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Research article

Positivity and monotonicity results for discrete fractional operators involving the exponential kernel

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Abstract: This work deals with the construction and analysis of convexity and nabla positivity for discrete fractional models that includes singular (exponential) kernel. The discrete fractional differences are considered in the sense of Riemann and Liouville, and the υ_1-monotonicity formula is employed as our initial result to obtain the mixed order and composite results. The nabla positivity is discussed in detail for increasing discrete operators. Moreover, two examples with the specific values of the orders and starting points are considered to demonstrate the applicability and accuracy of our main results.

Keywords: discrete fractional calculus; discrete fractional operators with exponential kernel; monotonicity; positivity

1. Introduction

Discrete fractional calculus is of central importance in many fields of research and monotonicity analysis. In the past few decades, studies of discrete fractional operators have attracted significant attention in different fields of pure mathematics such as stability analysis, mathematical modelling, and topological spaces, see [1–5], and applied mathematics such as bioscience, numerical analysis, statistics, the system of difference equations, and calculus of variations, see [6–10] for more details.

Analyzing the discrete fractional operators for monotonicity and positivity is one of the most interesting studies in the discrete analysis. Various studies have dealt with this issue in discrete fractional calculus. There are certain limitations that researchers have shown [11–14].

On the other hand, a common strategy for analysing discrete fractional operators is simplifying their summations by using the forward difference operator $(\Delta\, \mathtt{y})(\mathtt{x}) = \mathtt{y}(\mathtt{x}+1) - \mathtt{y}(\mathtt{x})$ or backward difference operator $(\nabla\, \mathtt{y})(\mathtt{x}) = \mathtt{y}(\mathtt{x}) - \mathtt{y}(\mathtt{x}-1)$, which reduces the components of the summations to zero. Monotonicity and positivity analyses have been found for different discrete fractional difference operators, such as nabla/delta Riemann-Liouville and Caputo fractional differences [15–18], nabla/delta Caputo-Fabrizio fractional differences involving exponential kernels [19, 20], nabla/delta Atangana-Baleanu fractional differences involving Mittag-Leffler kernels [21–24].

In addition, several studies have examined the monotonicity and positivity for various fractional difference operators of mixed order, including Caputo-Fabrizio fractional differences involving exponential kernels [25–27], Atangana-Baleanu fractional differences involving Mittag-Leffler kernels [28]. Most of these studies have used one or two initial conditions to compute other positivity and monotonicity of the functions.

Motivated by those researches and results, we are interested in two types of results in the present study:

- Monotonicity-type results for variation operators of discrete nabla fractional differences with exponential kernels and their commutators on the time scale $\mathbb{N}_{t_0}$;
- Delta positivity results for the single operator and mixed operators of discrete nabla fractional differences with exponential kernels on the time scale $\mathbb{N}_{t_0}$ via two basic lemmas.

The rest of this paper is composed of the following sections: The first section (Section 2) includes two basic definitions of discrete fractional calculus, which we will use in our work. The next section (Section 3) includes our main results, which are separated into two subsections: In the first subsection (Subsection 3.1), we will prove some auxiliary lemmas, including the discrete operators, which we will use them in proving other results in the second subsection (Subsection 3.2). We will complete our article by summarizing our results and providing the future directions for the interested reader in Section 4.

2. Basic definitions

We first give the definitions of discrete nabla Caputo-Fabrizio of Caputo-type and Caputo-Fabrizio of Riemann-type fractional differences with discrete exponential function kernels.

Definition 2.1. [2, 19] For any function $\mathtt{y}$ defined on $\mathbb{N}_{t_0}$ the discrete nabla Caputo-Fabrizio of Riemann-type and Caputo-Fabrizio of Riemann-type fractional differences are defined, respectively, as follows:

$$\left({}^{CFC}_{t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \Lambda(\upsilon_1)\left[\sum_{\mathtt{z}=t_0+1}^{\mathtt{x}} (\nabla_{\mathtt{z}}\mathtt{y})(\mathtt{z})(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}}\right] \tag{2.1}$$

and

$$\left({}^{CFR}_{t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \Lambda(\upsilon_1)\,\nabla_{\mathtt{x}}\left[\sum_{\mathtt{z}=t_0+1}^{\mathtt{x}} \mathtt{y}(\mathtt{z})(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}}\right] \tag{2.2}$$

for all $\upsilon_1 \in (0,1), t_0 \in \mathbb{R}$, and $\mathtt{x}$ in $\mathbb{N}_{t_0+1}$, where $\Lambda(\upsilon_1)$ denotes a normalizing positive constant.

Definition 2.2. [29] The nth order discrete nabla Caputo-Fabrizio of Caputo-type and Caputo-Fabrizio of Riemann-type fractional differences can be expressed, respectively, as follows:

$$\left({}^{CFC}_{t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \left({}^{CFR}_{t_0}\nabla^{\upsilon_1-M}\nabla^{M}\mathtt{y}\right)(\mathtt{x}) = \Lambda(\upsilon_1-M)\left[\sum_{\mathtt{z}=t_0+1}^{\mathtt{x}} \left(\nabla_{\mathtt{z}}^{M+1}\mathtt{y}\right)(\mathtt{z})(M+1-\upsilon_1)^{\mathtt{x}-\mathtt{z}}\right] \tag{2.3}$$

and

$$\left({}^{CFR}_{t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \left({}^{CFR}_{t_0}\nabla^{\upsilon_1-M}\nabla^{M}\mathtt{y}\right)(\mathtt{x}) = \Lambda(\upsilon_1-M)\,\nabla_{\mathtt{x}}\left[\sum_{\mathtt{z}=t_0+1}^{\mathtt{x}} \left(\nabla_{\mathtt{z}}^{M}\mathtt{y}\right)(\mathtt{z})(M+1-\upsilon_1)^{\mathtt{x}-\mathtt{z}}\right] \tag{2.4}$$

for all $\mathtt{x}$ in $\mathbb{N}_{a+M}$ and $\upsilon_1 \in (M, M+1]$.

Remark 2.1. *Throughout the rest of this paper, for $\ell_1 > \ell_2$, we consider the classical convention that*

$$\sum_{\mathtt{z}=\ell_1}^{\ell_2} B_{\mathtt{z}} := 0.$$

3. Discrete operators analyses

In this section, we mainly give some preliminaries and do the analysis of discrete operators. Hereafter, for any function $\mathtt{y}$ defined on $\mathbb{N}_{t_0}$ and satisfying $\mathtt{y}(t_0) \geqslant 0$ the υ_1-monotonicity increasing function on $\mathbb{N}_{t_0}$ can satisfy $\mathtt{y}(\mathtt{x}+1) \geqslant \upsilon_1\,\mathtt{y}(\mathtt{x})$, and the υ_1-monotonicity decreasing function on $\mathbb{N}_{t_0}$ can satisfy $\mathtt{y}(\mathtt{x}+1) \leqslant \upsilon_1\,\mathtt{y}(\mathtt{x})$ for all $\mathtt{x}$ in $\mathbb{N}_{t_0}$. This section is separated into two subsections.

3.1. Monotonicity analysis part

In this subsection, we will present the analyses of the υ_1-monotonicity for the aforementioned discrete operators.

Lemma 3.1. *Assume that a function $\mathtt{y} : \mathbb{N}_{t_0} \to \mathbb{R}$ satisfies $\mathtt{y}(t_0) \geqslant 0$ and*

$$\left({}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) \geqslant 0 \tag{3.1}$$

for $\upsilon_1 \in (0,1)$ and $\mathtt{x}$ in $\mathbb{N}_{t_0+1}$. Then, $\mathtt{y}(\mathtt{x})$ is positive and υ_1–monotone increasing on $\mathbb{N}_{t_0}$.

Proof. In view of Definition 2.1, one can have for all $x \in \mathbb{N}_{t_0+1}$:

$$\begin{aligned}\left({}^{CFR}_{t_0-1}\nabla^{\upsilon_1}y\right)(x) &= \Lambda(\upsilon_1)\nabla_x\left[\sum_{z=t_0}^{x} y(z)(1-\upsilon_1)^{x-z}\right]\\ &= \Lambda(\upsilon_1)\left[\sum_{z=t_0}^{x} y(z)(1-\upsilon_1)^{x-z} - \sum_{z=t_0}^{x-1} y(z)(1-\upsilon_1)^{x-z-1}\right]\\ &= \Lambda(\upsilon_1)\left[y(x) - \upsilon_1\sum_{z=t_0}^{x-1} y(z)(1-\upsilon_1)^{x-z-1}\right]\\ &= \Lambda(\upsilon_1)\left[y(x) - \upsilon_1\,(1-\upsilon_1)^{x-t_0-1}y(t_0) - \upsilon_1\sum_{z=t_0+1}^{x-1} y(z)(1-\upsilon_1)^{x-z-1}\right]. \qquad (3.2)\end{aligned}$$

Since $\Lambda(\upsilon_1) > 0$ and $\left({}^{CFR}_{t_0-1}\nabla^{\upsilon_1}y\right)(x) \geqslant 0$ by assumption, then we can express (3.2) as follows

$$y(x) \geqslant \upsilon_1\,(1-\upsilon_1)^{x-t_0-1}y(t_0) + \upsilon_1\sum_{z=t_0+1}^{x-1} y(z)(1-\upsilon_1)^{x-z-1}. \qquad (3.3)$$

For $x = t_0 + 1$ and by assumption, then (3.3) gives

$$y(t_0+1) \geqslant \upsilon_1\, y(t_0) \geqslant 0.$$

For $x = t_0 + 2$ and by assumption and the above result, then (3.3) leads to

$$y(t_0+2) \geqslant \upsilon_1(1-\upsilon_1)\, y(t_0) + \upsilon_1\, y(t_0+1) \geqslant 0.$$

We can proceed in the same way to get $y(x) \geqslant 0$ for each $x \in \mathbb{N}_{t_0}$.

To do the proof of the υ_1–monotonicity increasing of y, we reuse (3.3) in the form:

$$y(x) \geqslant \upsilon_1\, y(x-1) + \upsilon_1\,(1-\upsilon_1)^{x-t_0-1}y(t_0) + \upsilon_1\sum_{z=t_0+1}^{x-2} y(z)(1-\upsilon_1)^{x-z-1}. \qquad (3.4)$$

We just proved that $y(x) \geqslant 0$ for each $x \in \mathbb{N}_{t_0}$, and $\upsilon_1 \in (0,1), y(t_0) \geqslant 0$ by assumption and $(1-\upsilon_1)^{x-t_0-1} > 0$ for each $x \in \mathbb{N}_{t_0+1}$. So, we can deduce from (3.4) that

$$y(x) \geqslant \upsilon_1\, y(x-1) \quad \text{for all } x \in \mathbb{N}_{t_0+1},$$

and this gives the υ_1–monotonicity increasing of y on $\mathbb{N}_{t_0}$. Thus, our results are shown.

Our second and third main results regarding sequential operators depend on the above lemma as follows. For further details on the main concepts of sequential operators, one may read [30].

Theorem 3.1. *Assume that a function* $y : \mathbb{N}_{t_0} \to \mathbb{R}$ *satisfies* $y(t_0+1) \geqslant y(t_0) \geqslant 0$ *and*

$$\left({}^{CFR}_{t_0}\nabla^{\upsilon_2}\ {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}y\right)(x) \geqslant 0 \qquad (3.5)$$

for $\upsilon_1, \upsilon_2 \in (0,1)$ *with* $0 < \upsilon_1 + \upsilon_2 \leqslant 1$ *and* x *in* $\mathbb{N}_{t_0+2}$. *Then,* $y(x)$ *is positive and* $(\upsilon_1+\upsilon_2)$*–monotone increasing on* $\mathbb{N}_{t_0}$.

Proof. Let us denote for $\mathtt{x} \in \mathbb{N}_{t_0+1}$:

$$\left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) := \mathtt{y}_2(\mathtt{x}).$$

This enables us to write

$$\left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_2}\ {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_2}\ \mathtt{y}_2\right)(\mathtt{x}).$$

Since $\mathtt{y}(t_0+1) \geqslant \mathtt{y}(t_0) \geqslant 0$, $\left({}^{CFR}_{t_0+1}\nabla^{\upsilon_2}\ \mathtt{y}_2\right)(\mathtt{x}) \geqslant 0$ for each $\mathtt{x} \in \mathbb{N}_{t_0+2}$ by assumption, and

$$\mathtt{y}_2(t_0+1) = \left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(t_0+1) = \Lambda(\upsilon_1)(\nabla_{\mathtt{z}}\mathtt{y})(t_0+1) \geqslant 0,$$

by Definition 2.1, we can deduce that $\mathtt{y}_2$ is positive and υ_2–monotone increasing on $\mathbb{N}_{t_0+1}$ by Lemma 3.1. That is,

$$\mathtt{y}_2(\mathtt{x}) \geqslant 0 \quad \text{for all } \mathtt{x} \in \mathbb{N}_{t_0+1},$$

and

$$\mathtt{y}_2(\mathtt{x}) \geqslant \upsilon_2\, \mathtt{y}_2(\mathtt{x}-1) \quad \text{for all } \mathtt{x} \in \mathbb{N}_{t_0+2}. \tag{3.6}$$

Now, by our claim, we see that

$$\left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \mathtt{y}_2(\mathtt{x}) \geqslant 0$$

for all $\mathtt{x} \in \mathbb{N}_{t_0+1}$, and since $\mathtt{y}(t_0) \geqslant 0$ by assumption, then Lemma 3.1 guarantees that $\mathtt{y}$ is positive and υ_1–monotone increasing on $\mathbb{N}_{t_0}$. That is,

$$\mathtt{y}(\mathtt{x}) \geqslant 0 \quad \text{for all } \mathtt{x} \in \mathbb{N}_{t_0}, \tag{3.7}$$

and

$$\mathtt{y}(\mathtt{x}) \geqslant \upsilon_1\, \mathtt{y}(\mathtt{x}-1) \quad \text{for all } \mathtt{x} \in \mathbb{N}_{t_0+1}. \tag{3.8}$$

To prove the rest of the theorem, we need to have

$$\mathtt{y}(\mathtt{x}) \geqslant (\upsilon_1+\upsilon_2)\, \mathtt{y}(\mathtt{x}-1) \quad \text{for all } \mathtt{x} \in \mathbb{N}_{t_0+1}. \tag{3.9}$$

First, we see that (3.9) is true for $\mathtt{x} = t_0+1$ as follows:

$$\mathtt{y}(t_0+1) \geqslant (\upsilon_1+\upsilon_2)\, \mathtt{y}(t_0+1) \geqslant (\upsilon_1+\upsilon_2)\, \mathtt{y}(t_0), \tag{3.10}$$

since $0 < \upsilon_1 + \upsilon_2 \leqslant 1$.

By using (3.6) in (3.2) by replacing $t_0 - 1$ with t_0, we get for all $\mathtt{x} \in \mathbb{N}_{t_0+2}$:

$$\begin{aligned}
0 \leqslant \mathtt{y}_2(\mathtt{x}) - \upsilon_2\, \mathtt{y}_2(\mathtt{x}-1) &= \left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) - \upsilon_2\, \left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}-1) \\
&= \Lambda(\upsilon_1)\left[\mathtt{y}(\mathtt{x}) - \upsilon_1\,(1-\upsilon_1)^{\mathtt{x}-t_0}\mathtt{y}(t_0+1) - \upsilon_1 \sum_{\mathtt{z}=t_0+2}^{\mathtt{x}-1} \mathtt{y}(\mathtt{z})(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}-1}\right] \\
&\quad - \upsilon_2\, \Lambda(\upsilon_1)\left[\mathtt{y}(\mathtt{x}-1) - \upsilon_1\,(1-\upsilon_1)^{\mathtt{x}-t_0-1}\mathtt{y}(t_0+1) - \upsilon_1 \sum_{\mathtt{z}=t_0+2}^{\mathtt{x}-2} \mathtt{y}(\mathtt{z})(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}-2}\right]
\end{aligned}$$

$$= \Lambda(\upsilon_1)\Bigg[(\mathtt{y}(\mathtt{x})-\upsilon_2\,\mathtt{y}(\mathtt{x}-1)) - \upsilon_1(1-\upsilon_1-\upsilon_2)\,(1-\upsilon_1)^{\mathtt{x}-t_0-1}\mathtt{y}(t_0+1)$$
$$-\upsilon_1\,\mathtt{y}(\mathtt{x}-1) - \upsilon_1(1-\upsilon_1-\upsilon_2)\sum_{\mathtt{z}=t_0+2}^{\mathtt{x}-2}\mathtt{y}(\mathtt{z})(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}-2}\Bigg]. \tag{3.11}$$

Since $\Lambda(\upsilon_1) > 0$, it follows from (3.11) that

$$\mathtt{y}(\mathtt{x}) - (\upsilon_1+\upsilon_2)\,\mathtt{y}(\mathtt{x}-1) \geqslant \underbrace{\upsilon_1(1-\upsilon_1-\upsilon_2)}_{\geqslant 0}\,\underbrace{(1-\upsilon_1)^{\mathtt{x}-t_0-1}}_{>0}\,\underbrace{\mathtt{y}(t_0+1)}_{\geqslant 0}$$
$$+\underbrace{\upsilon_1(1-\upsilon_1-\upsilon_2)}_{\geqslant 0}\sum_{\mathtt{z}=t_0+2}^{\mathtt{x}-2}\underbrace{\mathtt{y}(\mathtt{z})}_{\geqslant 0 \text{ by (3.7)}}\,\underbrace{(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}-2}}_{>0}, \tag{3.12}$$

which together with (3.12) rearrange to $\mathtt{y}(\mathtt{x}) \geqslant (\upsilon_1+\upsilon_2)\,\mathtt{y}(\mathtt{x}-1)$ for each $\mathtt{x} \in \mathbb{N}_{t_0+1}$, as desired.

Corollary 3.1. *Assume that a function* $\mathtt{y} : \mathbb{N}_{t_0} \to \mathbb{R}$ *satisfies* $\mathtt{y}(t_0+1) \geqslant \mathtt{y}(t_0) \geqslant 0$ *and*

$$\left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_2}\ {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) \geqslant 0 \tag{3.13}$$

for $\upsilon_1, \upsilon_2 \in (0,1)$ *and* $\mathtt{x}$ *in* $\mathbb{N}_{t_0+2}$. *Then,* $\mathtt{y}$ *is positive and* υ_1*–monotone increasing on* $\mathbb{N}_{t_0}$. *Moreover, if* $0 < \upsilon_1+\upsilon_2 \leqslant 1$, *then* $\mathtt{y}$ *is* υ_2*–monotone strictly increasing on* $\mathbb{N}_{t_0}$.

Proof. It is evidence from (3.7) and (3.8) that $\mathtt{y}$ is positive and υ_1–monotone increasing on $\mathbb{N}_{t_0}$. Now, if $0 < \upsilon_1+\upsilon_2 \leqslant 1$, then by using Theorem 3.1, we have that $\mathtt{y}$ is $\upsilon_1+\upsilon_2$–monotone increasing on $\mathbb{N}_{t_0}$. Hence,

$$\mathtt{y}(\mathtt{x}) \geqslant (\upsilon_1+\upsilon_2)\,\mathtt{y}(\mathtt{x}-1) > \upsilon_2\,\mathtt{y}(\mathtt{x}-1),$$

for each $\mathtt{x} \in \mathbb{N}_{t_0+1}$. This implies that $\mathtt{y}$ is υ_2–monotone strictly increasing on $\mathbb{N}_{t_0}$. Hence the proof is done.

Example 3.1. Considering the definition of Caputo-Fabrizio of Riemann-type fractional difference (2.2), we have

$$\left({}^{CFR}_{t_0+1}\nabla^{\upsilon_1}\ {}^{CFR}_{\ \ t_0}\nabla^{\upsilon_2}\mathtt{y}\right)(\mathtt{x}) = \Lambda(\upsilon_1)\,\nabla_{\mathtt{x}}\sum_{\mathtt{z}=t_0+2}^{\mathtt{x}}\left({}^{CFR}_{\ \ t_0}\nabla^{\upsilon_2}\mathtt{y}\right)(\mathtt{z})(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}} \tag{3.14}$$

for $\upsilon_1, \upsilon_2 \in (0,1)$ and $\mathtt{x} \in \mathbb{N}_{t_0+1}$.

The chosen $t_0 = 1$ and $\mathtt{x} = t_0+2$ leads to

$$\left({}^{CFR}_{\ \ 1}\nabla^{\upsilon_1}\ {}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(3) = \Lambda(\upsilon_1)\left\{\sum_{\mathtt{z}=2}^{3}\left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(\mathtt{z})(1-\upsilon_1)^{3-\mathtt{z}} - \sum_{\mathtt{z}=1}^{2}\left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(\mathtt{z})(1-\upsilon_1)^{2-\mathtt{z}}\right\}$$
$$= \Lambda(\upsilon_1)\left\{(1-\upsilon_1)\left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(2) + \left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(3) - \left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(2)\right\}$$
$$= \Lambda(\upsilon_1)\left\{\left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(3) - \upsilon_1\left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(2)\right\}. \tag{3.15}$$

For the chosen $\mathtt{y}_1 = 0.005, \mathtt{y}_2 = 0.01, \mathtt{y}_3 = 0.05$, $\upsilon_1 = 0.25$ and $\upsilon_2 = 0.25$, we have

$$\left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2}\mathtt{y}\right)(3) = \Lambda(\upsilon_2)\,\nabla\sum_{\mathtt{z}=1}^{3}\mathtt{y}(\mathtt{z})(1-\upsilon_2)^{3-\mathtt{z}}$$

$$= \Lambda(\upsilon_2)\left\{\sum_{z=1}^{3} y(z)(1-\upsilon_2)^{3-z} - \sum_{z=1}^{2} y(z)(1-\upsilon_2)^{2-z}\right\}$$
$$= \Lambda(0.25)\{y(3) - \upsilon_2 y(2) - \upsilon_2(1-\upsilon_2)y(1)\}$$
$$= 0.0466\,\Lambda(0.25), \tag{3.16}$$

and similarly,

$$\left({}^{CFR}_{\ \ 0}\nabla^{\upsilon_2} y\right)(2) = \Lambda(\upsilon_2)\,\nabla \sum_{z=1}^{2} y(z)(1-\upsilon_2)^{2-z}$$
$$= \Lambda(0.25)\{y(2) - \upsilon_2 y(1)\}$$
$$= 0.0088\,\Lambda(0.25). \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15) for $\upsilon_1 = \upsilon_2 = 0.5$, we get

$$\left({}^{CFR}_{\ \ 1}\nabla^{\frac{1}{4}}\ {}^{CFR}_{\ \ 0}\nabla^{\frac{1}{4}} y\right)(3) = \Lambda(0.25)\left\{\left({}^{CFR}_{\ \ 0}\nabla^{\frac{1}{4}} y\right)(3) - \frac{1}{4}\left({}^{CFR}_{\ \ 0}\nabla^{\frac{1}{4}} y\right)(2)\right\}$$
$$= \Lambda^2(0.25)\left\{0.0466 + \frac{0.0088}{4}\right\}$$
$$= 0.0444\,\Lambda^2(0.25) > 0.$$

Also, it is clear that $y(2) > y(1)$. Thus, we find that y is positive and $\frac{1}{2}$–monotone increasing on $\mathbb{N}_1$ by Theorem 3.1.

3.2. Positivity analysis part

In this section, we first prove two essential lemmas for which we need a new nabla condition. Then, we will extend the result obtained to the above Caputo-Fabrizio of Riemann-type operators via a Caputo-Fabrizio of Riemann-type operator of another Caputo-Fabrizio of Riemann-type operator with two different starting points.

Lemma 3.2. *Assume that a function* $y : \mathbb{N}_{t_0} \to \mathbb{R}$ *satisfies* $y(t_0+1) \geqslant y(t_0) \geqslant 0$ *and*

$$\nabla\left({}^{CFR}_{t_0-1}\nabla^{\upsilon_1} y\right)(x) \geqslant 0 \tag{3.18}$$

for $\upsilon_1 \in (0,1)$ *and* x *in* $\mathbb{N}_{t_0+2}$. *Then,* $(\nabla y)(x) \geqslant 0$ *for all* x *in* $\mathbb{N}_{t_0+1}$.

Proof. We will proceed by using (3.2):

$$\nabla_x\left({}^{CFR}_{t_0-1}\nabla^{\upsilon_1} y\right)(x) = \Lambda(\upsilon_1)\nabla_x\left[y(x) - \upsilon_1 \sum_{z=t_0}^{x-1} y(z)(1-\upsilon_1)^{x-z-1}\right]$$
$$= \Lambda(\upsilon_1)\left[(\nabla y)(x) - \upsilon_1 \sum_{z=t_0}^{x-1} y(z)(1-\upsilon_1)^{x-z-1} + \upsilon_1 \sum_{z=t_0}^{x-2} y(z)(1-\upsilon_1)^{x-z-2}\right]$$
$$= \Lambda(\upsilon_1)\left[(\nabla y)(x) - \upsilon_1\, y(t_0)(1-\upsilon_1)^{x-t_0-1} - \upsilon_1 \sum_{z=t_0+1}^{x-1} y(z)(1-\upsilon_1)^{x-z-1}\right.$$

$$+\upsilon_1 \sum_{z=t_0+1}^{x-1} y(z-1)(1-\upsilon_1)^{x-z-1}\Bigg]$$

$$= \Lambda(\upsilon_1)\left[(\nabla y)(x) - \upsilon_1\, y(t_0)(1-\upsilon_1)^{x-t_0-1} - \upsilon_1 \sum_{z=t_0+1}^{x-1} (\nabla y)(z)(1-\upsilon_1)^{x-z-1}\right], \quad (3.19)$$

for all $x \in \mathbb{N}_{t_0+2}$. Since $\Lambda(\upsilon_1) > 0$ and $\nabla\left({}^{CFR}_{t_0-1}\nabla^{\upsilon_1} y\right)(x) \geqslant 0$, for all $x \in \mathbb{N}_{t_0+2}$, from (3.19) we get

$$(\nabla y)(x) \geqslant \upsilon_1\, y(t_0)(1-\upsilon_1)^{x-t_0-1} + \upsilon_1 \sum_{z=t_0+1}^{x-1} (\nabla y)(z)(1-\upsilon_1)^{x-z-1}. \quad (3.20)$$

We will proceed by induction to complete the proof. We know from the assumption that $(\nabla y)(t_0 + 1) \geqslant 0$. Assume that $(\nabla y)(t_0+k) \geqslant 0$ for some $k \in \mathbb{N}_1$. Then, we need to show that $(\nabla y)(t_0+k+1) \geqslant 0$. By using (3.20) at $x = t_0 + k + 1$, and our claim, we have

$$\begin{aligned}(\nabla y)(t_0+k+1) &\geqslant \underbrace{\upsilon_1\, y(t_0)(1-\upsilon_1)^k}_{\geqslant 0} + \underbrace{\upsilon_1 \sum_{z=t_0+1}^{a+k} \underbrace{(\nabla y)(z)}_{\geqslant 0}\, \underbrace{(1-\upsilon_1)^{a+k-z}}_{>0}}_{\geqslant 0} \\ &\geqslant 0,\end{aligned}$$

which rearranges to the required proof.

Lemma 3.3. *Assume that a function* $y : \mathbb{N}_{t_0} \to \mathbb{R}$ *satisfies* $(\nabla y)(t_0 + 1) \geqslant 0$ *and*

$$\left({}^{CFR}_{t_0}\nabla^{\upsilon_1} \nabla y\right)(x) \geqslant 0 \quad (3.21)$$

for $\upsilon_1 \in (0, 1)$ *and* x *in* $\mathbb{N}_{t_0+2}$. *Then,* $(\nabla y)(x) \geqslant 0$ *for all* x *in* $\mathbb{N}_{t_0+1}$.

Proof. By replacing $t_0 - 1$ with t_0, and y with (∇y) in (3.2), we get for all $x \in \mathbb{N}_{t_0+2}$:

$$\begin{aligned}&\left({}^{CFR}_{t_0}\nabla^{\upsilon_1} \nabla y\right)(x) \\ &= \Lambda(\upsilon_1)\left[(\nabla y)(x) - \upsilon_1 \sum_{z=t_0+1}^{x-1} (\nabla y)(z)(1-\upsilon_1)^{x-z-1}\right] \\ &= \Lambda(\upsilon_1)\left[(\nabla y)(x) - \upsilon_1(1-\upsilon_1)^{x-t_0-2}(\nabla y)(t_0+1) - \upsilon_1 \sum_{z=t_0+2}^{x-1} (\nabla y)(z)(1-\upsilon_1)^{x-z-1}\right].\end{aligned} \quad (3.22)$$

Since $\Lambda(\upsilon_1) > 0$ and $\left({}^{CFR}_{t_0}\nabla^{\upsilon_1} \nabla y\right)(x) \geqslant 0$ for all $x \in \mathbb{N}_{t_0+2}$, then (3.22) can give us

$$(\nabla y)(x) \geqslant \upsilon_1(1-\upsilon_1)^{x-t_0-2}(\nabla y)(t_0+1) + \upsilon_1 \sum_{z=t_0+2}^{x-1} (\nabla y)(z)(1-\upsilon_1)^{x-z-1}. \quad (3.23)$$

Again, by induction we will show that $(\nabla y)(t_0+k+1) \geqslant 0$, where we assumed that $(\nabla y)(t_0+k) \geqslant 0$ for some $k \in \mathbb{N}_1$. From assumption, we have that $(\nabla y)(t_0 + 1) \geqslant 0$. But then from the lower bound for

$(\nabla \mathrm{y})(t_0+k+1)$ in (3.23) and our claim, we have

$$(\nabla \mathrm{y})(t_0+k+1) \geqslant \underbrace{\upsilon_1(1-\upsilon_1)^{k-1}(\nabla \mathrm{y})(t_0+1)}_{\geqslant 0} + \underbrace{\upsilon_1 \sum_{\mathrm{z}=t_0+2}^{a+k} \underbrace{(\nabla \mathrm{y})(\mathrm{z})(1-\upsilon_1)^{a+k-\mathrm{z}}}_{\geqslant 0}}_{\geqslant 0}$$
$$\geqslant 0,$$

which completes the proof.

Next, we apply the lemmas so far obtained to prove a couple of results in the sequential setting for the discrete Caputo-Fabrizio of Riemann-type fractional operators perturbed by two different starting points.

Theorem 3.2. *Assume that a function* $\mathrm{y}: \mathbb{N}_{t_0} \to \mathbb{R}$ *satisfies* $(\nabla \mathrm{y})(t_0+2) \geqslant (\nabla \mathrm{y})(t_0+1) \geqslant \upsilon_1 \, \mathrm{y}(t_0) \geqslant 0$ *and*

$$\left({}^{CFR}_{t_0+1}\nabla^{\upsilon_2} \; {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathrm{y}\right)(\mathrm{x}) \geqslant 0 \tag{3.24}$$

for $\upsilon_1 \in (0,1), \upsilon_2 \in (1,2)$ *and* x *in* $\mathbb{N}_{t_0+3}$. *Then,* $(\nabla \mathrm{y})(\mathrm{x}) \geqslant 0$ *for all* x *in* $\mathbb{N}_{t_0+1}$.

Proof. Let us denote

$$\left({}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathrm{y}\right)(\mathrm{x}) := \mathrm{y}_2(\mathrm{x})$$

for all $\mathrm{x} \in \mathbb{N}_{t_0+1}$.Then, by using Definition 2.2 with $n=2$, we see that

$$\left({}^{CFR}_{t_0+1}\nabla^{\upsilon_2} \; {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathrm{y}\right)(\mathrm{x}) = \left({}^{CFR}_{t_0+1}\nabla^{\upsilon_2-1} \, \nabla \; {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathrm{y}\right)(\mathrm{x}) = \left({}^{CFR}_{t_0+1}\nabla^{\upsilon_2-1} \, \nabla \, \mathrm{y}_2\right)(\mathrm{x}).$$

We know by assumption that $\left({}^{CFR}_{t_0+2}\nabla^{\upsilon_2-1} \, \nabla \, \mathrm{y}_2\right)(\mathrm{x}) \geqslant 0$, for every $\mathrm{x} \in \mathbb{N}_{t_0+3}$. Then, (3.11) helps us to write

$$\begin{aligned}(\nabla \, \mathrm{y}_2)(t_0+2) &= \left(\nabla \; {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathrm{y}\right)(t_0+2) \\ &= \Lambda(\upsilon_1)\left[(\nabla \mathrm{y})(t_0+2) - \upsilon_1(\nabla \mathrm{y})(t_0+1) - \upsilon_1(1-\upsilon_1)\mathrm{y}(t_0)\right] \\ &= \Lambda(\upsilon_1)\underbrace{\left[(\nabla \mathrm{y})(t_0+2) - \upsilon_1\left(\mathrm{y}(t_0+1) - \upsilon_1\,\mathrm{y}(t_0)\right)\right]}_{\geqslant 0 \text{ by assumption}} \geqslant 0.\end{aligned} \tag{3.25}$$

Therefore, it follows from Lemma 3.3 that

$$(\nabla \, \mathrm{y}_2)(\mathrm{x}) = \left(\nabla \; {}^{CFR}_{t_0-1}\nabla^{\upsilon_1}\mathrm{y}\right)(\mathrm{x}) \geqslant 0$$

for all $\mathrm{x} \in \mathbb{N}_{t_0+2}$. Also, we know that $(\nabla \mathrm{y})(t_0+1) \geqslant 0$, then by Lemma 3.2 we deduce that $(\nabla \mathrm{y})(\mathrm{x}) \geqslant 0$ for all $\mathrm{x} \in \mathbb{N}_{t_0+1}$. This completes the proof.

Theorem 3.3. *Assume that a function* $\mathrm{y}: \mathbb{N}_{t_0} \to \mathbb{R}$ *satisfies* $(\nabla \mathrm{y})(t_0+2) \geqslant (\nabla \mathrm{y})(t_0+1) \geqslant 0$ *and*

$$\left({}^{CFR}_{t_0+1}\nabla^{\upsilon_2} \; {}^{CFR}_{t_0}\nabla^{\upsilon_1}\mathrm{y}\right)(\mathrm{x}) \geqslant 0 \tag{3.26}$$

for $\upsilon_1 \in (1,2), \upsilon_2 \in (0,1)$ *and* x *in* $\mathbb{N}_{t_0+3}$. *Then,* $(\nabla \mathrm{y})(\mathrm{x}) \geqslant 0$ *for all* x *in* $\mathbb{N}_{t_0+1}$.

Proof. Let us denote

$$\left({}_{t_0+1}^{CFR}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) := (\nabla\,\mathtt{y}_2)(\mathtt{x})$$

for all $\mathtt{x} \in \mathbb{N}_{t_0+2}$. Hence,

$$\left({}_{t_0+1}^{CFR}\nabla^{\upsilon_2}\ {}_{t_0}^{CFR}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \left({}_{t_0+1}^{CFR}\nabla^{\upsilon_2}\,\nabla\,\mathtt{y}_2\right)(\mathtt{x}).$$

Since we have $\left({}_{t_0+1}^{CFR}\nabla^{\upsilon_2}\,\nabla\,\mathtt{y}_2\right)(\mathtt{x}) \geqslant 0$, for every $\mathtt{x} \in \mathbb{N}_{t_0+3}$ by assumption, and from (3.11), we have

$$\begin{aligned}(\nabla\,\mathtt{y}_2)(t_0+2) &= \left({}_{t_0+1}^{CFR}\nabla^{\upsilon_1}\mathtt{y}\right)(t_0+2) = \left({}_{t_0+1}^{CFR}\nabla^{\upsilon_1-1}\,\nabla\mathtt{y}\right)(t_0+2)\\ &= \Lambda(\upsilon_1-1)\underbrace{[(\nabla\mathtt{y})(t_0+2)-(\upsilon_1-1)(\nabla\mathtt{y})(t_0+1)]}_{\geqslant 0 \text{ by assumption}}\\ &\geqslant 0, \qquad (3.27)\end{aligned}$$

where we have used that $(\nabla\mathtt{y})(t_0+2) \geqslant (\nabla\mathtt{y})(t_0+1 \geqslant (\upsilon_1-1)(\nabla\mathtt{y})(t_0+1)$. Then, from Lemma 3.3 we get

$$(\nabla\,\mathtt{y}_2)(\mathtt{x}) = \left({}_{t_0+1}^{CFR}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) \geqslant 0$$

for all $\mathtt{x} \in \mathbb{N}_{t_0+2}$. That is,

$$0 \leqslant \left({}_{t_0+1}^{CFR}\nabla^{\upsilon_1}\mathtt{y}\right)(\mathtt{x}) = \left({}_{t_0+1}^{CFR}\nabla^{\upsilon_1-1}\,\nabla\,\mathtt{y}\right)(\mathtt{x})$$

for all $\mathtt{x} \in \mathbb{N}_{t_0+2}$. Meanwhile, $(\nabla\mathtt{y})(t_0+1) \geqslant 0$, so Lemma 3.3 confirms that $(\nabla\mathtt{y})(\mathtt{x}) \geqslant 0$ for each $\mathtt{x} \in \mathbb{N}_{t_0+1}$. Hence, the proof is complete.

We conclude by presenting a concrete example of a possible Caputo-Fabrizio of Riemann-type fractional difference problem that can be considered.

Example 3.2. Considering the definition of Caputo-Fabrizio of Riemann-type fractional difference (2.2), we have

$$\left({}_{t_0+1}^{CFR}\nabla^{\upsilon_1}\ {}_{t_0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(\mathtt{x}) = \Lambda(\upsilon_1)\,\nabla_{\mathtt{x}}\sum_{\mathtt{z}=t_0+2}^{\mathtt{x}}({}_{t_0}^{CFR}\nabla^{\upsilon_2}\mathtt{y})(\mathtt{z})(1-\upsilon_1)^{\mathtt{x}-\mathtt{z}} \qquad (3.28)$$

for $\upsilon_1 \in (0,1), \upsilon_2 \in (1,2)$ and $\mathtt{x} \in \mathbb{N}_{t_0+1}$.

The chosen $t_0 = 0$ and $\mathtt{x} = t_0 + 3$ leads to

$$\begin{aligned}\left({}_{1}^{CFR}\nabla^{\upsilon_1}\ {}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(3) &= \Lambda(\upsilon_1)\left\{\sum_{\mathtt{z}=2}^{3}\left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(\mathtt{z})(1-\upsilon_1)^{3-\mathtt{z}} - \sum_{\mathtt{z}=1}^{2}\left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(\mathtt{z})(1-\upsilon_1)^{2-\mathtt{z}}\right\}\\ &= \Lambda(\upsilon_1)\left\{(1-\upsilon_1)\left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(2) + \left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(3) - \left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(2)\right\}\\ &= \Lambda(\upsilon_1)\left\{\left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(3) - \upsilon_1\left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(2)\right\}. \qquad (3.29)\end{aligned}$$

Calculating the inside terms for the chosen $\mathtt{y}_0 = 0.001, \mathtt{y}_1 = 1.001, \mathtt{y}_2 = 1.01, \mathtt{y}_3 = 1.05$, $\upsilon_1 = 0.5$ and $\upsilon_2 = 1.5$, we have

$$\begin{aligned}\left({}_{0}^{CFR}\nabla^{\upsilon_2}\mathtt{y}\right)(3) = \left({}_{0}^{CFR}\nabla^{\upsilon_2-1}\,\nabla\mathtt{y}\right)(3) &= \Lambda(\upsilon_2-1)\,\nabla\sum_{\mathtt{z}=1}^{3}(\nabla\mathtt{y})(\mathtt{z})(2-\upsilon_2)^{3-\mathtt{z}}\\ &= \Lambda(\upsilon_2-1)\left\{\sum_{\mathtt{z}=1}^{3}(\nabla\mathtt{y})(\mathtt{z})(2-\upsilon_2)^{3-\mathtt{z}} - \sum_{\mathtt{z}=1}^{2}(\nabla\mathtt{y})(\mathtt{z})(2-\upsilon_2)^{2-\mathtt{z}}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \Lambda(0.5)\left\{(\nabla \mathrm{y})(3) + (1-\upsilon_2)(\nabla \mathrm{y})(2) + (2-\upsilon_2)(1-\upsilon_2)(\nabla \mathrm{y})(1)\right\} \\
&= -0.2145\,\Lambda(0.5),
\end{aligned}
\tag{3.30}
$$

and similarly,

$$
\begin{aligned}
\left({}^{CFR}_{\;\;0}\nabla^{\upsilon_2}\mathrm{y}\right)(2) &= \left({}^{CFR}_{\;\;0}\nabla^{\upsilon_2-1}\,\nabla\mathrm{y}\right)(2) = \Lambda(\upsilon_2-1)\,\nabla\sum_{\mathrm{z}=1}^{2}(\nabla\mathrm{y})(\mathrm{z})(2-\upsilon_2)^{2-\mathrm{z}} \\
&= \Lambda(0.5)\left\{(\nabla \mathrm{y})(2) + (1-\upsilon_2)(\nabla \mathrm{y})(1)\right\} \\
&= -0.4910\,\Lambda(0.5).
\end{aligned}
\tag{3.31}
$$

Substituting (3.30) and (3.31) into (3.29) for $\upsilon_1 = 0.5$ and $\upsilon_2 = 1.5$, we get

$$
\begin{aligned}
\left({}^{CFR}_{\;\;1}\nabla^{\frac{1}{2}}\;{}^{CFR}_{\;\;0}\nabla^{\frac{3}{2}}\mathrm{y}\right)(3) &= \Lambda(0.5)\left\{\left({}^{CFR}_{\;\;0}\nabla^{\frac{3}{2}}\mathrm{y}\right)(3) - \frac{1}{2}\left({}^{CFR}_{\;\;0}\nabla^{\frac{3}{2}}\mathrm{y}\right)(2)\right\} \\
&= \Lambda^2(0.5)\left\{-0.2145 + \frac{0.4910}{2}\right\} \\
&= 0.0310\,\Lambda^2(0.5) > 0.
\end{aligned}
$$

Therefore, we conclude that $(\nabla \mathrm{y})(3) \geqslant 0$ by Theorem 3.3.

4. Concluding remarks with future directions

The conclusion of our results are as follows:

- The main result has been proved in Lemma 3.1. This lemma has shown that the function is positive and υ_1-monotone increasing at the same time.
- Lemma 3.1 helped us to obtain the $(\upsilon_1 + \upsilon_2)$-monotonicity and υ_2-monotonicity of the composite discrete operators with different orders.
- Two positivity results have been proved in the next section. The first lemma considered the nabla of a discrete Caputo-Fabrizio of Riemann-type fractional difference. The other one considered the discrete Caputo-Fabrizio of Riemann-type fractional difference of a nabla function. Those both lemmas gave a nabla positivity result.
- Lemmas 3.2 and 3.3 enabled us to get the nabla positivity of the composite discrete operators with different orders.
- Examples 3.1 and 3.2 confirmed the applicability and validity of the main results on positivity.

As we know, one- and multi-parameter discrete Mittag-Liffler functions have been determined for some initial values in the references [21–24]. These basic results together with the results presented in this study, it will be helpful for the interested researchers to modify and extend our present results and obtain new results for the discrete fractional operators with non-singular (Mittag-Liffler) kernels.

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Conflict of interest

The authors declare there is no conflict of interest.

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