise definition). After proofs, we follow up with the conclusion; moreover, for readers' convenience, we leave some preliminaries and computations in the Appendix.

3. Proof of Theorem 2.1

Besides the results from List 1.1, Lemmas 1.1 and 1.3, let us recall some results from [17] that will be needed to prove Theorem 2.1.

Lemma 3.1. ([17, Lemmas 3.5 and 3.6]) The semiflow $\Phi_t : X^+ \mapsto X^+$ defined by $\Phi_t(\phi) = u(t)$ guaranteed by List 1.1 (a) is a κ-contraction on X^+; i.e., there exists a constant $\tilde{c} > 0$ such that $\kappa(\Phi_t B) \leq e^{-\tilde{c}t}\kappa(B)$ for all bounded sets $B \subset X^+$ such that $\kappa(B) > 0$ so that $\lim_{t\to\infty} \kappa(\Phi_t B) = 0$. Moreover, Φ_t is κ-condensing and consequently asymptotically smooth. Finally, Φ_t admits a global connected attractor A.

The following result is an improvement of Lemma 1.3. In particular, it shows that a solution to the non-diffusive equation, namely u_4 that solves (1.2d), starting from some $t_0^4 \geq 0$ at which it does not vanish, does indeed become and remain strictly positive for all $t > t_0$. Not only that, it has the ability to spread such strict positivity to the other infection-related compartment, namely u_2.

Proposition 3.1. *Suppose that $u(x, t, \phi)$ is the solution to* (1.2) *such that $u(\cdot, 0, \phi) = \phi(\cdot) \in X^+$.*

(1) If there exists $t_0^i \geq 0$ such that $u_i(\cdot, t_0^i, \phi) \not\equiv 0$ for $i = 2$ or $i = 4$, then $u_k(x, t, \phi) > 0$ for all $(x, t) \in \bar{\Omega} \times (t_0^i, \infty)$ and all $k \in \{2, 4\}$.

(2) Moreover, $\Phi_t(\mathbb{W}_0) \subseteq \mathbb{W}_0$ for all $t \geq 0$.

Proof of Proposition 3.1. We assume the existence of t_0^i by hypothesis. First, consider the case $i = 2$ so that $u_2(\cdot, t_0^2, \phi) \not\equiv 0$. Lemma 1.3 (1) immediately implies that $u_2(x, t, \phi) > 0$ for all $x \in \bar{\Omega}$ and all $t > t_0^2$; in turn, Lemma 1.3 (3) now implies that $u_4(x, t, \phi) > 0$ for all $x \in \bar{\Omega}$ and $t > t_0^2$.

Next, consider the other case of $u_4(\cdot, t_0^4, \phi) \not\equiv 0$. This implies that there exists $x^* \in \bar{\Omega}$ such that $u_4(x^*, t_0^4, \phi) > 0$. By continuity in space and time, we can assume that $x^* \in \Omega$ and $t_0^4 > 0$ by relabeling if necessary. Then, suppose that $u_2(\cdot, t_0^4, \phi) \equiv 0$ on $\bar{\Omega}$. Substituting this in (1.2b) and referencing the strict positivity of u_1 for all $x \in \bar{\Omega}$ and all $t > 0$ from Lemma 1.3 (2), we see

$$\partial_t u_2(x^*, t_0^4) = \beta_2 u_1(x^*, t_0^4) u_4(x^*, t_0^4) > 0.$$

By the same argument in the proof of Lemma 1.4, this contradicts the non-negativity of u_2. Hence, instead, we must have $u_2(\cdot, t_0^4, \phi) \not\equiv 0$. Thus, by Lemma 1.3 (1) we have $u_2(x, t, \phi) > 0$ for all $x \in \bar{\Omega}$ and all $t > t_0^4$. In turn, this implies $u_4(x, t, \phi) > 0$ for all $(x, t) \in \bar{\Omega} \times (t_0^4, \infty)$ by Lemma 1.3 (3). This completes the proof of Proposition 3.1 (1).

Lastly, we will show that $\Phi_t(\mathbb{W}_0) \subseteq \mathbb{W}_0$ for all $t \geq 0$; the case $t = 0$ is immediate because Φ_0 is an identity and thus we focus on $t > 0$. In fact, the proof is an immediate consequence of Proposition 3.1 (1) that we just verified. Let $\phi \in \mathbb{W}_0$ so that $\phi_2 \not\equiv 0$ or $\phi_4 \not\equiv 0$. Proposition 3.1 (1) immediately implies that $u_2(x, t, \phi)$ and $u_4(x, t, \phi)$ are both strictly positive on $\bar{\Omega} \times (0, \infty)$, and this is more than enough to conclude that $\Phi_t(\phi) \in \mathbb{W}_0$ for all $t > 0$. This concludes the proof of Proposition 3.1 (2).

The following result does not necessarily rely on Proposition 3.1; nevertheless, because we redefined $\mathbb{W}_0$ and consequently $\partial\mathbb{W}_0$ and M_∂ in (1.18) and (2.1) differently from [17], we need to reprove it.

Proposition 3.2. *The omega limit set of* (1.2) *satisfies $\omega(\phi) = \{(m_A^*, 0, 0, 0)\}$ for all $\phi \in M_\partial$.*

Proof of Proposition 3.2. Let $\phi \in M_\partial$ so that we have $\Phi_t(\phi) \in \partial\mathbb{W}_0$ for all $t \geq 0$. Thus, $u_2(\cdot, t, \phi) \equiv u_4(\cdot, t, \phi) \equiv 0$ for all $t \geq 0$. Then, from (1.2c) we see

$$\partial_t u_3 = D\Delta u_3 - (\eta + d)u_3 \quad \forall\ x \in \Omega$$

and thus $\lim_{t\to\infty} u_3(x, t, \phi) = 0$ uniformly for $x \in \bar{\Omega}$ by Lemma A.2. Therefore, (1.2a) is asymptotic to

$$\partial_t Y = D\Delta Y + \lambda - dY \quad \forall\ x \in \Omega,$$

and hence by the theory of asymptotically autonomous semiflows (see for example [33, Corollary 4.3]) we have $\lim_{t\to\infty} u_1(x, t, \phi) = m^*_{\mathrm{A}}$ uniformly for $x \in \bar{\Omega}$. Thus, we have $\lim_{t\to\infty} u(x, t, \phi) = (m^*_{\mathrm{A}}, 0, 0, 0)$, and it follows that $\omega(\phi) = \{(m^*_{\mathrm{A}}, 0, 0, 0)\}$ for all $\phi \in M_\partial$.

The next result is the verification that the DFE is a uniform weak repeller for (1.2) that we described in Section 1.3 to be difficult with choice of $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ in (1.18).

Proposition 3.3. *If $\mathcal{R}_0 > 1$, then the DFE $(m^*_A, 0, 0, 0)$ is a uniform weak repeller for $\mathbb{W}_0$ in the sense that there exists $\delta_0 > 0$ such that*

$$\limsup_{t\to\infty} \|\Phi_t(\phi) - (m^*_A, 0, 0, 0)\|_{C(\bar{\Omega})} \geq \delta_0\ \forall\ \phi \in \mathbb{W}_0. \tag{3.1}$$

Proof of Proposition 3.3. By hypothesis, $\mathcal{R}_0 > 1$ so that by Lemma 1.1 we have $\theta(m^*_{\mathrm{A}}) > 0$. Suppose for purposes of contradiction that there exists $\phi^* \in \mathbb{W}_0$ such that for all $\delta_0 > 0$, and hence for $\delta_0 \in (0, m^*_{\mathrm{A}})$, we have

$$\limsup_{t\to\infty} \|\Phi_t(\phi^*) - (m^*_{\mathrm{A}}, 0, 0, 0)\|_{C(\bar{\Omega})} < \delta_0. \tag{3.2}$$

Then, there exists $t_1 > 0$ sufficiently large so that $u_1(x, t, \phi) > m^*_{\mathrm{A}} - \delta_0$ for all $(x, t) \in \bar{\Omega} \times [t_1, \infty)$. Thus, (1.2b) satisfies

$$\partial_t u_2 \geq D\Delta u_2 + \beta_1(m^*_{\mathrm{A}} - \delta_0)u_2 + \beta_2(m^*_{\mathrm{A}} - \delta_0)u_4 - (\gamma + d)u_2 \ \ \forall\ (x, t) \in \Omega \times [t_1, \infty).$$

We consider simultaneously

$$\begin{cases} \partial_t u_2 \geq D\Delta u_2 + \beta_1(m^*_{\mathrm{A}} - \delta_0)u_2 + \beta_2(m^*_{\mathrm{A}} - \delta_0)u_4 - (\gamma + d)u_2, \ x \in \Omega, \\ \partial_t u_4 = \alpha u_2 - c(x)u_4, \ x \in \Omega, \\ \nabla u_2 \cdot \nu = \nabla u_4 \cdot \nu = 0, \ x \in \partial\Omega, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \partial_t \hat{u}_2 = D\Delta \hat{u}_2 + \beta_1(m^*_{\mathrm{A}} - \delta_0)\hat{u}_2 + \beta_2(m^*_{\mathrm{A}} - \delta_0)\hat{u}_4 - (\gamma + d)\hat{u}_2, \ x \in \Omega, \\ \partial_t \hat{u}_4 = \alpha \hat{u}_2 - c(x)\hat{u}_4, \ x \in \Omega, \\ \nabla \hat{u}_2 \cdot \nu = \nabla \hat{u}_4 \cdot \nu = 0, \ x \in \partial\Omega, \end{cases} \tag{3.4}$$

for $t \geq t_1$ where ν is the outward unit normal vector on $\partial\Omega$. Consider them to have the same values at t_1; i.e., $(u_2, u_4)(\cdot, t_1) = (\hat{u}_2, \hat{u}_4)(\cdot, t_1)$. The second equations of (3.3) and (3.4) can be solved directly as

$$u_4(\cdot, t) = e^{-c(\cdot)(t-t_1)}u_4(\cdot, t_1) + \int_{t_1}^{t} \alpha u_2(\cdot, s)e^{c(\cdot)(s-t)}ds, \tag{3.5a}$$

$$\hat{u_4}(\cdot,t) = e^{-c(\cdot)(t-t_1)}\hat{u_4}(\cdot,t_1) + \int_{t_1}^{t} \alpha\hat{u_2}(\cdot,s)e^{c(\cdot)(s-t)}ds. \tag{3.5b}$$

Thus, we reduce (3.3) and (3.4) to the following systems for $t \geq t_1$:

$$\begin{cases} \partial_t u_2 \geq D\Delta u_2 + \beta_1(m_A^* - \delta_0)u_2 \\ \qquad + \beta_2(m_A^* - \delta_0)\left(e^{-c(t-t_1)}u_4(t_1) + \int_{t_1}^{t} \alpha u_2(s)e^{c(s-t)}ds\right) - (\gamma + d)u_2 \;\; \forall\, x \in \Omega, \\ \nabla u_2 \cdot \nu = 0 \;\; \forall\, x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} \partial_t \hat{u_2} = D\Delta \hat{u_2} + \beta_1(m_A^* - \delta_0)\hat{u_2} \\ \qquad + \beta_2(m_A^* - \delta_0)\left(e^{-c(t-t_1)}\hat{u_4}(t_1) + \int_{t_1}^{t} \alpha \hat{u_2}(s)e^{c(s-t)}ds\right) - (\gamma + d)\hat{u_2} \;\; \forall\, x \in \Omega, \\ \nabla \hat{u_2} \cdot \nu = 0 \;\; \forall\, x \in \partial\Omega. \end{cases}$$

By the comparison principle ([32, Theorem 7.3.4]), it then follows that

$$u_2(x,t) \geq \hat{u_2}(x,t) \;\; \forall\, (x,t) \in \bar{\Omega} \times [t_1, \infty). \tag{3.6}$$

Concerning the second equations of (3.3) and (3.4), (3.6) and (3.5) show that $u_4(x,t) \geq \hat{u_4}(x,t)$ for all $(x,t) \in \bar{\Omega} \times [t_1,\infty)$. As $m_A^* - \delta_0 > 0$, we may rely on Lemma 1.1 with $H \equiv m_A^* - \delta_0$ to deduce that the eigenvalue problem corresponding to (3.4) has principle eigenvalue $\theta(m_A^* - \delta_0)$ with corresponding eigenfunction $\tilde{\psi} := (\tilde{\psi_2}, \tilde{\psi_4}) \gg 0$. Because $\theta(m_A^*) > 0$, by taking δ_0 smaller if needed we obtain $\theta(m_A^* - \delta_0) > 0$. Now, by assumption, we have $\phi^* \in \mathbb{W}_0$ so that $\phi_i^* \not\equiv 0$ for $i = 2$ or $i = 4$. By Proposition 3.1, this implies that $u_k(x,t,\phi^*) > 0$ for all $(x,t) \in \bar{\Omega} \times (0,\infty)$ for both $k \in \{2,4\}$. Thus, there exists $\epsilon_0 > 0$ sufficiently small such that

$$(u_2(x,t_1,\phi^*), u_4(x,t_1,\phi^*)) \geq \epsilon_0 e^{\theta(m_A^* - \delta_0)t_1}\tilde{\psi}.$$

Finally, as $\epsilon_0 e^{\theta(m_A^* - \delta_0)(t-t_1)}\tilde{\psi}$ solves (3.4), we see

$$(u_2(x,t,\phi^*), u_4(x,t,\phi^*)) \geq \epsilon_0 e^{\theta(m_A^* - \delta_0)(t-t_1)}\tilde{\psi} \;\; \forall\, (x,t) \in \bar{\Omega} \times [t_1,\infty).$$

Since $\theta(m_A^* - \delta_0) > 0$ and $\tilde{\psi} \gg 0$, it follows that $u_2(x,t,\phi^*), u_4(x,t,\phi^*) \to \infty$ as $t \to \infty$, which contradicts (3.2), thereby proving our claim.

Our next task is a verification that p defined by (1.13) is a generalized distance function for Φ_t that we described to be difficult in Section 1.3 with our choice of $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ in (1.18) for the AI model (1.2) .

Proposition 3.4. *Define p by* (1.13). *If $\mathcal{R}_0 > 1$, then*

$$p^{-1}((0,\infty)) \subset \mathbb{W}_0; \tag{3.7}$$

moreover, p is a generalized distance function for Φ_t.

Proof of Proposition 3.4. First, if $\psi \in p^{-1}((0,\infty))$, then $p(\psi) \in (0,\infty)$ so that clearly $\psi \in \mathbb{W}_0$ by (1.18), concluding (3.7).

Next, recall from Definition 1.2 that, in order to verify that p is a generalized distance function for Φ_t, we must prove that

$$p(\Phi_t(\phi)) = \min\{\min_{x\in\bar{\Omega}} u_2(x,t), \min_{x\in\bar{\Omega}} u_3(x,t), \min_{x\in\bar{\Omega}} u_4(x,t)\} > 0 \quad \forall\, t > 0$$

whenever either $p(\phi) = 0$ and $\phi \in \mathbb{W}_0$ or $p(\phi) > 0$.

First, suppose $p(\phi) = 0$ and $\phi \in \mathbb{W}_0$. From (1.18), we deduce that $\phi_2 \not\equiv 0$ or $\phi_4 \not\equiv 0$. By Proposition 3.1, we see that $u_2(x,t,\phi), u_4(x,t,\phi) > 0$ for all $(x,t) \in \bar{\Omega}\times(0,\infty)$. We wish to prove that $u_3(x,t,\phi) > 0$ for all $(x,t) \in \bar{\Omega}\times(0,\infty)$. Due to Lemma 1.3 (1), we see that this is achieved if we verify that $u_3(t) \not\equiv 0$ for all $t > 0$. Thus, suppose that $u_3(\cdot,t_0,\phi) \equiv 0$ for some $t_0 > 0$. Then, (1.2c) gives us

$$\partial_t u_3(x,t_0) = \gamma u_2(x,t_0) > 0 \;\; \forall\, x \in \Omega,$$

allowing us to find $\epsilon \in (0,t_0)$ sufficiently small such that $u_3(x,t_0-\epsilon) < 0$ for any $x \in \Omega$, which contradicts the non-negativity of the solution. Hence, instead, we must have $u_3(\cdot,t,\phi) \not\equiv 0$ for all $t > 0$ due to the arbitrariness of $t_0 > 0$. Therefore, we have proven that $u_i(x,t,\phi) > 0$ for all $x \in \bar{\Omega}$, all $t > 0$, and all $i \in \{2,3,4\}$; by (1.13), this implies $p(\Phi_t(\phi)) > 0$ for all $t > 0$.

Finally, suppose that $p(\phi) > 0$ so that

$$\min\{\min_{x\in\bar{\Omega}} \phi_2(x), \min_{x\in\bar{\Omega}} \phi_3(x), \min_{x\in\bar{\Omega}} \phi_4(x)\} > 0$$

by (1.13). This implies by Lemma 1.3 (1) that $u_2(x,t,\phi), u_3(x,t,\phi) > 0$ for all $x \in \bar{\Omega}$ and all $t > 0$; consequently, by Lemma 1.3 (3), we have $u_4(x,t,\phi) > 0$ for all $x \in \bar{\Omega}$ and all $t > 0$. Thus, we have that $p(\Phi_t(\phi)) > 0$ for all $t > 0$. Hence, we conclude that p is indeed a generalized distance function for Φ_t.

At last, we are ready to conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. With all the results we have obtained thus far, the proof of Theorem 2.1 follows the same line of reasoning used in previous works (see [17,18]). By Proposition 3.2 we know that any bounded orbit of Φ_t in M_∂ converges to the DFE $(m^*_{\mathrm{A}},0,0,0)$, which is isolated in X^+. If we denote the stable set of the DFE by $W^s((m^*_{\mathrm{A}},0,0,0))$, we now see that $W^s((m^*_{\mathrm{A}},0,0,0)) \cap \mathbb{W}_0 = \emptyset$. Therefore, considering Proposition 3.2, it follows by [31, Lemma 3] (see also [16, Theorem 1.3.2]) that there exists $\sigma > 0$ such that for all compact chain transitive sets L such that $L \not\subset \{(m^*_{\mathrm{A}},0,0,0)\}$, we have $\min_{\phi\in L} p(\phi) > \sigma$. By Lemma 1.3 (2), taking $\sigma > 0$ smaller if necessary to satisfy $\sigma \le \frac{\lambda}{\Upsilon_A}$ for Υ_A from (1.21), we see then that

$$\liminf_{t\to\infty} u_i(\cdot,t,\phi) \ge \sigma \quad \forall\, \phi \in \mathbb{W}_0,\ \forall\, i \in \{1,2,3,4\}, \tag{3.8}$$

which is precisely (1.14). Finally, by Lemma 3.1, we know that $\Phi_t : X^+ \to X^+$ has a global connected attractor A, so from [34, Theorem 3.7 and Remark 3.10], we see that $\Phi_t : \mathbb{W}_0 \to \mathbb{W}_0$ also has a global attractor A_0. Because $\Phi_t(\mathbb{W}_0) \subseteq \mathbb{W}_0$ from Proposition 3.1 (2), and Φ_t is also κ-condensing by Lemma 3.1, it follows from [34, Theorem 4.7] that Φ_t has an equilibrium $\tilde{u} \in A_0$ and hence $\tilde{u} \in \mathbb{W}_0$. It follows immediately from (3.8) that $\tilde{u}$ is a positive steady state of (1.2). This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

Besides the results from List 1.1, Lemmas 1.2 and 1.4, let us recall some results from [18] that will be needed to prove Theorem 2.2.

Lemma 4.1. ([18, Propositions 4.5 and 4.6 and their proof]) Let $D_1 = D_2 =: \bar{D}$ in (1.4). Then, its semiflow $\Phi_t : X^+ \mapsto X^+$ defined by $\Phi_t(\phi) = u(t)$ guaranteed by List 1.1 (a) is a κ-contraction of order e^{-bt} on X^+; i.e., $\kappa(\Phi_t B) \leq e^{-bt}\kappa(B)$ for all bounded sets $B \subset X^+$ such that $\kappa(B) > 0$ so that $\lim_{t\to\infty} \kappa(\Phi_t B) = 0$. Moreover, Φ_t is κ-condensing and consequently asymptotically smooth. Finally, Φ_t admits a global connected attractor A.

The following result is an improvement of Lemma 1.4.

Proposition 4.1. *Let* $D_1 = D_2 =: \bar{D}$ *in* (1.4). *Suppose that* $u(x, t, \phi)$ *is the solution to* (1.4) *such that* $u(\cdot, 0, \phi) = \phi(\cdot) \in X^+$.

(1) If there exists $t_0^i \geq 0$ *such that* $u_i(\cdot, t_0^i, \phi) \not\equiv 0$ *for any* $i \in \{2, 4, 5\}$, *then* $u_k(x, t, \phi) > 0$ *for all* $(x, t) \in \bar{\Omega} \times (t_0^i, \infty)$ *and all* $k \in \{2, 4, 5\}$.

(2) Moreover, $\Phi_t(\mathbb{W}_0) \subseteq \mathbb{W}_0$ *for all* $t \geq 0$.

Proof of Proposition 4.1. The proof is similar to that of Proposition 3.1. First, consider the case $u_2(\cdot, t_0^2) \not\equiv 0$. It immediately follows by Lemma 1.4 (1) that $u_2(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times (t_0^2, \infty)$ and hence by Lemma 1.4 (3) that $u_5(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times (t_0^2, \infty)$. Moreover, suppose that $u_4(\cdot, t_0^2) \equiv 0$. The hypothesis that $u_2(\cdot, t_0^2) \not\equiv 0$ implies that there exists $x^* \in \bar{\Omega}$ at which $u_2(x^*, t_0^2) > 0$, relabeling if necessary. By continuity in space and time, we may assume that $x^* \in \Omega$ and $t_0^2 > 0$. Then, we have from (1.4d)

$$\partial_t u_4(x^*, t_0^2) = \xi u_2(x^*, t_0^2) + \alpha u_5(x^*, t_0^2) > 0,$$

indicating that we can find $\epsilon \in (0, t_0^2)$ sufficiently small so that $u_4(x, t_0^2 - \epsilon) < 0$, which is a contradiction. Therefore, $u_4(\cdot, t_0^2) \not\equiv 0$, and by Lemma 1.4 (1) this implies that $u_4(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times (t_0^2, \infty)$.

Now, consider the case $u_i(\cdot, t_0^i) \not\equiv 0$ for $i = 4$ or $i = 5$. This implies that there exists $x^* \in \bar{\Omega}$ such that $u_i(x^*, t_0^i) > 0$. By spatial and temporal continuity again, we can assume that $x^* \in \Omega$ and $t_0^i > 0$. Suppose that $u_2(\cdot, t_0^i) \equiv 0$. Then, from (1.4b)

$$\partial_t u_2(x^*, t_0^i) = (\beta_2 u_5(x^*, t_0^i) + \lambda u_4(x^*, t_0^i)) u_1(x^*, t_0^i) > 0,$$

where the inequality follows by the hypothesis that at least one of u_4 or u_5 at (x^*, t_0^i) is strictly positive, and $u_1(x, t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$ due to Lemma 1.4 (2). Therefore, we can find $\epsilon \in (0, t_0^i)$ sufficiently small so that $u_2(x^*, t_0^i - \epsilon) < 0$, which is again a contradiction. Hence, $u_2(\cdot, t_0^i) \not\equiv 0$, which implies by Lemma 1.4 (1) that $u_2(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times (t_0^i, \infty)$. By the proof in the case $u_2(\cdot, t_0^2) \not\equiv 0$, we now see that $u_j(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times (t_0^i, \infty)$ and $j \in \{4, 5\} \setminus \{i\}$ too. All necessary cases have been demonstrated; hence, Proposition 4.1 (1) was proven.

Lastly we must show that $\Phi_t(\mathbb{W}_0) \subseteq \mathbb{W}_0$ for all $t \geq 0$; it suffices to prove the case $t > 0$ as Φ_0 is an identity. Let $\phi \in \mathbb{W}_0$ so that $\phi_2 \not\equiv 0$ or $\phi_4 \not\equiv 0$ or $\phi_5 \not\equiv 0$. Proposition 4.1 (1) implies that $u_i(x, t, \phi) > 0$ on $\bar{\Omega} \times (0, \infty)$ for all $i \in \{2, 4, 5\}$. This is more than enough to deduce that $\Phi_t(\phi) \in \mathbb{W}_0$ for all $t > 0$, concluding the proof of Proposition 4.1 (2).

The following result does not necessarily rely on Proposition 4.1; nevertheless, because we redefined $\mathbb{W}_0$ and consequently $\partial\mathbb{W}_0$ and M_∂ in (1.19) and (2.1) differently from [18], it needs to be proven.

Proposition 4.2. *Let $D_1 = D_2 =: \bar{D}$ in (1.4). Then, the omega limit set of (1.4) satisfies $\omega(\phi) = \{(m_E^*, 0, 0, 0, 0)\}$ for all $\phi \in M_\partial$.*

Proof of Proposition 4.2. The proof is similar to that of Proposition 3.2; thus, we leave it in the Appendix for completeness.

The next result is a verification that the DFE is a uniform weak repeller for (1.4) that we described in Section 1.3 to be difficult with the choice of $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ in (1.18).

Proposition 4.3. *If $D_1 = D_2 =: \bar{D}$ and $\mathcal{R}_0 > 1$, then the DFE $(m_E^*, 0, 0, 0, 0)$ is a uniform weak repeller for $\mathbb{W}_0$ in the sense that there exists $\delta_0 > 0$ such that*

$$\limsup_{t\to\infty} ||\Phi_t(\phi) - (m_E^*, 0, 0, 0, 0)||_{C(\bar{\Omega})} \geq \delta_0 \ \forall \ \phi \in \mathbb{W}_0. \tag{4.1}$$

Proof of Proposition 4.3. Proof is similar to that of Proposition 3.3. By hypothesis, $\mathcal{R}_0 > 1$, so by Lemma 1.2 we have that $\theta(m_{\mathrm{E}}^*) > 0$. Suppose for purposes of contradiction that there exists $\phi^* \in \mathbb{W}_0$ such that, for all $\delta_0 > 0$, and hence for $\delta_0 \in (0, m_{\mathrm{E}}^*)$, we have

$$\limsup_{t\to\infty} ||\Psi_t(\phi^*) - (m_{\mathrm{E}}^*, 0, 0, 0, 0)||_{C(\bar{\Omega})} < \delta_0. \tag{4.2}$$

Then, there exists $t_1 > 0$ sufficiently large so that $u_1(x, t, \phi) > m_{\mathrm{E}}^* - \delta_0$ for all $(x, t) \in \bar{\Omega} \times [t_1, \infty)$. Thus, (1.4b) leads us to consider simultaneously

$$\begin{cases} \partial_t u_2 \geq \bar{D}\Delta u_2 + (\beta_1 u_2 + \beta_2 u_5 + \lambda u_4)(m_{\mathrm{E}}^* - \delta_0) - (\mu + \delta + \gamma)u_2, \ x \in \Omega, \\ \partial_t u_4 = D_4 \Delta u_4 + \xi u_2 + \alpha u_5 - \eta u_4, \ x \in \Omega, \\ \partial_t u_5 = \delta u_2 - b u_5, \ x \in \Omega, \\ \nabla u_2 \cdot \nu = \nabla u_4 \cdot \nu = \nabla u_5 \cdot \nu = 0, \ x \in \partial\Omega, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \partial_t \hat{u}_2 = \bar{D}\Delta \hat{u}_2 + (\beta_1 \hat{u}_2 + \beta_2 \hat{u}_5 + \lambda \hat{u}_4)(m_{\mathrm{E}}^* - \delta_0) - (\mu + \delta + \gamma)\hat{u}_2, \ x \in \Omega, \\ \partial_t \hat{u}_4 = D_4 \Delta \hat{u}_4 + \xi \hat{u}_2 + \alpha \hat{u}_5 - \eta \hat{u}_4, \ x \in \Omega, \\ \partial_t \hat{u}_5 = \delta \hat{u}_2 - b \hat{u}_5, \ x \in \Omega, \\ \nabla \hat{u}_2 \cdot \nu = \nabla \hat{u}_4 \cdot \nu = \nabla \hat{u}_5 \cdot \nu = 0, \ x \in \partial\Omega, \end{cases} \tag{4.4}$$

with $t \geq t_1$, starting from same values at t_1; i.e., $(u_2, u_4, u_5)(\cdot, t_1) = (\hat{u}_2, \hat{u}_4, \hat{u}_5)(\cdot, t_1)$. Solving for u_5 and $\hat{u}_5$ as

$$u_5(\cdot, t) = u_5(\cdot, t_1)e^{b(t_1 - t)} + \int_{t_1}^{t} \delta u_2(\cdot, s)e^{b(s-t)} ds,$$

$$\hat{u}_5(\cdot, t) = \hat{u}_5(\cdot, t_1)e^{b(t_1 - t)} + \int_{t_1}^{t} \delta \hat{u}_2(\cdot, s)e^{b(s-t)} ds,$$

and using the comparison theorem argument give us

$$u_2(x,t) \geq \hat{u}_2(x,t), u_4(x,t) \geq \hat{u}_4(x,t), \text{ and } u_5(x,t) \geq \hat{u}_5(x,t) \ \ \forall\ (x,t) \in \bar{\Omega} \times [t_1, \infty). \tag{4.5}$$

As $m^*_{\mathrm{E}} - \delta_0 > 0$, we may rely on Lemma 1.2 with $H \equiv m^*_{\mathrm{E}} - \delta_0$ to deduce that the eigenvalue problem corresponding to (4.4) has principle eigenvalue $\theta(m^*_{\mathrm{E}} - \delta_0)$ with corresponding eigenfunction $\tilde{\psi} := (\tilde{\psi}_2, \tilde{\psi}_4, \tilde{\psi}_5) \gg 0$. Because $\theta(m^*_{\mathrm{E}}) > 0$, by taking δ_0 smaller if needed we obtain $\theta(m^*_{\mathrm{E}} - \delta_0) > 0$. By assumption, we have $\phi^* \in \mathbb{W}_0$ so that $\phi^*_i \not\equiv 0$ for $i = 2$ or $i = 4$ or $i = 5$. By Proposition 4.1, this implies that $u_k(x,t,\phi^*) > 0$ for all $(x,t) \in \bar{\Omega} \times (0,\infty)$ for all $k \in \{2,4,5\}$. Therefore, there exists $\epsilon_0 > 0$ sufficiently small such that

$$(u_2(x,t_1,\phi^*), u_4(x,t_1,\phi^*), u_5(x,t_1,\phi^*)) \geq \epsilon_0 e^{\theta(m^*_{\mathrm{E}} - \delta_0)t_1} \tilde{\psi}.$$

Finally, as $\epsilon_0 e^{\theta(m^*_{\mathrm{E}} - \delta_0)(t-t_1)} \tilde{\psi}$ solves (4.4), we see

$$(u_2(x,t,\phi^*), u_4(x,t,\phi^*), u_5(x,t,\phi^*)) \geq \epsilon_0 e^{\theta(m^*_{\mathrm{E}} - \delta_0)(t-t_1)} \tilde{\psi} \ \ \forall\ (x,t) \in \bar{\Omega} \times [t_1, \infty).$$

Since $\theta(m^*_{\mathrm{E}} - \delta_0) > 0$ and $\tilde{\psi} \gg 0$, it follows that $u_2(x,t,\phi^*), u_4(x,t,\phi^*), u_5(x,t,\phi^*) \to \infty$ as $t \to \infty$, which contradicts (4.2), thereby proving our claim.

Our next task is a verification that p defined by (1.16) is a generalized distance function for Φ_t that we described in Section 1.3 to be difficult with our choice of $\mathbb{W}_0$ and $\partial\mathbb{W}_0$ in (1.19) for the EVD model (1.4).

Proposition 4.4. *Define p by* (1.16). *If $D_1 = D_2 =: \bar{D}$ and $\mathcal{R}_0 > 1$, then*

$$p^{-1}((0,\infty)) \subset \mathbb{W}_0; \tag{4.6}$$

moreover, p is a generalized distance function for Φ_t.

Proof of Proposition 4.4. The proof is similar to that of Proposition 3.4. First, if $\psi \in p^{-1}((0,\infty))$, then $p(\psi) \in (0,\infty)$ so that clearly $\psi \in \mathbb{W}_0$ by (1.19), concluding (4.6).

Next, we must prove that $p(\Phi_t(\phi)) > 0$ for all $t > 0$ whenever either $p(\phi) = 0$ and $\phi \in \mathbb{W}_0$ or $p(\phi) > 0$.

First, suppose $p(\phi) = 0$ and $\phi \in \mathbb{W}_0$. From (1.19), we deduce that $\phi_2 \not\equiv 0$ or $\phi_4 \not\equiv 0$ or $\phi_5 \not\equiv 0$. By Proposition 4.1 we see that $u_2(x,t,\phi), u_4(x,t,\phi), u_5(x,t,\phi) > 0$ for all $(x,t) \in \bar{\Omega} \times (0,\infty)$. To prove that $u_3(x,t,\phi) > 0$ for all $(x,t) \in \bar{\Omega} \times (0,\infty)$, due to Lemma 1.4 (1), it suffices to verify that $u_3(t) \not\equiv 0$ for all $t > 0$. Suppose that $u_3(\cdot, t_0, \phi) \equiv 0$ for some $t_0 > 0$. Then, (1.4c) gives us

$$\partial_t u_3(x,t_0) = \gamma u_2(x,t_0) > 0 \ \ \forall\ x \in \Omega,$$

allowing us to find $\epsilon \in (0, t_0)$ sufficiently small such that $u_3(x, t_0 - \epsilon) < 0$ for any $x \in \Omega$, which contradicts the non-negativity of the solution. Hence, instead, we must have $u_3(\cdot, t, \phi) \not\equiv 0$ for all $t > 0$ due to the arbitrariness of $t_0 > 0$. Therefore, we have proven that $u_i(x,t,\phi) > 0$ for all $x \in \bar{\Omega}$, all $t > 0$ and all $i \in \{2,3,4,5\}$ so that $p(\Phi_t(\phi)) > 0$ for all $t > 0$ by (1.16).

At last, suppose that we have $p(\phi) > 0$. Then, by Lemma 1.4 (1), we can obtain $u_2(x,t,\phi), u_3(x,t,\phi), u_4(x,t,\phi) > 0$ for all $x \in \bar{\Omega}$ and all $t > 0$. Consequently by Lemma 1.4 (3), we have $u_5(x,t,\phi) > 0$ for all $x \in \bar{\Omega}$ and all $t > 0$. Thus, we have that $p(\Phi_t(\phi)) > 0$ for all $t > 0$. Therefore, we conclude that p is indeed a generalized distance function for Φ_t.

As we mentioned, with all the results obtained thus far, the proof of Theorem 2.2 follows the same line of reasoning in the literature, and it is similar to the proof of Theorem 2.1. Thus, we leave this in the Appendix for completeness.

5. Conclusions

As PDE models of infectious diseases continue to attract attention from researchers, taking into account distinct mobilities among the solution components and therefore considering partially diffusive systems of PDEs seems inevitable. In this work, we improved uniform persistence results of such systems of PDEs, namely the AI model from [17] and the EVD model from [18]. To do so, we weaken the assumption of initial data so that any (rather than all) of the infection-related components must be initially non-vanishing. The mathematical novelty is the discovery of the ability of the solution to a non-diffusive equation that does not vanish at time $t_0 \geq 0$ to induce strict positivity on $\bar{\Omega}$ for all $t > t_0$ on not only itself but all other infection-related components (see Propositions 3.1 and 4.1). This takes the place of the minimum principle, which is standard to apply in a fully diffusive system. Specifically, the key in both cases is to make use of the fact that $S(x,t) > 0$ for all $x \in \bar{\Omega}$ and all $t > 0$ and verify that, as a consequence, $I(x,t) > 0$ for all $x \in \bar{\Omega}$ and all $t > 0$. Ultimately, this closes the gap between the uniform persistence results of the fully diffusive systems of PDEs and the partially diffusive systems of PDEs, with such results having previously been known for the former but not the latter. Our approach seems general and applies to other partially diffusive models such as the COVID-19 model that needs to take into account of the environmental modes of transmission.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Yamazaki is supported by the Simons Foundation (962572, KY). The authors thank Prof. Jin Wang for motivating discussions.

Conflict of interest

The authors declare no conflict of interest.

References

1. P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, **180** (2002), 29–48. https://doi.org/10.1016/S0025-5564(02)00108-6
2. H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, *SIAM J. Appl. Math.*, **70** (2009), 188–211. https://doi.org/10.1137/080732870

3. W. Wang, X. Q. Zhao, Basic reproduction numbers for reaction-diffusion epidemic models, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 1652–1673. https://doi.org/10.1137/120872942
4. K. Yamazaki, Global well-posedness of infectious disease models without life-time immunity: the cases of cholera and avian influenza, *Math. Med. Biol.*, **35** (2018), 427–445. https://doi.org/10.1093/imammb/dqx016
5. H. M. Yin, On a reaction-diffusion system modelling infectious diseases without lifetime immunity, *Eur. J. Appl. Math.*, **33** (2021), 803–827. https://doi.org/10.1017/S0956792521000231
6. W. Wang, X. Q. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, *SIAM J. Appl. Math.*, **71** (2011), 147–168. https://doi.org/10.1137/090775890
7. Y. Lou, X. Q. Zhao, A reaction-diffusion malaria model with incubation period in the vector population, *J. Math. Biol.*, **62** (2011), 543–568. https://doi.org/10.1007/s00285-010-0346-8
8. X. Q. Zhao, Global dynamics of a reaction and diffusion model for Lyme disease, *J. Math. Biol.*, **65** (2012), 787–808. https://doi.org/10.1007/s00285-011-0482-9
9. K. Yamazaki, X. Wang, Global well-posedness and asymptotic behavior of solutions to a reaction-convection-diffusion cholera epidemic model, *Discrete Contin. Dyn. Syst. Ser. -B*, **21** (2016), 1297–1316. https://doi.org/10.3934/dcdsb.2016.21.1297
10. K. Yamazaki, X. Wang, Global stability and uniform persistence of the reaction-convection-diffusion cholera epidemic model, *Math. Biosci. Eng.*, **14** (2017), 559–579. https://doi.org/10.3934/mbe.2017033
11. K. Yamazaki, Zika virus dynamics partial differential equations model with sexual transmission route, *Nonlinear Anal. Real World Appl.*, **50** (2019), 290–315. https://doi.org/10.1016/j.nonrwa.2019.05.003
12. S. B. Hsu, J. Jiang, F. B. Wang, On a system of reaction-diffusion equations arising from competition with internal storage in an unstirred chemostat, *J. Differ. Equations*, **248** (2010), 2470–2496. https://doi.org/10.1016/j.jde.2009.12.014
13. S. B. Hsu, F. B. Wang, X. Q. Zhao, Dynamics of a periodically pulsed bio-reactor model with a hydraulic storage zone, *J. Dyn. Differ. Equations*, **23** (2011), 817–842. https://doi.org/10.1007/s10884-011-9224-3
14. J. P. Grover, S. B. Hsu, F. B. Wang, Competition and coexistence in flowing habitats with a hydraulic storage zone, *Math. Biosci.*, **222** (2009), 42–52. https://doi.org/10.1016/j.mbs.2009.08.006
15. S. B. Hsu, F. B. Wang, X. Q. Zhao, Global dynamics of zooplankton and harmful algae in flowing habitats, *J. Differ. Equations*, **255** (2013), 265–297. https://doi.org/10.1016/j.jde.2013.04.006
16. X. Q. Zhao, *Dynamical Systems in Population Biology*, Springer-Verlag, New York, Inc., 2003. https://doi.org/10.1007/978-0-387-21761-1
17. N. K. Vaidya, F. B. Wang, X. Zou, Avian influenza dynamics in wild birds with bird mobility and spatial heterogeneous environment, *Discrete Contin. Dyn. Syst. Ser. -B*, **17** (2012), 2829–2848. https://doi.org/10.3934/dcdsb.2012.17.2829
18. K. Yamazaki, Threshold dynamics of reaction-diffusion partial differential equations model of Ebola virus disease, *Int. J. Biomath.*, **11** (2018), 1850108. https://doi.org/10.1142/S1793524518501085

19. Z. J. Cheng, J. Shan, Novel coronavirus: where we are and what we know, *Infection*, **48** (2020), 155–163. https://doi.org/10.1007/s15010-020-01401-y
20. C. Yang, J. Wang, A mathematical model for the novel coronavirus epidemic in Wuhan, China, *Math. Biosci. Eng.*, **17** (2020), 2708–2724. https://doi.org/10.3934/mbe.2020148
21. J. D. Brown, G. Goekjian, R. Poulson, S. Valeika, D. E. Stallknecht, Avian influenza virus in water: Infectivity is dependent on pH, salinity and temperature, *Vet. Microbiol.*, **136** (2009), 20–26. https://doi.org/10.1016/j.vetmic.2008.10.027
22. V. S. Hinshaw, R. G. Webster, B. Turner, The perpetuation of orthomyxoviruses and paramyxoviruses in Canadian waterfowl, *Can. J. Microbiol.*, **26** (1980), 622–629. https://doi.org/10.1139/m80-108
23. R. G. Webster, M. Yakhno, V. S. Hinshaw, W. J. Bean, K. G. Murti, Intestinal influenza: replication and characterization of influenza viruses in ducks, *Virology*, **84** (1978), 268–278. https://doi.org/10.1016/0042-6822(78)90247-7
24. *World Health Organization*, Ebola virus disease, 2023. Available from: https://www.who.int/news-room/fact-sheets/detail/ebola-virus-disease.
25. L. Evans, *Partial Differential Equations*, American Mathematics Society, Providence, Rhode Island, 1998.
26. R. Covington, S. Patton, E. Walker, K. Yamazaki, Stability analysis on a partially diffusive model of the coronavirus disease of 2019, submitted.
27. W. Puryear, K. Sawatzki, N. Hill, A. Foss, J. J. Stone, L. Doughty, et al., Pathogenic Avian Influenza A(H5N1) virus outbreak in New England Seals, United States, *Emerging Infect. Dis.*, **29** (2023), 786–791. https://doi.org/10.3201/eid2904.221538
28. T. Berge, J. M. S. Luburma, G. M. Moremedi, N. Morris, R. Kondera-Shava, A simple mathematical model for Ebola in Africa, *J. Biol. Dyn.*, **11** (2017), 42–74. https://doi.org/10.1080/17513758.2016.1229817
29. *World Health Organization*, Ebola disease caused by Sudan ebolavirus - Uganda, 2023. Available from: https://www.who.int/emergencies/disease-outbreak-news/item/2023-DON433.
30. C. Freeman, Meet the world's bravest undertakers - Liberia's Ebola burial squad, The Telegraph, 2014. Available from: https://www.telegraph.co.uk/news/worldnews/ebola/11024042/Meet-the-worlds-bravest-undertakers-Liberias-Ebola-burial-squad.html.
31. H. L. Smith, X. Q. Zhao, Robust persistence for semidynamical systems, *Nonlinear Anal. Theory Methods Appl.*, **47** (2001), 6169–6179. https://doi.org/10.1016/S0362-546X(01)00678-2
32. H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, **41** (1995).
33. H. R. Thieme, Convergence results and a Poincaré-Bendixson trichotomy for asymptotically autonomous differential equations, *J. Math. Biol.*, **30** (1992), 755–763. https://doi.org/10.1007/BF00173267
34. P. Magal, X. Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM J. Math. Anal.*, **37** (2005), 251–275. https://doi.org/10.1137/S0036141003439173

35. K. Deimling, *Nonlinear Functional Analysis*, Dover Publications, Inc., Mineola, New York, 1985. https://doi.org/10.1007/978-3-662-00547-7
36. J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, American Mathematics Society, Providence, Rhode Island, 1988. https://doi.org/10.1090/surv/025
37. J. Wang, C. Yang, K. Yamazaki, A partially diffusive cholera model based on a general second-order differential operator, *J. Math. Anal. Appl.*, **501** (2021), 125181. https://doi.org/10.1016/j.jmaa.2021.125181

Appendix

A. Preliminaries

The following well-known fact proved to be useful.

Lemma A.1. *Let $f : U \to \mathbb{R}$ be twice differentiable, where U is an open subset of $\mathbb{R}^n$. If f has a local minimum at $a \in U$, then $\Delta f(a) \geq 0$.*

We recall the definition of Kuratowski measure of non-compactness:

Definition A.1. ([16, p. 3]) Given any metric space Y, the Kuratowski measure of non-compactness for any bounded set B of Y is defined by

$$\kappa(B) := \inf\{r : B \text{ has a finite cover of diameter } r\}.$$

We refer to [16, p. 3], [35, Proposition 7.2] and [36, Lemma 2.3.5] for various properties concerning κ.

Lemma A.2. *([37, Proposition 4.1]) Consider a spatial domain $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ that is bounded with smooth boundary $\partial\Omega$ and the following equation:*

$$\partial_t w(x,t) = \tilde{D}(x)\Delta w(x,t) - (\overline{U}(x) \cdot \nabla)w(x,t) + g(x) - \lambda w(x,t), \tag{A.1a}$$

$$(n \cdot \nabla)w(x,t)|_{\partial\Omega} = 0 \text{ for } t > 0, \text{ and } w(x,0) = \psi(x) \text{ for } x \in \overline{\Omega}, \tag{A.1b}$$

where $\bar{U} \in C^2(\bar{\Omega})$, $\tilde{D}(x)$ is continuous and $\tilde{D}(x) \geq M > 0$ for all $x \in \overline{\Omega}$, $g(x) > 0$ is a continuous function, and n is an outward unit normal vector. Then, for all $\psi \in C(\overline{\Omega}, \mathbb{R}_+)$, there exists a unique positive steady state w^* which is globally attractive in $C(\overline{\Omega}, \mathbb{R})$. Moreover, if $g(x) \equiv g$, then $w^* \equiv \frac{g}{\lambda}$.

B. Proof of Proposition 4.2

Let $\phi \in M_\partial$ so that we have $\Phi_t(\phi) \in \partial\mathbb{W}_0$ for all $t \geq 0$. Thus, $u_2(\cdot,t,\phi) \equiv u_4(\cdot,t,\phi) \equiv u_5(\cdot,t,\phi) \equiv 0$ for all $t \geq 0$. From (1.4c), we see

$$\partial_t u_3 = D_3\Delta u_3 - \mu u_3$$

and thus $\lim_{t\to\infty} u_3(x,t,\phi) = 0$ uniformly for $x \in \bar{\Omega}$ due to Lemma A.2. Therefore, now (1.4a) is asymptotic to

$$\partial_t Y = \bar{D}\Delta Y + \pi - \mu Y$$

and hence by the theory of asymptotically autonomous semiflows (see [33, Corollary 4.3]) we have $\lim_{t\to\infty} u_1(x,t,\phi) = m_{\mathrm{E}}^*$ uniformly for all $x \in \bar{\Omega}$. Thus, we see $\lim_{t\to\infty} u(x,t,\phi) = (m_{\mathrm{E}}^*, 0, 0, 0, 0)$, and it follows that $\omega(\phi) = \{(m_{\mathrm{E}}^*, 0, 0, 0, 0)\}$.

C. Proof of Theorem 2.2

By Proposition 4.2 we know that any bounded orbit of Φ_t in M_∂ converges to the DFE $(m_E^*, 0, 0, 0, 0)$, which is isolated in X^+. Considering Proposition 4.2, it follows by [31, Lemma 3] (see also [16, Theorem 1.3.2]) that there exists $\sigma > 0$ such that for all compact chain transitive sets L such that $L \not\subset \{(m_E^*, 0, 0, 0, 0)\}$, we have $\min_{\phi \in L} p(\phi) > \sigma$. By Lemma 1.4 (2), taking $\sigma > 0$ smaller if necessary to satisfy $\sigma \leq \frac{\pi}{\Upsilon_E}$ for Υ_E from (1.23), we see then that

$$\liminf_{t\to\infty} u_i(\cdot, t, \phi) \geq \sigma \;\; \forall\, \phi \in \mathbb{W}_0, \;\; \forall\, i \in \{1, 2, 3, 4, 5\}, \tag{C.1}$$

which is precisely (1.17). Finally, by Lemma 4.1 we know that $\Phi_t : X^+ \to X^+$ has a global connected attractor A. Thus, we see from [34, Theorem 3.7 and Remark 3.10] that $\Phi_t : \mathbb{W}_0 \to \mathbb{W}_0$ also has a global attractor A_0. Because $\Phi_t(\mathbb{W}_0) \subseteq \mathbb{W}_0$ from Proposition 4.1 (2), and Φ_t is also κ-condensing by Lemma 4.1, it follows from [34, Theorem 4.7] that Φ_t has an equilibrium $\tilde{u} \in A_0$ and hence $\tilde{u} \in \mathbb{W}_0$. It follows from (C.1) that $\tilde{u}$ is a positive steady state of (1.4).