



Research article

Global dynamics of a diffusive phytoplankton-zooplankton model with toxic substances effect and delay

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Abstract: This paper examines a diffusive toxic-producing plankton system with delay. We first show the global attractivity of the positive equilibrium of the system without time-delay. We further consider the effect of delay on asymptotic behavior of the positive equilibrium: when the system undergoes Hopf bifurcation at some points of delay by the normal form and center manifold theory for partial functional differential equations. Global existence of periodic solutions is established by applying the global Hopf bifurcation theory.

Keywords: reaction-diffusion; delay; phytoplankton-zooplankton model; Hopf bifurcation; global periodic solutions

1. Introduction

In aquatic systems such as rivers, lakes and oceans, harmful algal blooms (HABs) have attracted considerable scientific attention in recent years. The effect of toxin-producing phytoplankton (TPP) on zooplankton is one reason for such attention. Researchers have made great progress in mathematical modeling in this field [1–6]. To fully understand the mechanism of planktonic blooms and to formulate reasonable control measures, Chattopadhyay et al. [3] build a ODE phytoplankton-zooplankton model with toxin effect, using data gathered from the coastal region of West Bengal and part of Orissa, India. These researchers investigated plankton models with different predational response functions and toxin liberation processes to obtain rich dynamics. They also considered the diffusivity of plankton influenced by ocean currents and tides, and the time delay effect of the phytoplankton toxin on zooplankton. Wang et al. [7] consider a ODE plankton system with Holling II response function and linear toxin processes. These authors concluded that the system undergoes Hopf bifurcation at positive equilibrium and Hopf-transcritical bifurcation when the parameters satisfy a particular condition.

In spatial plankton models, for example, Chaudhuri et al. [4] consider that the system includes both non-toxic and toxic phytoplankton and the system shows toxic phytoplankton-induced spatiotemporal patterns when one phytoplankton releases toxin. Further, to describe the reduction of zooplankton due to toxin-producing phytoplankton, plankton systems with discrete delay are presented in References [2, 8] to show the effect of delay. There is a large body of literatures describing the dynamics of aquatic models [6, 9–11].

In fact, the problem we analyze is based on Chen et al. [1], which considers a model as follows:

$$\begin{cases} \frac{du(t)}{dt} = (1 - u(t))u(t) - \frac{\alpha v(t)u(t)}{1 + cu(t)}, & t > 0, \\ \frac{dv(t)}{dt} = \frac{\beta u(t)v(t)}{1 + cu(t)} - \theta u(t)v(t) - rv(t), & t > 0. \end{cases} \quad (1.1)$$

They conclude that the plankton model (1.1) occurs as a bistable phenomenon.

In real-world conditions, plankton affected by tides and turbulence may move and diffuse across lakes and seas. Thus, such diffusion should be considered in studying the dynamics of plankton models. Meanwhile, the effect of phytoplankton toxin on zooplankton is in the form of a time-delay. We assume that no plankton species enter or leave at the boundaries of their environments. Based on these considerations, we present the plankton model as follows:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) + (1 - u(x, t))u(x, t) - \frac{v(x, t)u(x, t)}{1 + cu(x, t)}, \\ \frac{\partial v(x, t)}{\partial t} = \Delta v(x, t) + \frac{\beta v(x, t)u(x, t)}{1 + cu(x, t)} - \theta u(x, t - \tau)v(x, t) - rv(x, t), \end{cases} \quad (1.2)$$

for $t > 0, x \in \Omega = [0, l\pi]$. u represents the TPP population, and v represents the zooplankton population. Here, parameters β, r, θ and c are positive. β denotes the conversion ratio, r denotes the mortality rate of zooplankton due to natural death and higher predation, θ denotes the rate of toxin liberation by TPP population, and c denotes the half-saturation constant, here, we consider the case $c > 1$. d_1 and d_2 represent the diffusion coefficients of phytoplankton and zooplankton, respectively.

Here, the initial conditions of system (1.2) are considered as

$$u_0(x) = \varphi_1(x, \vartheta) \geq 0, \quad v_0(x) = \varphi_2(x, \vartheta) \geq 0, \quad x \in \Omega, \quad \vartheta \in [-\tau, 0] \quad (1.3)$$

where $\varphi = (\varphi_1, \varphi_2)$ is uniformly continuous, and the homogeneous Neumann boundary conditions are imposed as

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial\Omega, \quad (1.4)$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$.

This paper proceeds as follows. Section 2 gives the uniform boundedness of solutions. Section 3, by constructing the upper and lower solutions, establishes the global attractivity of positive equilibrium of the system without time-delay. Further, taking time delay as a branch parameter, we give the sufficient condition for which the system undergoes Hopf bifurcation at the interior equilibrium. In Section 4, under the assumption condition, we analyze the existence of Hopf bifurcation. Section 5 considers the global existence of these bifurcating periodic solutions. Finally, in Section 6 presents several numerical simulations.

2. Basic properties of solutions

This section shows the boundedness of the solutions of system (1.2).

Lemma 2.1. *Under the initial boundary conditions (1.3)–(1.4), all nontrivial solutions of system (1.2) are positive and uniformly bounded.*

Proof. For $v(x, t)$ note that, on $0 \leq t \leq \tau$, $v(x, t - \tau) = \varphi_2(x, t - \tau)$ and $u(x, t)$ is bounded on $x \in \bar{\Omega}$. For further proof, we consider the following auxiliary system

$$\begin{cases} m_t = \Delta m + \frac{\beta um}{1 + cu} - \theta M_1 m - rm, & x \in \Omega, \quad t > 0, \\ m_v = 0, & x \in \partial\Omega, \quad t > 0, \\ m(x, 0) = \varphi_2(x, 0), & x \in \Omega. \end{cases}$$

where $M_1 = \max_{x \in \bar{\Omega}, t \in [-\tau, 0]} u(x, t)$. By comparison principle, we know $v(x, t) \geq m(x, t) \geq 0$ for $x \in \Omega$, $t > 0$. To proceed in this way, for $x \in \Omega$, $t \in [\tau, 2\tau]$, we have $v(x, t) \geq m(x, t) \geq 0$. Then, by mathematical induction, we obtain the positivity of $v(x, t)$ in $x \in \Omega$, $t > 0$. Similarly, we can prove the positivity of $u(x, t)$.

Let

$$u(x_1, t_1) = \max_{x \in \Omega, t > 0} u(x, t), \quad \beta u(x_2, t_2) + v(x_2, t_2) = \max_{x \in \bar{\Omega}, t > 0} u(x, t).$$

Then, in view of the Hopf boundary lemma and the boundary condition, it follows that

$$u(x_1, t_1)(1 - u(x_1, t_1)) - \frac{u(x_1, t_1)v(x_1, t_1)}{1 + cu(x_1, t_1)} \geq 0 \quad \text{for } x_1 \in \Omega.$$

This implies

$$u(x_1, t_1) < 1.$$

By adding the two equations in system (1.2) with the form $\beta u + v$, it follows that

$$\begin{aligned} (\partial_t - \Delta)(\beta u + v) &= \beta u(1 - u) - rv - \theta u_t v \\ &= \beta u(1 - u) + r\beta u - r(\beta u + v) - \theta u_t v \\ &\leq \beta u(1 - u) + r\beta u - r(\beta u + v). \end{aligned}$$

By the maximum principle in Reference [12], this implies that

$$\beta u + v < \frac{\beta + 4r\beta}{4r}.$$

This completes the proof. □

Clearly, system (1.2) has two boundary equilibria, $E_0 = (0, 0)$ and $E_1 = (1, 0)$.

3. Stability of the positive equilibrium

Let us now analyze the stability of plankton system (1.2) at the interior equilibrium. By referring to [1, Lemma 2.3], we know system (1.2) has a unique positive equilibrium $E^*(u_-, v_-)$ when the parameters satisfy the following condition

$$(H1) \quad c > 1, \quad \sqrt{\frac{r}{c\theta}} > 1, \quad \text{and } \beta > (c+1)(r+\theta).$$

When $\tau = 0$, system (1.2) becomes

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) + (1 - u(x, t))u(x, t) - \frac{v(x, t)u(x, t)}{1 + cu(x, t)}, \\ \frac{\partial v(x, t)}{\partial t} = \Delta v(x, t) + \frac{\beta v(x, t)u(x, t)}{1 + cu(x, t)} - \theta u(x, t)v(x, t) - rv(x, t). \end{cases} \quad (3.1)$$

Applying the upper and lower solutions method, we show that the interior equilibrium $E^*(u_-, v_-)$ for system (3.1) is globally attractive under assumption (H1). We denote $E^*(u_-, v_-) = (\delta, v_\delta)$, where

$$\delta = \frac{\beta - (cr + \theta) - \sqrt{(cr + \theta - \beta)^2 - 4cr\theta}}{2c\theta},$$

$$v_\delta = (1 - \delta)(1 + c\delta) \quad (0 < \delta < 1).$$

In this section, note that $(u_1, v_1) > (u_2, v_2)$ means $u_1 > u_2$ and $v_1 > v_2$.

Theorem 3.1. *Under assumption (H1), the interior equilibrium E^* for the plankton system (3.1) is globally attractive.*

Proof. Firstly, we can know from system (3.1)

$$\frac{\partial u}{\partial t} - \Delta u = (1 - u)u - \frac{vu}{1 + cu} \leq u(1 - u),$$

by the convergence of the logistic equation, the comparison principle of parabolic equation and Lemma 2.1, then for a sufficiently small positive number ε , there must be a $t_1 > 0$ such that, for $t \geq t_1$,

$$u < 1 + \frac{\varepsilon}{2} \quad \text{and} \quad v < \frac{\beta(1 + 4r)}{4r} - \frac{\varepsilon}{2}.$$

Moreover, for any initial value $(u_0, v_0) > (0, 0)$, there exists a $t_2 > t_1$ such that, for $t \geq t_2$, $(u, v) > (\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) > (0, 0)$. Then the solutions (u, v) of system (3.1) satisfy

$$\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) < (u, v) < \left(1 + \frac{\varepsilon}{2}, \frac{\beta(1 + 4r)}{4r} - \frac{\varepsilon}{2}\right).$$

Denote $(\tilde{c}_1, \tilde{c}_2) = (1 + \frac{\varepsilon}{2}, \frac{\beta(1+4r)}{4r} - \frac{\varepsilon}{2})$ and $(\hat{c}_1, \hat{c}_2) = (\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, then $(0, 0) < (\hat{c}_1, \hat{c}_2) < (\tilde{c}_1, \tilde{c}_2)$ and

$$\begin{aligned} \tilde{c}_1(1 - \tilde{c}_1) - \frac{\tilde{c}_1\tilde{c}_2}{1 + c\tilde{c}_1} &\leq 0, & \frac{\beta\hat{c}_1\tilde{c}_2}{1 + c\hat{c}_1} - r\tilde{c}_2 - \theta\hat{c}_1\tilde{c}_2 &\leq 0, \\ \hat{c}_1(1 - \hat{c}_1) - \frac{\hat{c}_1\tilde{c}_2}{1 + c\hat{c}_1} &\geq 0, & \frac{\beta\tilde{c}_1\hat{c}_2}{1 + c\tilde{c}_1} - r\hat{c}_2 - \theta\tilde{c}_1\hat{c}_2 &\geq 0, \end{aligned}$$

and it is evident that there exists a $K > 0$ such that

$$\left| u_1(1 - u_1) - \frac{u_1 v_1}{1 + cu_1} - \left(u_2(1 - u_2) - \frac{u_2 v_2}{1 + cu_2} \right) \right| \leq K(|u_1 - u_2| + |v_1 - v_2|),$$

$$\left| \frac{\beta u_1 v_1}{1 + cu_1} - rv_1 - \theta u_1 v_1 - \left(\frac{\beta u_2 v_2}{1 + cu_2} - rv_2 - \theta u_2 v_2 \right) \right| \leq K(|u_1 - u_2| + |v_1 - v_2|).$$

In applying the upper and lower method in References [13, 14], we define the two sequences $(\tilde{c}_1^m, \tilde{c}_2^m)$ and $(\hat{c}_1^m, \hat{c}_2^m)$ as follows:

$$\begin{aligned} \tilde{c}_1^m &= \tilde{c}_1^{m-1} + \frac{1}{K} \left((1 - \tilde{c}_1^{m-1}) \tilde{c}_1^{m-1} - \frac{\tilde{c}_1^{m-1} \tilde{c}_2^{m-1}}{1 + c \tilde{c}_1^{m-1}} \right), \\ \tilde{c}_2^m &= \tilde{c}_2^{m-1} + \frac{1}{K} \left(\frac{\beta \tilde{c}_1^{m-1} \tilde{c}_2^{m-1}}{1 + c \tilde{c}_1^{m-1}} - r \tilde{c}_2^{m-1} - \theta \tilde{c}_1^{m-1} \tilde{c}_2^{m-1} \right), \\ \hat{c}_1^m &= \hat{c}_1^{m-1} + \frac{1}{K} \left(\hat{c}_1^{m-1} (1 - \hat{c}_1^{m-1}) - \frac{\hat{c}_1^{m-1} \hat{c}_2^{m-1}}{1 + c \hat{c}_1^{m-1}} \right), \\ \hat{c}_2^m &= \hat{c}_2^{m-1} + \frac{1}{K} \left(\frac{\beta \hat{c}_1^{m-1} \hat{c}_2^{m-1}}{1 + c \hat{c}_1^{m-1}} - r \hat{c}_2^{m-1} - \theta \hat{c}_1^{m-1} \hat{c}_2^{m-1} \right), \end{aligned}$$

where $(\tilde{c}_1^0, \tilde{c}_2^0) = (\tilde{c}_1, \tilde{c}_2)$, $(\hat{c}_1^0, \hat{c}_2^0) = (\hat{c}_1, \hat{c}_2)$.

Therefore, it follows that

$$(\hat{c}_1, \hat{c}_2) \leq (\hat{c}_1^m, \hat{c}_2^m) \leq (\hat{c}_1^{m+1}, \hat{c}_2^{m+1}) \leq (\tilde{c}_1^{m+1}, \tilde{c}_2^{m+1}) \leq (\tilde{c}_1^m, \tilde{c}_2^m) \leq (\tilde{c}_1, \tilde{c}_2)$$

then there exist $(\bar{c}_1, \bar{c}_2) > (0, 0)$ and $(\check{c}_1, \check{c}_2) > (0, 0)$ satisfying

$$\begin{aligned} \lim_{m \rightarrow \infty} \tilde{c}_1^m &= \bar{c}_1, & \lim_{m \rightarrow \infty} \tilde{c}_2^m &= \bar{c}_2, \\ \lim_{m \rightarrow \infty} \hat{c}_1^m &= \check{c}_1, & \lim_{m \rightarrow \infty} \hat{c}_2^m &= \check{c}_2, \end{aligned}$$

and

$$\begin{aligned} \bar{c}_1(1 - \bar{c}_1) - \frac{\bar{c}_1 \bar{c}_2}{1 + c \bar{c}_1} &= 0, & \frac{\beta \check{c}_1 \bar{c}_2}{1 + c \check{c}_1} - r \bar{c}_2 - \theta \check{c}_1 \bar{c}_2 &= 0, \\ \check{c}_1(1 - \check{c}_1) - \frac{\check{c}_1 \bar{c}_2}{1 + c \check{c}_1} &= 0, & \frac{\beta \bar{c}_1 \check{c}_2}{1 + c \bar{c}_1} - r \check{c}_2 - \theta \bar{c}_1 \check{c}_2 &= 0. \end{aligned}$$

Since (δ, v_δ) is the unique equilibrium of system (3.1), we observe $(\bar{c}_1, \bar{c}_2) = (\check{c}_1, \check{c}_2)$. Thus, by employing in Reference [14, Theorem 2.2], the solutions (u, v) of system (3.1) converge uniformly to E^* as $t \rightarrow \infty$. \square

Further, for the dynamics of system (1.2), we fix the parameters d_1, d_2, r, θ and c , and pick delay τ as the bifurcation parameter.

Firstly, in the phase space $C = C([-\tau, 0], X)$, we linearize system (1.2) to analyze the stability of the positive equilibrium (δ, v_δ) , and have

$$\dot{Z}(t) = D\Delta Z(t) + L(Z_t), \quad (3.2)$$

where $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ and define $L : C \rightarrow X$ as

$$L(\varphi_t) = L_1 \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} + L_2 \begin{pmatrix} \varphi_1(-\tau) \\ \varphi_2(-\tau) \end{pmatrix},$$

and the corresponding L_1 and L_2 are

$$L_1 = \begin{pmatrix} 1 - 2\delta - B_1 & -A_1 \\ \beta B_1 & 0 \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 0 & 0 \\ -\theta v_\delta & 0 \end{pmatrix},$$

where $A_1(\delta) = \frac{\delta}{1+c\delta}$ and $B_1(\delta) = \frac{1-\delta}{1+c\delta}$. Then by a simple derivation, the characteristic equation of the system (1.2) at (u_-, v_-) is

$$\lambda^2 - \lambda T_n + D_n - e^{-\lambda\tau} \theta \delta (1 - \delta) = 0, \quad n = 0, 1, 2, \dots, \quad (3.3)$$

where

$$\begin{cases} T_n = 1 - 2\delta - B_1 - \frac{(d_1 + d_2)n^2}{l^2}, \\ D_n = \frac{d_1 d_2 n^4}{l^4} - \frac{d_2 n^2}{l^2} (1 - 2\delta - B_1) + \beta A_1 B_1. \end{cases} \quad (3.4)$$

When $\tau = 0$, Eq (3.3) becomes

$$\lambda^2 - \lambda T_n + D_n - \theta \delta (1 - \delta) = 0, \quad n = 0, 1, 2, \dots, \quad (3.5)$$

and then Eq (3.5) has two roots given by

$$\lambda_n^\pm = \frac{T_n \pm \sqrt{T_n^2 - 4(D_n - \theta \delta (1 - \delta))}}{2}, \quad n = 0, 1, 2, \dots.$$

Here, we establish the definition:

$$(H2) \quad 1 - 2\delta - B_1 < 0.$$

Therefore, if (H1) holds, then $\beta A_1 B_1 - \theta \delta (1 - \delta) > 0$, and adding the condition (H2), and we obtain that $D_n - \theta \delta (1 - \delta) > 0$. Thus, we obtain the following results.

Lemma 3.2. *Under the assumptions (H1) and (H2), the positive equilibrium E^* of system (1.2) is locally asymptotically stable when $\tau = 0$.*

Remark 3.3. *When the parameters of system (1.2) satisfy the condition (H1) and (H2), $\lambda = 0$ is not the characteristic root of Eq (3.5).*

In the following, with the help from Ruan and Wei [15], we analyze the existence of purely imaginary eigenvalues $\lambda = \pm i\omega$ ($\omega > 0$) to study the stability of the positive equilibrium E^* . Plugging $\lambda = i\omega$ into Eq (3.3), we obtain

$$-\omega^2 - T_n i \omega + D_n - e^{-i\omega\tau} \theta \delta (1 - \delta) = 0, \quad n = 0, 1, 2, \dots,$$

then it follows from separating the real and imaginary parts that

$$\omega^4 + (T_n^2 - 2D_n)\omega^2 + D_n^2 - \theta^2\delta^2(1 - \delta)^2 = 0, \quad n = 0, 1, 2, \dots, \quad (3.6)$$

we rewrite the above Eq (3.6) by $z = \omega^2$, and get

$$z^2 + (T_n^2 - 2D_n)z + D_n^2 - \theta^2\delta^2(1 - \delta)^2 = 0, \quad n = 0, 1, 2, \dots, \quad (3.7)$$

and Eq (3.7) has a pair of roots given by

$$z_n^\pm = \frac{2D_n - T_n^2 \pm \sqrt{T_n^4 - 4T_n^2D_n + 4\theta^2\delta^2(1 - \delta)^2}}{2}.$$

When assumptions (H1) and (H2) hold, we find that

$$D_n^2 - \theta^2\delta^2(1 - \delta)^2 > 0, \quad n = 0, 1, 2, \dots.$$

Define

$$\mathfrak{N} := \{n_0 \in \mathbb{N}_0 | \text{Eq}(3.7) \text{ has two positive roots with } n = n_0\}.$$

By the above calculation, we have

$$\tau_{n,k}^\pm = \frac{1}{\omega_n^\pm} \left(-\arccos \frac{(\omega_n^\pm)^2 - D_n}{\theta\delta(\delta - 1)} + 2(k + 1)\pi \right), \quad n \in \mathfrak{N}, k \in \mathbb{N}_0. \quad (3.8)$$

Lemma 3.4. *When the parameters of system (1.2) satisfy conditions (H1) and (H2), the following conclusions hold.*

(i) *If $2D_n - T_n^2 < 0$ for all $n \in \mathbb{N}_0$, then all the roots of Eq (3.3) have negative real parts.*

(ii) *If $2D_n - T_n^2 > 0$ and $T_n^4 - 4T_n^2D_n + 4\theta^2\delta^2(1 - \delta)^2 \geq 0$, then the $(n + 1)$ th of Eq (3.3) has a pair of simple pure imaginary roots $\pm i\omega_n^\pm$ ($\pm i\omega_n^-$) at $\tau = \tau_{n,k}^\pm$ ($\tau = \tau_{n,k}^-$), $n \in \mathfrak{N}, k \in \mathbb{N}_0$.*

Lemma 3.5. *When the parameters of system (1.2) satisfy conditions (H1) and (H2), if*

$$T_n^2 - 2D_n \leq 0 \quad \text{and} \quad T_n^4 - 4T_n^2D_n + 4\theta^2\delta^2(1 - \delta)^2 > 0,$$

then

$$\text{Re}\lambda'(\tau_{n,k}^+) > 0, \quad \text{Re}\lambda'(\tau_{n,k}^-) < 0 \quad \text{for } n \in \mathfrak{N}, k \in \mathbb{N}_0.$$

Proof. Taking the differential of both sides of Eq (3.3) with respect to τ , we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{T_n - 2\lambda - \tau\theta\delta(1 - \delta)e^{-\lambda\tau}}{\theta\delta(1 - \delta)\lambda e^{-\lambda\tau}},$$

then we simply calculate to obtain

$$\begin{aligned} \text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \Big|_{\tau=\tau_{n,k}^\pm} &= \text{Re}\left(\frac{(T_n - 2i\omega_n^\pm)(\cos \omega_n^\pm\tau + i \sin \omega_n^\pm\tau) - \tau\theta\delta(1 - \delta)}{\theta\delta(1 - \delta)i\omega_n^\pm}\right) \\ &= \frac{T_n^2 - 2(D_n - (\omega_n^\pm)^2)}{\theta^2\delta^2(1 - \delta)^2} \\ &= \frac{\pm \sqrt{T_n^4 - 4T_n^2D_n + 4\theta^2\delta^2(1 - \delta)^2}}{\theta^2\delta^2(1 - \delta)^2}. \end{aligned}$$

Completing the proof. □

Thus, it is straightforward that

$$\tau_{n,0}^+ \leq \tau_{n,0}^- \text{ for all } n \in \mathfrak{N},$$

and we know that $\tilde{\tau} = \tau_{n,0}^+$ is the smallest point changing the stability of the linearized system (1.2) when the other parameters are fixed.

Theorem 3.6. *Assume that (H1) and (H2) hold.*

(i) *If $2D_n - T_n^2 < 0$ for all $n \in \mathbb{N}_0$, then the positive equilibrium E^* of system (1.2) is locally asymptotically stable.*

(ii) *If $2D_n - T_n^2 > 0$ and $T_n^4 - 4T_n^2 D_n + 4\theta^2 \delta^2 (1 - \delta)^2 < 0$, then the system (1.2) undergoes a Hopf bifurcation at E^* for $\tau = \tau_{n,k}^+$ ($\tau = \tau_{n,k}^-$), $n \in \mathfrak{N}, k \in \mathbb{N}_0$. Further,*

$$\operatorname{Re} \lambda'(\tau_{n,k}^+) > 0, \quad \operatorname{Re} \lambda'(\tau_{n,k}^-) < 0 \quad \text{for } n \in \mathfrak{N}, k \in \mathbb{N}_0.$$

4. Analysis of local Hopf bifurcation

In this section, by employing the bifurcation theory in References [16, 17], we analyze the property of local Hopf bifurcation near the interior equilibrium E^* . For fixed $k \in \mathbb{N}_0$ and $n \in \mathfrak{N}$, we denote $\hat{\tau} = \tau_{n,k}^\pm$. Setting $\tau = \hat{\tau} + \mu$, then $\mu = 0$ is the Hopf bifurcation value of system (1.2). We rescale the time by $t \rightarrow \frac{t}{\hat{\tau}}$ to normalize the delay. Meanwhile, let

$$z_1(x, t) = u(x, \tau t) - \delta, \quad z_2(x, t) = v(x, \tau t) - v_\delta.$$

System (1.2) becomes

$$\begin{cases} \frac{\partial z_1}{\partial t} = \hat{\tau}(\Delta z_1 + (1 - 2\delta - B_1)z_1 - A_1 z_2 + f_1((z_1)_t, (z_2)_t)), \\ \frac{\partial z_2}{\partial t} = \hat{\tau}(\Delta z_2 + \beta B_1 z_1 - \theta v_\delta z_2 + f_2((z_1)_t, (z_2)_t)), \end{cases} \quad (4.1)$$

furthermore, for simply research, system (4.1) is again transformed abstractly into

$$\frac{dZ(t)}{dt} = \hat{\tau} \Delta Z(t) + \hat{\tau} L(Z_t) + G(Z_t, \mu), \quad (4.2)$$

where

$$\begin{aligned} L(\varphi) &= \begin{pmatrix} (1 - 2\delta - B_1)\varphi_1(0) - A_1\varphi_2(0) \\ \beta B_1\varphi_1(0) - \theta v_\delta\varphi_1(-1) \end{pmatrix}, \\ G(\varphi, \mu) &= \mu \Delta Z(t) + \mu \hat{\tau} L(Z_t) + (\hat{\tau} + \mu) F_0(\varphi), \\ F_0(\varphi) &= \begin{pmatrix} f_1(\varphi_1, \varphi_2) \\ f_2(\varphi_1, \varphi_2) \end{pmatrix} \\ &= \begin{pmatrix} (\varphi_1(0) + \delta) - (\varphi_1(0) + \delta)^2 - \frac{(\varphi_1(0) + \delta)(\varphi_2(0) + v_\delta)}{1 + c(\varphi_1(0) + \delta)} - (1 - 2\delta - B_1)\varphi_1(0) + A_1\varphi_2(0) \\ \frac{\beta(\varphi_1(0) + \delta)(\varphi_2(0) + v_\delta)}{1 + c(\varphi_1(0) + \delta)} - r v_\delta - \theta(\varphi_1(-1) + \delta)(\varphi_2(0) + v_\delta) - \beta B_1\varphi_1(0) + \theta v_\delta\varphi_1(-1) \end{pmatrix} \end{aligned} \quad (4.3)$$

for $\varphi = (\varphi_1, \varphi_2)^T \in C([-1, 0], X)$.

We know that $\pm i\omega_{n_0}^\pm \hat{\tau}$ are two pairs of simple, purely imaginary eigenvalues corresponding to the following linear system at $(0,0)$

$$\frac{dZ(t)}{dt} = \hat{\tau}D\Delta Z(t) + \hat{\tau}L(Z_t), \quad (4.4)$$

and the linear functional differential equation

$$\frac{dZ(t)}{dt} = \hat{\tau}L(Z_t). \quad (4.5)$$

By Riesz representation theorem, there exists a 2×2 matrix $\eta(\vartheta, \hat{\tau})$ ($\vartheta \in [-1, 0]$), we know that the elements of $\eta(\vartheta, \hat{\tau})$ are of bounded variation functions such that

$$(\hat{\tau} + \mu)L(\varphi) = \int_{-1}^0 d\eta(\vartheta, \mu)\varphi(\vartheta) \text{ for } \varphi(\vartheta) \in C([-1, 0], \mathbb{R}^2), \quad (4.6)$$

and we have

$$d\eta(\vartheta, \mu) = (\hat{\tau} + \mu)L_1\phi(\vartheta) + (\hat{\tau} + \mu)L_2\phi(\vartheta + 1),$$

where L_1 and L_2 are defined in the previous section.

Here, for the sake of derivation, we give the following notation:

$$a_n = \frac{\cos(nx/l)}{\|\cos(nx/l)\|_{L^2}}, \quad \xi_n = \{a_n(1, 0)^T, a_n(0, 1)^T\}$$

and

$$\varphi_n = \langle \varphi, \xi_n \rangle = \left(\langle \varphi_1, a_n \rangle, \langle \varphi_2, a_n \rangle \right)^T$$

for $\varphi = (\varphi_1, \varphi_2)^T \in C$.

Define a bilinear form

$$(\psi, \varphi) = \sum_{i,j=0}^{\infty} (\psi_i, \varphi_j) \int_{\Omega} a_i a_j dx,$$

where

$$\psi = \sum_{n=0}^{\infty} \psi_n a_n \in C^* = C([0, 1], X^*), \quad \varphi = \sum_{n=0}^{\infty} \varphi_n a_n \in C,$$

and

$$\psi_n = C([0, 1], (\mathbb{R}^2)^*), \quad \varphi_n = C([-1, 0], \mathbb{R}^2).$$

Since

$$\int_{\Omega} a_i a_j dx = 0 \text{ for } i \neq j,$$

then

$$(\psi, \varphi) = \sum_{n=0}^{\infty} (\psi_n, \varphi_n) |a_n|^2,$$

with the bilinear form

$$(\psi_n, \varphi_n) = \bar{\psi}_n^T(0)\varphi_n(0) - \int_{-1}^0 \int_{\xi=0}^{\vartheta} \bar{\psi}_n^T(\xi - \vartheta) d\eta_n(0, \vartheta) \varphi_n(\xi) d\xi.$$

For $\varphi(\vartheta) \in C^1([-1, 0], \mathbb{R}^2)$, denote

$$A(0)(\varphi(\vartheta)) = \begin{cases} \frac{d\varphi(\vartheta)}{d\vartheta}, & \vartheta \in [-1, 0), \\ \int_{-1}^0 d\eta(\vartheta, 0)\varphi(\vartheta), & \vartheta = 0, \end{cases}$$

for ψ , define A^* as

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \sum_{n=0}^{\infty} \int_{-1}^0 d\eta_n^T(0, t)\psi_n(-t)a_n, & s = 0. \end{cases}$$

Here,

$$q^*(s) = \frac{1}{M}(q_1^*, 1)e^{i\omega_{n_0}^+ \hat{\tau}s} (s \in [0, 1]) \text{ and } q(\vartheta) = (1, q_1)^T e^{i\omega_{n_0}^+ \hat{\tau}\vartheta} (\vartheta \in [-1, 0])$$

are the eigenvectors of A^* and $A(0)$ corresponding to the eigenvalue $-i\omega_{n_0}^+ \hat{\tau}$ and $i\omega_{n_0}^+ \hat{\tau}$, respectively, where

$$\begin{aligned} q_1 &= \frac{\beta(i\omega_{n_0}^+ - 1 + 2\delta + \frac{v_\delta}{(1+c\delta)^2} + d_1 \frac{n^2}{l^2})}{r + \theta\delta}, \\ q_1^* &= \frac{\beta(i\omega_{n_0}^+ + d_2 \frac{n^2}{l^2})}{r + \theta\delta}, \\ \bar{M} &= \bar{q}_1^* + q_1 - \theta v_\delta \hat{\tau} q_1 e^{-i\omega_{n_0}^+ \hat{\tau}}. \end{aligned}$$

In view of $\{\pm i\omega_{n_0}^+ \hat{\tau}\}$, the center subspace of linear system (4.4) is given as

$$P = \{zqb_{n_0} + \bar{z}\bar{q}b_{n_0} | z \in \mathbb{C}\},$$

and $C = P \oplus Q$, where Q is the stable subspace.

In the following, when we let $\mu = 0$ in Eq (4.2), there exists a center manifold

$$W(t, \vartheta) = W(z, \bar{z}, \vartheta) = W_{20}(\vartheta) \frac{z^2}{2} + W_{11}z\bar{z} + W_{02}(\vartheta) \frac{\bar{z}^2}{2} + \dots, \quad (4.7)$$

from the theory of [16, 17], system (4.4) can be rewritten as

$$Z_t = zq(\cdot)a_{n_0} + \bar{z}\bar{q}(\cdot)a_{n_0} + W(t, \cdot),$$

then it follows that

$$\dot{z} = i\omega_{n_0}z + \bar{q}^{*T}(0)\langle G(0, Z_t), \xi_{n_0} \rangle, \quad (4.8)$$

with $\langle G, \xi_n \rangle = (\langle G_1, a_n \rangle, \langle G_2, a_n \rangle)^T$. We can also write Eq (4.9) as

$$\dot{z} = i\omega_{n_0}z + g(z, \bar{z}), \quad (4.9)$$

with

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11}z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2\bar{z}}{2} + \dots$$

According to the Taylor formula,

$$\begin{aligned} G(\varphi, 0) &= \hat{\tau} \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \\ &= \hat{\tau} \begin{pmatrix} a_{20}\varphi_1^2(0) + a_{11}\varphi_1(0)\varphi_2(0) \\ b_{20}\varphi_1^2(0) + b_{11}\varphi_1(0)\varphi_2(0) + b'_{11}\varphi_1(-1)\varphi_2(0) + b_{30}\varphi_1^3(0) + b_{21}\varphi_1^2(0)\varphi_2(0) + O(4) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{20} &= -\left(1 + \frac{v_\delta}{(1+c\delta)^2}\right), \quad a_{11} = -\frac{1}{(1+c\delta)^2}, \quad b_{20} = -\frac{2cv_\delta}{(1+c\delta)^3}, \\ b_{11} &= \frac{\beta}{(1+c\delta)^2}, \quad b'_{11} = -\theta, \quad b_{30} = -\frac{2c^2v_\delta}{(1+c\delta)^4}, \quad b_{21} = -\frac{2c}{(1+c\delta)^3}. \end{aligned}$$

By simple calculation, we have

$$\begin{aligned} g_{20} &= \frac{2}{M} \left[\bar{q}_1^* (a_{20} + a_{11}q_1) + (b_{11}q_1 + b'_{11}e^{-2i\omega_{n_0}\hat{\tau}}q_1 + b_{20}) \right], \\ g_{02} &= \frac{2}{M} \left[\bar{q}_1^* (a_{20} + a_{11}\bar{q}_1) + (b_{11}\bar{q}_1 + b'_{11}e^{2i\omega_{n_0}\hat{\tau}}\bar{q}_1 + b_{20}) \right], \\ g_{11} &= \frac{1}{M} \left[\bar{q}_1^* (2a_{20} + a_{11}(q_1 + \bar{q}_1)) + (b_{11}(\bar{q}_1 + q_1) + b'_{11}(e^{-i\omega_{n_0}\hat{\tau}}\bar{q}_1 + e^{i\omega_{n_0}\hat{\tau}}q_1) + 2b_{20}) \right], \\ g_{21} &= \frac{1}{M} \left[\bar{q}_1^* \left(a_{20} \frac{1}{l\pi} \int_0^{l\pi} (2W_{11}^1 + W_{02}^1) dx \right. \right. \\ &\quad + a_{11} \frac{1}{l\pi} \int_0^{l\pi} \left(\frac{W_{20}^2}{2} + \frac{W_{20}^1}{2} + W_{11}^1 q_1 + W_{11}^2 \right) dx \\ &\quad + b_{11} \frac{1}{l\pi} \int_0^{l\pi} \left(\frac{W_{20}^2}{2} + \frac{W_{20}^1}{2} + W_{11}^1 q_1 + W_{11}^2 \right) dx \\ &\quad + b'_{11} \frac{1}{l\pi} \int_0^{l\pi} \left(e^{-i\omega_{n_0}\hat{\tau}} W_{11}^2 + e^{i\omega_{n_0}\hat{\tau}} \frac{W_{20}^2}{2} + \frac{W_{20}^1(-\tau)}{2} \bar{q}_1 + W_{11}^1(-\tau) q_1 \right) dx \\ &\quad \left. + b_{20} \frac{1}{l\pi} \int_0^{l\pi} (W_{20}^1 + 2W_{11}^2) dx + b_{30} \frac{1}{l\pi} \int_0^{l\pi} (3 + 3W_{11}^2) dx + b_{21}(2q_1 + \bar{q}_1) \right]. \end{aligned}$$

In the following, we further compute $W_{11}(\vartheta)$ and $W_{20}(\vartheta)$ to obtain g_{21} . Here,

$$\begin{aligned} \dot{W} &= \dot{Z}_t - \dot{z}q a_{n_0} - \dot{\bar{z}}\bar{q} a_{n_0} \\ &= AW + H(z, \bar{z}, \vartheta), \end{aligned} \tag{4.10}$$

where

$$H(z, \bar{z}, \vartheta) = H_{20}(\vartheta) \frac{z^2}{2} + H_{11}(\vartheta) z\bar{z} + H_{02}(\vartheta) \frac{\bar{z}^2}{2} + \dots$$

Therefore, we have

$$(2i\omega_{n_0}\hat{\tau}I - A_0)W_{20} = H_{20}, \quad -A_0W_{11} = H_{11}, \quad (-2i\omega_{n_0}\hat{\tau} - A_0)W_{02} = H_{02}. \tag{4.11}$$

and from Eq (4.11), obtain

$$W_{20}(\vartheta) = -\frac{g_{20}}{i\omega_{n_0}\hat{\tau}} \left(\frac{1}{q_1}\right) e^{i\omega_{n_0}\hat{\tau}\vartheta} a_{n_0} - \frac{\bar{g}_{02}}{3i\omega_{n_0}\hat{\tau}} \left(\frac{1}{\bar{q}_1}\right) e^{-i\omega_{n_0}\hat{\tau}\vartheta} a_{n_0} + E_1 e^{2i\omega_{n_0}\hat{\tau}\vartheta}, \quad (4.12)$$

and

$$W_{11}(\vartheta) = \frac{g_{11}}{i\omega_{n_0}\hat{\tau}} \left(\frac{1}{q_1}\right) e^{i\omega_{n_0}\hat{\tau}\vartheta} a_{n_0} - \frac{\bar{g}_{11}}{i\omega_{n_0}\hat{\tau}} \left(\frac{1}{\bar{q}_1}\right) e^{-i\omega_{n_0}\hat{\tau}\vartheta} a_{n_0} + E_2. \quad (4.13)$$

For Eq (4.13), when $\vartheta = 0$, we have

$$(2i\omega_{n_0}\hat{\tau}I - A_0)E_1 = -F_{zz}, \quad -A_0E_2 = -F_{z\bar{z}}, \quad (4.14)$$

where $E_1 = \sum_{n=0}^{\infty} E_1^n b_n$ and $E_2 = \sum_{n=0}^{\infty} E_2^n b_n$. Further, we obtain

$$\begin{aligned} (2i\omega_{n_0}\hat{\tau}I - A_0)E_1^n b_n &= -\langle F_{zz}, \xi_n \rangle a_n, \\ -A_0 E_2^n b_n &= -\langle F_{z\bar{z}}, \xi_n \rangle a_n, \quad n = 0, 1, \dots, \end{aligned}$$

where

$$\langle F_{zz}, \xi_n \rangle = \begin{cases} \frac{1}{\sqrt{l\pi}} F_{20}, & n_0 \neq 0, n = 0, \\ \frac{1}{\sqrt{2l\pi}} F_{20}, & n_0 \neq 0, n = 2n_0, \\ \frac{1}{\sqrt{l\pi}} F_{20}, & n_0 = 0, n = 0, \\ 0, & \text{other,} \end{cases}$$

$$\langle F_{z\bar{z}}, \xi_n \rangle = \begin{cases} \frac{1}{\sqrt{l\pi}} F_{11}, & n_0 \neq 0, n = 0, \\ \frac{1}{\sqrt{2l\pi}} F_{11}, & n_0 \neq 0, n = 2n_0, \\ \frac{1}{\sqrt{l\pi}} F_{11}, & n_0 = 0, n = 0, \\ 0, & \text{other.} \end{cases}$$

We then compute E_1^n and E_2^n given by

$$E_1^n = E_{11}^n \cdot E_{12}^n, \quad E_2^n = E_{21}^n \cdot E_{22}^n,$$

where

$$E_{11}^n = \begin{pmatrix} 2i\omega_{n_0}\hat{\tau} - (1 - 2\delta - B_1) & A_1 \\ -\beta B_1 + \theta v_\delta e^{-2i\omega_{n_0}\hat{\tau}} & 2i\omega_{n_0}\hat{\tau} \end{pmatrix}^{-1},$$

$$E_{12}^n = \begin{pmatrix} a_{20} + a_{11}q_1 \\ b_{11}q_1 + b'_{11}e^{-2i\omega_{n_0}\hat{\tau}}q_1 + b_{20} \end{pmatrix},$$

and

$$E_{21}^n = \begin{pmatrix} -(1 - 2\delta - B_1) & A_1 \\ -\beta B_1 + \theta v_\delta & 0 \end{pmatrix}^{-1},$$

$$E_{22}^n = \begin{pmatrix} 2a_{20} + a_{11}(q_1 + \bar{q}_1) \\ b_{11}(\bar{q}_1 + q_1) + b'_{11}(e^{-i\omega_{n_0}\hat{\tau}}\bar{q}_1 + e^{i\omega_{n_0}\hat{\tau}}q_1) + 2b_{20} \end{pmatrix}.$$

Thus, by a series of derivations and calculations, we can determine the value of g_{21} .

Then, we analyze the bifurcation property according to the following expression:

$$c_1(0) = \frac{i}{2\omega_{n_0}\hat{\tau}} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2}g_{21}, \quad \mu_2 = \frac{\operatorname{Re}(c_1(0))}{\operatorname{Re}(\lambda'(\hat{\tau}))},$$

$$\beta_2 = 2\operatorname{Re}(c_1(0)), \quad T_2 = -\frac{\operatorname{Im}(c_1(0)) + \mu_2\operatorname{Im}(\lambda'(\hat{\tau}))}{\omega_{n_0}\hat{\tau}}. \quad (4.15)$$

Theorem 4.1. For system (1.2), the following statements hold.

(i) If $\mu_2 > 0 (< 0)$, then the direction of the Hopf bifurcation is forward (backward), that is, there exist bifurcating periodic solutions for $\tau > \hat{\tau} (\tau < \hat{\tau})$;

(ii) If $\beta_2 < 0 (> 0)$, then the bifurcating periodic solutions on the center manifolds are orbitally stable (unstable);

(iii) If $T_2 > 0 (< 0)$, then the bifurcating periodic solutions are period increases (decreases).

Further, by Theorem 3.6, we obtain the following result.

Corollary 4.2. If $\operatorname{Re}(c_1(0)) < 0 (> 0)$, then $\mu_2 > 0 (< 0)$ and $\beta_2 < 0 (> 0)$.

5. Analysis of global Hopf bifurcation

In the above section, the sufficient condition for the occurrence of Hopf bifurcation at E^* is given. In this section, we analyze the global dynamics of system (1.2) near the equilibrium E^* . First, we cite the global Hopf bifurcation result in Reference [17].

Lemma 5.1. Let S denote the closure of the set $\{(z, \alpha, \beta) \in E \times \mathbb{R} \times (0, \infty)\}$; $u(t) = z(\beta t)$ is a nontrivial $2\pi/\beta$ periodic solutions. Then for each connected component C , at least one of the following holds:

(1) C is unbounded, i.e. $\sup\{\max_{t \in \mathbb{R}} |z(t)| + |\alpha| + \beta + \beta^{-1}; (z, \alpha, \beta) \in C\} = \infty$;

(2) $C \cap (M^* \times (0, \infty))$ is finite and for all $k \geq 1$, one has the equality $\Sigma_{(x_0, \alpha_0, \beta_0) \in C \cap (M^* \times (0, \infty))} \mu_k(x_0, \alpha_0, \beta_0) = 0$.

For convenience, let $z_t = (u_t, v_t)$, the system (1.2) is rewritten as follows

$$\dot{z}(t) = F(z_t, \tau, p), \quad (5.1)$$

where $z_t(\vartheta) = z(t + \vartheta) \in C([- \tau, 0], X)$, X is the Banach space. With the help of the bifurcation theory in Reference [17], we define

$$\mathbf{Y} = C([- \tau, 0], X),$$

$$\mathbf{\Sigma} = Cl\{(z, \tau, p) \in \mathbf{Y} \times \mathbb{R} \times \mathbb{R}^+ : z \text{ is a } p\text{-periodic solution of (1.2)}\},$$

$$\mathbf{N} = \{(\bar{z}, \bar{\tau}, \bar{p}) : F(\bar{z}, \bar{\tau}, \bar{p}) = 0\},$$

and $I_{(z^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+})}$ is the connected component of $(z^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+})$ in Σ , where $\tau_{n_0, k}^+$ and $\omega_{n_0}^+$ are defined in (3.8).

Lemma 5.2. *Assume condition (H1) holds. Then system (1.2) has no any nontrivial τ -periodic solution.*

Proof. By contradiction, assume that system (1.2) has τ -periodic solution, in other words, system (3.1) then has a periodic solution. We know that system (3.1) also has the same equilibria as system (1.2), that is,

$$E_0 = (0, 0), \quad E_1 = (1, 0),$$

and an interior equilibrium E^* . For system (3.1), it is straightforward that the u -axis and v -axis are the invariable manifold, and the orbits of the system do not intersect each other, thus, the solutions can not cross the coordinate axes. Hence, it follows that it must be the interior equilibrium E^* if there exists any periodic solution within the first quadrant. From the above discussion in Lemma 2.1, the positive equilibrium E^* is globally attractive, then system (3.1) has no nontrivial positive periodic solution, this means that system (1.2) has no nontrivial τ -periodic solution. This leads to contradiction. Completing the proof. \square

Theorem 5.3. *Assume condition (H1) holds and $2D_n - T_n^2 > 0$, $T_n^2 - 4D_n + 4\theta^2\delta^2(1 - \delta)^2 > 0$. When $\lambda = i\omega_{n_0}^+$, then for each $\tau > \tau_{n_0, k}^+$ ($k = 0, 1, \dots$) system (1.2) has at least $k + 1$ periodic solutions.*

Proof. It is straightforward to calculate that the characteristic matrix of system (1.2) at a constant steady state $\tilde{E} = (\tilde{u}, \tilde{v})$ is shown by

$$\Upsilon(\tilde{E}, \tau, p)(\lambda) = \begin{pmatrix} \lambda + \frac{n^2}{l^2} - (1 - 2\tilde{u} - B_1) & A_1 \\ \theta\tilde{v} - \beta B_1 & \lambda + \frac{n^2}{l^2} - (\frac{\beta\tilde{u}}{1+c\tilde{u}} - r - \theta\tilde{u}) \end{pmatrix}. \quad (5.2)$$

Then we find that system (1.2) has no the center of the form (E_0, τ, p) and (E_1, τ, p) .

For discussion of E^* , we first know that $(E^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+})$ is a isolated center (see [32, Definition 3.1]) by Lemma 3.4. When $\lambda(\tau_{n_0, k}^+) = i\omega_{n_0}^+$, we show the result that $\text{Re}\lambda'(\tau_{n_0, k}^+) > 0$ in Lemma 3.5. We define the smooth curve of the solution of the characteristic equation corresponding to the characteristic matrix (5.2) $\lambda : (\tau_{n_0, k}^+ - \zeta, \tau_{n_0, k}^+ + \zeta) \rightarrow \mathbb{C}$ satisfying as follows: there exist $\varepsilon > 0$, $\zeta > 0$ such that, for all $\tau \in [\tau_{n_0, k}^+ - \zeta, \tau_{n_0, k}^+ + \zeta]$

$$\det(\Upsilon(\lambda(\tau))) = 0, \quad |\lambda(\tau) - i\omega_{n_0}^+| < \varepsilon,$$

Furthermore, we define

$$\tilde{\Omega}_{\varepsilon, \frac{2\pi}{\omega_{n_0}^+}} = \left\{ (\varsigma, p) : 0 < \varsigma < \varepsilon, \left| p - \frac{2\pi}{\omega_{n_0}^+} \right| < \varepsilon \right\}.$$

and

$$H^\pm(E^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+})(\varsigma, p) = \det\left(\Upsilon(E^*, \tau_{n_0, k}^+ \pm \zeta, p)\left(\varsigma + i\frac{2\pi}{p}\right)\right).$$

Then we obtain that the crossing number of the centers $(E^*, \tau_{n_0, j}^+, \frac{2\pi}{\omega_{n_0}^+})$ is

$$\begin{aligned} \gamma\left(E^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+}\right) &= \deg\left(H^-\left(E^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+}\right), \widetilde{\Omega}_{\varepsilon, \frac{2\pi}{\omega_{n_0}^+}}\right) \\ &\quad - \deg\left(H^+\left(E^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+}\right), \widetilde{\Omega}_{\varepsilon, \frac{2\pi}{\omega_{n_0}^+}}\right) \\ &= -1. \end{aligned}$$

Consequently, by [32, Theorem 3.3], the connected component $l_{(z^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+})}$ of $(z^*, \tau_{n_0, k}^+, \frac{2\pi}{\omega_{n_0}^+})$ in Σ is unbounded. Then, from (3.3), we have

$$\tau_{n, k}^+ = \frac{1}{\omega_{n_0}^+} \left(-\arccos \frac{(\omega_{n_0}^+)^2 - D^n}{\theta \delta (1 - \delta)} + 2(k + 1)\pi \right), \quad n_0 \in \mathfrak{N}, \quad k \in \mathbb{N}_0,$$

and combining with Lemmas 2.1 and 5.2, it follows that for each $\tau > \tau_{n_0, k}^+$ ($k = 0, 1, \dots$) system (1.2) has at least $k + 1$ periodic solutions. \square

6. Simulations

In the following, let $\Omega = (0, \pi)$, and we choose the set of parameters as

$$c = 1.6, \quad \beta = 0.8, \quad r = 0.1, \quad \theta = 0.05$$

to simulate the dynamics of system (1.2). For the above values and $n = 0$, we can verify that conditions (H1) and (H2) hold, $2D_0 - T_0^2 \approx 0.0014 > 0$ and $T_0^4 - 4T_0^2 D_0 + 4\theta^2 \delta^2 (1 - \delta)^2 \approx 0.002 > 0$, by calculation, Eq (3.8) is obtained as

$$\tau_{0, k}^+ = 12.15 + 22.60k, \quad \tau_{0, k}^- = 24.13 + 24.98k, \quad k \in \mathbb{N}_0,$$

then the simulations are shown as (see Figures 1 and 2)

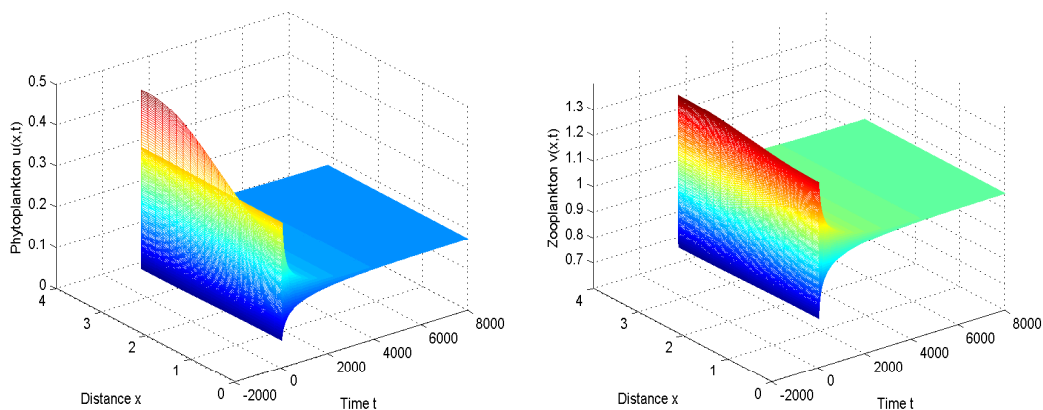


Figure 1. System (1.2) near $E^* = (0.17, 1.06)$ is asymptotically stable with $T_0^4 - 4T_0^2 D_0 + 4\theta^2 \delta^2 (1 - \delta)^2 \approx 0.002$ and $2D_0 - T_0^2 > 0 \approx 0.0014$ when $\tau = 7 < \tilde{\tau} \approx 12.15$.

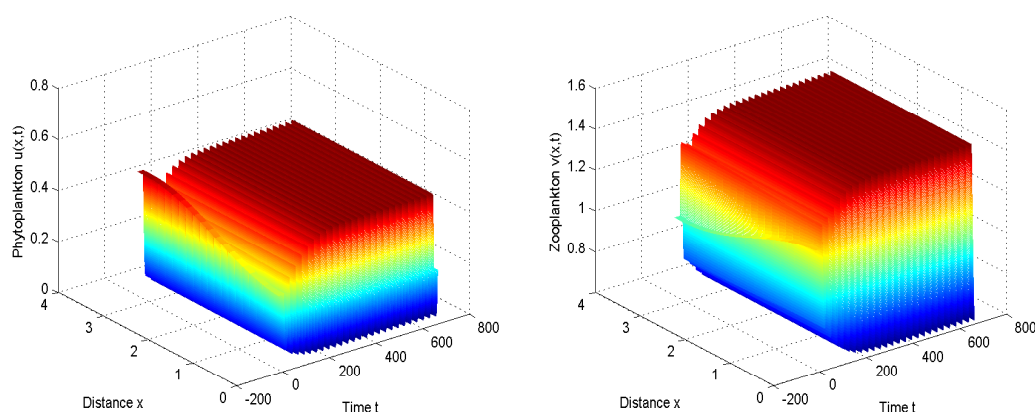


Figure 2. System (1.2) near $E^* = (0.17, 1.06)$ undergoes Hopf bifurcation with $T_0^4 - 4T_0^2D_0 + 4\theta^2\delta^2(1 - \delta)^2 = 0.002$ and $2D_0 - T_0^2 > 0 = 0.0014$ when $\tau = 22 > \tilde{\tau} \approx 12.15$.

7. Summary and discussion

This paper's main contribution is that it provides analytic results for the reaction-diffusion TPP-zooplankton model with Holling II response function. Here, for system (3.1), when time delay is considered, it is found that time delay is a factor that causes the dynamic behavior of the system to destabilize, that is, the positive equilibrium of system (1.2) changes from globally asymptotically stable into unstable. Furthermore, from a biological point of view, plankton populations fluctuates periodically over time. Finally, numerical simulation shows that the plankton system (1.2) with time delay discussed in this paper better describes real-world problems.

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Conflict of interest

The author declare there is no conflict of interest.

Declarations

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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