Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(1): 1376–1401.
DOI: 10.3934/mbe.2023062
Received: 16 July 2022
Revised: 19 September 2022
Accepted: 23 September 2022
Published: 28 October 2022

*Research article*

# Product outsourcing policy for a sustainable flexible manufacturing system with reworking and green investment

**Raj Kumar Bachar**[1], **Shaktipada Bhuniya**[2], **Santanu Kumar Ghosh**[1], **Ali AlArjani**[3], **Elawady Attia**[3,4], **Md. Sharif Uddin**[5,6] **and Biswajit Sarkar**[7,8]*

[1] Department of Mathematics, Kazi Nazrul University, Asansol 713340, West Bengal, India
[2] Department of Mathematics & Statistics, Banasthali Vidyapith, Rajasthan 304022, India
[3] Industrial Engineering Department, College of Engineering, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
[4] Mechanical Engineering Department, Faculty of Engineering (Shoubra), Benha University, Cairo, Egypt
[5] Department of Mechanical and Industrial Engineering, College of Engineering, Prince Sattam Bin Abdulaziz University, Al-kharj 11942, Saudi Arabia
[6] Department of Mathematics, Jahangirnagar University, Savar, Dhaka 1342, Bangladesh
[7] Department of Industrial Engineering, Yonsei University, 50 Yonsei-ro, Sinchon-dong, Seodaemun-gu, Seoul 03722, South Korea
[8] Center for Transdisciplinary Research (CFTR), Saveetha Institute of Medical and Technical Sciences, Saveetha University, Velappanchavadi, Tamil Nadu 600077, India

* **Correspondence:** Email: bsbiswajitsarkar@gmail.com; Tel: +821074981981.

**Abstract:** Production of defective products is a very general phenomenon. But backorder and shortages occur due to this defective product, and it hampers the manufacturer's reputation along with customer satisfaction. That is why, these outsourced products supply, a portion of required products for in-line production. This study develops a flexible production model that reworks repairable defective products and outsources products to prevent backlogging. A percentage of total in-line production is defective products, which is random, and those defective products are repairable. A green investment helps the reworking process, which has a direct impact on the market demand for products. A classical optimization solves the profit maximization model, and a numerical method proves the global optimal solutions. Sensitivity analysis, managerial insights, and discussions provide the highlights and decision-making strategies for the applicability of this model.

**Keywords:** flexible production; outsourcing; reworking; green investment; sustainability

## 1. Introduction

Any machinery system can go to *out-of-control* any time, and it causes defective products. Problems start with distribution planning when these defective products cause a shortage in the system. A smart production system (Sarkar and Bhuniya [1]) is more reliable than a traditional production system. But a machinery system can be defective at any time. Then the rate of defective products reduces, but it is still in the system. As the system is a smart production system, it has control power for different processes of the production system. Thus, it is assumed that the product's defectiveness is less than the defective products from a traditional production system. Thus, defective products from a smart production system are repairable, and there does not exist any scrap for defective products. Whenever defective products produce, there is a chance of backlogging and shortage. The problem with the shortage is that it causes backlogging (both partial and full backlogging). It not only affects the business immediately but also the long-term effect is more than the instant loss. It not only reduces the sales of products but also damages the brand image. Then, what policy should the manager consider for the production such that the smart production system becomes a sustainable smart production system (Sarkar et al. [2])? The term sustainable refers to a steady production system amid defective production. This implies that amid defective products, the smart production system manages to run without backlogging. Based on these phenomena, this study aims to use product outsourcing (Caterino et al. [3], Xia et al. [4]) with rework within the smart production system. The rework process for repairable defective products is necessary for the proposed system. It is necessary because a sustainable smart production system aims to reduce waste (Sarkar et al. [5]) from the system, along with reusing products and materials. Investment for this rework process becomes helpful for smooth execution. Thus, the research gaps for the sustainable smart production system are below.

### *1.1. Research gap*

The following research gap is addressed in this study.

- How does a smart production system deal with defective production than a flexible production system (Chiu et al. [6])? The flexible production system is a well-discussed topic in the literature on shortage and backlogging. A flexible production system is generally used to avoid such situations. But, dealing with defective products for a smart production system is not a widely discussed topic in literature.
- Defective products in any production system are not the discrete case. But, how do defective products of a smart production system differ from defective products of other production systems (De et al. [7])? How do these defective products support the product outsourcing policy in a smart production system?
- Most importantly, how do the two combinations of outsourcing and rework make the system sustainable? There are several studies in the literature based on the green manufacturing system (Lu et al. [8]) but very few discussions about the decision-making strategies for a sustainable smart production system.

### *1.2. Contributions*

This study contributes to the literature based on the following points:

- This study provides an decision-making scenarios for smart production system in the post COVID-19 era. Manufacturing industries are one of the most affected sectors due to this pandemic. Price hiking in manufacturing sectors is skyrocketing[1,2,3]. As a result, the production rate decreases for manufacturing sectors than previous. Thus, a partial policy for the production sector is really helpful for this situation to reduce manufacturing costs and collaborate with other industries for economic growth. This policy will help to overcome the crisis time in the manufacturing industry. Smart production sectors face a little bit more problems than traditional productions as their investments are on a larger scale.
- This study deals with defective products from a smart production system. The inspection process is not necessary for these defective products to find out scrap products. Because no scrap product exists among defective products. All defective products are repairable. This reworking is supported by the accuracy level of the smart production.
- The manufacturer uses partial outsourcing of products in combination with in-line production. These outsourced products are used for delivery instead of defective products. This maintains a balance between the profit and reputation of the manufacturer in a sustainable manner.

### *1.3. Structure of this study*

The rest of this paper is as follows: Section 2 gives a literature review, and Section 3 presents the purpose of the problem, related mathematical symbols, and associated hypotheses. Mathematical modelling is presented in Section 4, and Section 5 provides the methodology to determine the solution. Numerical discussions are described in Section 6, Section 7 presents a sensitivity analysis, Section 8 provides managerial insights, and Section 9 presents the conclusions.

## 2. Literature review

A detailed review of a few related studies in literature is given here.

### *2.1. Flexible production rate and variable demand*

Product outsourcing in a sustainable smart production system with repairable defective products is not a widely discussed research in the literature. A flexible production rate is very fruitful in order to fulfill the customer demand and regularize the market size. To make a sustainable production system more reliable, flexible production can be used to make the right decisions regarding variable demand. A lot of research papers have been published based on variable production rates. AlDurgam et al. [9] derived a single-vendor single-manufacturer integrated inventory model with variable production rate and stochastic demand. AlDurgam [10] developed an integrated inventory and workforce planning Markov decision process model with a flexible production rate. Nowadays, every company tries to sell their product faster compared to others. To fulfill this, companies adopt various promotional efforts such as trade credit policy and advertisement. Among all the existing promotional efforts, a discount

[1] https://www.reuters.com/markets/europe/mexican-manufacturing-tumbles-price-hikes-bite-2022-07-30/, 18th September, 2022; 9.10 am KST

[2] https://english.alarabiya.net/business/markets/2022/09/01/Steel-price-hikes-to-impact-Japan-s-manufacturers-as-costs-surge, 18th September, 2022; 9.15 am KST

[3] https://www.pwc.com/us/en/industries/industrial-products/library/inflation-supply-chain-manufacturing.html, 18th September, 2022; 9.20 am KST

on the selling price is one of the most attractive ways. Maiti and Giri [11] developed a model based on two-period pricing and decision strategies under price-dependent demand. Feng et al. [12] developed a model in which the demand depended on selling price, displayed stocks, and expiration date. Giri et al. [13] considered warranty period, selling price, and green-sensitive consumer demand under revenue-sharing contracts. Li and Teng [14] discussed a model where demand depended on selling price, reference price, product freshness, and displayed stocks. Das et al. [15] derived a model based on the application of preservation technology in inventory control systems with partial backlogging and price-dependent demand. Agrawal and Yadav [16] found profit for an integrated inventory model based on price structuring. They considered a single-manufacturer multi-buyer model with price-sensitive demand. Ruidas et al. [17] developed a production inventory model based on price-sensitive demand. They also considered interval-valued carbon emission parameters. Mahapatra et al. [18] studied a model on the impact of preservation for imprecise demand. They included time-dependent deterioration, promotional cost, and fuzzy learning. Sarkar et al. [19] studied an inventory model on artificial neural networks and multithreading. They developed the model under uncertainty and inflation.

### *2.2. Outsourcing in a smart production system*

Related research showed that every research included outsourcing in the manufacturing system. Still, no research considered the effect of product outsourcing on a sustainable smart production system under flexible demand and reworking. Ameknassi et al. [20] developed a green supply chain design with outsourcing. Li et al. [21] considered outsourcing decisions in production and transportation. They considered single and multiple carbon policies in the model. Chen et al. [22] considered outsourcing contracts and ordering decisions in a supply chain model. In that model, they considered multi-dimensional uncertainties. Cortinhal et al. [23] developed a model based on a multi-stage supply chain network design problem where they considered in-house production and partial product outsourcing. They proposed a non-linear programming problem (NLPP) for the location and capacity of warehouse and outsourcing. Heydari et al. [24] considered a two-echelon supply chain system with quantity flexibility contracts and outsourcing. They used a quantity of flexibility (QF) for the retailer to upgrade and downgrade the number of products. Lou et al. [25] studied a retailer-led supply chain and logistics service outsourcing. They used a profit-sharing contract for a double marginalization effect.

### *2.3. Defective products in a flexible production system*

It is clear from previous studies (Table 1) that flexible production produces defective products. Every earlier research focused on defective production, but no earlier research focused on the effect of defective production through partial outsourcing under variable demand. Screening of defective products is required. Pasandideh et al. [26] developed an economic production quantity (EPQ) model where they considered a multi-product produced by a single machine. They considered an imperfect production system of under warehouse construction costs. Khan and Sarkar [27] developed a model where they discussed risk in a supply chain that occurred due to shortage. Taleizadeh et al. [28] developed an economic ordered quantity (EOQ) inventory model, whereas they considered imperfect products and partial backordering. Tsao et al. [29] determined an inventory model with imperfect production, reworking, and trade-credit. They incorporated radio frequency identification (RFID) within the system to support trade-credit. Jani et al. [30] derived an inventory model where shortages could be prevented using preservation technology. They considered chilled products for testing their mathematical model. Panja and Mondal [31] derived a model based on an imperfect production inventory model where they introduced green products under a credit period. They used a tire-two fuzzy number for the retailer's trade-credit policy when the retailer faced defective products in the lot size. Malik and Sarkar [32] developed an inventory model based on an imperfect multi-product production system. They discussed a post-pandemic situation through the production model that produced defective products. Rout et al. [33] derived a sustainable supply chain model which included imperfect production, reworking, and carbon emissions. They tested the model with past data and found a trade-off between total cost and carbon emissions.

**Table 1.** Contribution of the authors.

| Author(s) | Greening Cost | Demand Rate | DI | OC | Rework | Model Type |
|---|---|---|---|---|---|---|
| Chiu et al. [6] | NA | Constant | Yes | Yes | Yes | Inventory |
| AlDurgam et al. [9] | NA | Stochastic | NA | NA | NA | Inventory |
| Maiti and Giri [11] | NA | SPDD | NA | NA | NA | SCM |
| Feng et al. [12] | NA | Price and Stock | NA | NA | NA | Inventory |
| Giri et al. [13] | Yes | SPDD | Yes | NA | Yes | SCM |
| Li and Teng [14] | NA | SPDD and Stock | NA | NA | NA | Inventory |
| Das et al. [15] | NA | SPDD | NA | NA | NA | Inventory |
| Agrawal and Yadav [16] | NA | SPDD | NA | NA | NA | Production-inventory |
| Ruidas et al. [17] | Yes | SPDD | Yes | NA | Yes | Inventory |
| Heydari et al. [24] | NA | Stochastic | NA | Yes | NA | SCM |
| Taleizadeh et al. [28] | NA | NA | Yes | NA | Yes | Inventory |
| Tsao et al. [29] | NA | NA | Yes | NA | Yes | Inventory |
| Chen [34] | NA | SPDD | Yes | NA | Yes | SCM |
| Chiu et al. [35] | NA | Constant | Yes | NA | Yes | EPQ |
| Nia et al. [36] | Yes | Constant | NA | NA | NA | Inventory |
| Tayyab et al. [37] | NA | Random | Yes | NA | Yes | Multi-stage production |
| This Paper | Yes | SPDD | Yes | Yes | Yes | Inventory |

SCM, supply chain management; EPQ, economic production quantity; DI, defective items; OC, Outsourcing; SPDD, selling price-dependent demand; NA, not applied.

### *2.4. Reworking within for defective products*

Due to the advancement of modern technology, the industry always tries to produce perfect products, but in reality, it is impossible to produce all the products perfectly. Due to technical error, machine breakdown, human error, and other causes, some defective products are produced in the system. Some products are repairable out of all the defective products. It is a better decision from the industry point of view to make defective products into perfect products after reworking. Keeping this in mind, new production industries introduce reworking processes in their production system. Chiu et al. [35] derived a simplified EOQ model for multi-item where they considered rework, scrap products, and multiple deliveries. Tayyab et al. [37] discussed an EPQ model that was considered a defective serial production system with reworking. They studied a multi-stage cleaner production system to achieve sustainability. Moussawi-Haidar et al. [38] developed an EPQ model where they included imperfect production and reworking. They worked on the production time and the relation with the economic valuation. Pal and Mahapatra [39] considered an imperfect production, reworking, stochastic demand, and shortages in the production model. Chen [34] determined a model where they considered imperfect production, reworking, and price-dependent demand. The retailer went through three different strategies for the inspection of defective products. Chiu et al. [6] developed a supply chain model with reworking and optimal batch size. They explained a fabricated production system to not outsource products from other industries. Sonntag and Kiesmüller [40] discussed an interesting topic on imperfect production and reworking in their production model. They trade-off between rework and disposal based on the priority of quality and time. Al-Salamah [41] derived an imperfect production model for 100% inspection. They discussed two procedures of reworking, namely, asynchronous and synchronous. Lin et al. [42] determined a production model with capacity constraints. They found the relationship between rework and on-time delivery. Sarkar et al. [43] developed a model for the complex multi-stage production model.

### *2.5. Green investment in a production model*

Green investment helps a production system in many ways. It helps to reduce carbon emissions for reworking and for remanufacturing. A sustainable production system has a great impact on the economy, society, and environment. Lu et al. [8] developed a green manufacturing model where they considered a vendor-managed inventory and effort-dependent demand. Nia et al. [36] determined a vendor-managed inventory model with shortage and greenhouse gas emissions. They used a hybrid methodology and competitive algorithm for a multi-constraint optimization problem. Sepehri et al. [44] developed a supply chain model where they invested in green technology to reduce carbon emissions. They established that investment is mandatory for carbon emissions reduction. Sarkar et al. [5] established a supply chain model for reducing waste and a circular economy. Waste nullification was the new concept in the circular economy for a green environment. Kumar et al. [45] studied a remanufacturing model on greening cost with advertising-dependent demand.

### *2.6. Sustainable smart production system*

Literature shows that the discussions about sustainability measure economic, social, and environmental development. But studies on a sustainable production system are different from the sustainable production system. There is not much research discussing the sustainability of a production system that can develop a long-run error-free production system. Recently, Sarkar and Bhuniya [1] discussed a sustainable supply chain under manufacturing-remanufacturing and service strategy. But their model did not consider partial outsourcing. Garai and Sarkar [46] studied a model considering sustainability, second-generation biofuel, returnable items, and customer satisfaction. Sarkar et al. [2] developed a model based on environment and economic sustainability. They included innovative green products by re-manufacturing in their study. Sarkar et al. [47] examined a model on renewable energy, smart multi-type production, variable production rate, sustainability, selling price-dependent demand, and autonomation. Kugele et al. [48] developed a smart production model for reliability and controlled carbon ejection, which was an extension of the concept of Moon et al.'s [49] study. studied a production model for controlled carbon emissions to improve reliability. They discussed on the degree of difficulty of a smart production system.

## 3. Problem description, symbols, and hypotheses

In this portion, the problem, along with symbols and hypotheses, is properly described. At first, the research problem is described elaborately, then symbols of the mathematical model, and finally, hypotheses are briefly described.

### *3.1. Problem description*

The proposed model gives a new direction for achieving sustainability of a smart production system using repairable defective products and outsourcing. through green investment. The market demand for the product is a function of selling price and green investment. The green investment is used for the rework process to maintain product quality. The unit production cost is variable depending on raw material cost, development cost, and tool/die cost. This study derives a decision-making strategy for achieving sustainability of a smart production system using partial outsourcing. The defective rate of products is a random variable. Two scenarios are explained for calculating cycle time: with product outsourcing and without product outsourcing. The objective is to find the optimum profit for the proposed model.

### *3.2. Symbols*

The following symbols are considered to illustrate the model (Table 2).

The next subsection gives related hypotheses for the model.

### *3.3. Hypotheses*

1) A fixed portion $\pi$ of the optimal production lot size $Q$ ($0 < \pi < 1$) is outsourced by the manufacturer, i.e., partial outsourcing is considered for the production system. The outsourced products are perfect in quality and delivered to the production system after finishing the reworking process. If $\pi = 0$, then the system becomes an in-house production system. If $\pi = 1$, then the system becomes a fully outsourcing system (Chiu et al. [6]).
2) Defective rate is applicable for the lot size of in-house production, and the defective rate $x$ is random. Among those defective products, only repairable products are reworked (Malik and Sarkar [32]). Rework is used to maintain the quality of reworked products as good as new. Green investment is used for maintaining the quality of reworked products.
3) The manufacturer has a production system with a variable production rate (VPR). Then the unit production cost (UPC) is variable and depending upon development cost, raw material cost, and tool/die cost, i.e., $C(P) = (\delta_1 + \frac{\delta_2}{P} + \delta_3 P)$ (Sarkar and Bhuniya [1]). The function of UPC clearly shows that the raw material cost is fixed, the development cost is inversely proportional to the VPR, and the tool/die cost is directly proportional to the VPR.
4) Market demand of the product is selling-price and green investment dependent, i.e., $D = \xi_1 \frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c$ (Giri et al. [13]).

**Table 2.** Notation for this model.

| Decision | variables |
|---|---|
| $P$ | flexible production rate (unit/year) |
| $Q$ | production lot size (units/cycle) |
| $p$ | average selling price (\$/unit) |
| $\theta_c$ | cost for green investment (\$/year) |
| Parameters | |
| $K$ | setup cost for in-house production (\$/setup) |
| $h$ | unit holding cost (\$/unit/unit time) |
| $h_c$ | unit holding cost for reworked products (\$/unit/unit time) |
| $\delta_1$ | scaling parameter of raw material cost for manufacturing system |
| $\delta_2$ | scaling parameter of the development cost for the product during the manufacturing |
| $\delta_3$ | scaling parameter of tool/die cost |
| $R_C$ | reworking cost (\$/unit) |
| $M_\pi$ | constant cost of outsourcing (\$/unit) |
| $N_\pi$ | unit variable outsourcing cost (\$/unit) |
| $R_1$ | reworking rate (units/year) |
| $\pi$ | outsourcing portion of the lot size ($0 < \pi < 1$) |
| $\tau_1$ | connecting variable between $K$ and $M_\pi$, where $M_\pi = [(1+\tau_1)K]$ and $-1 \leq \tau_1 \leq 0$, $K > M_\pi$ |
| $\tau_2$ | connecting variable between $C$ and $N_\pi$, where $N_\pi = [(1+\tau_2)C(P)]$ and $\tau_2 \geq 0$ |
| $T_\pi$ | replenishment cycle time (time unit) |
| $E_1$ | on-hand inventory level of new products when production ends |
| $E_2$ | on-hand inventory level when rework ends |
| $H$ | on-hand inventory level when outsourcing ends |
| $t_1$ | production uptime, if $\pi = 0$ (year) |
| $t_2$ | reworking time, if $\pi = 0$ (year) |
| $t_3$ | production downtime, if $\pi = 0$ (year) |
| $T$ | cycle time, if $\pi = 0$ (year) |
| $f_{1\pi}$ | production uptime for new products when outsourcing happens |
| $f_{2\pi}$ | reworking time when outsourcing happens |
| $f_{3\pi}$ | outsourcing duration and production downtime when outsourcing happens |
| $I(t)$ | good quality products (units) |
| $Id(t)$ | defective products (units) |
| $C(P)$ | unit production cost (\$/unit) |
| $TC$ | total operating cost per cycle (\$/year) |
| $l_{max}$ | maximum selling price of the product (\$/unit) |
| $l_{min}$ | minimum selling price of the product (\$/unit) |
| $\xi_i$ | scaling parameters ($i = 1, 2, 3$) |
| $x$ | portion of repairable defective products, which is random |
| $E[x]$ | expected value of $x$ |

## 4. Mathematical model

In this section, different costs are discussed for formulating the proposed model. In the present socioeconomic situation, outsourcing can play a vital role in a production system while fulfilling customer requirements. The production rate is variable with a selling price-dependent demand (SPDD). Reworking on defective items with partial outsourcing makes the model more profitable. At the end of each production cycle, the reworking of repairable defective products begins. From Figures 1 and 2, the following formulas are obtained.

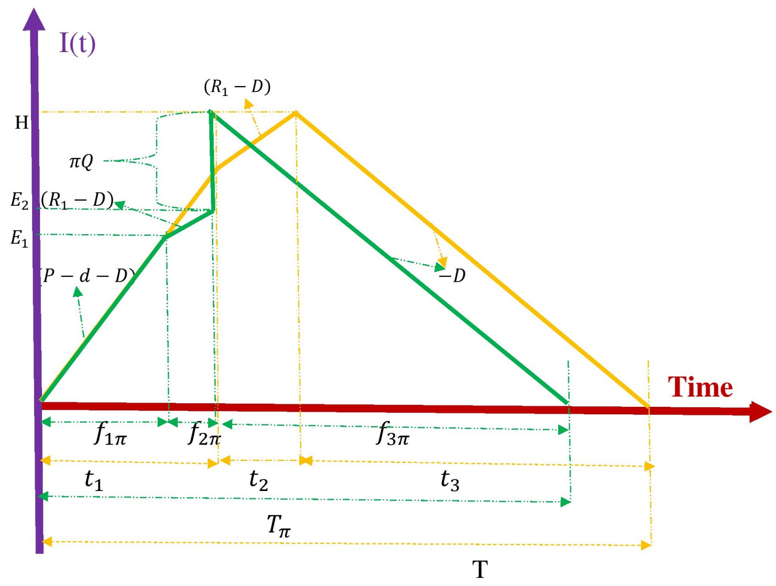


**Figure 1.** Inventory position for the proposed system with outsourcing (green line) versus inventory position of a traditional system without outsourcing plan (yellow line).

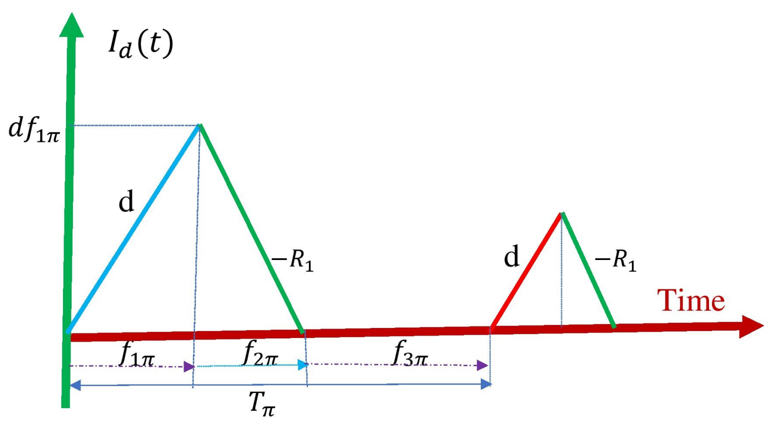


**Figure 2.** The level of on-hand repairable defective products in production and reworking system for the proposed system.

The level of perfect quality on-hand products after the completion of in-house production is obtained by subtracting the defective and demand of the products from the production rate using the following formula.

$$E_1 = (P - d - D)f_{1\pi}. \tag{4.1}$$

The level of the perfect quality on-hand products when the reworking process ends is obtained by the sum of the perfect quality products and the remaining reworked products, which cover the market demand simultaneously. The formula is

$$E_2 = E_1 + (R_1 - D)f_{2\pi}. \tag{4.2}$$

When the outsourced items are received, the maximum level of perfect quality on-hand products is obtained by the sum of the total perfect quality products after rework with the outsourcing products.

$$H = E_2 + \pi Q = Df_{3\pi}. \tag{4.3}$$

The following time indicates the production uptime, reworking time, and production downtime when the outsourcing continues to fulfill the market demand. Thus, the required formulas are as follows:

$$f_{1\pi} = \frac{E_1}{(P - d - D)} = \frac{(1-\pi)Q}{P} \tag{4.4}$$

$$f_{2\pi} = \frac{x[(1-\pi)Q]}{R_1} \tag{4.5}$$

$$f_{3\pi} = \frac{H}{D} = \frac{E_2 + \pi Q}{D}. \tag{4.6}$$

The cycle time is the sum of the production time (production uptime) of perfect quality products, reworking time of defective products, and production downtime. In general, the cycle time is calculated by dividing the lot sizes by market demand. Hence, the cycle time and repairable defective products formula are considered as follows:

$$T = f_{1\pi} + f_{2\pi} + f_{3\pi} = \frac{Q}{D} \tag{4.7}$$

$$df_{1\pi} = xPf_{1\pi} = x[(1-\pi)Q]. \tag{4.8}$$

### *4.1. Production setup cost (PSC)*

Production setup costs are associated with configuring a machine for a production system. Some examples of production setup costs are the scrap cost of test units run on the machine, the cost of the labor to configure the machine, etc. In this model, the production setup cost is

$$PSC = K. \tag{4.9}$$

### *4.2. Variable production cost (VPC)*

Variable Production cost is an expense that varies in proportion to the output. Variable production cost increases or decreases depending on the volume of the output of the production system. Some examples of variable production costs are the costs of raw materials, packaging, etc. Here, the variable production cost is

$$VPC = (\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi)Q. \tag{4.10}$$

### 4.3. Fixed outsourcing cost (FOC)

The cost, which is fixed and associated with the outsourcing of products, is called fixed outsourcing cost. Some examples of fixed outsourcing costs are the cost of management and coordination of suppliers, the cost of an outsourcing strategy, etc. In this model, the fixed outsourcing cost is

$$FOC = M_\pi = [(1+\tau_1)K], -1 \leq \tau_1 \leq 0. \tag{4.11}$$

### 4.4. Variable outsourcing cost (VOC)

The cost that varies from time to time and is associated with the outsourcing of products is called variable outsourcing cost. Some examples of variable outsourcing costs are the cost of unplanned logistics activities and premium freight, the cost of poor or substandard quality, the cost of the warranty, returns, and allowances. etc. In this paper, variable outsourcing cost is as follows:

$$VOC = N_\pi(\pi Q) = [(1+\tau_2)C(P)](\pi Q), \tau_2 \geq 0. \tag{4.12}$$

### 4.5. Reworking cost (RC)

Rework cost is an expense incurred in manufacturing and other productive works. When a newly manufactured product finds defective, a rework process executes to fix them and make them salable in this model; the reworking cost is

$$RC = R_C x[(1-\pi)Q]. \tag{4.13}$$

### 4.6. Holding cost for reworked items (HCR)

Defective products undergo a rework process to make them perfect as new. These reworked products are then ready for sale. In this model, the holding cost for reworked items is

$$HCR = h_c \frac{d f_{1\pi}}{2}(f_{2\pi}). \tag{4.14}$$

### 4.7. Holding cost for perfect quality and defective items (HCPD)

Every production system produces a perfect product as well as some defective products. All the defective products have undergone a reworking process to make them perfect. in this model, holding costs for perfect quality and defective items are as follows:

$$HCPD = h\left[\frac{E_1 + d f_{1\pi}}{2}(f_{1\pi}) + \frac{E_1 + E_2}{2}(f_{2\pi}) + \frac{H}{2}(f_{3\pi})\right]. \tag{4.15}$$

### 4.8. Green investment for reworking procedure (IGP)

Management of a smart production system is much concerned with the reworking process and the quality of reworked products. In this model, this investment is used to maintain the quality of reworked products in the reworking process. This will help to reduce the waste from the smart production system. Thus, the green investment is

$$IGP = \frac{\xi_3 \theta_c^2}{2}. \tag{4.16}$$

*4.9. Total cost of the sustainable smart production system (TC)*

Thus, the total operating cost for the proposed production system, $TC(P, Q, p, \theta_c)$, includes PSC, VPR, FOC, VOC, RC, HCR, and HCPD in $f_{1\pi}, f_{2\pi}$ and $f_{3\pi}$ and IGP. Therefore, $TC(P, Q, p, \theta_c)$ is as follows:

$$\begin{aligned} TC(P, Q, p, \theta_c) &= (PSC + VPC + FOC + VOC + RC + HCR + HCPD + IGP) \\ &= K + (\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi)Q + M_\pi + N_\pi(\pi Q) + R_C x[(1-\pi)Q] \\ &+ h_c \frac{df_{1\pi}}{2}(f_{2\pi}) + h\left[\frac{E_1 + df_{1\pi}}{2}(f_{1\pi}) + \frac{E_1 + E_2}{2}(f_{2\pi}) + \frac{H}{2}(f_{3\pi})\right] + \frac{\xi_3 \theta_c^2}{2}. \end{aligned} \tag{4.17}$$

By substituting $M_\pi$ and $N_\pi$ in Eq (4.17), the operating cost $TC(P, Q, p, \theta_c)$ becomes

$$\begin{aligned} TC(P, Q, p, \theta_c) &= K + (\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi)Q + K(1+\tau_1) + (1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(\pi Q) \\ &+ R_C x[(1-\pi)Q] + h_c \frac{df_{1\pi}}{2}(f_{2\pi}) + h\left[\frac{E_1 + df_{1\pi}}{2}(f_{1\pi}) + \frac{E_1 + E_2}{2}(f_{2\pi}) + \frac{H}{2}(f_{3\pi})\right] \\ &+ \frac{\xi_3 \theta_c^2}{2}. \end{aligned} \tag{4.18}$$

Thus, the expected operating cost per cycle $E[TCU(P, Q, p, \theta_c)]$ reduces to

$$\begin{aligned} E[TCU(P, Q, p, \theta_c)] &= \frac{E[TC(Q, P)]}{E[T]} \\ &= \frac{D}{Q}\Big[K + Q(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + K(1+\tau_1) + Q\pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P) \\ &+ Q(1-\pi)\zeta R_C + \frac{Q^2(h_c - h)}{2}\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right) \\ &+ \frac{hQ^2}{2}\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right) + \frac{\xi_3 \theta_c^2}{2}\Big] \\ &= \frac{1}{Q}\left(\xi_1 \frac{(l_{max} - p)}{(p - l_{min})} + \xi_2 \theta_c\right)\Big[K + Q(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + K(1+\tau_1) \\ &+ Q\pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P) + Q(1-\pi)\zeta R_C + \frac{Q^2(h_c - h)}{2}\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right) \\ &+ \frac{hQ^2}{2}\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right) + \frac{\xi_3 \theta_c^2}{2}\Big] \end{aligned} \tag{4.19}$$

where $E[x] = \zeta$.

*4.10. Expected total profit (TEP)*

The revenue is calculated as Revenue= $pD$. Thus the expected total profit per cycle becomes

$$
\begin{aligned}
TEP(P,Q,p,\theta_c) &= p\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)-\frac{1}{Q}\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[K+Q(\delta_1+\frac{\delta_2}{P}+\delta_3 P)(1-\pi)\\
&+ K(1+\tau_1)+Q\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3 P)+Q(1-\pi)\zeta R_C+\frac{Q^2(h_c-h)}{2}\\
&\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)+\frac{hQ^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{R_1}(-2\pi)\Big)+\frac{\xi_3\theta_c^2}{2}\Big]\\
&= p\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\\
&- \frac{1}{Q}\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[K+Q(\delta_1+\frac{\delta_2}{P}+\delta_3 P)(1-\pi)+K(1+\tau_1)\\
&+ Q\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3 P)+Q(1-\pi)\zeta R_C+\frac{Q^2(h_c-h)}{2}\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)\\
&+ \frac{hQ^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{R_1}(-2\pi)\Big)+\frac{\xi_3\theta_c^2}{2}\Big]. \qquad (4.20)
\end{aligned}
$$

## 5. Solution methodology

To solve the mathematical model, the classical optimization method is considered. The decision variables $P$, $Q$, $p$, and $\theta_c$ are optimized using a continuous optimization technique. As there are multiple decision variables, the Hessian matrix is used to test the global optimal solution. At first, the expected total profit is partially differentiated with respect to the decision variables and equated to zero. Thus, optimum values $P^*$, $Q^*$, $p^*$, $\theta_c^*$ are

$$
P^* = \sqrt{\frac{2\delta_2(1-\pi)+2\pi(1+\tau_2)\delta_2-hQ(1-\pi^2)}{\delta_3(1-\pi)+\pi\delta_3(1+\tau_2)}} \qquad (5.1)
$$

$$
Q^* = \frac{\Psi-\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[(\delta_1+\frac{\delta_2}{P}+\delta_3P)(1-\pi)+\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3P)+(1-\pi)\zeta R_C\Big]}{\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[Q(h_c-h)\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)+hQ\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{R_1}(-2\pi)\Big)\Big]} \qquad (5.2)
$$

$$
p^* = \frac{\sqrt{\Upsilon^2-4\zeta p^2(\xi_1+\xi_2\theta_c)\Big[1-\frac{h\xi_1 Q}{D^2}\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\Big]}-\Upsilon}{2p^2(\xi_1+\xi_2\theta_c)\Big[1-\frac{h\xi_1 Q}{D^2}\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\Big]} \qquad (5.3)
$$

$$
\theta_c^* = \frac{\sqrt{\Big[\xi_1\frac{(l_{max}-p)}{(p-l_{min})}\frac{1}{Q}\Big(\frac{h\xi_2Q^2}{2D^2}-\xi_3\theta_c\Big)\Big]^2-8\xi_2\theta_c\frac{1}{Q}\Big(\frac{h\xi_2Q^2}{2D^2}-\xi_3\theta_c\Big)(p-\Psi)\xi_2}-\xi_1\frac{(l_{max}-p)}{(p-l_{min})}\frac{1}{Q}\Big(\frac{h\xi_2Q^2}{2D^2}-\xi_3\theta_c\Big)}{2\xi_2\theta_c\frac{1}{Q}\Big(\frac{h\xi_2Q^2}{2D^2}-\xi_3\theta_c\Big)} \qquad (5.4)
$$

See Appendix A for the calculations of first-order derivatives.

**5.1 proposition** The expected total profit function is a convex at $P^*$, $Q^*$, $p^*$, $\theta_c^*$ if
$\psi < 0$
$\psi\chi > \sigma^2$
$\psi(\chi\varphi - \vartheta^2) + \rho(\sigma\vartheta - \rho\chi) < \sigma(\sigma\varphi - \rho\vartheta)$
$\psi[\chi(\varphi\tau - \Omega^2) - \vartheta(\vartheta\tau - \theta\Omega) + \theta(\vartheta\Omega - \varphi\theta)] + \rho[\sigma(\vartheta\tau - \theta\Omega) - \chi(\rho\tau - \kappa\Omega) + \theta(\rho\theta - \kappa\vartheta)]$
$> \sigma[\sigma(\varphi\tau - \Omega^2) - \vartheta(\rho\tau - \kappa\Omega) + \theta(\rho\Omega - \kappa\varphi)] + \kappa[\sigma(\vartheta\Omega - \theta\varphi) - \chi(\rho\Omega - \kappa\varphi) + \vartheta(\rho\theta - \kappa\vartheta)]$.
**Proof.** See Appendix B.

## 6. Numerical examples

Two numerical examples are given here to validate the model. Numerical data are taken from Sarkar and Bhuniya [1] and Malik and Sarkar [32]. Data have been modified due to the convergence of the algorithm.

### 6.1. Example 1

The mathematical model is tested numerically to validate the theoretical solution. The following input parametric values are considered to illustrate the numerical example. Here $K$ = 5000 (\$/setup); $\delta_1$ = 320; $\delta_2$ = 11,910; $\delta_3$ = 0.009; $R_C$ = 50 (\$/unit); $\tau_1$ = -0.3; $\tau_2$ = 0.3; $l_{max}$ = 900 (\$/unit); $l_{min}$ = 400 (\$/unit); $\xi_1$ = 20; $\xi_2$ = 6; $\xi_3$ = 500; $h_c$ = 25.01 (\$/unit/unit time); h = 15.9(\$/unit/unit time); $E[x]$ = 0.2; $\pi$ = 0.05; $R_1$ = 110 (units/year);

The optimal result of decision variables are $P^*$ = 908.55 (unit/year); $Q^*$ = 573.53 (units/cycle); $p^*$ = 496.02 (\$/unit); $\theta_c^*$ = 6.35 (\$/year), and the expected total profit (TEP) = 9,211.85 (\$/year).

The global optimality of the result is checked analytically as well as numerically. For the numerical test, values of principal minors are $H_{11}$ = -0.00245769 < 0; $H_{22}$ = 0.0000577413 > 0; $H_{33}$ = -0.0000339684 < 0; $H_{44}$ = 0.00538134 > 0.

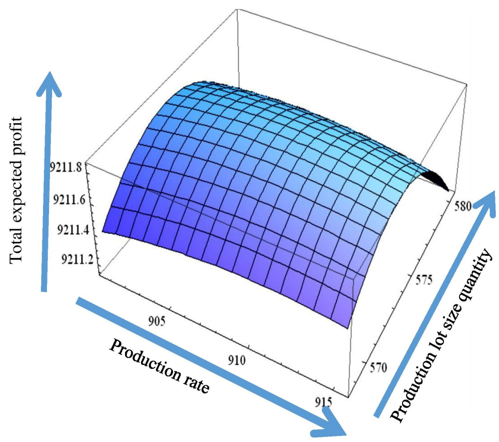


**Figure 3.** TEP versus variable production rate and production lot size.

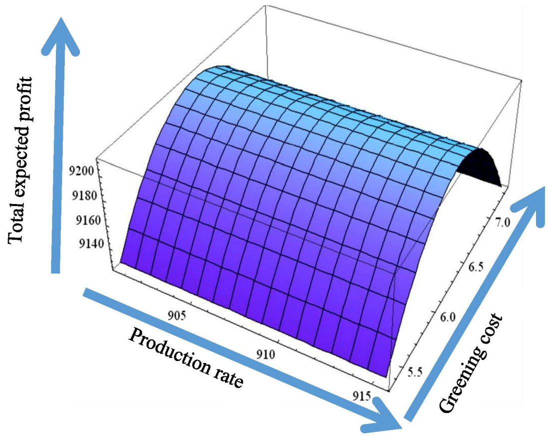


**Figure 4.** TEP versus variable production rate and green investment.

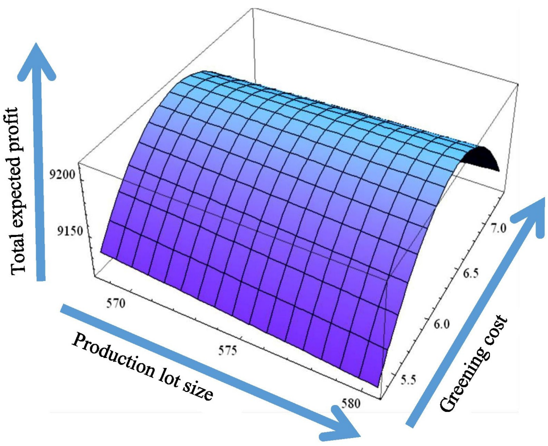


**Figure 5.** TEP versus production lot size and green investment.

### *6.2. Example 2*

The input values for Example 2 are as follows: $K$ = 500 (\$/setup); $\delta_1$ = 320 ; $\delta_2$ = 900 ; $\delta_3$ = 0.02; $R_C$ = 100 (\$/unit); $\tau_1$ = -0.3; $\tau_2$ = 0.3; $l_{max}$ = 900 (\$/unit); $l_{min}$ = 400 (\$/unit); $\xi_1$ = 10; $\xi_2$ = 3; $\xi_3$ = 300; $h_c$ = 9 (\$/unit/unit time); h = 0.1 (\$/unit/unit time); $E[x]$ = 0.62; $\pi$ = 0.05; $R_1$ = 50 (units/year).

The optimal results of decision variables are $P^*$ = 211.01 (unit/year); $Q^*$ = 192.97 (units); $p^*$ = 461.19 (\$/unit); $\theta_c^*$ = 1.45 (\$/year); and the expected total profit (TEP) is 4,321.14 (\$/year).

The global optimality of the result is checked analytically as well as numerically. For numerical test, $H_{11}$ = -0.0146317 < 0; $H_{22}$ = 0.000360228 > 0; $H_{33}$ = -0.0000666809 < 0; $H_{44}$ = 0.00618225 > 0 (Figures 3–5).

### *6.3. Special observations*

Some special observations are examined and discussed here based on the proposed model. The actual results and comparative studies validate the present model.

6.3.1. Fixed production rate

A special observation is made for the $TEP$ for a fixed production rate instead of a $VPR$. Keeping the parametric values of Example 1 fixed and using the fixed production rate of $P$ = 200 units per year, the optimum outcomes are $Q^*$ = 139.52 (units); $p^*$ = 667.03 ($/unit); and $TEP$ = 1,061.87 ($/year). Here the $TEP$ is less than that of the proposed result. Hence, it can be concluded that a $VPR$ is much better for the sustainable smart production system than a fixed production rate. Here, a statistical analysis is considered to achieve the results with confidence. The values of the principal minor are $H_{11} = -0.109231 < 0$ and $H_{22} = +0.0122797 > 0$. Hence, the TEP is maximum as the signs of the Hessian are alternative in sign.

6.3.2. Fixed demand and selling price

Another special observation is made for the $TEP$ for a fixed selling price and demand instead of a variable type. Keeping the parametric values of Example 1 fixed and using the fixed selling price of $p$ = 500 units per product and demand $D$ = 500 per year, the optimum outcomes are $P^*$ = 1129.69 (units); $q^*$ = 356.74 ($/unit); and $TEP$ = 1963.49 ($/year).Here the $TEP$ is less than the originally proposed model. Hence, it is concluded that variable selling price and variable demand are better for the sustainable smart production system than the fixed demand and selling price. Here, a statistical analysis is considered to achieve the results with confidence. The values of the principal minor are $H_{11} = -0.00202357 < 0$, $H_{22} = +0.0000757578 > 0$. Hence, the $TEP$ is maximum, as the values of the principal minors are alternate in sign.

6.3.3. Discussions

From the above numerical experiments and their special observations, it is concluded that the $TEP$ is the maximum for the originally proposed model. All special observations are numerically expressed using MATHEMATICA 11.3.0 software. Even if the manager needs any constant selection mode between fixed production rate and fixed demand & selling price, the latter is more profitable. A thorough discussion of this study reveals that the variable production rate and variable selling price under variable demand increase the profit margins. Moreover, it is concluded that the proposed model makes a higher profit in both cases than in the other cases. Hence, the special observations help validate the original research idea.

*6.4. Sensitivity analysis*

Remarkable observations for costs and scaling parameters are numerically calculated, and the changes in these parametric effects are described in Table 3 and Figure 6.

Table 2 indicates how cost parameters and scaling parameters effects the expected total profit due to changes such as (− 50%, − 25%, + 25%, + 50% ). From the following sensitivity table, conclusions are as follows:

1) Holding cost for new products is the most sensitive parameter with a notable impact on TEP. The TEP decreases when the holding cost increases and increases when the holding cost decreases. The table shows that to get a maximum profit, holding cost has a significant role.

2) Second most sensitive parameter is the setup cost. It has a great impact on TEP. Decreasing the value of this parameter increases the TEP, and increasing its value decreases TEP. Thus the setup cost plays a vital role in TEP.
3) Holding cost for reworked products affects TEP significantly. Thus, the holding cost for reworked products can not be ignored during the calculation of TEP.
4) Reworking cost has little effect on the TEP. Though it does not play a vital role in calculating the TEP, its effect can not be ignored.

**Table 3.** Sensitivity analysis table.

| Parameters | change (%) | TEP (%) | Parameters | change ( %) | TEP (%) |
|---|---|---|---|---|---|
| | -50% | +10.39 | | -50% | +25.98 |
| | -25% | +05.04 | | -25% | +11.17 |
| $K$ | +25% | -49.77 | $h$ | +25% | -09.03 |
| | +50% | -51.48 | | +50% | -45.75 |
| | -50% | +06.36 | | -50% | +1.62 |
| | -25% | +03.16 | | -25% | +0.31 |
| $R_C$ | +25% | -03.14 | $h_c$ | +25% | -44.49 |
| | +50% | -06.26 | | +50% | -1.50 |

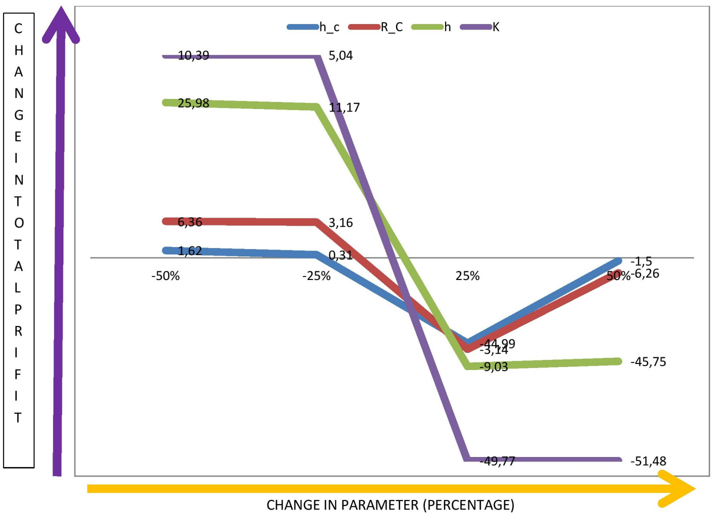


**Figure 6.** Effects of changes in parametric values versus expected total profit.

### *6.5. Managerial insights*

The followings are the main insights of this study. The study is very relevant to the post-pandemic situation of the COVID-19 era. This study provides a few insights into a sustainable smart production system.

1) As a smart production system is considered for the technological development of industry 4.0,

the machine inspects automatically products. If the experimental value of inspection exceeds the threshold value of the machinery inspection, then those products are good. Then, other products are considered repairable. Defective products are repairable because the quality of the newly manufactured products is high, and thus the level of defectiveness is less. Thus, the reworking process for defective products is fruitful for a sustainable smart production system.

2) Whenever the management concentrates on the reworking of process, it reduces the manufacturing cost along with disposal cost Most importantly, amid the post-COVID-19 pandemic era, price hiking is one of the most concerning economic problems. Reworking servers has two purposes. First, it helps to reduce the raw material use, and thus, the management can cut the raw material and post-processing cost. Second, job security and placement are the second most discussed topic in the post-pandemic situation. The reworking sector provides an opportunity for workers to work more and have job security.
3) The concept of partial outsourcing helps to handle the entire situation for defective products and reworking. Besides, when some industries outsource, this outsourcing supports other industries. That gave back-and-forth support for interactive industries and helps them grow economically. This policy supports post-pandemic industry regression and price inflation.

## 7. Conclusions

The market demand for a product highly fluctuated and volatile as it was depended on the selling price and rework process. Any minor changes in these factors would have an impact on the profit and revenue of the industry. This study considered selling price and green investment-dependent demand for smart products in a sustainable production system. The TEP for various cases was separately optimized using decision variables, both analytically and numerically. It was already proved that the sustainable smart production system could easily provide a significant profit with the facility of global outsourcing and reworking. In addition, depending on the variable customer demand, products can cover the entire competitive market with replacement, warranty, buyback, and reworking facilities. This model can be further extended by considering a demand that is a stock, discount, advertisement, promotion, and trade-credit-dependent (Mahapatra et al. [18]). In the future, this model can be expanded by preservative technology for deteriorating products (Sarkar et al. [50]). Alternatively, the production of green products may be considered, which is closely related to sustainable outsourcing (Rinaldi et al. [51], Ahi and Searcy [52]). A cost-effective subsidy policy, bio-fuels, and animal fat-based biodiesel may be considered for future extension (Garai and Sarkar [46]). In the future, this model can be expanded by considering. In the present COVID-19 situation, the global business procedure easily fulfills and satisfies customer demand through online or online shopping systems or e-supply chain management. Another direction for development is to incorporate the inspection cost and errors during the inspection and back-ordering cost.

## Acknowledgments

The authors extend their appreciation to the Deputyship for Research & In-novation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IF-PSAU-2021/01/18925).

## Conflict of interest

The authors declare no conflict of interest.

## References


1. B. Sarkar, S. Bhuniya, A sustainable flexible manufacturing-remanufacturing model with improved service and green investment under variable demand, *Expert Syst. Appl.*, **202** (2022), 117154. https://doi.org/10.1016/j.eswa.2022.117154
2. B. Sarkar, M. Ullah, M. Sarkar, Environmental and economic sustainability through innovative green products by remanufacturing, *J. Cleaner Prod.*, **332** (2022), 129813. https://doi.org/10.1016/j.jclepro.2021.129813
3. M. Caterino, M. Fera, R. Macchiaroli, D. T. Pham, Cloud remanufacturing: remanufacturing enhanced through cloud technologies, *J. Manuf. Syst.*, **64** (2022), 133–148. https://doi.org/10.1016/j.jmsy.2022.06.003
4. X. Xia, M. Li, B. Li, H. Wang, The impact of carbon trade on outsourcing remanufacturing, *Int. J. Environ. Res. Public Health*, **18** (2021), 10804. https://doi.org/10.3390/ijerph182010804
5. B. Sarkar, A. Debnath, A. S. Chiu, W. Ahmed, Circular economy-driven two-stage supply chain management for nullifying waste, *J. Cleaner Prod.*, **339** (2022), 130513. https://doi.org/10.1016/j.jclepro.2022.130513
6. Y. S. P. Chiu, C. J. Liu, M. H. Hwang, Optimal batch size considering partial outsourcing plan and rework, *Jordan J. Mech. Ind. Eng.*, **11** (2017), 195–200.
7. M. De, B. Das, M. Maiti, Green logistics under imperfect production system: a rough age based multi-objective genetic algorithm approach, *Comput. Ind. Eng.*, **119** (2018), 100–113. https://doi.org/10.1016/j.cie.2018.03.021
8. X. Lu, J. Shang, S. Y. Wu, G. G. Hegde, L. Vargas, D. Zhao, Impacts of supplier hubris on inventory decisions and green manufacturing endeavors, *Eur. J. Oper. Res.*, **245** (2015), 121–132. https://doi.org/10.1016/j.ejor.2015.02.051
9. M. AlDurgam, K. Adegbola, C. H. Glock, A single-vendor single-manufacturer integrated inventory model with stochastic demand and variable production rate, *Int. J. Prod. Econ.*, **191** (2017), 335–350. https://doi.org/10.1016/j.ijpe.2017.05.017
10. M. M. AlDurgam, An integrated inventory and workforce planning Markov decision process model with a variable production rate, *IFAC-PapersOnLine*, **52** (2019), 2792–2797. https://doi.org/10.1016/j.ifacol.2019.11.631
11. T. Maiti, B. C. Giri, Two-period pricing and decision strategies in a two-echelon supply chain under price-dependent demand, *Appl. Math. Modell.*, **42** (2017), 655–674. https://doi.org/10.1016/j.apm.2016.10.051
12. L. Feng, Y. L. Chan, L. E. C*á*rdenas-Barr*á*n, Pricing and lot-sizing polices for perishable goods when the demand depends on selling price, displayed stocks, and expiration date, *Int. J. Prod. Econ.*, **185** (2017), 11–20. https://doi.org/10.1016/j.ijpe.2016.12.017

13. B. C. Giri, C. Mondal, T. Maiti, Analysing a closed-loop supply chain with selling price, warranty period and green sensitive consumer demand under revenue sharing contract, *J. Cleaner Prod.*, **190** (2018), 822–837. https://doi.org/10.1016/j.jclepro.2018.04.092

14. R. Li, J. T. Teng, Pricing and lot-sizing decisions for perishable goods when demand depends on selling price, reference price, product freshness, and displayed stocks, *Eur. J. Oper. Res.*, **270** (2018), 1099–1108. https://doi.org/10.1016/j.ejor.2018.04.029

15. S. C. Das, A. M. Zidan, A. K. Manna, A. A. Shaikh, A. K. Bhunia, An application of preservation technology in inventory control system with price dependent demand and partial backlogging, *Alexandria Eng. J.*, **59** (2020), 1359–1369. https://doi.org/10.1016/j.aej.2020.03.006

16. A. K. Agrawal, S. Yadav, Price and profit structuring for single manufacturer multi-buyer integrated inventory supply chain under price-sensitive demand condition, *Comput. Ind. Eng.*, **139** (2020), 106208. https://doi.org/10.1016/j.cie.2019.106208

17. S. Ruidas, M. R. Seikh, P. K. Nayak, A production inventory model with interval-valued carbon emission parameters under price-sensitive demand, *Comput. Ind. Eng.*, **154** (2021), 107154. https://doi.org/10.1016/j.cie.2021.107154

18. A. S. Mahapatra, M. S. Mahapatra, B. Sarkar, S. K. Majumder, Benefit of preservation technology with promotion and time-dependent deterioration under fuzzy learning, *Expert Syst. Appl.*, **201** (2022), 117169. https://doi.org/10.1016/j.eswa.2022.117169

19. A. Sarkar, R. Guchhait, B. Sarkar, Application of the artificial neural network with multithreading within an inventory model under uncertainty and inflation, *Int. J. Fuzzy Syst.*, **24** (2022), 2318–2332. https://doi.org/10.1007/s40815-022-01276-1

20. L. Ameknassi, D. Aït-Kadi, N. Rezg, Integration of logistics outsourcing decisions in a green supply chain design: a stochastic multi-objective multi-period multi-product programming model, *Int. J. Prod. Econ.*, **182** (2016), 165–184. https://doi.org/10.1016/j.ijpe.2016.08.031

21. J. Li, Q. Su, L. Ma, Production and transportation outsourcing decisions in the supply chain under single and multiple carbon policies, *J. Cleaner Prod.*, **141** (2017), 1109–1122. https://doi.org/10.1016/j.jclepro.2016.09.157

22. K. Chen, H. Zhao, T. Xiao, Outsourcing contracts and ordering decisions of a supply chain under multi-dimensional uncertainties, *Comput. Ind. Eng.*, **130** (2019), 127–141. https://doi.org/10.1016/j.cie.2019.02.010

23. M. J. Cortinhal, M. J. Lopes, M. T. Melo, A multi-stage supply chain network design problem with in-house production and partial product outsourcing, *Appl. Math. Modell.*, **70** (2019), 572–594. https://doi.org/10.1016/j.apm.2019.01.046

24. J. Heydari, K. Govindan, H. R. E. Nasab, A. A. Taleizadeh, Coordination by quantity flexibility contract in a two-echelon supply chain system: Effect of outsourcing decisions, *Int. J. Prod. Econ.*, **225** (2020), 107586. https://doi.org/10.1016/j.ijpe.2019.107586

25. Y. Lou, L. Feng, S. He, Z. He, X. Zhao, Logistics service outsourcing choices in a retailer-led supply chain, *Transp. Res. Part E Logist. Transp. Rev.*, **141** (2020), 101944. https://doi.org/10.1016/j.tre.2020.101944

26. S. H. R. Pasandideh, S. T. A. Niaki, A. H. Nobil, L. E. C*á*rdenas-Barr*á*n, A multiproduct single machine economic production quantity model for an imperfect production system under warehouse construction cost, *Int. J. Prod. Econ.*, **169** (2015), 203–214. https://doi.org/10.1016/j.ijpe.2015.08.004
27. I. Khan, B. Sarkar, Transfer of risk in supply chain management with joint pricing and inventory decision considering shortages, *Mathematics*, **9** (2021), 638. https://doi.org/10.3390/math9060638
28. A. A. Taleizadeh, M. P. S. Khanbaglo, L. E. C*á*rdenas-Barr*á*n, An EOQ inventory model with partial backordering and reparation of imperfect products, *Int. J. Prod. Econ.*, **182** (2016), 418–434. https://doi.org/10.1016/j.ijpe.2016.09.013
29. Y. C. Tsao, Q. Zhang, F. C. Chang, L. Vu-Thuy, An imperfect production model under radio frequency identification adoption and trade credit, *Appl. Math. Modell.*, **42** (2017), 493–508. https://doi.org/10.1016/j.apm.2016.10.009
30. M. Y. Jani, M. R. Betheja, U. Chaudhari, B. Sarkar, Optimal investment in preservation technology for variable demand under trade-credit and shortages, *Mathematics*, **9** (2021), 1301. https://doi.org/10.3390/math9111301
31. S. Panja, S. K. Mondal, Analyzing a four-layer green supply chain imperfect production inventory model for green products under type-2 fuzzy credit period, *Comput. Ind. Eng.*, **129** (2019), 435–453. https://doi.org/10.1016/j.cie.2019.01.059
32. A. I. Malik, B. Sarkar, Disruption management in a constrained multi-product imperfect production system, *J. Manuf. Syst.*, **56** (2020), 227–240. https://doi.org/10.1016/j.jmsy.2020.05.015
33. C. Rout, A. Paul, R. S. Kumar, D. Chakraborty, A. Goswami, Cooperative sustainable supply chain for deteriorating item and imperfect production under different carbon emission regulations, *J. Cleaner Prod.*, **272** (2020), 122170. https://doi.org/10.1016/j.jclepro.2020.122170
34. T. H. Chen, Optimizing pricing, replenishment and rework decision for imperfect and deteriorating items in a manufacturer-retailer channel, *Int. J. Prod. Econ.*, **183** (2017), 539–550. https://doi.org/10.1016/j.ijpe.2016.08.015
35. Y. S. Chiu, M. F. Wu, S. W. Chiu, H. H. Chang, A simplified approach to the multi-item economic production quantity model with scrap, rework, and multi-delivery, *J. Appl. Res. Technol.*, **13** (2015), 472–476. https://doi.org/10.1016/j.jart.2015.09.004
36. A. R. Nia, M. H. Far, S. T. A. Niaki, A hybrid genetic and imperialist competitive algorithm for green vendor managed inventory of multi-item multi-constraint EOQ model under shortage, *Appl. Soft Comput.*, **30** (2015), 353–364. https://doi.org/10.1016/j.asoc.2015.02.004
37. M. Tayyab, M. S. Habib, M. S. S. Jajja, B. Sarkar, Economic assessment of a serial production system with random imperfection and shortages: a step towards sustainability, *Comput. Ind. Eng.*, **177** (2022), 108398. https://doi.org/10.1016/j.cie.2022.108398
38. L. Moussawi-Haidar, M. Salameh, W. Nasr, Production lot sizing with quality screening and rework, *Appl. Math. Modell.*, **40** (2016), 3242–3256. https://doi.org/10.1016/j.apm.2015.09.095

39. S. Pal, G. S. Mahapatra, A manufacturing-oriented supply chain model for imperfect quality with inspection errors, stochastic demand under rework and shortages, *Comput. Ind. Eng.*, **106** (2017), 299–314. https://doi.org/10.1016/j.cie.2017.02.003

40. D. Sonntag, G. P. Kiesmüller, Disposal versus rework–inventory control in a production system with random yield, *Eur. J. Oper. Res.*, **267** (2018), 138–149. https://doi.org/10.1016/j.ejor.2017.11.019

41. M. Al-Salamah, Economic production quantity in an imperfect manufacturing process with synchronous and asynchronous flexible rework rates, *Oper. Res. Perspect.*, **6** (2019), 100103. https://doi.org/10.1016/j.orp.2019.100103

42. F. Lin, T. Jia, R. Y. Fung, P. Wu, Impacts of inspection rate on integrated inventory models with defective items considering capacity utilization: rework-versus delivery-priority, *Comput. Ind. Eng.*, **156** (2021), 107245. https://doi.org/10.1016/j.cie.2021.107245

43. B. Sarkar, J. Joo, Y. Kim, H. Park, M. Sarkar, Controlling defective items in a complex multi-phase manufacturing system, *RAIRO-Oper. Res.*, **56** (2022), 871–889. https://doi.org/10.1051/ro/2022019

44. A. Sepehri, U. Mishra, M. L. Tseng, B. Sarkar, Joint pricing and inventory model for deteriorating items with maximum lifetime and controllable carbon emissions under permissible delay in payments, *Mathematics*, **9** (2021), 470. https://doi.org/10.3390/math9050470

45. S. Kumar, M. Sigroha, K. Kumar, B. Sarkar, Manufacturing/remanufacturing based supply chain management under advertisements and carbon emission process, *RAIRO-Oper. Res.*, **56** (2022), 831–851. https://doi.org/10.1051/ro/2021189

46. A. Garai, B. Sarkar, Economically independent reverse logistics of customer-centric closed-loop supply chain for herbal medicines and biofuel, *J. Cleaner Prod.*, **334** (2022), 129977. https://doi.org/10.1016/j.jclepro.2021.129977

47. B. Sarkar, B. Mridha, S. Pareek, A sustainable smart multi-type biofuel manufacturing with the optimum energy utilization under flexible production, *J. Cleaner Prod.*, **332** (2022), 129869. https://doi.org/10.1016/j.jclepro.2021.129869

48. A. S. H. Kugele, W. Ahmed, B. Sarkar, Geometric programming solution of second degree difficulty for carbon ejection controlled reliable smart production system, *RAIRO-Oper. Res.*, **56** (2022), 1013–1029. https://doi.org/10.1051/ro/2022028

49. I. Moon, W. Y. Yun, B. Sarkar, Effects of variable setup cost, reliability, and production costs under controlled carbon emissions in a reliable production system, *Eur. J. Ind. Eng.*, **16** (2022), 371–397. https://doi.org/10.1504/EJIE.2022.123748

50. B. Sarkar, S. Kar, K. Basu, R. Guchhait, A sustainable managerial decision-making problem for a substitutable product in a dual-channel under carbon tax policy, *Comput. Ind. Eng.*, **172** (2022) 108635. https://doi.org/10.1016/j.cie.2022.108635

51. M. Rinaldi, M. Caterino, M. Fera, P. Manco, R. Macchiaroli, Technology selection in green supply chains-the effects of additive and traditional manufacturing, *J. Cleaner Prod.*, **282** (2021), 124554. https://doi.org/10.1016/j.jclepro.2020.124554

52. P. Ahi, C. Searcy, A comparative literature analysis of definitions for green and sustainable supply chain management, *J. Cleaner Prod.*, **52** (2013), 329–341. https://doi.org/10.1016/j.jclepro.2013.02.018

**Appendix A**

$TEP(.) = TEP(P, Q, p, \theta_c)$

The first order partial derivatives of the objective function with respect to the decision variables are

$$\begin{aligned}
\frac{\partial TEP(.)}{\partial P} &= -\frac{1}{Q}\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[Q\Big(\frac{-\delta_2}{P^2}+\delta_3\Big)(1-\pi)+\pi Q(1+\tau_2)\Big(\frac{-\delta_2}{P^2}+\delta_3\Big)+\frac{hQ^2(1-\pi^2)}{2P^2}\Big]\\
\frac{\partial TEP(.)}{\partial Q} &= \frac{1}{Q^2}\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[K+Q(\delta_1+\frac{\delta_2}{P}+\delta_3 P)(1-\pi)+K(1+\tau_1)\\
&+ Q\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3 P)+Q(1-\pi)\zeta R_C+\frac{Q^2(h_c-h)}{2}\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)\\
&+ \frac{hQ^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{R_1}(-2\pi)\Big)+\frac{\xi_3\theta_c^2}{2}\Big]\\
&- \frac{1}{Q}\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[(\delta_1+\frac{\delta_2}{P}+\delta_3 P)(1-\pi)+\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3 P)\\
&+ (1-\pi)\zeta R_C+Q(h_c-h)\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)+hQ\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{R_1}(-2\pi)\Big)\Big]\\
\frac{\partial TEP(.)}{\partial p} &= \xi_1\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\Big[p-\frac{1}{Q}\Big\{K+Q(\delta_1+\frac{\delta_2}{P}+\delta_3 P)(1-\pi)+K(1+\tau_1)\\
&+ Q\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3 P)+Q(1-\pi)\zeta R_C+\frac{Q^2(h_c-h)}{2}\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)\\
&+ \frac{hQ^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{R_1}(-2\pi)\Big)+\frac{\xi_3\theta_c^2}{2}\Big\}\Big]\\
&+ \Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[1-\frac{h\xi_1 Q}{D^2}\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\Big]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial TEP(.)}{\partial \theta_c} &= \frac{1}{Q}\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big(\frac{h\xi_2 Q^2}{2D^2}-\xi_3\theta_c\Big)+\xi_2\Big[p-\frac{1}{Q}\Big\{K+Q(\delta_1+\frac{\delta_2}{P}+\delta_3 P)(1-\pi)\\
&+ K(1+\tau_1)+Q\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3 P)+Q(1-\pi)\zeta R_C+\frac{Q^2(h_c-h)}{2}\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)\\
&+ \frac{hQ^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{R_1}(-2\pi)\Big)+\frac{\xi_3\theta_c^2}{2}\Big\}\Big]
\end{aligned}$$

where

$$\begin{aligned}
\Psi &= \frac{1}{Q}\Big(\xi_1\frac{(l_{max}-p)}{(p-l_{min})}+\xi_2\theta_c\Big)\Big[K+Q(\delta_1+\frac{\delta_2}{P}+\delta_3 P)(1-\pi)+K(1+\tau_1)\\
&+ Q\pi(1+\tau_2)(\delta_1+\frac{\delta_2}{P}+\delta_3 P)+Q(1-\pi)\zeta R_C+\frac{Q^2(h_c-h)}{2}\Big(\frac{\zeta^2(1-\pi)^2}{R_1}\Big)
\end{aligned}$$

$$+ \quad \frac{hQ^2}{2}\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right) + \frac{\xi_3\theta_c^2}{2}\Big]$$

and

$$\Upsilon = p\xi_1(l_{min} - l_{max}) + p\Big\{\xi_1(l_{max} + l_{min}) - 2\xi_2\theta_c l_{min}\Big\}\Big[1 - \frac{h\xi_1 Q}{D^2}\frac{(l_{min} - l_{max})}{(p - l_{min})^2}\Big]$$

$$\zeta = \xi_1(l_{min} - l_{max})\Psi + l_{min}(\xi_1 l_{max} + 2\theta_c\xi_2 l_{min})\Big[1 - \frac{h\xi_1 Q}{D^2}\frac{(l_{min} - l_{max})}{(p - l_{min})^2}\Big].$$

## Appendix B

The second order partial derivatives of the objective functions with respect to the decision variables are

$$\begin{aligned}
\frac{\partial^2 TEP(.)}{\partial P^2} &= -\frac{\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)}{P^3}\Big[2\delta_2(1-\pi) + 2(1+\tau_2)\delta_2 - hQ(1-\pi^2)\Big] = \psi(say)\\
\frac{\partial^2 TEP(.)}{\partial Q^2} &= -\frac{2}{Q^3}\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)\Big[K + Q(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + K(1+\tau_1)\\
&+ Q\pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P) + Q(1-\pi)\zeta R_C + \frac{Q^2(h_c - h)}{2}\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right)\\
&+ \frac{hQ^2}{2}\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right) + \frac{\xi_3\theta_c^2}{2}\Big]\\
&+ \frac{2\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)}{Q^2}\Big[(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + \pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)\\
&+ (1-\pi)\zeta R_C + Q(h_c - h)\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right) + hQ\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right)\Big]\\
&- \frac{\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)}{Q}\Big[\frac{(h_c - h)\zeta^2(1-\pi)^2}{R_1} + h\left(\frac{1}{D} - \frac{(1-\pi^2)}{P} + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right)\Big]\\
&= \chi(say)\\
\frac{\partial^2 TEP(.)}{\partial p^2} &= -2\xi_1\frac{(l_{min} - l_{max})}{(p - l_{min})^3}\Big[p - \frac{1}{Q}\Big\{K + Q(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + K(1+\tau_1)\\
&+ Q\pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P) + Q(1-\pi)\zeta R_C + \frac{Q^2(h_c - h)}{2}\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right)\\
&+ \frac{hQ^2}{2}\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right) + \frac{\xi_3\theta_c^2}{2}\Big\}\Big]\\
&+ 2\xi_1\frac{(l_{min} - l_{max})}{(p - l_{min})^2}\Big[1 - \frac{h\xi_1 Q}{D^2}\frac{(l_{min} - l_{max})}{(p - l_{min})^2}\Big]\\
&+ \left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)\frac{2h\xi_1 Q(l_{min} - l_{max})}{D^2(p - l_{min})^3} = \varphi(say)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 TEP(.)}{\partial \theta_c^2} &= -\frac{1}{Q}\left(\xi_3 + \frac{hQ^2\xi_2^2}{D^3}\right)\left(\xi_1 \frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right) - \frac{\xi_2}{Q}\left(\xi_3\theta_c - \frac{h\xi_2 Q^2}{2D^2}\right) = \tau\ (say)\\
\frac{\partial^2 TEP(.)}{\partial P\partial Q} &= \frac{\partial^2 TEP(.)}{\partial Q\partial P} = \frac{1}{Q^2}\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)\left[Q\left(\frac{-\delta_2}{P^2} + \delta_3\right)(1-\pi)\right.\\
&+ \left.\pi Q(1+\tau_2)\left(\frac{-\delta_2}{P^2} + \delta_3\right) + \frac{hQ^2(1-\pi^2)}{2P^2}\right] - \frac{1}{Q}\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)\left[\left(\frac{-\delta_2}{P^2} + \delta_3\right)(1-\pi)\right.\\
&+ \left.\pi(1+\tau_2)\left(\frac{-\delta_2}{P^2} + \delta_3\right) + \frac{hQ(1-\pi^2)}{P^2}\right] = \sigma\ (say)\\
\frac{\partial^2 TEP(.)}{\partial P\partial p} &= \frac{\partial^2 TEP(.)}{\partial p\partial P} = -\frac{\xi_1}{Q}\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\left[Q\left(\frac{-\delta_2}{P^2} + \delta_3\right)(1-\pi)\right.\\
&+ \left.\pi Q(1+\tau_2)\left(\frac{-\delta_2}{P^2} + \delta_3\right) + \frac{hQ^2(1-\pi^2)}{2P^2}\right] = \rho\ (say)\\
\frac{\partial^2 TEP(.)}{\partial P\partial \theta_c} &= \frac{\partial^2 TEP(.)}{\partial \theta_c\partial P} = -\frac{\xi_2}{Q}\left[Q\left(\frac{-\delta_2}{P^2} + \delta_3\right)(1-\pi) + \pi Q(1+\tau_2)\left(\frac{-\delta_2}{P^2} + \delta_3\right) + \frac{hQ^2(1-\pi^2)}{2P^2}\right]\\
&= \kappa(say)\\
\frac{\partial^2 TEP(.)}{\partial Q\partial p} &= \frac{\partial^2 TEP(.)}{\partial p\partial Q} = \xi_1\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\frac{1}{Q^2}\left[K + Q(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + K(1+\tau_1)\right.\\
&+ Q\pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P) + Q(1-\pi)\zeta R_C + \frac{Q^2(h_c-h)}{2}\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right)\\
&+ \left.\frac{hQ^2}{2}\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right) + \frac{\xi_3\theta_c^2}{2}\right]\\
&- \frac{\xi_1}{Q}\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\left[(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + \pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P) + (1-\pi)\zeta R_C\right.\\
&+ \left.\frac{(h_c-h)}{2}\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right) + hQ\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right) + \frac{\xi_3\theta_c^2}{2}\right]\\
&- \left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)\frac{h\xi_1}{D^2}\frac{(l_{min}-l_{max})}{(p-l_{min})^2} = \vartheta(say)\\
\frac{\partial^2 TEP(.)}{\partial Q\partial \theta_c} &= \frac{\partial^2 TEP(.)}{\partial \theta_c\partial Q} = \left(\frac{\xi_3\theta_c}{Q^2} + \frac{h}{2D^2}\right)\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right) + \frac{\xi_2}{Q^2}\left\{K + Q(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi)\right.\\
&+ K(1+\tau_1) + Q\pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P) + Q(1-\pi)\zeta R_C\\
&+ \frac{Q^2(h_c-h)}{2}\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right) + \frac{hQ^2}{2}\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right)\\
&+ \left.\frac{\xi_3\theta_c^2}{2}\right\} - \frac{\xi_2}{Q}\left[(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)(1-\pi) + \pi(1+\tau_2)(\delta_1 + \frac{\delta_2}{P} + \delta_3 P)\right.\\
&+ (1-\pi)\zeta R_C + Q(h_c-h)\left(\frac{\zeta^2(1-\pi)^2}{R_1}\right)\\
&+ \left.hQ\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{R_1}(-2\pi)\right)\right] = \theta\ (say)
\end{aligned}
$$

$$\frac{\partial^2 TEP(.)}{\partial p \partial \theta_c} = \frac{\partial^2 TEP(.)}{\partial \theta_c \partial p} = \frac{\xi_1}{Q}\left(\frac{h\xi_2 Q^2}{2D^2} - \xi_3\theta_c\right)\frac{(l_{min}-l_{max})}{(p-l_{min})^2} + \xi_2\left[1 - \frac{h\xi_1 Q}{D^2}\frac{(l_{min}-l_{max})}{(p-l_{min})^2}\right]$$
$$+ \left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)\frac{2h\xi_1\xi_2 Q}{D^3}\frac{(l_{min}-l_{max})}{(p-l_{min})^2} = \Omega\ (say)$$

$$|H_{11}| = \left|\frac{\partial^2 TEP(.)}{\partial P^2}\right| = -\frac{\left(\xi_1\frac{(l_{max}-p)}{(p-l_{min})} + \xi_2\theta_c\right)}{P^3}\left[2\delta_2(1-\pi) + 2(1+\tau_2)\delta_2 - hQ(1-\pi^2)\right] = \psi$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 TEP(.)}{\partial P^2} & \frac{\partial^2 TEP(.)}{\partial P\partial Q} \\ \frac{\partial^2 TEP(.)}{\partial Q\partial P} & \frac{\partial^2 TEP(.)}{\partial Q^2} \end{vmatrix} = \frac{\partial^2 TEP(.)}{\partial P^2}\frac{\partial^2 TEP(.)}{\partial Q^2} - \left(\frac{\partial^2 TEP(.)}{\partial P\partial Q}\right)^2$$
$$= \psi\chi - \sigma^2$$

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 TEP(.)}{\partial P^2} & \frac{\partial^2 TEP(.)}{\partial P\partial Q} & \frac{\partial^2 TEP(.)}{\partial P\partial p} \\ \frac{\partial^2 TEP(.)}{\partial Q\partial P} & \frac{\partial^2 TEP(.)}{\partial Q^2} & \frac{\partial^2 TEP(.)}{\partial Q\partial p} \\ \frac{\partial^2 TEP(.)}{\partial p\partial P} & \frac{\partial^2 TEP(.)}{\partial p\partial Q} & \frac{\partial^2 TEP(.)}{\partial p^2} \end{vmatrix} = \psi(\chi\varphi - \vartheta^2) - \sigma(\sigma\varphi - \rho\vartheta) + \rho(\sigma\vartheta - \rho\chi)$$

$$|H_{44}| = \begin{vmatrix} \frac{\partial^2 TEP(.)}{\partial P^2} & \frac{\partial^2 TEP(.)}{\partial P\partial Q} & \frac{\partial^2 TEP(.)}{\partial P\partial p} & \frac{\partial^2 TEP(.)}{\partial P\partial \theta_c} \\ \frac{\partial^2 TEP(.)}{\partial Q\partial P} & \frac{\partial^2 TEP(.)}{\partial Q^2} & \frac{\partial^2 TEP(.)}{\partial Q\partial p} & \frac{\partial^2 TEP(.)}{\partial Q\partial \theta_c} \\ \frac{\partial^2 TEP(.)}{\partial p\partial P} & \frac{\partial^2 TEP(.)}{\partial p\partial Q} & \frac{\partial^2 TEP(.)}{\partial p^2} & \frac{\partial^2 TEP(.)}{\partial p\partial \theta_c} \\ \frac{\partial^2 TEP(.)}{\partial \theta_c\partial P} & \frac{\partial^2 TEP(.)}{\partial \theta_c\partial Q} & \frac{\partial^2 TEP(.)}{\partial \theta_c\partial p} & \frac{\partial^2 TEP(.)}{\partial \theta_c^2} \end{vmatrix} = \psi[\chi(\varphi\tau - \Omega^2) - \vartheta(\vartheta\tau - \theta\Omega) + \theta(\vartheta\Omega - \varphi\theta)]$$
$$+\rho[\sigma(\vartheta\tau - \theta\Omega) - \chi(\rho\tau - \kappa\Omega) + \theta(\rho\theta - \kappa\vartheta)] - \sigma[\sigma(\varphi\tau - \Omega^2) - \vartheta(\rho\tau - \kappa\Omega)$$
$$+\theta(\rho\Omega - \kappa\varphi)] - \kappa[\sigma(\vartheta\Omega - \theta\varphi) - \chi(\rho\Omega - \kappa\varphi) + \vartheta(\rho\theta - \kappa\vartheta)]$$