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Research article

An SEIR model for information propagation with a hot search effect in complex networks

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Abstract: We formulate an SEIR model for information propagation with the effect of a hot search in complex networks. Mathematical analysis is conducted in both a homogeneous network and heterogeneous network. The results reveal that the dynamics are completely determined by the basic propagation number if the effect of a hot search is absent. On the other hand, when the effect of a hot search is taken into account, there exists no information-free equilibrium, and the information-propagating equilibrium is stable if the threshold is greater than 1. Numerical simulations were performed to examine the sensitivity of the parameters to the basic propagation number and the propagable nodes. Furthermore, the proposed model has been applied to fit the collected data for two types of information spreading in Sina Weibo, which confirmed the validity of our model and simulated the dynamical behaviors of information propagation.

Keywords: information propagation; online social networks; hot search effect; dynamics; case study

1. Introduction

With the rapid development of mobile smart terminals, online social networks (OSNs) have become the main platform for Internet users to communicate with each other. The popular OSNs, such as Facebook, Twitter and Sina Weibo, enable people to share information simultaneously with a large number of peers and thus play an important role in information propagation. Modeling the process of information spreading in OSNs has attracted more and more attention to understanding the way information is disseminated and the network structures [1–6]. One of the primary means of information dissemination in OSNs is through connections between users of the network. When original real-time information is released on the web, it is possible that a very large number of users have the interest to read, comment and retweet it; then, one user becomes the follower of another and, eventually, millions of individual users can share the information.

In the new media era, a hot search that tends to focus on people's livelihood and current affairs has become an important source to understand what Internet citizens are concerned about, and it serves as another way to affect information diffusion significantly. A hot search is generally calculated according to the real behaviors of the users in OSNs, and it a real-time list is formed based on data indicators such as search volume, blog volume and interaction volume. Whenever a piece of information is pushed to the hot search list, it is highlighted from a amout of network information and is more accessible to online users within a certain period of time, which encourages people to read and forward it. From this perspective, a mathematical model with a hot search effect may be more reasonable to reveal the important consequences of it in the process of information propagation. To our knowledge, there are few results related to hot searches to describe their effect on the time evolution of information spreading. However, some other characteristics emerged in OSNs have been incorporated into models to improve the understanding of information propagation mechanisms, such as the homepage effect in [7] and super spreader in [8].

Due to the similarity between the way information spreads and the transmission of infectious diseases, classical epidemic models have been adopted and extended to understand information propagation on social network platforms [7, 9–11]. For instance, the influence of opinions involving opinion leaders on delayed information propagation dynamics was investigated by establishing the OD-SFI (opinion-delay susceptible-forwarding-immunized) model [10]. In recent years, some scholars have applied complex networks to describe the characteristics of information dissemination and refine the establishing models. Assuming that each node has the same probability of contacting with the other nodes, homogeneous networks have been used in some models to investigate the evolution of different states in information dissemination. Huo et al. proposed a dynamic model of I2SR for the rumor spreading in [12], in which the state nodes were divided into the high rate of active spreaders and the low rate of the ones. To examine the effects of interactions between two rumors, the authors in [13] detailed the propagating dynamics of a nine-state model in homogeneous networks under symmetric conditions. The delays in expert intervention and government action were included in the model of malicious news in [14], which were demonstrated to be a cause of unstable situations of the homogeneous social network. To address the heterogeneity in the information spreading, some studies have focused on modeling the propagation in heterogeneous networks, which contribute to revealing the underlying mechanisms in the spreading process [15–17] and identifying control strategies to optimize the network performance [18, 19].

Considering the different factors that affect information dissemination, a variety of models have been proposed to study the dynamic behaviors of the social network system. Zhu et al. investigated the influence of a forced silence function, time delay and network topology on rumor propagation in OSNs comprehensively [20], and they also established a susceptible-believed-denied type model for rumor spreading with a nonlinear incidence rate and time delay in complex networks [21]. To help with designing effective communication strategies during a public hot event, Yin et al. proposed the cross-transmission susceptible-forwarding-immune model to describe the dynamics of co-propagation, particularly with focus on the cross-transmission effects [22]. A multi-scale model was formulated in [23], which explicitly modeled both the disease transmission with a saturated recovery rate and information transmission to evaluate the effect of information transmission on dynamic properties. These studies have characterized the information diffusion and improved the understanding of the dynamics in time evolution.

In order to study the possible effect of a hot search list in OSNs, we developed a mathematical model for information propagation in complex networks to describe the spreading dynamics, which is different from the existing models. In this model, the hot search is regarded as a propagation source that plays a crucial role in accelerating the spreading and enlarging the influenced range. Particularly, the parameters, including the presence rate and the effect of a hot search, have been introduced to examine the potential impacts of it on information diffusion. The stabilities of the information-free equilibrium and the information-propagating equilibrium were analyzed in homogeneous networks and heterogeneous networks respectively. And, a sensitivity analysis was conducted to investigate the correlation between the concerned parameters and the basic propagation number. In addition, two specific datasets of information dissemination in OSNs were collected for a case study, which validates our proposed SEIR model and illustrates the capacity of it to simulate the dynamical behaviors of information propagation.

The organization of the remaining part is as follows. In Section 2, we introduce the model framework. In Section 3, the model in homogeneous networks is presented and analyzed in two cases, i.e., with or without the hot search effect. Section 4 focuses on the dynamical behaviors of the model in heterogeneous networks in the same two cases as in Section 3. Numerical simulations are presented in Section 5 to describe the sensitive analysis. In Section 6, the model is demonstrated to fit two types of datasets collected from Sina Weibo. Section 7 gives the summary and conclusions.

2. Model framework

Based on the compartmental model of the classical SEIR type, we formulate a susceptible-hesitant-propagable-rejected model that describes the evolution of information dissemination in OSNs, such as Weibo. All users are divided into classes of susceptible (S), hesitant (E), propagable (I) and rejected (R), respectively. According to the characteristics of online information propagation, the susceptible users refer to the population that have not heard of or known about the target information. When being informed of the related propagation through hotspots, tweets or retweets from other users, but not retweeting or forwarding the topic promptly due to hesitation, the susceptible state shifts subsequently to the hesitant one. Otherwise, if the susceptible have been convinced of the information and choose to retweet it actively, then they progress to the propagable class I. In Class R, the population is in the state of never having access to or no longer being interested in the relevant information. Let $S(t)$, $E(t)$, $I(t)$ and $R(t)$ be the population at time t in the corresponding classes, respectively.

In OSNs, it is common for users to subscribe or unsubscribe to some topic or login or logout of a certain application program. Consequently, we suppose that all of the new users in the network are susceptible and the individuals in each state of the SEIR model can leave the information propagation progress at the same rate. Let Λ be the recruitment rate into the susceptible compartment and μ be the removal rate from the network. Different from the exposed state in disease transmission, it is optional for the hesitant individuals in information dissemination to forward it and become propagable subjectively, or to neglect it and transfer to the rejected compartment, proportions of which are denoted by ε and γ_1, respectively.

Considering the role of the hot search effect in spreading the information, we introduce a constant rate p for the hot search presence for some topic information in a certain period of time, such as the Weibo Hot Topic List, thus, $\frac{1}{p}$ denotes the average leaving time from S due to the information posted on the hot search list. We assume that there is a proportion of q among pS that is interested in the information and then transfers to the hesitant class. This means that the parameter q reflects the effect of a hot search on susceptible users. The other individuals of the proportion $1 - q$ show disinterest and shift to the rejected state. Although the hot search list has an obvious impact on promoting the propagation, it is still possible that direct contacts in OSNs induce a higher proportion of the susceptible population to retweet some topics, since a number of individuals are more likely to trust the people they follow on social platforms.

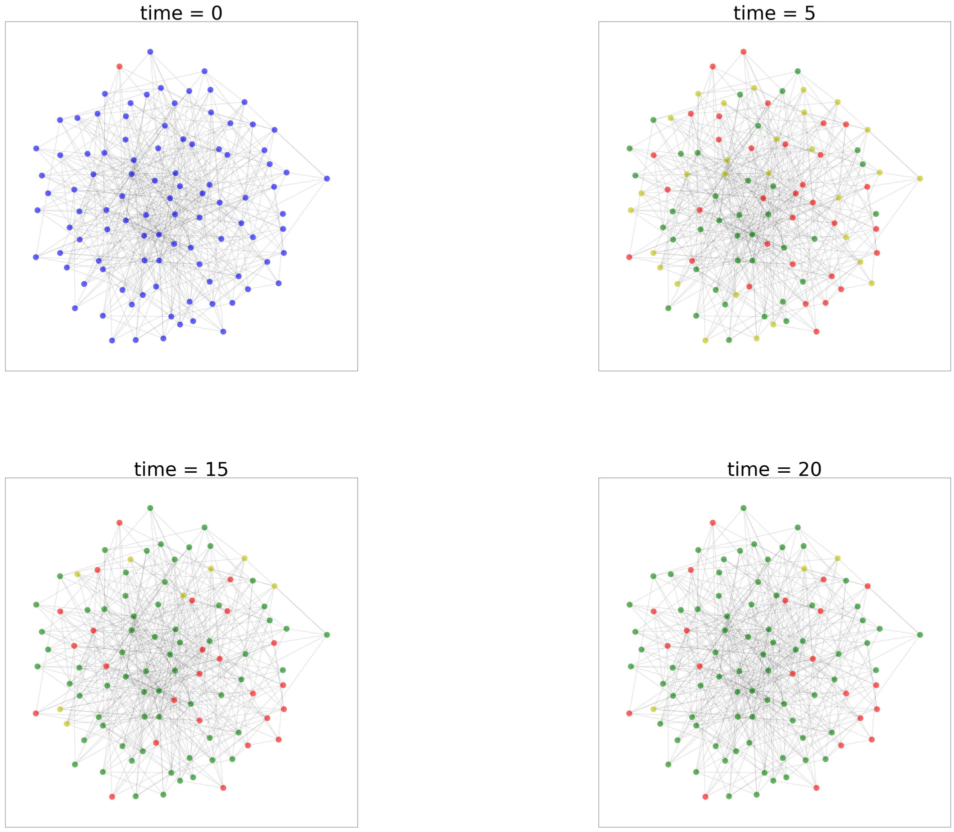

Figure 1. Schematic diagram of the state transitions on BA network.

The mixed group of users in online social media is considered as a complex network, where each user can be seen as a node in the corresponding state, and their connections are represented by the edges along which the information can spread. The topology of the network has a great influence in the overall behavior of information spreading, and the connectivity fluctuations of the network play a major role by enhancing the incidence of the propagation. Many complex social networks can be classified into homogeneous networks and heterogeneous networks according to their degree distribution. Homogeneous networks are a more idealized type of network. More generally, heterogeneous networks with a more complex topology structure are much closer to real social networks. The process of state transitions in time steps of 0, 5, 15 and 20 is shown graphically in Figure 1, which is on a Barabási-Albert (BA) scale-free network containing 100 nodes. At the very beginning, we place a red node that has forwarded a message in propagable state. The susceptible, hesitant and rejected states are denoted by nodes of blue, yellow and green respectively.

Table 1. Definitions of parameters in SEIR model.

Parameter	Description
Λ	rate of recruitment into the susceptible class
μ	per capital removal rate from the network
β	information transmitting rate
α	proportion of the incidence moving to hesitant class
ε	transition rate from E to I
γ_1	transition rate from E to R
γ_2	transition rate from I to R
p	rate of hot search presence
q	the effect of hot search

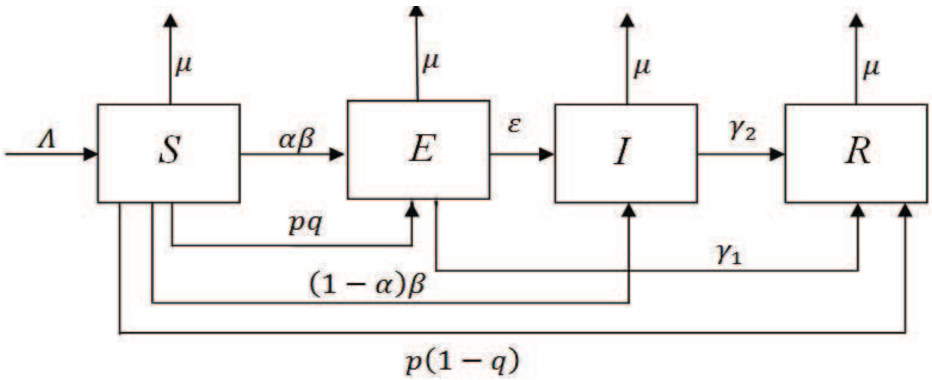

Figure 2. Propagation structure diagram of SEIR model.

To clarify the transition process for the status of each user in our model, the flow diagram is illustrated in Figure 2 and the definitions of those parameters are listed in Table1. All parameters are constants and remain nonnegative.

3. Model in homogeneous networks

3.1. Model formulation

First, according to the proposed framework of information propagation in Figure 2 , we consider the SEIR model in a homogeneous network, which is described by the mean-field theory, as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta\overline{k}S(t)I(t) - \mu S(t) - pS(t), \\ \frac{dE(t)}{dt} = \alpha\beta\overline{k}S(t)I(t) - \varepsilon E(t) - \gamma_1 E(t) - \mu E(t) + pqS(t), \\ \frac{dI(t)}{dt} = (1-\alpha)\beta\overline{k}S(t)I(t) + \varepsilon E(t) - \gamma_2 I(t) - \mu I(t), \\ \frac{dR(t)}{dt} = p(1-q)S(t) + \gamma_1 E(t) + \gamma_2 I(t) - \mu R(t), \end{cases} \tag{3.1}$$

where $\overline{k}$ denotes the average degree of the network. The initial conditions of System (3.1) are given as

$$S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0.$$

For the initial conditions, there exists a unique solution for Model (3.1) by the existence and uniqueness theory of ordinary differential equations. In addition, the solution with nonnegative initial conditions is nonnegative. Indeed, $\frac{dS(t)}{dt}|_{S=0} = \Lambda > 0$, $\frac{dE(t)}{dt}|_{E=0} = \alpha\beta\bar{k}SI + pqS > 0$ and the similar inequalities hold for I and R.

Denote the total number of users as N, i.e., $N(t) = S(t) + E(t) + I(t) + R(t)$. Adding up all equations in (3.1) gives

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t),$$

which implies $\lim\limits_{t\to+\infty} N(t) = \frac{\Lambda}{\mu}$. Obviously,

$$\mathcal{D} = \left\{ (S, E, I, R) \middle| S, E, I, R \geq 0, S + E + I + R \leq \frac{\Lambda}{\mu} \right\}$$

is the positively invariant domain of System (3.1).

To reveal the hot search effect on the process of information spreading, we conduct dynamic analysis in two cases, i.e., $q = 0$ and $q \neq 0$.

3.2. Model without hot search effect

For the case of $q = 0$, System (3.1) becomes

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta\bar{k}S(t)I(t) - \mu S(t) - pS(t), \\ \frac{dE(t)}{dt} = \alpha\beta\bar{k}S(t)I(t) - \varepsilon E(t) - \gamma_1 E(t) - \mu E(t), \\ \frac{dI(t)}{dt} = (1-\alpha)\beta\bar{k}S(t)I(t) + \varepsilon E(t) - \gamma_2 I(t) - \mu I(t), \\ \frac{dR(t)}{dt} = pS(t) + \gamma_1 E(t) + \gamma_2 I(t) - \mu R(t). \end{cases} \tag{3.2}$$

For an epidemic model, there is a typical threshold parameter that determines whether the infectious disease will spread, i.e., the basic reproductive number, which gives the expected number of secondary cases which are produced in a completely susceptible population by an infective individual in his infectious period. Due to the similarity, such a threshold, i.e., the basic propagation number, exists in the information propagation model, and the information spread in the network is determined by the magnitude of this parameter.

By the next generation matrix approach [24], we can calculate the basic propagation number as

$$\mathcal{R}_0^1 = \rho(FV^{-1}) = \frac{\varepsilon\alpha\beta\Lambda\bar{k} + (1-\alpha)\beta\Lambda\bar{k}(\varepsilon+\gamma_1+\mu)}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)(\mu+p)},$$

where ρ denotes the spectral radius of FV^{-1} and

$$F = \begin{pmatrix} 0 & \frac{\alpha\beta\Lambda\bar{k}}{\mu+p} \\ 0 & \frac{(1-\alpha)\beta\Lambda\bar{k}}{\mu+p} \end{pmatrix}, V = \begin{pmatrix} \varepsilon + \gamma_1 + \mu & 0 \\ -\varepsilon & \gamma_2 + \mu \end{pmatrix}.$$

The value of $\mathcal{R}_0^1$ can be seen as the sum of two separate terms representing secondary infections, which is produced by the hesitant and propagable individuals, respectively.

It is obvious that System (3.2) has an information-free equilibrium:

$$Q_0 = \left(\frac{\Lambda}{\mu+p}, 0, 0, \frac{p\Lambda}{\mu(\mu+p)}\right).$$

As for the information-propagating equilibrium $Q_1 = (S^*, E^*, I^*, R^*)$, it exists if $\mathcal{R}_0^1 > 1$, where

$$S^* = \frac{\Lambda}{\beta\bar{k}I^*+\mu+p}, \quad E^* = \frac{\alpha\beta\bar{k}S^*I^*}{\varepsilon+\gamma_1+\mu}, \quad I^* = -\frac{b}{a}, \quad R^* = \frac{pS^*+\gamma_1E^*+\gamma_2I*}{\mu},$$

with

$$\begin{aligned} a &= (\varepsilon + \gamma_1 + \mu)(\gamma_2 + \mu)\beta\bar{k} > 0, \\ b &= (\varepsilon + \gamma_1 + \mu)\left[(\mu + p)(\gamma_2 + \mu) - (1 - \alpha)\beta\Lambda\bar{k}\right] - \Lambda\varepsilon\alpha\beta\bar{k} < 0. \end{aligned}$$

Theorem 3.1. *For System* (3.2)*, if* $\mathcal{R}_0^1 < 1$*, then the information-free equilibrium point* Q_0 *is globally asymptotically stable, and it is unstable if* $\mathcal{R}_0^1 > 1$.

Proof. Note that the users in the state R do not appear in other states, thus, the R equation can be neglected in the remaining analysis. We first discuss the local stability of Q_0 when $\mathcal{R}_0^1 < 1$ and its instability when $\mathcal{R}_0^1 > 1$.

The characteristic equation of System (3.2) at Q_0 is

$$f_0(\lambda_0) := (\lambda_0 + \mu + p)\left(a_0\lambda_0^2 + b_0\lambda_0 + c_0\right) = 0,$$

with

$$\begin{aligned} a_0 &= \mu + p, \\ b_0 &= (\mu + p)(\varepsilon + \gamma_1 + \mu) + (\alpha - 1)\beta\Lambda\bar{k} + (\gamma_2 + \mu)(\mu + p), \\ c_0 &= \left[(\alpha - 1)\beta\Lambda\bar{k} + (\gamma_2 + \mu)(\mu + p)\right](\varepsilon + \gamma_1 + \mu) - \varepsilon\alpha\beta\Lambda\bar{k}. \end{aligned}$$

It is easy to see that $b_0 > 0$ and $c_0 > 0$ if $\mathcal{R}_0^1 < 1$, then, $f_0(\lambda_0) = 0$ has only roots with the negative real part, which certifies the local stability of Q_0 by the Hurwitz criterion [25]. Otherwise, if $\mathcal{R}_0^1 > 1$, then $c_0 < 0$, so Q_0 is unstable.

Then, consider the Lyapunov function of the following form:

$$V(t) = \varepsilon E(t) + (\varepsilon + \gamma_1 + \mu) I(t).$$

Calculating the time derivative of $V(t)$, along with System (3.2), gives

$$\frac{dV(t)}{dt}\Big|_{(3.2)} = (\gamma_2 + \mu)(\varepsilon + \gamma_1 + \mu)\left(\mathcal{R}_0^1 - 1\right) I(t) \ .$$

If $\mathcal{R}_0^1 < 1$, it follows that $\frac{dV}{dt} \leq 0$ and the equality $\frac{dV}{dt} = 0$ holds only at Q_0. According to LaSalle's invariance principle [26], the information-free equilibrium Q_0 is globally asymptotically stable.

Theorem 3.2. *If* $\mathcal{R}_0^1 > 1$*, the information-propagating equilibrium* Q_1 *is locally stable.*

Proof. The Jacobi matrix of System (3.2) at Q_1 is

$$J_1(Q_1) = \begin{pmatrix} -\beta\bar{k}I^* - \mu - p & 0 & -\beta\bar{k}S^* \\ \alpha\beta\bar{k}I^* & -\varepsilon - \gamma_1 - \mu & \alpha\beta\bar{k}S^* \\ (1-\alpha)\beta\bar{k}I^* & \varepsilon & (1-\alpha)\beta\bar{k}S^* - \gamma_2 - \mu \end{pmatrix},$$

which leads to the characteristic equation

$$f_1(\lambda_1) := A_3\lambda_1^3 + A_2\lambda_1^2 + A_1\lambda_1 + A_0 = 0, \tag{3.3}$$

where

$$\begin{aligned}
A_3 &= \beta\bar{k}I^* + \mu + p,\\
A_2 &= (\alpha-1)\beta\Lambda\bar{k} + \left(\beta\bar{k}I^* + 3\mu + p + \varepsilon + \gamma_1 + \gamma_2\right)\left(\beta\bar{k}I^* + \mu + p\right),\\
A_1 &= \left(\beta\bar{k}I^* + 2\mu + p + \varepsilon + \gamma_1\right)\left[(\alpha-1)\beta\Lambda\bar{k} + (\gamma_2+\mu)\left(\beta\bar{k}I^* + \mu + p\right)\right]\\
&\quad + (1-\alpha)\beta\bar{k}I^*\beta\Lambda\bar{k} - \varepsilon\alpha\beta\Lambda\bar{k} + \left(\beta\bar{k}I^* + \mu + p\right)^2(\varepsilon + \gamma_1 + \mu),\\
A_0 &= \left[(\alpha-1)\beta\Lambda\bar{k} + (\gamma_2+\mu)\left(\beta\bar{k}I^* + \mu + p\right)\right]\left(\beta\bar{k}I^* + \mu + p\right)(\varepsilon + \gamma_1 + \mu)\\
&\quad + (1-\alpha)\beta\bar{k}I^*\beta\Lambda\bar{k}(\varepsilon + \gamma_1 + \mu) - \varepsilon\alpha\beta\Lambda\bar{k}(\mu + p).
\end{aligned}$$

It is easy to obtain that $A_i > 0$ $(i = 0,1,2,3)$ and $A_1A_2 - A_0A_3 > 0$ when $\mathcal{R}_0^1 > 1$. The Hurwitz criteria implies that the characteristic equation at Q_1 only has roots with the negative real part, which ensures the local stability of Q_1.

3.3. Model with hot search effect

Note that Model (3.1) with $q \neq 0$ does not contain an information-free equilibrium and consequently has no basic propagation number. It is observed that when the threshold $\mathcal{R}_0^1 > 1$, there exists an information-propagating equilibrium for System (3.1), denoted by $Q_2 = (S^{**}, E^{**}, I^{**}, R^{**})$, where

$$S^{**} = \frac{\Lambda}{\beta\bar{k}I^{**} + \mu + p}, \quad E^{**} = \frac{\alpha\beta\bar{k}S^{**}I^{**} + pqS^{**}}{\varepsilon + \gamma_1 + \mu},$$
$$I^{**} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad R^{**} = \frac{p(1-q)S^{**} + \gamma_1E^{**} + \gamma_2I^{**}}{\mu},$$

with the same a and b as in Q_1 and $c = -\Lambda\varepsilon pq < 0$. Assume $\alpha > q$, which implies that direct contacts in the network attract a higher proportion of the susceptible population to follow some information; then, we obtain the following local stability of Q_2.

Theorem 3.3. *For System* (3.1)*, if the threshold* $\mathcal{R}_0^1 > 1$*, then the information-propagating equilibrium* $Q_2 = (S^{**}, E^{**}, I^{**}, R^{**})$ *is locally asymptotically stable.*

Proof. When neglecting the R equation, the Jacobi matrix of System (3.1) at Q_2 is

$$J_2 = \begin{pmatrix} -\beta\bar{k}I^{**} - \mu - p & 0 & -\beta\bar{k}S^{**} \\ \alpha\beta\bar{k}I^{**} + pq & -\varepsilon - \gamma_1 - \mu & \alpha\beta\bar{k}S^{**} \\ (1-\alpha)\beta\bar{k}I^{**} & \varepsilon & (1-\alpha)\beta\bar{k}S^{**} - \gamma_2 - \mu \end{pmatrix}.$$

Then, the characteristic equation of J_2 follows

$$f_2(\lambda_2) := B_3\lambda_2^3 + B_2\lambda_2^2 + B_1\lambda_2 + B_0 = 0,$$

where B_i can be obtained by substituting I^{**} for I^* in the expression of $A_i (i = 3,2,1)$ in Eq (3.3), and $B_0 = A_0 + \varepsilon\beta\Lambda pq\bar{k}$. Notice that $I^{**} > I^*$ leads to $B_i > A_i$ for $i = 0,1,2,3$. If $\alpha > q$, it can be verified that $B_1B_2 - B_0B_3 > 0$. According to the Hurwitz criteria [24], we have the local stability of Q_2 when $\mathcal{R}_0^1 > 1$.

The results of numerical simulation (see Figure 3) indicate that the hot search in a social network of information propagation has an obvious effect on the evolution of state density in Model (3.1), which is illustrated when $q = 0$ and $q = 0.3$ for the homogenous network with size $N = 1000$. From the comparison, we can see that it is necessary to take the effect of a hot search into consideration in network propagation modeling.

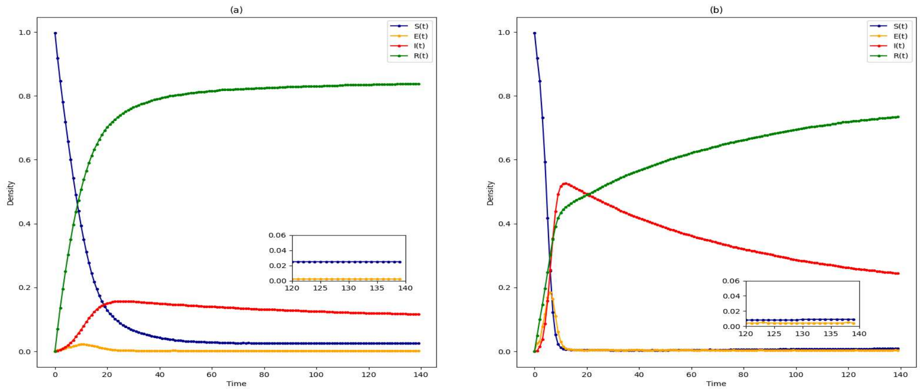

Figure 3. Plots of state density with and without the hot search effect. (a): $q = 0$. (b): $q = 0.3$.

4. Model in heterogeneous networks

4.1. Model formulation

Due to the characteristics in real world networks, specifically in OSNs, heterogeneous networks are used to describe the pattern of information propagation more accurately. In this subsection, in order to take into account the heterogeneity induced by the presence of nodes with different connectivity, all users in the network are partitioned into n groups according to their connections from 1 to n, with n being the maximum number of connectivity. Assuming that the users in the group i have the same degree $k_i (i = 1, 2, \ldots, n)$, the densities of the susceptible, hesitant, propagable and rejected nodes of connectivity k_i at time t are denoted as $S_{k_i}(t), E_{k_i}(t), I_{k_i}(t)$ and $R_{k_i}(t)$, respectively. In addition, we assume that the recruitment rate equals the removal rate in the heterogeneous network, i.e., $\Lambda = \mu$, and that these state variables of time are connected by means of the normalization condition, $S_{k_i}(t) + E_{k_i}(t) + I_{k_i}(t) + R_{k_i}(t) = 1$. Based on the mean field theory, we formulate the SEIR model of information dissemination in a heterogeneous network as follows:

$$\begin{cases} \frac{dS_{k_i}(t)}{dt} = \Lambda - \beta k_i S_{k_i}(t)\Theta(t) - \mu S_{k_i}(t) - pS_{k_i}(t), \\ \frac{dE_{k_i}(t)}{dt} = \alpha\beta k_i S_{k_i}(t)\Theta(t) - \varepsilon E_{k_i}(t) - \gamma_1 E_{k_i}(t) - \mu E_{k_i}(t) + pqS_{k_i}(t), \\ \frac{dI_{k_i}(t)}{dt} = (1-\alpha)\beta k_i S_{k_i}(t)\Theta(t) + \varepsilon E_{k_i}(t) - \gamma_2 I_{k_i}(t) - \mu I_{k_i}(t), \\ \frac{dR_{k_i}(t)}{dt} = p(1-q)S_{k_i}(t) + \gamma_1 E_{k_i}(t) + \gamma_2 I_{k_i}(t) - \mu R_{k_i}(t), \end{cases} \tag{4.1}$$

with the initial conditions

$$S_{k_i}(0) > 0, \ E_{k_i}(0) \geq 0, I_{k_i}(0) \geq 0, R_{k_i}(0) \geq 0$$

and $S_{k_i}(0) + E_{k_i}(0) + I_{k_i}(0) + R_{k_i}(0) = 1$. Define $\Theta(t) = \frac{1}{\langle k \rangle}\sum_{i=1}^{n} k_i p(k_i) I_{k_i}(t)$, which denotes the probability that any given node is connected to a node in the propagable state. $p(k_i)$ denotes the connectivity distribution of a node with the degree k_i and $\langle k \rangle = \sum_i k_i p(k_i)$ denotes the finite average degree of all nodes in the network, which address the effects of contact heterogeneity in information spreading.

For System (4.1), it is claimed that the solution with the above initial condition is nonnegative for all $t \geq 0$. Actually, if there exists $t_0 > 0$ such that $S_{k_i}(t_0) = 0$ and $S_{k_i}(t) > 0$ for $t \in (0, t_0)$, then $S'_{k_i}(t_0) = \Lambda > 0$, which implies that $S_{k_i}(t) \geq 0$ for all $t \geq 0$. By the expression of $\Theta(t)$, it is known that if there is k_i^* satisfying $I_{k_i^*}(0) > 0$, then $\Theta(0) > 0$. By the continuity of $\Theta(t)$, there exists t_1 such that $\Theta(t) > 0$ for any $t \in (0, t_1)$. Then, for $E_{k_i}(t)$, we have

$$E_{k_i}(t) \geq E_{k_i}(0)e^{-(\varepsilon+\gamma_1+\mu)t} \geq 0$$

for $t \in (0, t_2)$. If there exists $t_2 > t_1$ such that $\Theta(t_2) = 0$ and $\Theta(t) > 0$ for $t \in (0, t_2)$, then

$$\left.\frac{d\Theta(t)}{dt}\right|_{t=t_2} = \frac{1}{\langle k \rangle}\sum_{i=1}^{n} k_i p(k_i)\left[(1-\alpha)\beta k_i S_{k_i}(t)\Theta(t) + \varepsilon E_{k_i} - (\gamma_2+\mu)I_{k_i}(t)\right]\Big|_{t=t_2} \geq 0,$$

leading to a contradiction. Thus, $\Theta(t) > 0$ is valid for all $t > 0$.

In the following, we will discuss the model in two cases of $q = 0$ and $q \neq 0$, respectively, as in Section 3.2. The dynamic analysis is conducted theoretically and the effect of a hot search on the time evolution of state variables is demonstrated numerically.

4.2. Model without hot search effect

If the hot search effect is not considered ($q = 0$), then System (4.1) is reduced as

$$\begin{cases} \frac{dS_{k_i}(t)}{dt} = \Lambda - \beta k_i S_{k_i}(t)\Theta(t) - \mu S_{k_i}(t) - pS_{k_i}(t), \\ \frac{dE_{k_i}(t)}{dt} = \alpha\beta k_i S_{k_i}(t)\Theta(t) - \varepsilon E_{k_i}(t) - \gamma_1 E_{k_i}(t) - \mu E_{k_i}(t), \\ \frac{dI_{k_i}(t)}{dt} = (1-\alpha)\beta k_i S_{k_i}(t)\Theta(t) + \varepsilon E_{k_i}(t) - \gamma_2 I_{k_i}(t) - \mu I_{k_i}(t), \\ \frac{dR_{k_i}(t)}{dt} = pS_{k_i}(t) + \gamma_1 E_{k_i}(t) + \gamma_2 I_{k_i}(t) - \mu R_{k_i}(t). \end{cases} \tag{4.2}$$

By the next generation matrix approach, similar to $\mathcal{R}_0^1$, the basic propagation number for System (4.2) is defined as

$$\mathcal{R}_0^2 = \frac{\varepsilon\alpha\beta\Lambda + (1-\alpha)\beta\Lambda(\varepsilon+\gamma_1+\mu)}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)(\mu+p)}\frac{\langle k^2 \rangle}{\langle k \rangle},$$

in which the term $\langle k^2 \rangle = \sum_i k_i^2 p(k_i)$ presents the connectivity fluctuations.

It is easy to obtain that System (4.2) always has the information-free equilibrium

$$G_0 = \left\{\left(S_{k_1}^0, E_{k_1}^0, I_{k_1}^0, R_{k_1}^0\right), \ldots, \left(S_{k_n}^0, E_{k_n}^0, I_{k_n}^0, R_{k_n}^0\right)\right\},$$

where $S_{k_i}^0 = \frac{\Lambda}{\mu+p}, E_{k_i}^0 = 0, I_{k_i}^0 = 0, R_{k_i}^0 = \frac{p\Lambda}{\mu(\mu+p)}$ for $i = 1, 2, \ldots, n$. If $\mathcal{R}_0^2 > 1$, there exists an information-propagating equilibrium

$$G_1 = \left\{\left(S_{k_1}^*, E_{k_1}^*, I_{k_1}^*, R_{k_1}^*\right) \ldots \left(S_{k_n}^*, E_{k_n}^*, I_{k_n}^*, R_{k_n}^*\right)\right\},$$

where

$$\begin{aligned} S_{k_i}^* &= \frac{\Lambda}{\beta k_i\Theta^*+\mu+p}, \qquad E_{k_i}^* = \frac{\alpha\beta k_i\Lambda\Theta^*}{(\beta k_i\Theta^*+\mu+p)(\varepsilon+\gamma_1+\mu)}, \\ I_{k_i}^* &= \frac{(1-\alpha)\beta k_i\Lambda\Theta^*}{(\beta k_i\Theta^*+\mu+p)(\gamma_2+\mu)} + \frac{\varepsilon\alpha\beta k_i\Lambda\Theta^*}{(\beta k_i\Theta^*+\mu+p)(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)}, \\ R_{k_i}^* &= \frac{pS_{k_i}^*+\gamma_1 E_{k_i}^*+\gamma_2 I_{k_i}^*}{\mu} \end{aligned}$$

for $i = 1, 2, \ldots, n$, with $\Theta^* = \frac{1}{\langle k \rangle}\sum_{i=1}^{n} k_i p(k_i) I_{k_i}^*(t)$. In fact, we have

$$\Theta^* = \frac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p\left(k_i\right)\left[\frac{(1-\alpha)\beta k_i}{\gamma_2+\mu} + \frac{\varepsilon\alpha\beta k_i}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)}\right]\frac{\Lambda\Theta^*}{\beta k_i\Theta^*+\mu+p};$$

then, denote

$$F_1(\Theta) = 1 - \frac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p\left(k_i\right)\left[\frac{(1-\alpha)\beta k_i}{\gamma_2+\mu} + \frac{\varepsilon\alpha\beta k_i}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)}\right]\frac{\Lambda}{\beta k_i\Theta^*+\mu+p}.$$

It follows that

$$F_1(1) = 1 - \frac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p\left(k_i\right)\frac{\Lambda[(\varepsilon+\gamma_1+\mu)-\alpha(\gamma_1+\mu)]}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)}\frac{\beta k_i}{\beta k_i\Theta^*+\mu+p} > 0,$$

$$F_1(0) = 1 - \frac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p\left(k_i\right)\left[\frac{1-\alpha}{\gamma_2+\mu} + \frac{\varepsilon\alpha}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)}\right]\frac{\Lambda\beta k_i}{\mu+p} = 1 - \mathcal{R}_0^2,$$

and

$$F_1'(\Theta) = \frac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p\left(k_i\right)\left[\frac{(1-\alpha)\beta k_i}{\gamma_2+\mu} + \frac{\varepsilon\alpha\beta k_i}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)}\right]\frac{\Lambda\beta k_i}{(\beta k_i\Theta^*+\mu+p)^2} > 0.$$

When the basic propagation number $\mathcal{R}_0^2 > 1$, from above, it can be seen that there is only one positive Θ^* such that $F_1(\Theta^*) = 0$. Consequently, there exists a unique information-propagating equilibrium G_1.

In the following, we discuss the stability of the equilibria by the Hurwitz criteria; it is necessary to utilize the Jacobi matrix at the equilibria, which is a $3n \times 3n$ order matrix when the recovery state is not considered.

Theorem 4.1. *If $\mathcal{R}_0^2 < 1$, then the information-free equilibrium G_0 of System* (4.2) *is globally asymptotically stable, and it is unstable when $\mathcal{R}_0^2 > 1$.*

Proof. We will first establish the local stability of G_0 by using the Hurwitz criterion and then the global stability by using the Lyapunov function.

Let $A_{11} = -(\mu + p)$, $A_{22} = -(\varepsilon + \gamma_1 + \mu)$, $A_{33} = -(\gamma_2 + \mu)$ and $B_{ij} = -\frac{\beta k_i S_{k_i}^0}{\langle k\rangle} k_j p(k_j)$; then, we consider the following matrix $\tilde{J}_0 = (m_{ij})_{3n\times 3n}$ at the component

$$\left(S_{k_1}^0, E_{k_1}^0, I_{k_1}^0, \cdots, S_{k_n}^0, E_{k_n}^0, I_{k_n}^0\right)$$

of G_0 for $k_i = 1, 2, \cdots, n$:

$$\tilde{J}_0 = \begin{pmatrix}
A_{11} & \cdots & 0 & 0 & \cdots & 0 & B_{11} & \cdots & B_{1n} \\
0 & \cdots & 0 & 0 & \cdots & 0 & B_{21} & \cdots & B_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & A_{11} & 0 & \cdots & 0 & B_{n1} & \cdots & B_{nn} \\
0 & \cdots & 0 & A_{22} & \cdots & 0 & -\alpha B_{11} & \cdots & -\alpha B_{1n} \\
0 & \cdots & 0 & 0 & \cdots & 0 & -\alpha B_{21} & \cdots & -\alpha B_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & A_{22} & -\alpha B_{n1} & \cdots & -\alpha B_{nn} \\
0 & \cdots & 0 & \varepsilon & \cdots & 0 & A_{33} - (1-\alpha)B_{11} & \cdots & -(1-\alpha)B_{1n} \\
0 & \cdots & 0 & 0 & \cdots & 0 & -(1-\alpha)B_{21} & \cdots & -(1-\alpha)B_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \varepsilon & -(1-\alpha)B_{n1} & \cdots & A_{33} - (1-\alpha)B_{nn}
\end{pmatrix}.$$

Then the characteristic equation of $\tilde{J}_0$ has the following form:

$$g(\lambda) := (\lambda + \mu + p)^n P_{2n}(\lambda) = 0,$$

with $P_{2n}(\lambda)$ being a characteristic polynomial of degree 2n that corresponds to the following matrix $\tilde{J}_1 = (\tilde{m}_{ij})_{2n\times 2n}$:

$$\tilde{J}_1 = \begin{pmatrix} A_{22} & 0 & \cdots & 0 & -\alpha B_{11} & \cdots & -\alpha B_{1n} \\ 0 & A_{22} & \cdots & 0 & -\alpha B_{21} & \cdots & -\alpha B_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{22} & -\alpha B_{n1} & \cdots & -\alpha B_{nn} \\ \varepsilon & 0 & \cdots & 0 & A_{33} - (1-\alpha) B_{11} & \cdots & -(1-\alpha) B_{1n} \\ 0 & \varepsilon & \cdots & 0 & -(1-\alpha) B_{21} & \cdots & -(1-\alpha) B_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \varepsilon & -(1-\alpha) B_{n1} & \cdots & A_{33} - (1-\alpha) B_{nn} \end{pmatrix}.$$

Noting that $S^0_{k_1} = S^0_{k_2} = \ldots = S^0_{k_n}$, and then making a similar transformation on $\tilde{J}_1$ leads to the following upper triangular matrix $\tilde{J}_2$, which has the same eigenvalues as $\tilde{J}_1$:

$$\tilde{J}_2 = \begin{pmatrix} \tilde{m}_{11} & 0 & \cdots & 0 & -\alpha B_{11} & -\alpha B_{12} & \cdots & -\alpha B_{1n} \\ 0 & A_{22} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & A_{33} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & A_{33} \end{pmatrix},$$

with $\tilde{m}_{11} = A_{22} - \frac{(1-\alpha)(\varepsilon+\gamma_1+\mu)+\alpha\varepsilon}{A_{33}} \sum_{i=1}^{n} B_{ii}$. Therefore, the characteristic equation of $\tilde{J}_2$ has the eigenvalue A_{22} with multiplicity $n-1$, the eigenvalue A_{33} with multiplicity n and a single eigenvalue $\tilde{m}_{11}$. In order to show the sign of $\tilde{m}_{11}$, it can be further expressed as

$$\begin{aligned} \tilde{m}_{11} &= -(\varepsilon + \gamma_1 + \mu) + \tfrac{\beta S^0_{k_i}}{\gamma_2+\mu} \tfrac{\langle k^2 \rangle}{\langle k \rangle} ((1-\alpha)(\varepsilon + \gamma_1 + \mu) + \alpha\varepsilon) \\ &= (\varepsilon + \gamma_1 + \mu)(\mathcal{R}_0^2 - 1). \end{aligned}$$

If $\mathcal{R}_0^2 < 1$, then all eigenvalues of $\tilde{J}_0$ are negative, which ensures the local stability of G_0. Otherwise, if $\mathcal{R}_0^2 > 1$, there exists an eigenvalue with a positive real part, which implies that G_0 is unstable.

Then,we consider a Lyapunov function of the following form:

$$V(t) = \sum_{i=1}^{n} [\varepsilon E_{k_i}(t) + (\varepsilon + \gamma_1 + \mu) I_{k_i}(t)].$$

Calculating the derivative of $V(t)$ along System (4.2) gives

$$\begin{aligned}\frac{dV(t)}{dt} &= \sum_{i=1}^{n} \varepsilon\Big[(\alpha\beta S_{k_i}(t)\,\Theta(t) - (\varepsilon+\gamma_1+\mu)E_{k_i}(t))\\ &\quad + (\varepsilon+\gamma_1+\mu)\big((1-\alpha)\beta k_i S_{k_i}(t)\,\Theta(t) + \varepsilon E_{k_i}(t) - (\gamma_2+\mu)I_{k_i}(t)\big)\Big]\\ &\le (\varepsilon\alpha + (\varepsilon+\gamma_1+\mu)(1-\alpha))\tfrac{\beta\Lambda}{\mu+p}\tfrac{\langle k^2\rangle}{\langle k\rangle}\sum_{i=1}^{n} I_{k_i} - (\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)\sum_{i=1}^{n} I_{k_i}\\ &\le (\gamma_2+\mu)(\varepsilon+\gamma_1+\mu)\left(\mathcal{R}_0^2 - 1\right)\sum_{i=1}^{n} I_{k_i}.\end{aligned}$$

If $\mathcal{R}_0^2 < 1$, then we can see that $\frac{dV(t)}{dt} \le 0$, and it can be verified that the largest invariant set where $\frac{dV(t)}{dt} = 0$ is the singleton $\{G_0\}$. By the LaSalle invariance principle, the theorem is proved.

Remark. In Model (4.1), the parameter p is the presence rate of the hot search. If there is no such source emerging in the process of information spreading, i.e., $p = 0$, then by the method in [27, 28], the information-propagating equilibrium G_1 is globally asymptotically stable when $\mathcal{R}_0^2 > 1$.

4.3. Model with hot search effect

Similarly, suppose $\alpha > q$ in the case of $q \neq 0$. For System (4.1), due to the hot search effect, there exists no information-free equilibrium. However, there exists the information-propagating equilibrium

$$G_2 = \left\{\left(S_{k_1}^{**}, E_{k_1}^{**}, I_{k_1}^{**}, R_{k_1}^{**}\right)\ldots\left(S_{k_n}^{**}, E_{k_n}^{**}, I_{k_n}^{**}, R_{k_n}^{**}\right)\right\},$$

where

$$\begin{aligned}S_{k_i}^{**} &= \tfrac{\Lambda}{\beta k_i\Theta^{**}+\mu+p}, \quad E_{k_i}^{**} = \tfrac{\Lambda(\alpha\beta k_i\Theta^{**}+pq)}{(\varepsilon+\gamma_1+\mu)(\beta k_i\Theta^{**}+\mu+p)},\\ I_{k_i}^{**} &= \tfrac{(1-\alpha)\Lambda\beta k_i\Theta^{**}}{(\gamma_2+\mu)(\beta k_i\Theta^{**}+\mu+p)} + \tfrac{\varepsilon\Lambda(\alpha\beta k_i\Theta^{**}+pq)}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)(\beta k_i\Theta^{**}+\mu+p)},\\ R_{k_i}^{**} &= \tfrac{p(1-q)S_{k_i}^{**}+\gamma_1 E_{k_i}^{**}+\gamma_2 I_{k_i}^{**}}{\mu},\end{aligned}$$

for $i = 1, 2, \ldots, n$, and the unique Θ^{**} satisfying

$$\Theta^{**} = \tfrac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p(k_i)\left[\tfrac{(1-\alpha)\Lambda\beta k_i\Theta^{**}}{(\gamma_2+\mu)(\beta k_i\Theta^{**}+\mu+p)} + \tfrac{\varepsilon\Lambda(\alpha\beta k_i\Theta^{**}+pq)}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)(\beta k_i\Theta^{**}+\mu+p)}\right].$$

In fact, let

$$F_2(\Theta) = \Theta - \tfrac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p(k_i)\left[\tfrac{(1-\alpha)\Lambda\beta k_i\Theta^{**}}{(\gamma_2+\mu)(\beta k_i\Theta^{**}+\mu+p)} + \tfrac{\varepsilon\Lambda(\alpha\beta k_i\Theta^{**}+pq)}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)(\beta k_i\Theta^{**}+\mu+p)}\right];$$

then, the positive equilibrium Θ^{**} is given by $F_2(\Theta^{**}) = 0$. We have that

$$\begin{aligned}&\lim_{\Theta\to 0^+} F_2(\Theta) = -\tfrac{\varepsilon\Lambda pq}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)(\mu+p)} < 0,\\ &\lim_{\Theta\to 1} F_2(\Theta) = 1 - \tfrac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p(k_i)\,\tfrac{\varepsilon\Lambda(\beta k_i+pq)+\beta k_i\Lambda(1-\alpha)(\gamma_1+\mu+\varepsilon)}{(\varepsilon+\gamma_1+\mu)(\gamma_2+\mu)(\beta k_i+\mu+p)} > 0.\end{aligned}$$

Now, we examine the derivative of $F_2(\Theta)$; then,

$$F_2'(\Theta) = 1 + \tfrac{1}{\gamma_2+\mu}\tfrac{1}{\langle k\rangle}\sum_{i=1}^{n} k_i p(k_i)\left[\tfrac{(1-\alpha)\beta k_i\Lambda(\mu+p)}{(\beta k_i\Theta^{**}+\mu+p)^2} + \tfrac{\varepsilon\Lambda}{\varepsilon+\gamma_1+\mu}\tfrac{\alpha\beta k_i(\mu+p)-\beta k_i pq}{(\beta k_i\Theta^{**}+\mu+p)^2}\right] > 1,$$

implying that the function $F_2(\Theta)$ is strictly increasing. Therefore, there is only one positive Θ^{**} such that $F_2(\Theta^{**}) = 0$, and the existence and uniqueness of the positive equilibrium G_2 follows. We have the following result.

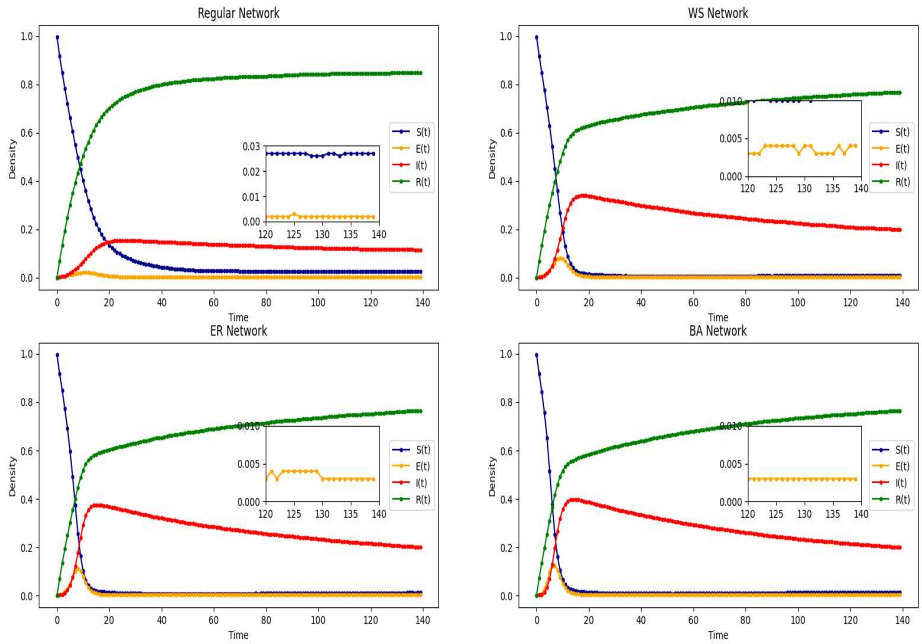

Figure 4. Plots of state density with $q = 0$ in four networks.

Theorem 4.2. *For System* (4.1)*, there exists a unique information-propagating equilibrium G_2.*

The numerical investigations in Figures 4 and 5, performed on the four networks (regular network, Watts-Strogatz (WS) network, Erdös-Rényi (ER) network and BA network) with size $N = 1000$, confirm the theoretical analysis about the influence of q on the evolution of state density in Model (4.1). In the figures, with the other parameters taken as the same values, the state variables are depicted when q is fixed to 0 and 0.65, respectively, From Figure 4, it can be seen that the curves of the propagable density $I(t)$ is always lower than that of the rejected density $R(t)$ when $q = 0$. However, in the case of $q = 0.65$, the curves of $I(t)$ are obviously higher than those of $R(t)$ in the initial time steps, and then they decline gradually to be lower than the density of $R(t)$, as shown in Figure 5. In addition, it can be observed that the density of $I(t)$ in Figure 4 is lower than that in Figure 5 at the same time steps, although it is not particularly obvious. Therefore, the comparison results in the fact that the hot search effect is not ignorable in the progress of information propagation.

5. Numerical simulations

In this section, we describe the numerical simulations of Models (3.1) and (4.1) to examine the sensitivity analysis, which demonstrate the effects of some parameters on the basic propagation number, as well as the density of propagable nodes, allowing us to improve the understanding of information spreading.

For convenience, in the following simulations, let K be the average degree $\bar{k}$ in the homogeneous network, as well as $\frac{\langle k^2 \rangle}{\langle k \rangle}$ in the heterogeneous network. First, we consider the SEIR model on the regular

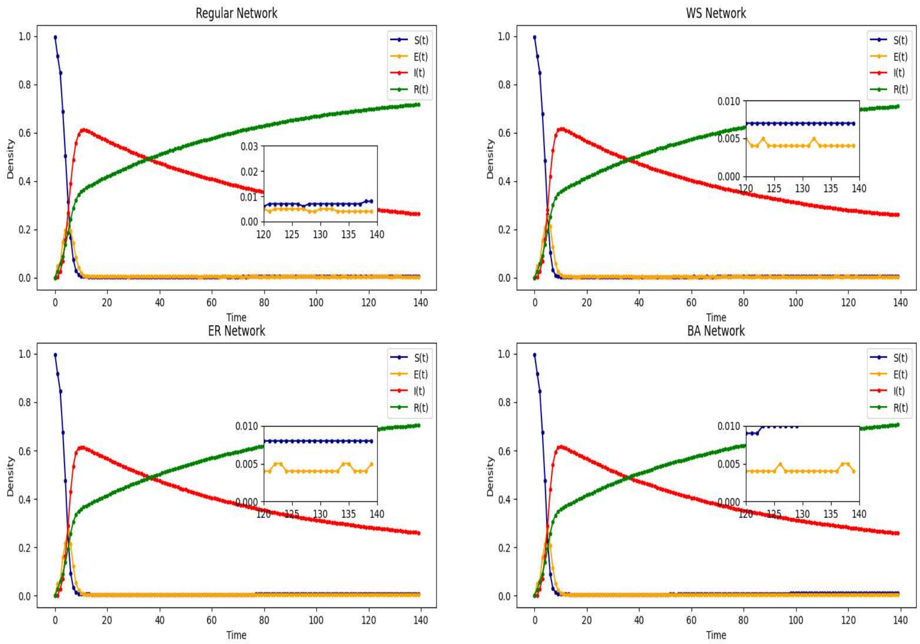

Figure 5. Plots of state density with $q = 0.65$ in four networks.

network and the ER network in the absence of the hot search effect ($q = 0$). In this scenario, there exists a basic propagation number $\mathcal{R}_0$ (denoting $\mathcal{R}_0^1$ and $\mathcal{R}_0^2$). To explore the sensitivity of parameter variations on $\mathcal{R}_0$ due to the uncertainty in estimating the input parameter, the partial rank correlation coefficients (PRCCs) was calculated based on the Latin hypercube sampling scheme.

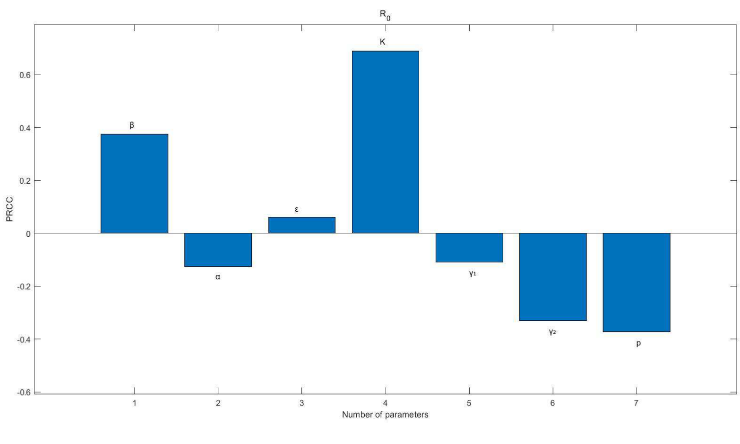

Figure 6. Sensitivity analysis of PRCCs for input parameters and $\mathcal{R}_0$ (outcome variable).

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For our performance, the sample size was taken as $n = 2000$ and seven parameters were chosen as the input variables. The values of the PRCCs were calculated for each of the seven parameters, as shown in Figure 6, which can be used to observe the contribution made by each input parameter

Table 2. PRCC values for input parameters and $\mathcal{R}_0$.

Input parameters	The basic propagation number $\mathcal{R}_0$	
	PRCC	p-value
β	0.3757	0
α	-0.1253	0
ε	0.0615	0.0059
K	0.6898	0
γ_1	-0.1087	0
γ_2	-0.3300	0
p	-0.3716	0

to the variability of $\mathcal{R}_0$. The positive or negative value corresponding to each parameter reveals the correlation that is generally evaluated as being important, moderate or not significantly different from zero. In Table 2, the specific values of the PRCCs and p-values are presented, noting that p-values are regarded as zero when smaller than 0.0001. If we consider that absolute values of PRCCs that are greater than 0.4 indicate an important correlation between the input parameters and $\mathcal{R}_0$, the values in $(0.2, 0.4)$ are moderate correlations and the values in $(0, 0.2)$ are not significantly different from zero, it can be seen that K (PRCC=0.6898) has an important impact on $\mathcal{R}_0$, and β, γ_2 and p have a moderate impact, with the PRCC values being 0.3757, -0.3300 and -0.3716 respectively. Compared with them, the remaining parameters affect $\mathcal{R}_0$ slightly.

We focus on the sensitive parameters to examine the variation in $\mathcal{R}_0$ due to the change of the concerned parameters. In Figure 7, the contour plots on the regular network and ER network are presented in terms of parameter pairs (β,p) and (β,γ_2) respectively. For comparison, the degree in the regular network is taken as $\langle k \rangle = 4$ and $\langle k \rangle \approx 4$ in the ER network, so that $\frac{\langle k^2 \rangle}{\langle k \rangle} \approx 7.03$ when $\left\langle k^2 \right\rangle \approx 28.1327$. The effects of β and p are shown in the first row in Figure 7. It can be seen that the basic propagation number $\mathcal{R}_0$ is less than 1 only when β is approximately in the interval of $[0, 0.4]$ in the regular network, compared with the interval of $[0, 0.2]$ in the ER network. In particular, if $\beta = 0.2$ and $p = 0.6$, then $\mathcal{R}_0$ is less than 1 in the regular network; however, it is greater than 1 in the ER network. In addition, for a fixed p and varying β, the basic propagation number $\mathcal{R}_0$ experiences more variation in its value in the ER network than in the regular network. For the pair (β,γ_2), the contour plot in the second row presents the numerical result similar to that for the parameter pair (β,p).

If the hot search effect is taken into account, it is previously referred that there exists no information-free equilibrium in this case and $\mathcal{R}_0$ is only a critical parameter related to the stability of the information-propagating equilibrium. To further analyze the impact of different connectivity K on the information propagation process, we consider the model on four networks with size $N = 1000$ and connectivity ranging from 4 to 16. The profile for the propagable population is depicted in Figure 8. From the curves, we can see that, starting on nodes with different connectivity K, the higher the connectivity, the higher the peak of the density $I(t)$, resulting in a higher number of propagable individuals in information spreading. On the contrary, the classes of nodes with few connections have a relatively small density at its peak and a slight shift of the peak time, which suggests that a lower prevalence in information propagation can be achieved by managing the connected nodes to some extent.

For OSNs, the hot search list can serve as a source of hot information that is more easily accessible

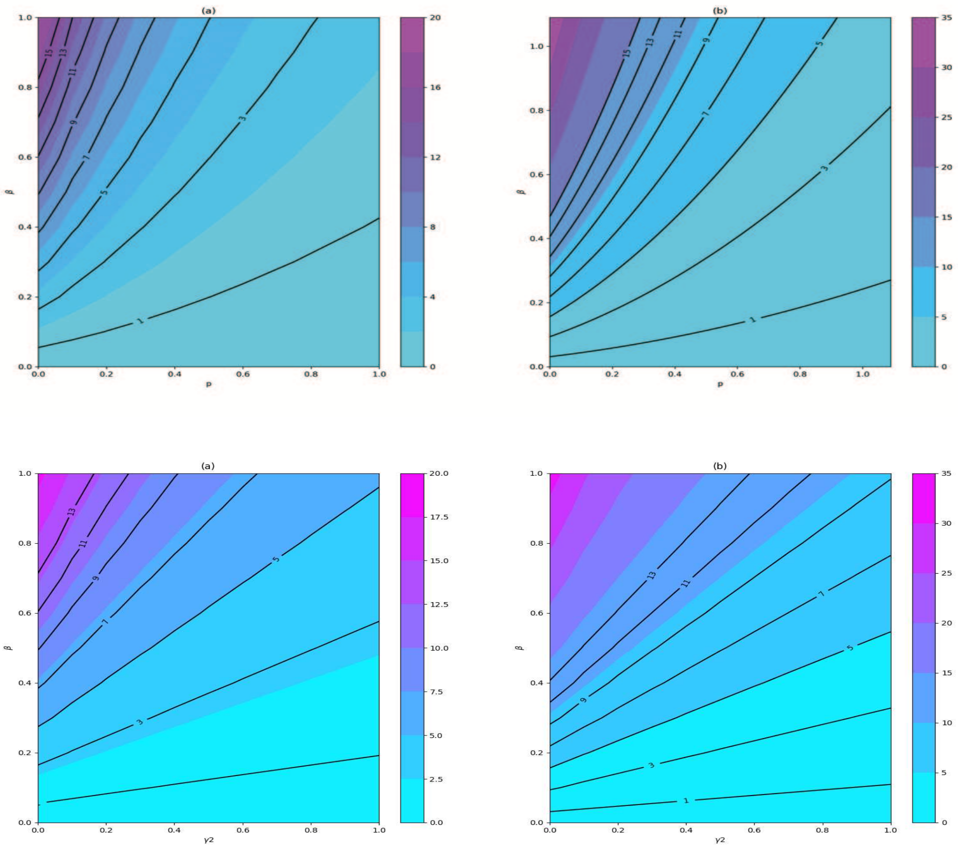

Figure 7. Contour plots of the basic propagation number on the regular network (Left) and ER network (Right), revealing the estimated effects of the parameter pairs (β,p) and (β,γ_2).

and attracts more attention. Once becoming interested, the internet users have the potential to retweet and spread the information actively. In order to provide further insight into the hot search effect, the density of propagable nodes as a function of time was investigated for the four types of networks in Figure 9. When the hot search effect is absent, i.e., $q = 0$, it is shown that there is a very small variation in the density of $I(t)$. With the increase of q from 0 to 0.6, there is a much higher number of propagable individuals in each type of the network, which provides striking evidence of the peculiar influence of the hot search on the process of information propagation.

6. Case study

To validate the proposed SEIR model, real data from Sina Weibo (the largest online social media in China) were collected through Python software. Then, the model was employed to fit the data, with the aim of minimizing the difference between model simulation and the collected data. In the case study, two datasets, named Information A and Information B, were used to illustrate the capability of our model to simulate the dynamical behaviors of information propagation.

Information A is a promotional tweet from an actress for her starring TV series. Table 3 lists the number of retweets in the first 13 hours after the microblog was posted. From the initial time, the data

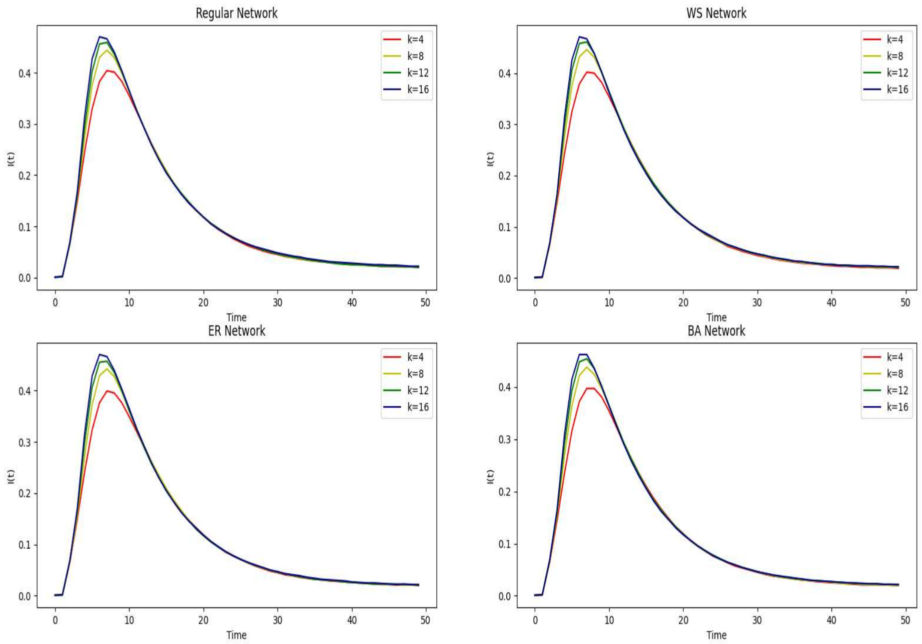

Figure 8. Effect of the parameter K on the density of $I(t)$ on four networks.

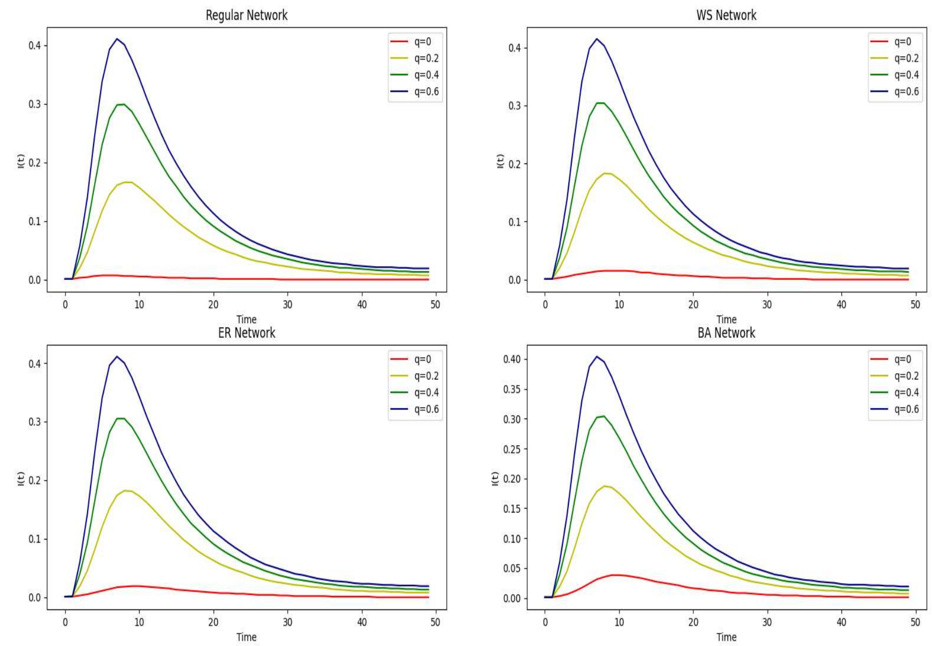

Figure 9. Hot search effect on the density of $I(t)$ on four networks.

Table 3. Data for Information A.

Time	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Number	1	76	97	144	89	93	55	29	22	12	14	17	16	6
Time	15	16	17	18	19	20	21	22	23	24	25	26	27	
Number	4	5	4	4	3	1	1	1	4	3	2	1	1	

Table 4. Data for Information B.

Time	1	2	3	4	5	6	7	8	9	10	11	12
Number	1	332	660	704	352	251	213	127	69	66	40	30
Time	13	14	15	16	17	18	19	20	21	22	23	24
Number	25	26	30	30	17	20	19	13	13	18	9	16
Time	25	26	27	28	29	30	31					
Number	20	18	12	18	7	1	1					

were recorded in half-hour time intervals; then, a total of 27 data points was obtained. Information B is a tweet about the performance of Chinese women players in a World Table Tennis Championship. The collecting time lasted 15 hours, and the data of the forwarding after information release is listed in Table 4, in which the data were also collected every half an hour, the same as with Information A. In these two tables, the first data point denotes the publisher's release, counted as 1. In data collecting, when the retweet of the concerned information decreased to 1 and experienced no fluctuations in the next hour, it was regarded as the end time, and the last data point was 1.

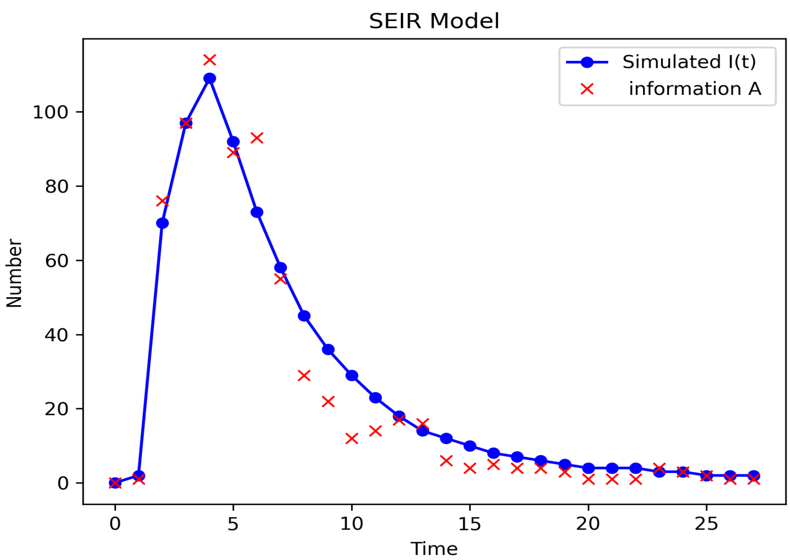

Figure 10. Data fitting for Information A.

In our numerical simulations, we use the BA network to fit the collected data for Information A and Information B, according to the experimental results in [4], revealing the phenomenon that some real networks, including social networks, have a power-law exponent that also exists in microblog net-

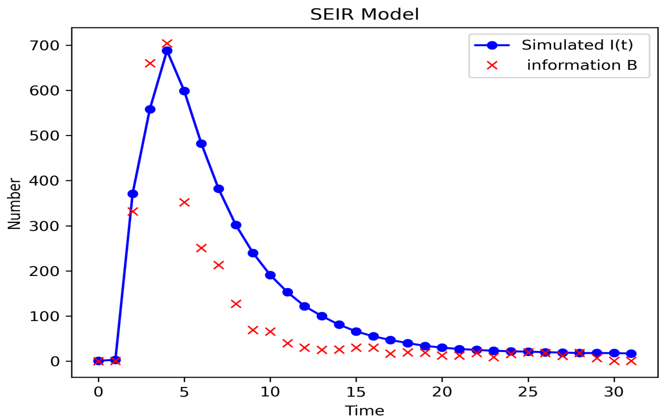

Figure 11. Data fitting for Information B.

works. In fitting the data, we determine the values of parameters by using the least squares method to minimize their root mean square error (RMSE) and mean absolute error (MAE) for better accuracy; then, the model was applied on the BA network with an average degree of 12. The simulation results and the collected data are plotted in Figures 10 and 11. The experimental illustrations show that the whole process for the two cases can be divided into three stages. The retweet of the Weibo information increased dramatically at the initial stage, implying that most users interested in it were involved in forwarding the information shortly after its release. Then, the popularity of it decayed and the forwarders of the microblog decreased in the following second stage. At the last stage, the number of retweets approached a very low level eventually.

Table 5. List of parameters and variables values used in case study.

Parameter & variable	Λ	β	α	ε	μ	γ_1	γ_2	p	q	$S(0)$
Information A	4	0.89	0.68	0.78	0.004	0.12	0.22	0.65	0.66	277
Information B	4	0.85	0.68	0.76	0.004	0.20	0.22	0.58	0.65	1997

Table 6. RMSE and MAE.

Case	RMSE	MAE
Information A	7.29	4.85
Information B	92.53	59.68

The parameters and initial values used in Figures 10 and 11 are listed in Table 5. In these two cases, we set $E(0) = 1$, $I(0) = 1$ and $R(0) = 1$. From the comparison of the model simulation and the data of Information A, it can be observed from Figure 10 that the proposed model captured the trend of the data accurately, although there exists some error in the fitting. In Figure 11, there is an obvious deviation of the simulation results from the real data of forwarding Information B in the decay period, and the possible reason may be that the sharp decline in the data occurred at midnight when there are relatively few users active on Weibo, which is not considered in the simulation. The values of the RMSE and MAE are given in Table 6.

Regarding the case study, it is worth noting that the best-fit parameter q came out to be almost the same, i.e., $q = 0.66$ for Information A and $q = 0.65$ for Information B. The possible reason is that these two kinds of data had similar structural characteristics and three stages, as mentioned before. Particularly, from Figures 10 and 11, it can be seen that a single wave of forwarding occurred in the propagating process. Although the fitting results were not accurate enough to match the collected data, they may give us some hint that the hot search exerts a similar effect on information propagation with these types of characteristics.

7. Conclusions

In this paper, an extended SEIR model was investigated to capture the dynamic property of information propagation in complex networks. In particular, the effect of a hot search in OSNs was incorporated into our model, with aim of demonstrating its performance in information diffusion. Based on the proposed model framework, we conducted dynamical analysis of the model in homogeneous networks and heterogeneous networks, including the stability of the information-free equilibrium and the information-propagating equilibrium. In addition, numerical simulations were carried out to validate the theoretical results and analyze the sensitivity of the concerned parameters. Furthermore, we focused on two types of specific information spreading in Weibo and performed data fitting on the BA network, which confirmed the validity of our model in application. The presented results allow us to have a better knowledge of the process of information propagation in OSNs.

In fact, real-world networks in which information is disseminated are more complex and dynamic; it is reasonable to expect that the proposed model can characterize the behaviors of information diffusion more accurately, and consequently have better performance in practical realization. In this perspective, the incorporation of many elements of realism, such as a time delay in forwarding, followers' awareness and competition of different online information, is important in modeling to acquire a better understanding of information propagation. Other than the case with one single wave of forwarding, numerical examples with more characteristics should be considered to improve the level of agreement between the model and real data, which will be left for us to explore in future.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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