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Research article

Upper and lower bounds for the pull-in voltage and the pull-in distance for a generalized MEMS problem

Yan-Hsiou Cheng[1], **Kuo-Chih Hung**[2,*], **Shin-Hwa Wang**[3] **and Jhih-Jyun Zeng**[3]

[1] Department of Mathematics and Information Education, National Taipei University of Education, Taipei 106, Taiwan

[2] Fundamental General Education Center, National Chin-Yi University of Technology, Taichung 411, Taiwan

[3] Department of Mathematics, National Tsing Hua University, Hsinchu 300, Taiwan

* **Correspondence:** Email: kchung@ncut.edu.tw.

Abstract: We study upper and lower bounds for the pull-in voltage and the pull-in distance for the one-dimensional prescribed mean curvature problem arising in MEMS

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \frac{\lambda}{(1-u)^p}, \ u < 1, \quad -L < x < L, \\ u(-L) = u(L) = 0, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter, and $p, L > 0$ are two evolution parameters. We further study monotonicity properties and asymptotic behaviors for the pull-in voltage and pull-in distance with respect to positive parameters p and L.

Keywords: prescribed mean curvature problem; global bifurcation diagram; positive solution; pull-in voltage; pull-in distance; MEMS

1. Introduction

This paper is a continuation of Cheng, Hung and Wang [1]. In this paper, we study some structures of bifurcation diagrams of positive solutions $u \in C^2(-L, L) \cap C[-L, L]$ for the one-dimensional prescribed mean curvature problem arising in electrostatic MEMS (Micro-Electro-Mechanical Systems)

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \dfrac{\lambda}{(1-u)^p}, \ \ u<1, \quad -L<x<L, \\ u(-L)=u(L)=0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a *bifurcation* parameter, and $p, L > 0$ are two *evolution* parameters. In particular, we study upper and lower bounds, monotonicity properties and asymptotic behaviors for the pull-in voltage and the pull-in distance. The singular nonlinearity in (1.1)

$$f(u) \equiv \frac{1}{(1-u)^p}, \ p > 0$$

satisfies

$$f(0) = 1, \ \lim_{u\to 1^-} f(u) = \infty, \text{ and } f'(u), f''(u) > 0 \text{ on } [0,1). \tag{1.2}$$

Notice that the improper integral of f over $[0,1)$ satisfies

$$\int_0^1 f(u)du = \begin{cases} \infty & \text{if } p \geq 1, \\ \frac{1}{1-p} < \infty & \text{if } 0 < p < 1. \end{cases}$$

The one-dimensional prescribed mean curvature problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda \tilde{f}(u), \quad -L<x<L, \\ u(-L)=u(L)=0, \end{cases} \tag{1.3}$$

and n-dimensional problem of it, with general nonlinearity $\tilde{f}(u)$ or with many different types nonlinearities, like u^p $(p>0)$, $u^p + u^q$ $(0 \leq p < q < \infty)$, $(1+u)^p$ $(p>0)$, $\exp(u)$, $\exp(u)-1$, $\exp\left(\frac{au}{a+u}\right)$ $(a>0)$, $\exp\left(\frac{au}{a+u}\right)-1$ $(a>0)$, a^u $(a>0)$, $u-u^3$, and $(1-u)^{-p}$ $(p>0)$ have been investigated intensively since 1990, see, e.g., [1–15].

A solution $u \in C^2(-L,L) \cap C[-L,L]$ of (1.3) with $u' \in C([-L,L],[-\infty,\infty])$ is called classical if $|u'(\pm L)| < \infty$, and it is called non-classical if $u'(-L) = \infty$ or $u'(L) = -\infty$, see [8] . Notice that it can be shown that (see [2, 8]), for (1.3),

(i) Any non-trivial positive solution $u \in C^2(-L,L) \cap C[-L,L]$ is concave on $(-L,L)$ if $\tilde{f}(u) > 0$ for $u > 0$, since the equation in (1.3) can be written in the equivalent form

$$u''(x) = -\lambda(1+u'^2)^{3/2}\tilde{f}(u) < 0 \text{ on } (-L,L). \tag{1.4}$$

(ii) A positive solution $u \in C^2(-L,L) \cap C[-L,L]$ must be symmetric on $[-L,L]$. Thus $u'(-L) = -u'(+L)$.

In this paper for prescribed mean curvature problem (1.1), we simply consider *classical* positive solutions u. For any fixed $p, L > 0$, we define the bifurcation diagram $C_{p,L}$ of (1.1) by

$$C_{p,L} \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a classical positive solution of (1.1)}\}.$$

We say the bifurcation diagram $C_{p,L}$ is ⊃-shaped (see e.g., Figure 1(i) depicted below) on the $(\lambda, \|u\|_\infty)$-plane if there exists $\lambda^* > 0$ such that $C_{p,L}$ consists of a continuous curve with exactly one turning point at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ where the bifurcation diagram $C_{p,L}$ turns to the *left*.

Brubaker and Pelesko [3] studied existence and multiplicity of positive solutions of the n-dimensional prescribed mean curvature problem

$$\begin{cases} -\text{div}\dfrac{\nabla u(\mathbf{x})}{\sqrt{1+|\nabla u(\mathbf{x})|^2}} = \dfrac{\lambda}{(1-u)^2}, \quad u<1, \quad \mathbf{x}\in\Omega_L, \\ u=0, \quad \mathbf{x}\in\partial\Omega_L, \end{cases} \tag{1.5}$$

where $\lambda > 0$ is a bifurcation parameter and $\Omega_L \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain depending on some parameter $L > 0$. Problem (1.5) with an inverse square type nonlinearity $f(u) = (1-u)^{-p}$, $p = 2$ is a derived variant of a *canonical* model used in the modeling of electrostatic Micro-Electro Mechanical Systems (MEMS) device obeying the electrostatic Coulomb law with the Coulomb force satisfying the *inverse square law* with respect to the distance of the two charged objects, which is a function of the deformation variable (cf. [16, p. 1324].) The modeling of electrostatic MEMS device consists of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at +1. In (1.5), u is the unknown profile of the deflecting MEMS membrane, λ is the drop voltage between the ground plate and the deflecting membrane, and the term $|\nabla u|^2$ is called a fringing field (cf. [3]). When a voltage λ is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^*, referred to as the "pull-in voltage". (So if voltage λ exceeds pull-in voltage λ^*, an equilibrium defection is no longer attainable and the lower surface will touch up on the upper plate.) This creates a so-called "pull-in instability" which greatly affects the design and manufacture of MEMS devices. Also, in the actual design of a MEMS device, typically, one of the primary device design goals is to achieve the maximum possible stable steady-state deflection (that is, $\|u_{\lambda^*}\|_\infty$ (< 1), cf. Theorems 1.1–1.2 and Figures 1 and 2 below), referred to as the "pull-in distance", with a relatively small applied voltage. We refer to [3, 17] for detailed discussions on MEMS devices modeling. Notice that the physically relevant dimensions are $n = 1$ and $n = 2$. In the case for $n = 1$, Ω_L is a rectangular strip with two opposite edges at $x = \pm L$ fixed ($2L$ is the length of the strip) and the remaining two edges free, the deflection $u = u(x, y)$ may be assumed a function of x only. In the case for $n = 2$, Ω_L is a planar bounded domain with smooth boundary, and L is the characteristic length (diameter) of the domain. In particular, Ω_L could be a circular disk of radius L.

With general $p > 0$, (1.1) is a generalized MEMS problem under the assumption that the Coulomb force satisfies the *inverse p-th power law* with respect to the distance of the two charged objects, where $p > 0$ characterizes the *force strength*. See [18].

Brubaker and Pelesko [4] and Pan and Xing [11] studied global bifurcation diagrams and exact multiplicity of positive (classical) solutions for the one dimensional problem of (1.5),

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \dfrac{\lambda}{(1-u)^2}, \quad u<1, \quad -L<x<L, \\ u(-L)=u(L)=0. \end{cases} \tag{1.6}$$

Brubaker and Pelesko [4, Theorem 1.1] and Pan and Xing [11, Theorem 1.1] independently proved that, for (1.6), which corresponds to the case $p = 2$ in (1.1) with $L > 0$, there exists a positive number $L^* \approx 0.34997$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram $C_{2,L}$ consists of a (continuous) ⊃-shaped curve which emanates from the origin and has exactly one (left) turning point at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ when $L \geq L^*$, and as L transitions from greater than or equal to L^* to less than L^* the upper branch of the bifurcation diagram $C_{2,L}$ splits into two parts. See Figure 1 depicted below and see [4, Theorem 1.1] and [11, Theorem 1.1] for details. In addition, for (1.6), Brubaker and Pelesko [4, Theorem 1.1] proved that the pull-in voltage $\lambda^* = \lambda^*(L)$ satisfies

$$\lambda^* < \min\left\{L^{-1}, \tfrac{\pi^2}{27}L^{-2}\right\}, \tag{1.7}$$

and Pan and Xing [11, Theorem 1.1, part (4)] proved that the pull-in voltage $\lambda^*(L)$ is a strictly decreasing function of $L > 0$. Brubaker and Pelesko [4, Figure 1.2(b)] also gave numerical simulation of the pull-in distance $\|u_{\lambda^*}\|_\infty$ for $0.1 \leq L \leq 0.8$, which show that $\|u_{\lambda^*}\|_\infty$ is a strictly increasing function of $L \in [0.1, 0.8]$.

In following Theorems 1.1–1.2, Cheng, Hung and Wang [1, Theorems 2.1–2.3] extended and improved the results of Brubaker and Pelesko [4, Theorem 1.1] and Pan and Xing [11, Theorem 1.1] by generalizing the nonlinearity $f(u) = (1-u)^{-2}$ in (1.6) to $f(u) = (1-u)^{-p}$ with general $p \in (0, \infty)$. Theorems 1.1–1.2 show that p is a bifurcation parameter to prescribed mean curvature problem (1.1). Note that this result remains hold for the standard MEMS problem

$$\begin{cases} -u''(x) = \dfrac{\lambda}{(1-u)^p}, \quad u < 1, \quad -L < x < L, \\ u(-L) = u(L) = 0; \end{cases} \tag{1.8}$$

see [19, Section 2]. The standard MEMS problem (1.8) and n -dimensional (generalized) problems of it have been studied by numerous authors, see e.g., [20–22]. For semilinear problem (1.8) with any $p > 0$ and $L > 0$, by applying (1.2) and Laetsch [23, Theorems 2.5, 2.9 and 3.2] , we obtain that, on the $(\lambda, \|u\|_\infty)$ -plane, the bifurcation diagram of positive solutions consists of a (continuous) ⊃-shaped curve which emanates from the origin, initially continues to the right, and making a left turn at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ which is a bifurcation point with neutral stability, then continues to the left, and ends at $(0, 1)$, cf. Figure 1(i). For semilinear problem (1.8) with $p = 2$ and $L = 1$, Ghoussoub and Guoand [22, Theorem 1.1, part 4] proved that the pull-in voltage $\lambda^* \leq 9/8 = 1.125$. For semilinear problem (1.8) with $p = 2$ and $L = 1$, the value of the pull-in voltage was numerically determined as $\lambda^* \approx 1.400016469/4 \approx 0.350004$ in [21, Theorem 2.2] (see also [24, FIGURE 2(a)]), and the value of the pull-in distance was numerically determined as $\|u_{\lambda^*}\|_\infty \approx 0.38$ in [24, FIGURE 2(a)]. Cowan and Ghoussoub [20, Corollary 2.1, part 1] proved that, for Laplacian problem (1.8) with $p > 0$ and $L = 1$, the pull-in distance

$$(1 >)\ \|u_{\lambda^*}\|_\infty \geq \frac{1}{p+1}.$$

This implies that

$$\lim_{p \to 0^+} \|u_{\lambda^*}\|_\infty = 1 \ \text{ for (1.8) with } L = 1. \tag{1.9}$$

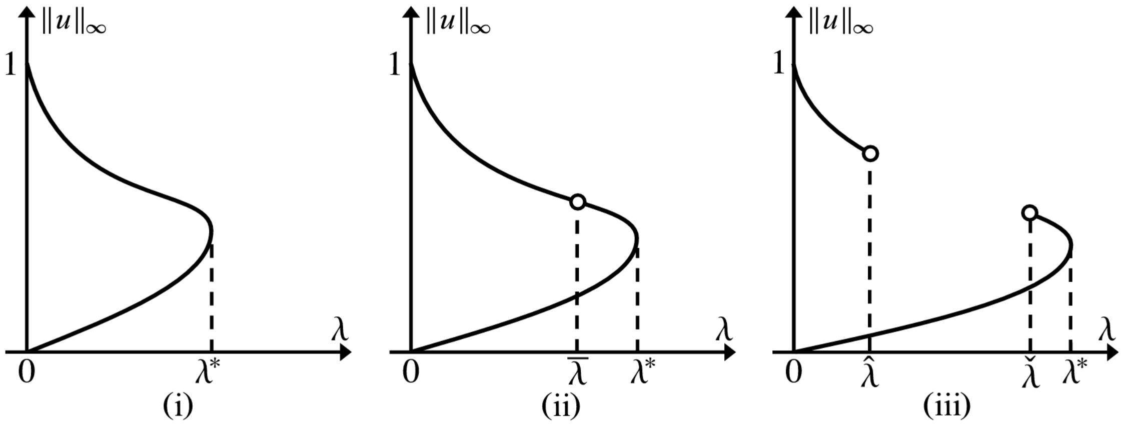

Figure 1. Global bifurcation diagrams $C_{p,L}$ of (1.1) with $p \geq 1$. (i) $L > L^*$. (ii) $L = L^*$. (iii) $0 < L < L^*$. This figure is from [1, Figure 1].

Theorem 1.1 (See Figure 1)**.** *Consider classical positive solutions u of (1.1) with $p \geq 1$. There exists $L^* = L^*(p) > 0$ such that the following assertions (i)–(iii) hold:*

- *(i)* (*See Figure 1(i).*) *If $L > L^*$, then there exists $\lambda^* > 0$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ for $0 < \lambda < \lambda^*$, exactly one positive solution u_λ for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*
- *(ii)* (*See Figure 1(ii).*) *If $L = L^*$, then there exist $0 < \bar{\lambda}\ (= \bar{\lambda}(p)) < \lambda^*$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ for $0 < \lambda < \bar{\lambda}$ and $\bar{\lambda} < \lambda < \lambda^*$, exactly one positive solution u_λ for $\lambda = \bar{\lambda}, \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*
- *(iii)* (*See Figure 1(iii).*) *If $0 < L < L^*$, then there exist $0 < \hat{\lambda} < \check{\lambda} < \lambda^*$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ for $0 < \lambda < \hat{\lambda}$ and $\check{\lambda} < \lambda < \lambda^*$, exactly one positive solution u_λ for $\hat{\lambda} \leq \lambda \leq \check{\lambda}$ and $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*

Theorem 1.2 (See Figure 2)**.** *Consider classical positive solutions u of (1.1) with $0 < p < 1$. There exist $0 < L_*\ (= L_*(p)) < L^*\ (= L^*(p))$ such that the following assertions (i)–(iv) hold:*

- *(i)* (*See Figure 2(i)–(ii).*) *If $L > L^*$, then there exist $0 < \lambda_* < \lambda^*$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ for $\lambda_* < \lambda < \lambda^*$, exactly one positive solution u_λ for $0 < \lambda \leq \lambda_*$ and $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*
- *(ii)* (*See Figure 2(iii).*) *If $L = L^*$, then there exist $0 < \lambda_* < \bar{\lambda}\ (= \bar{\lambda}(p)) < \lambda^*$ satisfying $\lambda_* < 1 - p < \bar{\lambda}$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ for $\lambda_* < \lambda < \bar{\lambda}$ and $\bar{\lambda} < \lambda < \lambda^*$, exactly one positive solution u_λ for $0 < \lambda \leq \lambda_*$ and $\lambda = \bar{\lambda}, \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*
- *(iii)* (*See Figure 2(iv).*) *If $L_* < L < L^*$, then there exist $0 < \lambda_* < \hat{\lambda} < \check{\lambda} < \lambda^*$ satisfying $\lambda_* < 1 - p < \hat{\lambda}$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ for $\lambda_* < \lambda < \hat{\lambda}$ and $\check{\lambda} < \lambda < \lambda^*$, exactly one positive solution u_λ for $0 < \lambda \leq \lambda_*$, $\hat{\lambda} \leq \lambda \leq \check{\lambda}$ and $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*
- *(iv)* (*See Figure 2(v).*) *If $0 < L \leq L_*$, then there exist $0 < \check{\lambda} < \lambda^*$ satisfying $1 - p < \check{\lambda}$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ for $\check{\lambda} < \lambda < \lambda^*$, exactly one positive solution u_λ for $0 < \lambda \leq \check{\lambda}$ and $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*

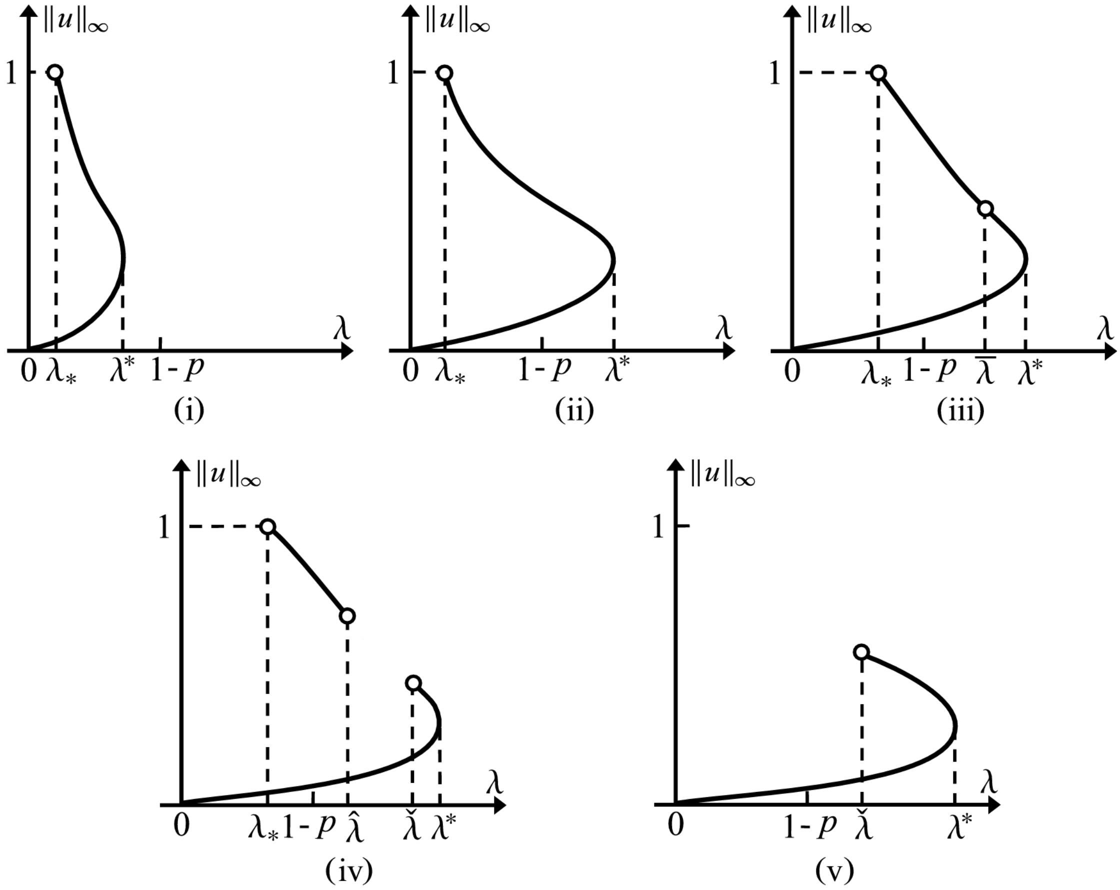

Figure 2. Global bifurcation diagrams $C_{p,L}$ of (1.1) with $0 < p < 1$. (i)–(ii) $L > L^*$. (iii) $L = L^*$. (iv) $L_* < L < L^*$. (v) $0 < L \leq L_*$. This figure is from [1, Figure 2].

The paper is organized as follows. Section 2 contains statements of the main results (Theorems 2.1–2.3). Section 3 contains several lemmas needed to prove Theorem 2.3. Section 4 contains the proofs of the main results.

2. Main results

The main results in this paper are next Theorems 2.1–2.3 for the generalized MEMS problem (1.1), in which we first study upper and lower bounds for the pull-in voltage λ^* and the pull-in distance $\|u_{\lambda^*}\|_\infty$. We further study monotonicity properties and asymptotic behaviors of the pull-in voltage λ^* and the pull-in distance $\|u_{\lambda^*}\|_\infty$ with respect to positive parameters p and L.

Theorem 2.1 (See Theorems 1.1–1.2 and Figures 1 and 2)**.** *Consider (1.1) with $p > 0$ and $L > 0$. Then the pull-in voltage $\lambda^* = \lambda^*(p, L)$ satisfies the following assertions (i)–(iii):*

(i) For $p > 0$ and $L > 0$,

$$\frac{2p^p}{(p+1)^{p+1}L^2\left[1+\frac{4}{(p+1)^2L^2}\right]^{3/2}} \leq \lambda^*(p, L) < \min\left\{L^{-1}, \frac{p^p}{4(p+1)^{p+1}}\pi^2 L^{-2}\right\} \tag{2.1}$$

$$\leq \min\left\{L^{-1}, \tfrac{1}{4}\pi^2 L^{-2}\right\}.$$

(ii) For any fixed $L > 0$, $\lambda^*(p, L)$ is a strictly decreasing function of $p > 0$, and

$$\frac{2}{L^2\left[1+\frac{4}{L^2}\right]^{3/2}} \leq \lim_{p\to 0^+} \lambda^*(p, L) \leq \min\left\{L^{-1}, \tfrac{1}{4}\pi^2 L^{-2}\right\},$$

$$\lim_{p\to\infty} \lambda^*(p, L) = 0.$$

(iii) For any fixed $p > 0$, $\lambda^*(p, L)$ is a strictly decreasing function of $L > 0$, $\lim_{L\to 0^+} \lambda^*(p, L) = \infty$, and $\lim_{L\to\infty} \lambda^*(p, L) = 0$.

Remark 2.2. *In (2.1) for (1.1) with $p > 0$, the upper bound $\min\left\{L^{-1}, \frac{p^p}{4(p+1)^{p+1}}\pi^2 L^{-2}\right\}$ for $\lambda^*(p, L)$ is reduced to $\min\left\{L^{-1}, \frac{\pi^2}{27}L^{-2}\right\}$ when $p = 2$; that is, our result for the upper bound for $\lambda^*(p, L)$ generalizes (1.7). In particular, when $p = 2$ and $L = 1$, then (2.1) is reduced to*

$$0.171 \approx \frac{8}{13\sqrt{13}} \leq \lambda^*(2, 1) < \frac{\pi^2}{27} \approx 0.366.$$

This suggests that (2.1) give suitable upper and lower bounds for the pull-in voltage $\lambda^*(p, L)$ with general $p, L > 0$.

Theorem 2.3 (See Theorems 1.1–1.2 and Figures 1–5)**.** *Consider (1.1) with $p > 0$ and $L > 0$. Then the pull-in distance $\|u_{\lambda^*}\|_\infty = \|u_{\lambda^*}\|_\infty(p, L)$ satisfies the following assertions (i)–(iv):*

(i) For $p > 1$ and $L > 0$,

$$\bar{L}(p, L) < \|u_{\lambda^*}\|_\infty < \bar{U}(p), \tag{2.2}$$

where

$$\begin{aligned}\bar{L}(p, L) &\equiv \min\left\{1-\left[1+\frac{1}{3}(p-1)L\right]^{\frac{1}{1-p}}, \frac{1}{9p+1}\right\}\\ &= \begin{cases}\dfrac{1}{9p+1} & \text{if } L \geq L_1(p) \equiv \dfrac{3^{3-2p}p\left(\frac{1+9p}{p}\right)^p - 27p - 3}{(p-1)(1+9p)},\\ 1-\left[1+\frac{1}{3}(p-1)L\right]^{\frac{1}{1-p}} & \text{if } 0 < L < L_1(p),\end{cases}\end{aligned}$$

and $(p, \bar{U}(p))$ is the unique positive solution pair of the equation

$$\Gamma(p, r) \equiv (p+1)r + 2(1-r)^p - 2 = 0, \ \ p > 1, \ 0 < r < 1,$$

and $\bar{U}(p)$ satisfies

$$\bar{U}(p) \leq U(p) \equiv \begin{cases} 1-2e^{-2} \ (\approx 0.729) & \text{for } 1 < p \leq \frac{1}{1-2e^{-2}} \approx 1.371,\\ 1/p & \text{for } \frac{1}{1-2e^{-2}} < p \leq 2,\\ 2/(p+2) & \text{for } p > 2;\end{cases} \tag{2.3}$$

see Figure 4.

(ii) For $p = 1$ *and* $L > 0$,

$$\min\left\{1 - e^{-\frac{L}{3}}, \frac{1}{10}\right\} < \|u_{\lambda^*}\|_\infty < 1 - 2e^{-2} \approx 0.729,$$

where

$$\min\left\{1 - e^{-\frac{L}{3}}, \frac{1}{10}\right\} = \begin{cases} \frac{1}{10} & \text{if } L \geq L_2 \equiv 3\ln(\frac{10}{9}) \approx 0.316, \\ 1 - e^{-\frac{L}{3}} & \text{if } 0 < L < L_2. \end{cases}$$

(iii) For $0 < p < 1$ *and* $L > 0$,

$$\hat{L}(p, L) < \|u_{\lambda^*}\|_\infty < 1,$$

where

$$\begin{aligned} \hat{L}(p, L) \quad &\equiv \quad \min\left\{1 - \left[1 - \frac{1}{k}(1-p)L\right]^{\frac{1}{1-p}}, \frac{1}{k^2 p + 1}\right\}, \; k = \max\{3, (1-p)L\} \\ &= \quad \begin{cases} \frac{1}{p(1-p)^2L^2+1} & \text{if } (1-p)L \geq 3, \\ \min\left\{1 - \left[1 - \frac{1}{3}(1-p)L\right]^{\frac{1}{1-p}}, \frac{1}{9p+1}\right\} & \text{if } 0 < (1-p)L < 3. \end{cases} \end{aligned} \tag{2.4}$$

(iv) (a) For fixed $L > 0$,

$$\lim_{p\to\infty} \|u_{\lambda^*}\|_\infty = 0. \tag{2.5}$$

In addition, for fixed $L \geq 3$,

$$\lim_{p\to 0^+} \|u_{\lambda^*}\|_\infty = 1, \tag{2.6}$$

and for fixed positive $L < 3$,

$$\lim_{p\to 0^+} \|u_{\lambda^*}\|_\infty \geq \frac{L}{3}. \tag{2.7}$$

(b) For fixed $p > 0$,

$$\lim_{L\to 0^+} \|u_{\lambda^*}\|_\infty = 0. \tag{2.8}$$

Remark 2.4. *In particular, when* $p = 2$*, then in (2.2) we have that* $\bar{U}(p = 2) = 1/2$ *and*

$$\bar{L}(p = 2, L) = \begin{cases} \frac{1}{19} \approx 0.0526 & \text{if } L \geq L_1(p = 2) = \frac{1}{6} \approx 0.167, \\ \frac{L}{L+3} & \text{if } 0 < L < L_1(p = 2) = \frac{1}{6}. \end{cases}$$

Hence, for $L \geq 1/6$*, the estimates give*

$$0.0526 \approx \frac{1}{19} < \|u_{\lambda^*}\|_\infty (p = 2, L) < \frac{1}{2} = 0.5,$$

and, for $0 < L < 1/6$*, the estimates give*

$$(0, \frac{1}{19}) \ni \frac{L}{L+3} < \|u_{\lambda^*}\|_\infty (p = 2, L) < \frac{1}{2} = 0.5.$$

Our analytic result in (2.2) also agrees with some numerical simulations obtained by Brubaker and Pelesko [4, Figure 1.2(b)–(d)] for $p = 2$ *and* $0.1 \leq L \leq 0.8$.

Remark 2.5. *We conjecture that, for any fixed* $L > 0$, $\lim_{p\to 0^+} \|u_{\lambda^*}\|_\infty = 1$*; cf. (2.6)–(2.7) and (1.9). In addition, for any fixed* $p > 0$ *and* $\lambda^* = \lambda^*(L)$, $\|u_{\lambda^*}\|_\infty$ *is a strictly increasing function of* $L > 0$*. Note that, when* $p = 2$*, our numerical simulation shows that* $\lim_{L\to\infty} \|u_{\lambda^*}\|_\infty = \zeta \approx 0.388$ *for some* ζ*, cf. [4, Figure 1.2(b)] and (2.8). Further investigations are needed.*

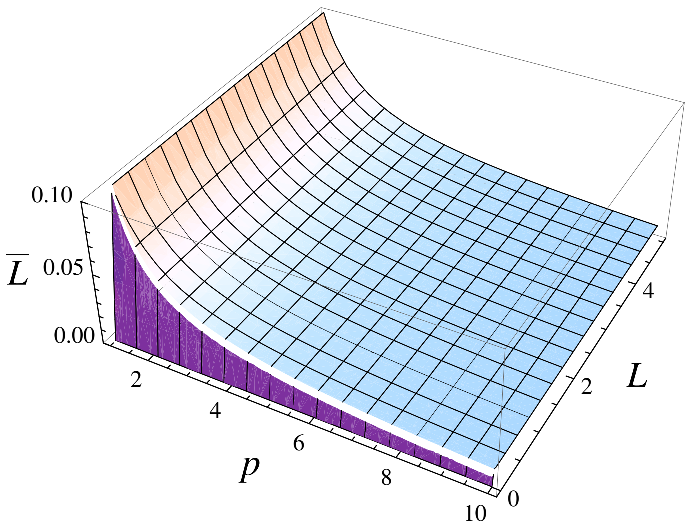

Figure 3. The graph of $\bar{L}(p, L)$ with $p \in (1, 10)$ and $L \in (0, 5)$.

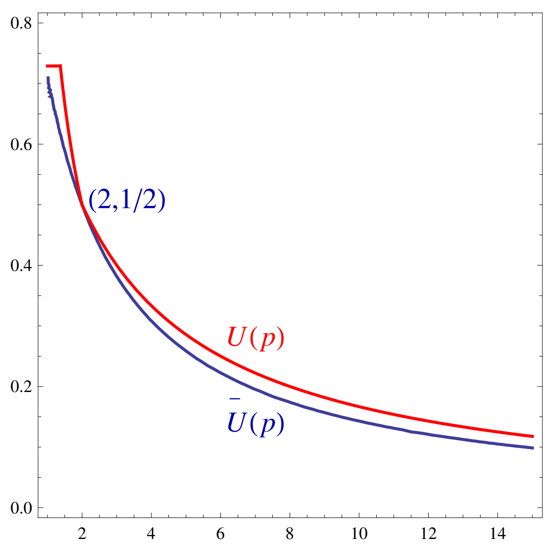

Figure 4. Graphs of $\bar{U}(p)$ and $U(p)$ with $p \in (1, 15)$. Note that $\bar{U}(2) = U(2) = \frac{1}{2}$ and $\lim_{p\to\infty} \bar{U}(p) = \lim_{p\to\infty} U(p) = \lim_{p\to\infty} \frac{2}{p+2} = 0$.

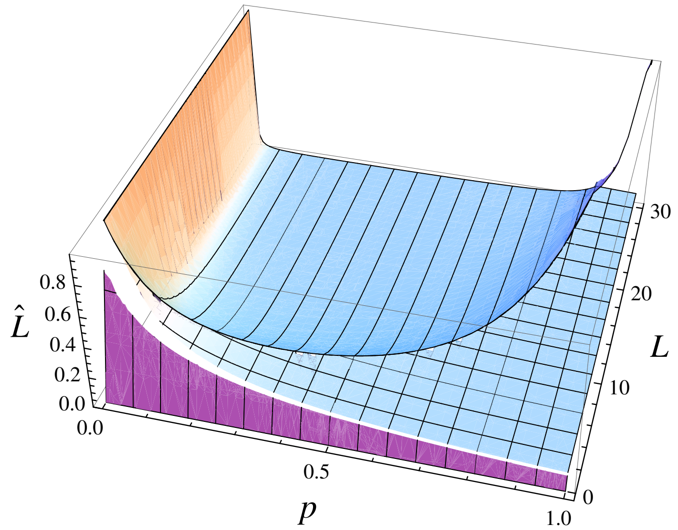

Figure 5. The graph of $\hat{L}(p, L)$ with $p \in (0, 1)$ and $L \in (0, 30)$.

3. Lemmas

To prove the upper and lower bounds for $\|u_{\lambda^*}\|_\infty$ in Theorem 2.3(i)–(iii) for (1.1) with $p > 0$, we need the following Lemmas 3.1–3.6. We first establish sufficient conditions on r and p such that $\|u_{\lambda^*}\|_\infty < r$ for all $L > 0$. To this purpose, we recall the time map formula $T_{p,\lambda}(r)$ for (1.1) as follows:

$$T_{p,\lambda}(r) = \int_0^r \frac{1 + \lambda F(u) - \lambda F(r)}{\sqrt{1 - [1 + \lambda F(u) - \lambda F(r)]^2}} du, \quad r = \|u\|_\infty \in I; \tag{3.1}$$

where

$$F(u) \equiv \int_0^u f(t)dt = \int_0^u \frac{1}{(1-t)^p} dt = \begin{cases} \dfrac{1 - (1-u)^{1-p}}{1-p} & \text{if } p > 0 \text{ and } p \neq 1, \\ -\ln(1-u) & \text{if } p = 1, \end{cases} \tag{3.2}$$

and I is the domain of $T_{p,\lambda}(r)$. Notice that the domain I of $T_{p,\lambda}(r)$ depends on the value p. We have that:

(I) If $p \geq 1$, $F : [0, 1) \to [0, \infty)$ is strictly increasing, and hence F^{-1} is well defined on $[0, \infty)$. Then for any $\lambda > 0$, the domain I of $T_{p,\lambda}(r)$ is

$$(0, F^{-1}(\frac{1}{\lambda})].$$

(II) If $0 < p < 1$, $F : [0, 1] \to [0, \frac{1}{1-p}]$ is strictly increasing, and hence F^{-1} is only defined on $[0, \frac{1}{1-p}]$. Then for any $\lambda > 0$, the domain I of $T_{p,\lambda}(r)$ is

$$\begin{cases} (0, F^{-1}(\frac{1}{\lambda})] & \text{if } \lambda > 1 - p, \\ (0, 1) & \text{if } 0 < \lambda \leq 1 - p. \end{cases}$$

See [1, p. 286].

Observe that positive solutions u_λ for (1.1) correspond to

$$\|u_\lambda\|_\infty = r \text{ and } T_{p,\lambda}(r) = L. \tag{3.3}$$

Thus, studying of the exact number of positive solutions of (1.1) for any fixed $\lambda > 0$ is equivalent to studying the shape of the time map $T_{p,\lambda}(r)$ on its domain I. Moreover, we observe that

$$\begin{aligned} & T'_{p,\lambda}(r) \\ = & \int_0^1 \frac{[1+\lambda F(rs)-\lambda F(r)]\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}+\lambda\left[rsf(rs)-rf(r)\right]}{\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}^{3/2}}ds \\ = & \int_0^1 \frac{\Psi(\lambda,r,s)}{\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}^{3/2}}ds, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} & \Psi(\lambda,r,s) \\ \equiv & [1+\lambda F(rs)-\lambda F(r)]\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}+\lambda\left[rsf(rs)-rf(r)\right]. \end{aligned} \tag{3.5}$$

See [1, (3.2)].

First, we have the next lemma which shows that $T_{p,\lambda}(r)$ for (1.1) has exactly one critical point, a local maximum, on its domain.

Lemma 3.1 ([1, Lemma 3.2])**.** *Consider $T_{p,\lambda}(r)$ for (1.1). The following assertions (i)–(iii) hold:*

(i) For fixed $p \ge 1$, $T_{p,\lambda}(r)$ has exactly one critical point, a local maximum, on $(0, F^{-1}(1/\lambda))$ for any $\lambda > 0$.
(ii) For fixed $p \in (0,1)$, $T_{p,\lambda}(r)$ has exactly one critical point, a local maximum, on $(0, F^{-1}(1/\lambda))$ for any $\lambda > 1-p$.
(iii) For fixed $p \in (0,1)$, $T_{p,\lambda}(r)$ has exactly one critical point, a local maximum, on $(0,1)$ for any $0 < \lambda \le 1-p$.

In the following Lemma 3.2 with $p > 1$, and Lemma 3.3 with $p = 1$, we prove that $T'_{p,\lambda}(r) < 0$ for $\lambda \in (0, 1/F(r))$, where r satisfies some conditions stated below. Thus $\|u_{\lambda^*}\|_\infty < r$ for $L > 0$ by applying Theorem 1.1.

Lemma 3.2. *Suppose that $0 < r < 1$ and $p > 1$. Then $T'_{p,\lambda}(r) < 0$ for $\lambda \in (0, 1/F(r))$ if (p, r) satisfies $\Gamma(p,r) = (p+1)r + 2(1-r)^p - 2 = 0$.*

Lemma 3.3. *Suppose that $p = 1$. Then $T'_{1,\lambda}(1-2e^{-2}) < 0$ for $\lambda \in (0, 1/F(1-2e^{-2}))$.*

Proof of Lemma 3.2. For $p > 1$, we have that $f(u) = (1-u)^{-p}$ and $F(u) = \frac{1-(1-u)^{1-p}}{1-p}$. Hence, by (3.5), we compute that

$$\Psi(\lambda, r, s) = \lambda\Phi(\lambda, r, s) \tag{3.6}$$

where

$$\begin{aligned}
&\Phi(\lambda, r, s)\\
\equiv\ & \frac{1}{(p-1)^3}\left[(1-r)^{1-p}-(1-rs)^{1-p}\right]^3\lambda^2-\frac{3}{(p-1)^2}\left[(1-r)^{1-p}-(1-rs)^{1-p}\right]^2\lambda\\
&+\frac{2}{p-1}\left[(1-r)^{1-p}-(1-rs)^{1-p}\right]+\left[rs(1-rs)^{-p}-r(1-r)^{-p}\right].
\end{aligned} \tag{3.7}$$

For any fixed positive $r, s < 1$, $\tilde{\Phi}(\lambda) \equiv \Phi(\lambda, r, s)$ is a quadratic polynomial in λ. To prove this lemma, by (3.1)–(3.6), for $p > 1$, it suffice to prove that $\tilde{\Phi}(\lambda) < 0$ on $[0, 1/F(r)]$ which follows by proving that $\tilde{\Phi}(\lambda)$ is convex on $[0, 1/F(r)]$, $\tilde{\Phi}(0) < 0$ and $\tilde{\Phi}(1/F(r)) < 0$.

(I) First, we consider the leading coefficient of the quadratic polynomial $\tilde{\Phi}(\lambda)$. Let

$$a_1(s) \equiv (1-r)^{1-p}-(1-rs)^{1-p}.$$

We have that $a_1(1) = 0$ and

$$a_1'(s) = (1-p)\,r\,(1-rs)^{-p} < 0 \ \text{ for } p > 1 \text{ and } 0 < r, s < 1.$$

Hence, $a_1(s) > 0$ for $0 < r, s < 1$. So $\tilde{\Phi}(\lambda)$ is convex on $[0, 1/F(r)]$.

(II) Secondly, we let

$$\begin{aligned}
a_2(s) &\equiv (1-p)\,(1-r)^p(1-rs)^p\tilde{\Phi}(0)\\
&= -(2-r-rp)\,(1-rs)^p+(2-rs-rsp)\,(1-r)^p.
\end{aligned}$$

We find that $a_2(1) = 0$, and

$$a_2(0) = (p+1)r + 2(1-r)^p - 2 = 0$$

by the assumption. On the other hand, since

$$a_2'(s) = rp\,(2-r-rp)\,(1-rs)^{p-1}-(r+rp)\,(1-r)^p$$

and

$$a_2''(s) = -p\,(p-1)\,r^2\,(2-r-rp)\,(1-rs)^{p-2},$$

we find that

$$\begin{aligned}
(1-rs)a_2''(s)+r\,(p-1)\,a_2'(s) &= -\left(p^2-1\right)r^2(1-r)^p\\
&< 0 \ \text{ for } p > 1 \text{ and } 0 < r, s < 1.
\end{aligned}$$

That is, for $s \in (0, 1)$, we find that $a_2''(s) < 0$ whenever $a_2'(s) = 0$. Hence $a_2(s) > 0$ for $s \in (0, 1)$. This implies that $\tilde{\Phi}(0) < 0$.

(III) Thirdly, we obtain that

$$\begin{aligned} a_3(s) &\equiv (p-1)\,\tilde{\Phi}(1/F(r)) \\ &= \frac{1}{\left[(1-r)^{1-p}-1\right]^2}\left[(1-r)^{1-p}-(1-rs)^{1-p}\right]^3 \\ &\quad -\frac{3}{(1-r)^{1-p}-1}\left[(1-r)^{1-p}-(1-rs)^{1-p}\right]^2 \\ &\quad -(2-rs-rsp)\,(1-rs)^{-p}+(2-r-rp)\,(1-r)^{-p}. \end{aligned}$$

We have that $a_3(1)=0$ and $a_3(0)=-r\,(p-1)\,(1-r)^{-p}<0$ for $p>1$ and $0<r<1$. Moreover, we compute that

$$\begin{aligned} \phi_1(s) &\equiv \frac{(1-rs)^p}{r}a_3'(s) \\ &= \frac{3(1-p)}{\left[(1-r)^{1-p}-1\right]^2}(1-rs)^{2-2p}-\frac{6(1-p)}{\left[(1-r)^{1-p}-1\right]^2}(1-rs)^{1-p} \\ &\quad -\frac{3(1-p)\left[(1-r)^{2-2p}-2(1-r)^{1-p}\right]}{\left[(1-r)^{1-p}-1\right]^2}+p\,(rsp+rs-2)\,(1-rs)^{-1}+p+1. \end{aligned}$$

Then we compute that $\phi_1(0)=2(p-1)>0$ and

$$\phi_1(1)=\frac{(p-1)\left[(p+1)r-1\right]}{1-r}. \tag{3.8}$$

We claim that $\phi_1(1)>0$ if (p,r) satisfies $\Gamma(p,r)=(p+1)\,r+2\,(1-r)^p-2=0$. We next give a proof of this claim. First, it is easy to see that

$$\begin{aligned} \hat{\Phi} &\equiv \{(p,r): p>1, 0<r<1,\ (p+1)\,r+2\,(1-r)^p-2\geq 0\} \\ &\quad \cap\{(p,r): p>1, 0<r<1,\ (p+1)\,r-1\geq 0\}\neq\emptyset \end{aligned} \tag{3.9}$$

since $(p,r)=(2,1/2)\in\hat{\Phi}$. Now, suppose (p,r) satisfies $(p+1)\,r-1=0$. We find that

$$(p+1)\,r+2\,(1-r)^p-2=2\,(1-r)^p-1=2\left(\frac{p}{p+1}\right)^p-1<0$$

since $2\left(\frac{p}{p+1}\right)^p-1=0$ when $p=1$ and it is a strictly decreasing function of $p\geq 1$. So, in addition to (3.9), for $p>1$, $0<r<1$, we obtain that $(p+1)\,r-1>0$ if $(p+1)\,r+2\,(1-r)^p-2\geq 0$; i.e.,

$$\begin{aligned} &\{(p,r): p>1, 0<r<1,\ (p+1)\,r+2\,(1-r)^p-2\geq 0\} \\ \subseteq\ &\{(p,r): p>1, 0<r<1,\ (p+1)\,r-1>0\}. \end{aligned} \tag{3.10}$$

So, by (3.8) and (3.10), $\phi_1(1)>0$ if (p,r) satisfies $(p+1)\,r+2\,(1-r)^p-2=0$.

We also compute that, if $p\geq\frac{3}{2}$,

$$\frac{(1-rs)}{r}\phi_1''(s)+(1-2p)\phi_1'(s)$$

$$= -\frac{6(p-1)^3 r}{\left[(1-r)^{1-p}-1\right]^2}(1-rs)^{-p} - rp(p-1)(2p-3)(1-rs)^{-2} < 0 \ \text{ for } s \in (0,1).$$

So we find that $\phi_1''(s) < 0$ whenever $\phi_1'(s) = 0$. This implies that $\phi_1(s) = \frac{(1-rs)^p}{r} a_3'(s) > 0$ for $s \in (0,1)$ (Observe $\phi_1(0) > 0$ and $\phi_1(1) > 0$). Therefore, $a_3(s) < 0$ for $s \in (0,1)$, and hence

$$\tilde{\Phi}(1/F(r)) < 0 \ \text{ if } p \geq \frac{3}{2}.$$

(IV) If $1 < p < \frac{3}{2}$, we claim that

$$\frac{(1-rs)}{r}\phi_1''(s) + (1-2p)\phi_1'(s) < 0 \ \text{ for } s \in (0,1).$$

We next give a proof of this claim. We compute that

$$\begin{aligned} \phi_2(s) &\equiv \frac{(1-rs)^2}{r(p-1)}\left[\frac{(1-rs)}{r}\phi_1''(s) + (1-2p)\phi_1'(s)\right] \\ &= p(3-2p) - \frac{6(p-1)^2}{[(1-r)^{1-p}-1]^2}(1-sr)^{2-p}. \end{aligned} \tag{3.11}$$

Then

$$\phi_2(1) = p(3-2p) - \frac{6(p-1)^2}{[(1-r)^{1-p}-1]^2}(1-r)^{2-p} \tag{3.12}$$

and

$$\phi_2'(s) = \frac{6r(p-1)^2(2-p)}{[(1-r)^{1-p}-1]^2}(1-sr)^{1-p} > 0. \tag{3.13}$$

So, if (p,r) satisfies

$$p(3-2p) - \frac{6(p-1)^2}{[(1-r)^{1-p}-1]^2}(1-r)^{2-p} \leq 0,$$

then

$$\frac{(1-rs)}{r}\phi_1''(s) + (1-2p)\phi_1'(s) \leq 0 \ \text{ for } s \in (0,1).$$

Next, we have that, for $1 < p < \frac{3}{2}$,

$$\begin{aligned} &\{(p,r): \ p \in (1,3/2), r \in (0,1), \ (p+1)r + 2(1-r)^p - 2 = 0\} \\ \subsetneq \ &\left\{(p,r): p \in (1,3/2), r \in (0,1), p(3-2p) - \frac{6(p-1)^2}{[(1-r)^{1-p}-1]^2}(1-r)^{2-p} \leq 0\right\}; \end{aligned} \tag{3.14}$$

see Figure 6. (Note that we can provide an analytic proof for (3.14) but we omit it here since it is too tedious.) If $p \in (1, 3/2)$, then

$$\frac{(1-rs)^2}{r(p-1)}\left\{\frac{(1-rs)}{r}\phi_1''(s) + (1-2p)\phi_1'(s)\right\} < 0 \text{ for } s \in (0,1)$$

by (3.11)–(3.14). So we find that $\phi_1''(s) < 0$ whenever $\phi_1'(s) = 0$. This implies that $\phi_1(s) = \frac{(1-rs)^p}{r}a_3'(s) > 0$ for $s \in (0,1)$ (Observe $\phi_1(0) > 0$ and $\phi_1(1) > 0$). Therefore, $a_3(s) < 0$ for $s \in (0,1)$, and hence

$$\tilde{\Phi}(1/F(r)) < 0 \text{ if } 1 < p < \frac{3}{2}.$$

We conclude that, by above parts (I)–(IV), $\tilde{\Phi}(\lambda)$ is convex on $[0, 1/F(r)]$, $\tilde{\Phi}(0) < 0$ and $\tilde{\Phi}(1/F(r)) < 0$. So $\tilde{\Phi}(\lambda) < 0$ on $[0, 1/F(r)]$ for $p > 1$. By (3.4)–(3.7), $T'_{p,\lambda}(r) < 0$ for $\lambda \in (0, 1/F(r))$ if $p > 1$, $r \in (0,1)$, and (p, r) satisfies $\Gamma(p, r) = (p+1)r + 2(1-r)^p - 2 = 0$.

The proof of Lemma 3.2 is complete.

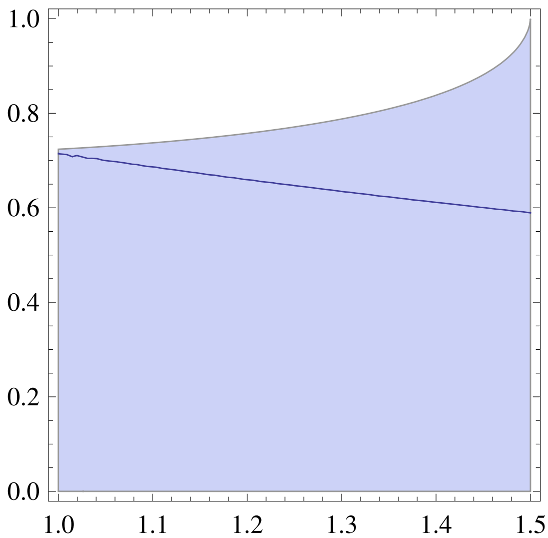

Figure 6. The curve $\{(p, r) : p \in (1, 3/2), r \in (0,1), (p+1)r + 2(1-r)^p - 2 = 0\}$ $\subsetneq$ The region $\left\{(p, r) : p \in (1, 3/2), r \in (0,1), p(3-2p) - \frac{6(p-1)^2}{\left[1-(1-r)^{1-p}\right]^2}(1-r)^{2-p} \le 0\right\}$. Note that the curve $\{(p, r) : p \in (1, 3/2), r \in (0,1), (p+1)r + 2(1-r)^p - 2 = 0\}$ emanates from the point $(1, r_2)$ with $r_2 \approx 0.715$ and the curve $\left\{(p, r) : p \in (1, 3/2), r \in (0,1), p(3-2p) - \frac{6(p-1)^2}{\left[1-(1-r)^{1-p}\right]^2}(1-r)^{2-p} = 0\right\}$ emanates from the point $(1, r_3)$ with $0.715 \approx r_2 < r_3 \approx 0.724$.

Proof of Lemma 3.3. For $p = 1$, we have that $f(u) = \frac{1}{1-u}$ and $F(u) = -\ln(1-u)$, and (3.5) can be reduced to

$$\Psi(\lambda, r, s) = \lambda\Theta(\lambda, r, s), \tag{3.15}$$

where

$$\Theta(\lambda, r, s) \equiv [\ln(1-rs) - \ln(1-r)]^3 \lambda^2 - 3[\ln(1-rs) - \ln(1-r)]^2 \lambda$$

$$+2\left[\ln(1-rs)-\ln(1-r)\right]-\frac{r\,(1-s)}{(1-r)\,(1-rs)}. \tag{3.16}$$

Let $r_0 \equiv 1-2e^{-2}$ (≈ 0.729), since $\ln(1-r_0 s)-\ln(1-r_0)>0$ for $s\in(0,1)$, $\tilde{\Theta}(\lambda)\equiv\Theta(\lambda,r_0,s)$ is a convex quadratic polynomial in λ. In the following, we will prove that

$$\tilde{\Theta}(0)=2\left[\ln(1-r_0 s)-\ln(1-r_0)\right]-\frac{r_0\,(1-s)}{(1-r_0)\,(1-r_0 s)}<0 \ \text{ for } s\in(0,1)$$

and

$$\tilde{\Theta}(1/F(r_0))=\frac{1}{[\ln(1-r_0)]^2}\left[\ln(1-r_0 s)\right]^3-\ln(1-r_0 s)-\frac{r_0\,(1-s)}{(1-r_0)\,(1-r_0 s)}<0 \ \text{ for } s\in(0,1).$$

Let

$$\theta_1(s)\equiv 2\left[\ln(1-r_0 s)-\ln(1-r_0)\right]-\frac{r_0\,(1-s)}{(1-r_0)\,(1-r_0 s)}.$$

Then $\theta_1(0)=-2\ln(1-r_0)-\frac{r_0}{1-r_0}\approx-0.0808<0$, $\theta_1(1)=0$,

$$\theta_1'(s)=\frac{-2r_0}{1-r_0 s}+\frac{r_0}{(1-r_0 s)^2}$$

and

$$\theta_1''(s)=\frac{-2r_0^2}{(1-r_0 s)^2}+\frac{2r_0^2}{(1-r_0 s)^3}.$$

We compute that

$$sr_0^2\theta_1'(s)+(1-r_0 s)^2\,\theta_1''(s)=\frac{r_0^3 s}{(1-r_0 s)^2}>0.$$

This implies that $\theta_1''(s)>0$ whenever $\theta_1'(s)=0$. Thus, $\theta_1(s)<0$ for all $s\in(0,1)$ and then $\tilde{\Theta}(0)<0$ for $s\in(0,1)$.

On the other hand, let

$$\theta_2(s)\equiv\frac{1}{[\ln(1-r_0)]^2}\left[\ln(1-r_0 s)\right]^3-\ln(1-r_0 s)-\frac{r_0\,(1-s)}{(1-r_0)\,(1-r_0 s)}.$$

Then $\theta_2(0)=\frac{-r_0}{1-r_0}$ (≈-2.695) <0, $\theta_2(1)=0$,

$$\theta_2'(s)=-\frac{3r_0}{[\ln(1-r_0)]^2\,(1-r_0 s)}\left[\ln(1-r_0 s)\right]^2+\frac{r_0\,(2-r_0 s)}{(1-r_0 s)^2}$$

and

$$\begin{aligned}\theta_2''(s) &= -\frac{3r_0^2}{[\ln(1-r_0)]^2\,(1-r_0 s)^2}\left[\ln(1-r_0 s)\right]^2\\ &\quad+\frac{6r_0^2}{[\ln(1-r_0)]^2\,(1-r_0 s)^2}\left[\ln(1-r_0 s)\right]+\frac{r_0^2\,(3-r_0 s)}{(1-r_0 s)^3}.\end{aligned}$$

We compute that $\theta_2'(1) = \frac{r_0}{(1-r_0)^2}(2r_0 - 1)$ (≈ 4.566) > 0, and

$$r_0\theta_2'(s) - (1 - r_0 s)\,\theta_2''(s) = -\frac{r_0^2}{(1-r_0 s)^2}\left\{\frac{6\,(1-r_0 s)}{[\ln(1-r_0)]^2}\,[\ln(1-r_0 s)] + 1\right\}. \tag{3.17}$$

Furthermore, let

$$\theta_3(s) \equiv \frac{6\,(1-r_0 s)}{[\ln(1-r_0)]^2}\,[\ln(1-r_0 s)] + 1. \tag{3.18}$$

Then $\theta_3(0) = 1 > 0$, $\theta_3(1) = \frac{6(1-r_0)}{\ln(1-r_0)} - 1$ (≈ -0.243) < 0, and

$$r_0\theta_3'(s) + [\ln(1-r_0 s)]\,(1-r_0 s)\theta_3''(s) = -\frac{6r_0^2}{[\ln(1-r_0)]^2} < 0.$$

This implies that $\theta_3''(s) > 0$ whenever $\theta_3'(s) = 0$. Then there exists $s_0 \in (0,1)$ such that

$$\theta_3(s)\begin{cases} > 0 & \text{on } (0, s_0), \\ = 0 & \text{when } s = s_0, \\ < 0 & \text{on } (s_0, 1). \end{cases} \tag{3.19}$$

By (3.17)–(3.19), we obtain that $\theta_2''(s) > 0$ (resp. $\theta_2''(s) = 0$, $\theta_2''(s) < 0$) whenever $\theta_2'(s) = 0$ and $s \in (0, s_0)$ (resp. $s = s_0$, $s \in (s_0, 1)$). We next show that $\theta_2(s) < 0$ for all $s \in (0,1)$. Observe $\theta_2(0) < 0$, $\theta_2(1) = 0$ and $\theta_2'(1) > 0$. Assume $\theta_2(s) \geq 0$ for some $s \in (0,1)$, then there exist $0 < s_1 < s_2 < 1$ such that $\theta_2(s_1) \geq 0$ is a local maximum of $\theta_2(s)$ and $\theta_2(s_2) < 0$ is a local minimum of $\theta_2(s)$. Thus $\theta_2''(s_1) \leq 0$ and $\theta_2(s_2) \geq 0$, which contradicts to (3.17)–(3.19). So $\theta_2(s) < 0$ for all $s \in (0,1)$.

Finally, by the above analyses with $r_0 = 1 - 2e^{-2}$, $\tilde{\Theta}(\lambda)$ is convex on $[0, 1/F(r_0)]$, $\tilde{\Theta}(0) < 0$ and $\tilde{\Theta}(1/F(r_0)) < 0$. So $\tilde{\Theta}(\lambda) < 0$ on $[0, 1/F(r_0)]$. By (3.4), (3.5), (3.15) and (3.16), $T'_{1,\lambda}(r_0) < 0$ for $\lambda \in (0, 1/F(r_0))$.

The proof of Lemma 3.3 is complete.

In the following Lemma 3.4 with $p > 1$, Lemma 3.5 with $p = 1$, and Lemma 3.6 with $0 < p < 1$, we prove that $T'_{p,\lambda}(r_p) > 0$ for $\lambda \in (0, 1/L]$, where r_p is defined below. Thus $\|u_{\lambda^*}\|_\infty > r_p$ for $L > 0$ by applying Theorems 1.1–1.2.

Lemma 3.4. *Consider $p > 1$. Then $T'_{p,\lambda}(r_p) > 0$ for $\lambda \in (0, 1/L]$, where*

$$r_p \equiv \min\left\{1 - \left[1 + \frac{1}{3}(p-1)\,L\right]^{\frac{1}{1-p}}, \frac{1}{9p+1}\right\}.$$

Lemma 3.5. *Consider $p = 1$. Then $T'_{1,\lambda}(r_1) > 0$ for $\lambda \in (0, 1/L]$, where $r_1 \equiv \min\left\{1 - e^{-\frac{L}{3}}, \frac{1}{10}\right\}$.*

Lemma 3.6. *Consider $0 < p < 1$. Then $T'_{p,\lambda}(r_p) > 0$ for $\lambda \in (0, 1/L]$, where*

$$r_p \equiv \hat{L}(p, L) = \min\left\{1 - \left[1 - \frac{1}{k}(1-p)\,L\right]^{\frac{1}{1-p}}, \frac{1}{k^2 p + 1}\right\}, \quad k = \max\{3, (1-p)L\}.$$

Proof of Lemma 3.4. For $p > 1$, we have that $f(u) = (1-u)^{-p}$ and $F(u) = \frac{(1-u)^{1-p}-1}{p-1}$. Let $\lambda = \frac{q}{L}$ with $q \in (0, 1]$. Hence by (3.5), we compute that

$$\begin{aligned} G(q,r,s) &\equiv (p-1)^3 L^3 \Psi(\frac{q}{L}, r, s) \\ &= q^3 \left[(1-r)^{1-p} - (1-rs)^{1-p}\right]^3 - 3q^2 (p-1) L \left[(1-r)^{1-p} - (1-rs)^{1-p}\right]^2 \\ &\quad + 2q (p-1)^2 L^2 \left[(1-r)^{1-p} - (1-rs)^{1-p}\right] \\ &\quad - q (p-1)^3 L^2 \left[r(1-r)^{-p} - rs(1-rs)^{-p}\right]. \end{aligned} \tag{3.20}$$

Assume that $M > 1$ is a given number. Then, for $0 < r, s < 1$ satisfying $\frac{1}{1-r} \leq M$, by applying Cauchy's Mean Value Theorem, it is easy to check that

$$0 < r(1-r)^{-p} - rs(1-rs)^{-p} \leq \left(\frac{pM}{p-1} - 1\right)\left[(1-r)^{1-p} - (1-rs)^{1-p}\right].$$

Therefore,

$$G(q,r,s) \geq q\left[(1-r)^{1-p} - (1-rs)^{1-p}\right] \tilde{G}(q,r,s), \tag{3.21}$$

where

$$\begin{aligned} &\tilde{G}(q,r,s) \\ \equiv\ & q^2 \left[(1-r)^{1-p} - (1-rs)^{1-p}\right]^2 \\ & -3q (p-1) L \left[(1-r)^{1-p} - (1-rs)^{1-p}\right] + (p-1)^2 L^2 (p+1-pM). \end{aligned} \tag{3.22}$$

Note that $0 < (1-r)^{1-p} - (1-rs)^{1-p} < (1-r)^{1-p} - 1 \leq M^{p-1} - 1$. Let

$$g_M(z) \equiv z^2 - 3(p-1) Lz + (p-1)^2 L^2 (p+1-pM). \tag{3.23}$$

We aim to find a number $M_0 > 1$ such that $g_{M_0}(z) > 0$ for $0 \leq z \leq M_0^{p-1} - 1$. This implies that

$$\tilde{G}(q,r,s) > 0 \ \text{ for } 0 < q \leq 1,\ 0 < r, s < 1$$

with $\frac{1}{1-r} \leq M_0$. Since $g'_M(0) = -3(p-1)L < 0$ and g_M is convex on $(0, \infty)$, we only need to prove that $g'_{M_0}(M_0^{p-1} - 1) \leq 0$ and $g_{M_0}(M_0^{p-1} - 1) \geq 0$ for some $M_0 > 1$. We have that

$$g'_M(z) = 2z - 3(p-1)L$$

and

$$g'_{M_0}(M_0^{p-1} - 1) = 2M_0^{p-1} - 2 - 3(p-1)L \leq 0 \ \text{ if and only if } \ M_0 \leq \left[1 + \frac{3}{2}(p-1)L\right]^{\frac{1}{p-1}}. \tag{3.24}$$

Thus we choose $M_0 \equiv \min\left\{\left[1 + \frac{1}{3}(p-1)L\right]^{\frac{1}{p-1}}, 1 + \frac{1}{9p}\right\}$. Then

$$g_{M_0}(M_0^{p-1} - 1) = (M_0^{p-1} - 1)^2 - 3(p-1) L(M_0^{p-1} - 1) + (p-1)^2 L^2 (p+1-pM_0)$$

$$
\begin{aligned}
&= \left[\frac{8}{3}(p-1)L-(M_0^{p-1}-1)\right]\left[\frac{1}{3}(p-1)L-(M_0^{p-1}-1)\right] \\
&\quad +(p-1)^2L^2\left(p+\frac{1}{9}-pM_0\right) \\
&\geq (p-1)^2L^2\left(p+\frac{1}{9}-pM_0\right)\geq 0. \qquad (3.25)
\end{aligned}
$$

So $g'_{M_0}(M_0^{p-1}-1)\leq 0$ and $g_{M_0}(M_0^{p-1}-1)\geq 0$ by (3.24) and (3.25).

Finally, we choose

$$
r_p \equiv 1-\frac{1}{M_0} = \min\left\{1-\left[1+\frac{1}{3}(p-1)L\right]^{\frac{1}{1-p}}, \frac{1}{9p+1}\right\}.
$$

Then we obtain that $T'_{p,\lambda}(r_p) > 0$ for $\lambda\in(0,1/L]$ by (3.4), (3.5), and (3.20)–(3.25). Observe that $r_p\in(0,F^{-1}(\frac{1}{\lambda}))$ for $\lambda\in(0,1/L]$.

The proof of Lemma 3.4 is complete.

Proof of Lemma 3.5. For $p=1$, we have that $f(u)=(1-u)^{-1}$ and $F(u)=-\ln(1-u)$. Let $\lambda=\frac{q}{L}$ with $q\in(0,1]$. Hence by (3.5), we compute that

$$
\begin{aligned}
J(q,r,s) &\equiv L^3\Psi(\frac{q}{L},r,s) \\
&= q^3[\ln(1-rs)-\ln(1-r)]^3-3q^2L[\ln(1-rs)-\ln(1-r)]^2 \\
&\quad +2qL^2[\ln(1-rs)-\ln(1-r)]-qL^2\left[\frac{r}{1-r}-\frac{rs}{1-rs}\right]. \qquad (3.26)
\end{aligned}
$$

Assume that $M>1$ is a given number. Then, for $0<r,s<1$ satisfying $\frac{1}{1-r}\leq M$, by applying Cauchy's Mean Value Theorem, it is easy to check that

$$
0<\frac{r}{1-r}-\frac{rs}{1-rs}\leq M[\ln(1-rs)-\ln(1-r)].
$$

Therefore,

$$
J(q,r,s)\geq q[\ln(1-rs)-\ln(1-r)]\tilde{J}(q,r,s), \qquad (3.27)
$$

where

$$
\tilde{J}(q,r,s)\equiv q^2[\ln(1-rs)-\ln(1-r)]^2-3qL[\ln(1-rs)-\ln(1-r)]+L^2(2-M). \qquad (3.28)
$$

Note that $0<\ln(1-rs)-\ln(1-r)<\ln\frac{1}{1-r}\leq\ln M$. Let

$$
j_M(z)\equiv z^2-3Lz+L^2(2-M). \qquad (3.29)
$$

We aim to find a number $M_0>1$ such that $j_{M_0}(z)>0$ for $0\leq z\leq\ln M_0$. This implies that

$$
\tilde{J}(q,r,s)>0 \ \text{ for } 0<q\leq 1,\ 0<r,s<1
$$

with $\frac{1}{1-r}\leq M_0$. Since $j'_M(0)=-3L<0$ and j_M is convex on $(0,\infty)$, we only need to prove that $j'_{M_0}(\ln M_0)\leq 0$ and $j_{M_0}(\ln M_0)\geq 0$ for some $M_0>1$. We have that

$$
j'_M(z)=2z-3L
$$

and

$$j'_{M_0}(\ln M_0) = 2\ln M_0 - 3L \le 0 \text{ if and only if } M_0 \le e^{\frac{3}{2}L}. \tag{3.30}$$

Thus we choose $M_0 \equiv \min\left\{e^{\frac{L}{3}}, \frac{10}{9}\right\}$. Then

$$\begin{aligned} j_{M_0}(\ln M_0) &= (\ln M_0)^2 - 3L(\ln M_0) + L^2(2 - M_0) \\ &= \left(\frac{8}{3}L - \ln M_0\right)\left(\frac{1}{3}L - \ln M_0\right) + L^2\left(\frac{10}{9} - M_0\right) \\ &\ge L^2\left(\frac{10}{9} - M_0\right) \ge 0. \end{aligned} \tag{3.31}$$

So $j'_{M_0}(\ln M_0) \le 0$ and $j_{M_0}(\ln M_0) \ge 0$ by (3.30) and (3.31).

Finally, we choose

$$r_1 \equiv 1 - \frac{1}{M_0} = \min\left\{1 - e^{-\frac{L}{3}}, \frac{1}{10}\right\}.$$

Then we obtain that $T'_{1,\lambda}(r_1) > 0$ for $\lambda \in (0, 1/L]$ by (3.4), (3.5), and (3.26)–(3.31). Observe that $r_1 \in (0, F^{-1}(\frac{1}{\lambda}))$ for $\lambda \in (0, 1/L]$.

The proof of Lemma 3.5 is complete.

Proof of Lemma 3.6. For $0 < p < 1$, we have that $f(u) = (1-u)^{-p}$ and $F(u) = \frac{1-(1-u)^{1-p}}{1-p}$. Let $\lambda = \frac{q}{L}$ with $q \in (0, 1]$. Hence by (3.5), we compute that

$$\begin{aligned} H(q, r, s) &\equiv (1-p)^3 L^3 \Psi(\frac{q}{L}, r, s) \\ &= q^3\left[(1-rs)^{1-p} - (1-r)^{1-p}\right]^3 - 3q^2(1-p)L\left[(1-rs)^{1-p} - (1-r)^{1-p}\right]^2 \\ &\quad +2q(1-p)^2L^2\left[(1-rs)^{1-p} - (1-r)^{1-p}\right] \\ &\quad -q(1-p)^3L^2\left[r(1-r)^{-p} - rs(1-rs)^{-p}\right]. \end{aligned} \tag{3.32}$$

Assume that $M > 1$ is a given number. Then, for $0 < r, s < 1$ satisfying $\frac{1}{1-r} \le M$, by applying Cauchy's Mean Value Theorem, it is easy to check that

$$0 < r(1-r)^{-p} - rs(1-rs)^{-p} \le \left(\frac{pM}{1-p} + 1\right)\left[(1-rs)^{1-p} - (1-r)^{1-p}\right].$$

Therefore,

$$H(q, r, s) \ge q\left[(1-rs)^{1-p} - (1-r)^{1-p}\right]\tilde{H}(q, r, s), \tag{3.33}$$

where

$$\begin{aligned} &\tilde{H}(q, r, s) \\ \equiv\ & q^2\left[(1-rs)^{1-p} - (1-r)^{1-p}\right]^2 \\ & -3q(1-p)L\left[(1-rs)^{1-p} - (1-r)^{1-p}\right] + (1-p)^2L^2(p + 1 - pM). \end{aligned} \tag{3.34}$$

Note that $0 < (1-rs)^{1-p} - (1-r)^{1-p} < 1 - (1-r)^{1-p} \le 1 - M^{p-1}$. Let

$$h_M(z) \equiv z^2 - 3(1-p)Lz + (1-p)^2L^2(p + 1 - pM). \tag{3.35}$$

We aim to find a number $M_0 > 1$ such that $h_{M_0}(z) > 0$ for $0 \le z \le 1 - M_0^{p-1}$. This implies that

$$\tilde{H}(q, r, s) > 0 \;\text{ for } 0 < q \le 1,\ 0 < r, s < 1$$

with $\frac{1}{1-r} \le M_0$. Since $h'_M(0) = -3\,(1-p)\,L < 0$ and h_M is convex on $(0, \infty)$, we only need to prove that $h'_{M_0}(1 - M_0^{p-1}) \le 0$ and $h_{M_0}(1 - M_0^{p-1}) \ge 0$ for some $M_0 > 1$. We have that

$$h'_M(z) = 2z - 3\,(1-p)\,L$$

and

$$h'_{M_0}\left(1 - M_0^{p-1}\right) = 2 - 2M_0^{p-1} - 3\,(1-p)\,L.$$

Thus we choose

$$M_0 \equiv \min\left\{\left[1 - \frac{1}{k}\,(1-p)\,L\right]^{\frac{1}{p-1}}, 1 + \frac{1}{k^2 p}\right\} \;\text{ with } k = \max\left\{3, (1-p)L\right\}.$$

Then

$$\begin{aligned}
h_{M_0}\left(1 - M_0^{p-1}\right) &= \left(1 - M_0^{p-1}\right)^2 - 3\,(1-p)\,L\left(1 - M_0^{p-1}\right) + (1-p)^2\,L^2\,(p + 1 - pM_0) \\
&= \left[\frac{1}{k}\,(1-p)\,L - \left(1 - M_0^{p-1}\right)\right]\left[(3 - \frac{1}{k})\,(1-p)\,L - \left(1 - M_0^{p-1}\right)\right] \\
&\quad + (1-p)^2\,L^2\left(\frac{1}{k^2} - \frac{3}{k} + p + 1 - pM_0\right) \\
&\ge (1-p)^2\,L^2\left(\frac{1}{k^2} - \frac{3}{k} + p + 1 - pM_0\right) \\
&\ge (1-p)^2\,L^2\left(\frac{1}{k^2} + p - pM_0\right) \ge 0 \qquad (3.36)
\end{aligned}$$

and

$$h'_{M_0}(1 - M_0^{p-1}) = 2 - 2M_0^{p-1} - 3\,(1-p)\,L \le \left(\frac{2}{k} - 3\right)(1-p)\,L < 0. \qquad (3.37)$$

In (3.36) and (3.37), notice $p - 1 < 0$ and hence $M_0^{p-1} \ge 1 - \frac{1}{k}\,(1-p)\,L$.

Finally, we choose

$$r_p \equiv 1 - \frac{1}{M_0} = \min\left\{1 - \left[1 - \frac{1}{k}\,(p-1)\,L\right]^{\frac{1}{1-p}}, \frac{1}{k^2 p + 1}\right\} \;\text{ with } k = \max\left\{3, (1-p)L\right\}.$$

Then we obtain that $T'_{p,\lambda}(r_p) > 0$ for $\lambda \in (0, 1/L]$ by (3.4), (3.5), and (3.32)–(3.37). Observe that $r_p \in (0, F^{-1}(\frac{1}{\lambda}))$ for $\lambda \in (0, 1/L]$.

The proof of Lemma 3.6 is complete.

4. Proofs of main results

Proof of Theorem 2.1.

(I) We prove Theorem 2.1(i). First, for $p > 0$, the upper bound $\min\left\{L^{-1}, \frac{p^p}{4(p+1)^{p+1}}\pi^2 L^{-2}\right\}$ for $\lambda^*(p, L)$ in (2.1) can be obtained by slightly modifying the proof of the upper bound for λ^* for $p = 2$ in (1.7) in [4, Theorem 1.1]; we omit the proof. Also, it is easy to see that

$$\min\left\{L^{-1}, \frac{p^p}{4(p+1)^{p+1}}\pi^2 L^{-2}\right\} \leq \min\left\{L^{-1}, \frac{1}{4}\pi^2 L^{-2}\right\}$$

since $\frac{p^p}{(p+1)^{p+1}}$ is a strictly decreasing function of $p > 0$ and $\lim_{p\to 0^+} \frac{p^p}{(p+1)^{p+1}} = 1$.
We then prove the lower bound for λ^* in (2.1) by modifying the proof of Wang and Ruan [25, Ineq. (2.10)] and by Pan and Xing [12, Theorem 3.1]. We first take the function

$$w(x) \equiv \frac{1}{p+1}\left(1 - \frac{x^2}{L^2}\right) < 1 \quad \text{for } x \in (-L, L),$$

which satisfies $w(x) > 0$ on $(-L, L)$ and $w(\pm L) = 0$. We then compute that, for $x \in (-L, L)$,

$$\begin{aligned}
-w''(x) &= \frac{2}{(p+1)L^2} = \frac{2\left[1 - \frac{1}{(p+1)}\right]^p}{(p+1)L^2} \frac{1}{\left[1 - \frac{1}{(p+1)}\right]^p} \\
&\geq \frac{2p^p}{(p+1)^{p+1}L^2} \frac{1}{\left[1 - \frac{1}{(p+1)}\left(1 - \frac{x^2}{L^2}\right)\right]^p} \\
&= \frac{2p^p}{(p+1)^{p+1}L^2} \frac{1}{[1 - w(x)]^p} \\
&= \frac{2p^p}{(p+1)^{p+1}L^2} \frac{\left[1 + (w'(x))^2\right]^{3/2}}{[1 - w(x)]^p} \frac{1}{\left[1 + (w'(x))^2\right]^{3/2}} \\
&= \frac{2p^p}{(p+1)^{p+1}L^2} \frac{\left[1 + (w'(x))^2\right]^{3/2}}{[1 - w(x)]^p} \frac{1}{\left[1 + \frac{4x^2}{(p+1)^2L^4}\right]^{3/2}} \\
&\geq \frac{2p^p}{(p+1)^{p+1}L^2} \frac{\left[1 + (w'(x))^2\right]^{3/2}}{[1 - w(x)]^p} \frac{1}{\left[1 + \frac{4L^2}{(p+1)^2L^4}\right]^{3/2}} \\
&= \frac{2p^p}{(p+1)^{p+1}L^2\left[1 + \frac{4}{(p+1)^2L^2}\right]^{3/2}} \frac{\left[1 + (w'(x))^2\right]^{3/2}}{[1 - w(x)]^p}.
\end{aligned}$$

So for

$$\lambda = \frac{2p^p}{(p+1)^{p+1}L^2\left[1 + \frac{4}{(p+1)^2L^2}\right]^{3/2}},$$

$w(x)$ is a supersolution of (1.1) on $(-L, L)$ as (1.1) can be written in the equivalent form

$$-u''(x) = \lambda \frac{\left[1 + (u'(x))^2\right]^{3/2}}{[1 - u(x)]^p} \quad \text{on } (-L, L);$$

see (1.4) with $\tilde{f}(u) = \frac{1}{(1-u)^p}$. Since $w_0(x) \equiv 0$ is a subsolution of (1.1) on $(-L, L)$ and $f(0) = 1 > w(x) > w_0(x) = 0$ on $(-L, L)$, by applying Pan and Xing [12, Theorem 3.1] obtained by the lower and upper solution method, there exists a (classical) solution $\tilde{w}(x) \in C^2[-L, L]$ of (1.1) satisfying $0 < \tilde{w}(x) \leq w(x)$ on $(-L, L)$. This proves that

$$\frac{2p^p}{(p+1)^{p+1}L^2\left[1+\frac{4}{(p+1)^2L^2}\right]^{3/2}} \leq \lambda^*(p, L).$$

(II) We prove Theorem 2.1(ii). Consider $L > 0$ be fixed. For any fixed $\lambda > 0$, $r \in I$ and $0 < u < r$, in $T_{p,\lambda}(r)$ in (3.1), the integrand $\frac{1+\lambda F(u)-\lambda F(r)}{\sqrt{1-[1+\lambda F(u)-\lambda F(r)]^2}}$ is strictly decreasing in $p > 0$ since

$$F(r) - F(u) = \int_u^r \frac{1}{(1-t)^p} dt$$

is increasing in $p > 0$. So $T_{p,\lambda}(r)$ is a strictly decreasing function of $p > 0$. Hence $\lambda^*(p, L)$ is a strictly decreasing function of $p > 0$. The rest of part (ii) follow from part (i) and by simple calculus with the fact that $\lim_{p\to 0^+} \frac{p^p}{(p+1)^{p+1}} = 1$ and $\lim_{p\to\infty} \frac{p^p}{(p+1)^{p+1}} = 0$.

(III) We prove Theorem 2.1(iii). Consider fixed $p > 0$. Let

$$h_p(\lambda) \equiv \begin{cases} \sup\left\{T_{p,\lambda}(r) : r \in (0, F^{-1}(\frac{1}{\lambda})]\right\}, & \text{if } (p \geq 1,\ \lambda > 0) \text{ or } (0 < p < 1,\ \lambda > 1-p), \\ \sup\left\{T_{p,\lambda}(r) : r \in (0, 1)\right\} & \text{if } 0 < p < 1,\ 0 < \lambda \leq 1-p. \end{cases}$$

See [1, (1.9) and (1.11)]. By [1, Lemma 3.3(ii)], $h_p(\lambda)$ is a continuous, strictly decreasing function of $\lambda > 0$, $\lim_{\lambda\to 0^+} h_p(\lambda) = \infty$ and $\lim_{\lambda\to\infty} h_p(\lambda) = 0$. Thus we obtain that

$$h_p(\lambda^*(p, L)) = L \text{ for all } L > 0; \tag{4.1}$$

see [1, Proofs of Theorems 2.1 and 2.2]. Moreover, we obtain that $\lambda^*(p, L)$ is a strictly decreasing function of $L > 0$, $\lim_{L\to 0^+} \lambda^*(p, L) = \infty$ and $\lim_{L\to\infty} \lambda^*(p, L) = 0$.

The proof of Theorem 2.1 is complete.

Proof of Theorem 2.3.

In (3.3), positive solutions u_λ for (1.1) correspond to

$$\|u_\lambda\|_\infty = r \text{ and } T_{p,\lambda}(r) = L.$$

Thus, studying of the exact number of positive solutions of (1.1) for any fixed $\lambda > 0$ is equivalent to studying the shape of the time map $T_{p,\lambda}(r)$ on its domain I. Moreover, $T_{p,\lambda}(r)$ has exactly one critical point, a local maximum, on its domain I by Lemma 3.1. Hence we have that:

(i) If there exists $r^* > 0$ such that $T'_{p,\lambda}(r^*) < 0$ for $\lambda \in (0, 1/F(r^*))$, then $\|u_{\lambda^*}\|_\infty < r^*$ and hence r^* is an upper bound of $\|u_{\lambda^*}\|_\infty$.

(ii) If there exists $r_* > 0$ such that $T'_{p,\lambda}(r_*) > 0$ for $\lambda \in (0, 1/L]$, then $\|u_{\lambda^*}\|_\infty > r_*$ and hence r_* is a lower bound of $\|u_{\lambda^*}\|_\infty$. Observe $\lambda^*(p, L) < 1/L$ by Theorem 2.1(i).

We are now in a position to prove Theorem 2.3(i)–(iii).

(I) We prove the upper bounds for $\|u_{\lambda^*}\|_\infty$ in Theorem 2.3(i)–(iii).

(A) For $p > 1$ and $r = \bar{U}(p)$, by Lemma 3.2, we have that $T'_{p,\lambda}(r) < 0$ for $\lambda \in (0, 1/F(r))$, and hence

$$\|u_{\lambda^*}\|_\infty < r = \bar{U}(p)\ (\leq U(p))$$

for $L > 0$ by applying (3.3) and Lemma 3.1. See also Theorem 1.1. It is easy to check that $\bar{U}(p)$ is bounded above by $U(p)$ in (2.3) for $p > 1$, we omit the proof.

(B) For $p = 1$, similarly, we have that $\|u_{\lambda^*}\|_\infty < 1 - 2e^{-2} \approx 0.729$ for $L > 0$ by applying (3.3), Lemmas 3.1 and 3.3. See also Theorem 1.1.

(C) For $0 < p < 1$, it is trivial that $\|u_{\lambda^*}\|_\infty < 1$ for $L > 0$.

By above (A)–(C), we obtain the upper bounds for $\|u_{\lambda^*}\|_\infty$ in Theorem 2.3(i)–(iii).

(II) We prove the lower bounds for $\|u_{\lambda^*}\|_\infty$ in Theorem 2.3(i)–(iii).

(A) For $p > 1$ and

$$r_p \equiv \bar{L}(p, L) \equiv \min\left\{1 - \left[1 + \frac{1}{3}(p-1)L\right]^{\frac{1}{1-p}}, \frac{1}{9p+1}\right\},$$

by Lemma 3.4, we have that $T'_{p,\lambda}(r_p) > 0$ for $\lambda \in (0, 1/L]$. Hence

$$\bar{L}(p, L) < \|u_{\lambda^*}\|_\infty$$

for $L > 0$ by applying (3.3) and Lemma 3.1. See also Theorem 1.1.

(B) For $p = 1$ and

$$r_1 \equiv \min\left\{1 - e^{-\frac{L}{3}}, \frac{1}{10}\right\},$$

by Lemma 3.5, we have that $T'_{1,\lambda}(r_1) > 0$ for $\lambda \in (0, 1/L]$. Hence

$$\min\left\{1 - e^{-\frac{L}{3}}, \frac{1}{10}\right\} < \|u_{\lambda^*}\|_\infty$$

for $L > 0$ by applying (3.3) and Lemma 3.1. See also Theorem 1.1.

(C) For $0 < p < 1$ and

$$r_p \equiv \hat{L}(p, L) = \min\left\{1 - \left[1 - \frac{1}{k}(1-p)L\right]^{\frac{1}{1-p}}, \frac{1}{k^2p+1}\right\},\quad k = \max\{3, (1-p)L\},$$

by Lemma 3.6, we have that $T'_{p,\lambda}(r_p) > 0$ for $\lambda \in (0, 1/L]$. Hence

$$\min\left\{1 - \left[1 - \frac{1}{k}(1-p)L\right]^{\frac{1}{1-p}}, \frac{1}{k^2p+1}\right\} < \|u_{\lambda^*}\|_\infty$$

for $L > 0$ by applying (3.3) and Lemma 3.1. See also Theorem 1.2.

By above (A)–(C) in this part (II), we obtain the lower bounds for $\|u_{\lambda^*}\|_\infty$ in Theorem 2.3(i)–(iii).

(III) We prove Theorem 2.3(iv). Assertion (2.5) follows immediately by (2.2)–(2.3) and since

$$\lim_{p\to\infty} \|u_{\lambda^*}\|_\infty \le \lim_{p\to\infty} U(p) = \lim_{p\to\infty} \frac{2}{p+2} = 0.$$

In addition, (2.6) and (2.7) follow easily by (2.4).

We then prove (2.8). For any fixed $L > 0$, by (4.1), we obtain that $h_p(\lambda^*) = h_p(\lambda^*(L)) = L$. Since $\|u_{\lambda^*}\|_\infty \in (0, F^{-1}(\frac{1}{\lambda^*})) = (0, F^{-1}(\frac{1}{\lambda^*(L)}))$, we have that

$$\begin{aligned} 0 &\le \lim_{L\to 0^+} \|u_{\lambda^*}\|_\infty \le \lim_{L\to 0^+} F^{-1}(\frac{1}{\lambda^*(L)}) \\ &= F^{-1}\left(\lim_{L\to 0^+} \frac{1}{\lambda^*(L)}\right) \\ &= F^{-1}(0) \ \ (\text{since } \lim_{L\to 0^+} \lambda^*(L) = \infty \text{ by Theorem 2.1(iii)}) \\ &= 0. \end{aligned}$$

Thus $\lim_{L\to 0^+} \|u_{\lambda^*}\|_\infty = 0$. So (2.8) holds.

The proof of Theorem 2.3 is complete.

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Conflict of interest

The authors declare there is no conflict of interest.

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