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Research article

Due date assignment scheduling with positional-dependent weights and proportional setup times

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Abstract: In this paper, we investigate the single-machine scheduling problem that considers due date assignment and past-sequence-dependent setup times simultaneously. Under common (slack and different) due date assignment, the objective is to find jointly the optimal sequence and optimal due dates to minimize the weighted sum of lateness, number of early and delayed jobs, and due date cost, where the weight only depends on it's position in a sequence (i.e., a position-dependent weight). Optimal properties of the problem are given and then the polynomial time algorithm is proposed to obtain the optimal solution.

Keywords: scheduling; positional-dependent weight; due date assignment; lateness; past-sequence-dependent setup times

1. Introduction

Scheduling models with setup times are widely used in manufacture and operational processes (see Allahverdi et al. [1] and Allahverdi [2]). Koulamas and Kyparisis [3,4] and Biskup and Herrmann [5] investigated single-machine scheduling with past-sequence-dependent setup times ($\widetilde{psdst}$). They showed that several regular objective function minimizations remain polynomially solvable. Wang [6] and Wang and Li [7] examined single-machine problems with learning effects and $\widetilde{psdst}$. Hsu et al. [8] studied unrelated parallel machine scheduling problems with learning effects and $\widetilde{psdst}$. They proved that the total completion time minimization remains polynomially solvable. Cheng et al. [9] investigated scheduling problems with $\widetilde{psdst}$ and deterioration effects in a single machine. Huang et al. [10] and Wang and Wang [11] studied scheduling jobs with $\widetilde{psdst}$, learning and deterioration effects. They showed that the single-machine makespan and the sum of the αth ($\alpha > 0$) power of job completion times minimizations remain polynomially solvable. Wang et al. [12] dealt with scheduling with $\widetilde{psdst}$ and deterioration effects. Under job rejection, they showed that the the sum of scheduling cost and rejection cost minimization can be solved in polynomial time.

In the real production scheduling, the jobs often have due dates (see Gordon et al. [13,14] and the recent survey papers Rolim and Nagano [15], and Sterna [16]). Recently, Wang [17] and Wang et al. [18] studied single-machine scheduling problems with $\widetilde{psdst}$ and due-date assignment. Under common, slack and different due-date assignment methods, Wang [17] proved that the linear weighted sum of earliness-tardiness, number of early and delayed jobs, and due date penalty minimization can be solved in polynomial time. Under common and slack due date assignment methods, Wang et al. [18] showed that the weighted sum of earliness, tardiness and due date minimization can be solved in polynomial time, where the weights are position-dependent weights. The real application of the position-dependent weights can be found in production services and resource utilization (see Brucker [19], Liu et al. [20] and Jiang et al. [21]). Hence, it would be interesting to investigate due date assignment scheduling with $\widetilde{psdst}$ and position-dependent weights. The purpose of this article is to determine the optimal due dates and job sequence to minimize the weight sum of generalized earliness-tardiness penalties, where the weights are position-dependent weights. The contributions of this study are given as follows:

• We focus on the due date assignment single-machine scheduling problems with $\widetilde{psdst}$ and position-dependent weights;

• We provide an analysis for the non-regular objective function (including earliness, tardiness, number of early and delayed jobs, and due date cost);

• We derive the structural properties of the position-dependent weights and show that three due date assignments can be solved in polynomial time, respectively.

The problem formulation is described in Section 2. Three due-date assignments are discussed in Section 3. An example is presented in Section 4. In Section 5, the conclusions are given.

2. Problem definition

The symbols used throughout the article are introduced in Table 1.

Suppose there are N independent jobs $\widetilde{V} = \{J_1, J_2, \ldots, J_N\}$ need to be processed on a single-machine. The $\widetilde{psdst}$ setup time $s_{[l]}$ of job $J_{[l]}$ is $s_{[l]} = \beta \sum_{j=1}^{l-1} p_{[j]}$, where $\beta \geq 0$ is a normalizing constant, $s_{[1]} = 0$, and $\beta \sum_{j=1}^{l-1} p_{[j]} + p_{[l]}$ is the total processing requirement of job $J_{[l]}$. Let $L_l = C_l - d_l$ denote the lateness of job J_l, U_l (V_l) be earliness (tardiness) indicator viable of job J_l, i.e., if $C_l < d_l$, $U_l = 1$, otherwise, $U_l = 0$; if $C_l > d_l$, $V_l = 1$, otherwise, $V_l = 0$.

For the common ($\widetilde{con}$) due date assignment, $d_l = d$ ($l = 1, 2, \ldots, N$) and d is a decision variable. For the slack ($\widetilde{slk}$) due date assignment, $d_l = s_l + p_l + q$ and q is a decision variable. For the different due date ($\widetilde{dif}$) assignment, d_l is a decision variable for $l = 1, 2, \ldots, N$. The target is to determine d_l and a sequence ϱ such that is minimized.

$$M = \sum_{l=1}^{N} \left(\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]} \right), \tag{1}$$

Table 1. Symbols used in this article.

Symbol	Meaning
N	number of jobs
J_l	index of job
p_l	processing time of J_l
$\widetilde{psdst}$	past-sequence-dependent setup times
s_l	setup time of $\widetilde{psdst}$ of J_l
C_l	completion time of J_l
β	a normalizing constant
d_l	due date of J_l
d	common due date
q	common flow allowance
$[l]$	lth position in a sequence
$L_l = C_l - d_l$	lateness of J_l
U_l	earliness indicator viable of J_l
V_l	tardiness indicator viable of job J_l
ζ_l	positional-dependent weight of lateness cost
η_l (θ_l)	positional-dependent weight of earliness (tardiness) indicator viable
ϑ_l	positional-dependent weight of due date cost
ϱ	sequence of all jobs
$\widetilde{con}$ ($\widetilde{slk}, \widetilde{dif}$)	common (slack, different) due date

where $\zeta_l \geq 0$, $\eta_l \geq 0$, $\eta_l \geq 0$ and $\delta_l \geq 0$ are given positional-dependent weight constants. From Pinedo [22], the problem can be defined as:

$$1|\widetilde{psdst}, H| \sum_{l=1}^{N} \left(\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]}\right), \tag{2}$$

where $H \in \{\widetilde{con}, \widetilde{slk}, \widetilde{dif}\}$. The literature review related to the scheduling problems with $\widetilde{psdst}$ and due date assignment is given in Table 2. For a given sequence $\varrho = (J_{[1]}, J_{[2]}, \ldots, J_{[N]})$, from (Wang [17]), we have

$$C_{[l]} = \sum_{j=1}^{l} (s_{[j]} + p_{[j]}) = \sum_{j=1}^{l} \left[1 + \beta(l - j)\right] p_{[j]}, l = 1, 2, \ldots, N. \tag{3}$$

Table 2. Problems with $\widetilde{psdst}$ and due date assignment.

Problem	Complexity	Reference
$1\vert\widetilde{psdst},\widetilde{con}\vert\sum_{l=1}^{N}\left(\tilde{\alpha}E_l+\tilde{\delta}T_l+\tilde{\eta}_lU_l+\tilde{\theta}_lV_l+\tilde{\vartheta}d\right)$	$O(N^4)$	Wang [17]
$1\vert\widetilde{psdst},\widetilde{con}\vert\sum_{l=1}^{N}\left(\tilde{\alpha}E_l+\tilde{\delta}T_l+\tilde{\vartheta}d\right)$	$O(N\log N)$	Wang [17]
$1\vert\widetilde{psdst},\widetilde{slk}\vert\sum_{l=1}^{N}\left(\tilde{\alpha}E_l+\tilde{\delta}T_l+\tilde{\eta}_lU_l+\tilde{\theta}_lV_l+\tilde{\vartheta}q\right)$	$O(N^4)$	Wang [17]
$1\vert\widetilde{psdst},\widetilde{slk}\vert\sum_{l=1}^{N}\left(\tilde{\alpha}E_l+\tilde{\delta}T_l+\tilde{\vartheta}q\right)$	$O(N\log N)$	Wang [17]
$1\vert\widetilde{psdst},\widetilde{dif}\vert\sum_{l=1}^{N}\left(\tilde{\alpha}E_l+\tilde{\delta}T_l+\tilde{\eta}_lU_l+\tilde{\theta}_lV_l+\tilde{\vartheta}d_j\right)$	$O(N\log N)$	Wang [17]
$1\vert\widetilde{psdst},\widetilde{con}\vert\sum_{l=1}^{N}\zeta_l\vert L_{[l]}\vert+\tilde{\vartheta}d$	$O(N\log N)$	Wang et al. [18]
$1\vert\widetilde{psdst},\widetilde{slk}\vert\sum_{l=1}^{N}\zeta_l\vert L_{[l]}\vert+\tilde{\vartheta}q$	$O(N\log N)$	Wang et al. [18]
$1\vert\widetilde{psdst},\widetilde{con}\vert\sum_{l=1}^{N}(\zeta_l\vert L_{[l]}\vert+\eta_lU_{[l]}+\theta_lV_{[l]}+\vartheta_ld_{[l]})$	$O(N^4)$	This paper
$1\vert\widetilde{psdst},\widetilde{con}\vert\sum_{l=1}^{N}(\zeta_l\vert L_{[l]}\vert+\vartheta_ld_{[l]})$	$O(N\log N)$	This paper
$1\vert\widetilde{psdst},\widetilde{slk}\vert\sum_{l=1}^{N}(\zeta_l\vert L_{[l]}\vert+\eta_lU_{[l]}+\theta_lV_{[l]}+\vartheta_ld_{[l]})$	$O(N^4)$	This paper
$1\vert\widetilde{psdst},\widetilde{slk}\vert\sum_{l=1}^{N}(\zeta_l\vert L_{[l]}\vert+\vartheta_ld_{[l]})$	$O(N\log N)$	This paper
$1\vert\widetilde{psdst},\widetilde{dif}\vert\sum_{l=1}^{N}(\zeta_l\vert L_{[l]}\vert+\eta_lU_{[l]}+\theta_lV_{[l]}+\vartheta_ld_{[l]})$	$O(N\log N)$	This paper

where $\tilde{\alpha},\tilde{\delta},\tilde{\vartheta}$ are given constants, $\tilde{\eta}_l$ ($\tilde{\theta}_l$) is the earliness (tardiness) penalty of job J_l, $E_l=\max\{0,d_l-C_l\}$ ($T_l=\max\{0,C_l-d_l\}$) is the earliness (tardiness) of job J_l.

3. Main results

Lemma 1. *For $1|\widetilde{psdst},H|\sum_{l=1}^{N}(\zeta_l|L_{[l]}|+\eta_lU_{[l]}+\theta_lV_{[l]}+\vartheta_ld_{[l]})$ ($H\in\{\widetilde{con},\widetilde{slk},\widetilde{dif}\}$), an optimal sequence exists such that the first job is processed at time zero and contains no machine idle time.*

Proof. The result is obvious (see Brucker [19] and Liu et al. [20]).

3.1. The $1|\widetilde{psdst},\widetilde{con}|\sum_{l=1}^{N}(\zeta_l|L_{[l]}|+\eta_lU_{[l]}+\theta_lV_{[l]}+\vartheta_ld_{[l]})$

Lemma 2. *For any given sequence ϱ, the optimal d is equal to the completion time of some job, i.e., $d=C_{[a]}$, $a=1,2,\ldots,N$.*

Proof. For any given sequence $\varrho=(J_{[1]},J_{[2]},\ldots,J_{[N]})$, suppose that d is not equal to the completion time of some job, i.e., $C_{[a]}<d<C_{[a+1]}$, $0\le a<n$, $C_{[0]}=0$, we have

$$M=\sum_{l=1}^{a}\zeta_l(d-C_{[l]})+\sum_{l=a+1}^{N}\zeta_l(C_{[l]}-d)+\sum_{j=1}^{a}\eta_l+\sum_{j=a+1}^{n}\theta_l+\sum_{l=1}^{N}d\vartheta_l.$$

(i) When $d=C_{[a]}$, we have

$$M_1=\sum_{l=1}^{a}\zeta_l(C_{[a]}-C_{[l]})+\sum_{l=a+1}^{N}\zeta_l(C_{[l]}-C_{[a]})+\sum_{l=1}^{a-1}\eta_l+\sum_{l=a+1}^{n}\theta_l+\sum_{l=1}^{N}C_{[a]}\vartheta_l.$$

(ii) When $d=C_{[a+1]}$, we have

$$M_2=\sum_{l=1}^{a}\zeta_l(C_{[a+1]}-C_{[l]})+\sum_{l=a+1}^{N}\zeta_l(C_{[l]}-C_{[a+1]})+\sum_{l=1}^{a}\eta_l+\sum_{l=a+2}^{n}\theta_l+\sum_{l=1}^{N}C_{[a+1]}\vartheta_l,$$

$$\begin{aligned} M - M_1 &= \sum_{l=1}^{a} \zeta_l (d - C_{[a]}) - \sum_{l=a+1}^{N} \zeta_l (d - C_{[a]}) + \eta_a + \sum_{l=1}^{N} \vartheta_l (d - C_{[a]}) \\ &= \left(\sum_{l=1}^{a} \zeta_l - \sum_{l=a+1}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \right) (d - C_{[a]}) + \eta_a \end{aligned}$$

and

$$\begin{aligned} M - M_2 &= \sum_{l=1}^{a} \zeta_l (d - C_{[a+1]}) - \sum_{l=a+1}^{N} \zeta_l (d - C_{[a+1]}) + \theta_{a+1} + \sum_{l=1}^{N} \vartheta_l (d - C_{[a+1]}) \\ &= \left(\sum_{l=1}^{a} \zeta_l - \sum_{l=a+1}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \right) (d - C_{[a+1]}) + \theta_{a+1}. \end{aligned}$$

If $\sum_{l=1}^{a} \zeta_l - \sum_{l=a+1}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \geq 0$ and $C_{[a]} < d < C_{[a+1]}$, then $M - M_1 \geq 0$; If $\sum_{l=1}^{a} \zeta_l - \sum_{l=a+1}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \leq 0$ and $C_{[a]} < d < C_{[a+1]}$, then $M - M_2 \geq 0$. Therefore, d is the completion time of some job.

Lemma 3. *For any given sequence $\varrho = (J_{[1]}, J_{[2]}, \ldots, J_{[N]})$, if $\theta_l = \vartheta_l = 0$ ($l = 1, 2, \ldots N$), there exists an optimal common due date $d = C_{[a]}$, where a is determined by*

$$\sum_{l=1}^{a-1} \zeta_l - \sum_{l=a}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \leq 0 \tag{4}$$

and

$$\sum_{l=1}^{a} \zeta_l - \sum_{l=a+1}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \geq 0. \tag{5}$$

Proof. From Lemma 2, when $d = C_{[a]}$, we have

$$M = \sum_{l=1}^{a-1} \zeta_l (C_{[a]} - C_{[l]}) + \sum_{l=a+1}^{N} \zeta_l (C_{[l]} - C_{[a]}) + \sum_{l=1}^{N} C_{[a]} \vartheta_l.$$

(i) When d reduces ε (i.e., $d = C_{[a]} - \varepsilon$), we have

$$M' = \sum_{l=1}^{a-1} \zeta_l (C_{[a]} - \varepsilon - C_{[l]}) + \sum_{l=a}^{N} \zeta_l (C_{[l]} - C_{[a]} + \varepsilon) + \sum_{l=1}^{N} (C_{[a]} - \varepsilon) \vartheta_l.$$

(ii) When d increases ε (i.e., $d = C_{[a]} + \varepsilon$), we have

$$M'' = \sum_{l=1}^{a} \zeta_l (C_{[a]} + \varepsilon - C_{[l]}) + \sum_{l=a+1}^{N} \zeta_l (C_{[l]} - C_{[a]} - \varepsilon) + \sum_{l=1}^{N} (C_{[a]} + \varepsilon) \vartheta_l.$$

Hence, we have

$$M - M' = \varepsilon \left(\sum_{l=1}^{a-1} \zeta_l - \sum_{l=a}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \right) \leq 0$$

$$M - M'' = -\varepsilon \left(\sum_{l=1}^{a} \zeta_l - \sum_{l=a+1}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \right) \leq 0,$$

i.e., a is determined by $\sum_{l=1}^{a-1}\zeta_l - \sum_{l=a}^{N}\zeta_l + \sum_{l=1}^{N}\vartheta_l \le 0$ and $\sum_{l=1}^{a}\zeta_l - \sum_{l=a+1}^{N}\zeta_l + \sum_{l=1}^{N}\vartheta_l \ge 0$.
From Lemma 2, if $d = C_{[a]}$, the objective function is:

$$\begin{aligned}
M &= \sum_{l=1}^{N}(\zeta_l|L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + d\vartheta_l) \\
&= \sum_{l=1}^{a-1}\zeta_l(C_{[a]} - C_{[l]}) + \sum_{l=a+1}^{N}\zeta_l(C_{[l]} - C_{[a]}) + \sum_{l=1}^{a-1}\eta_l + \sum_{l=a+1}^{N}\theta_l + \sum_{l=1}^{N}C_{[a]}\vartheta_l \\
&= \sum_{l=1}^{a-1}\zeta_l\left\{\sum_{j=1}^{a}[1+\beta(a-j)]\,p_{[j]} - \sum_{j=1}^{l}[1+\beta(l-j)]\,p_{[j]}\right\} \\
&\quad + \sum_{l=a+1}^{N}\zeta_l\left\{\sum_{j=1}^{l}[1+\beta(l-j)]\,p_{[j]} - \sum_{j=1}^{a}[1+\beta(a-j)]\,p_{[j]}\right\} \\
&\quad + \sum_{l=1}^{a-1}\eta_l + \sum_{l=a+1}^{N}\theta_l + \sum_{l=1}^{N}\vartheta_l\left\{\sum_{j=1}^{a}[1+\beta(a-j)]\,p_{[j]}\right\} \\
&= \sum_{l=1}^{N}\Psi_l p_{[l]} + \sum_{l=1}^{a-1}\eta_l + \sum_{l=a+1}^{N}\theta_l, \qquad (6)
\end{aligned}$$

where

$$\Psi_l = \begin{cases}
\beta(a-1)\zeta_1 + \beta(a-2)\zeta_2 + \beta(a-3)\zeta_3 + \ldots + \beta\zeta_{a-1} & \\
+\beta\zeta_{a+1} + 2\beta\zeta_{a+2} + \ldots + \beta(N-a)\zeta_N + [1+\beta(a-1)]\sum_{j=1}^{N}\vartheta_j, & l = 1, \\
(1+\beta(a-2))\zeta_1 + \beta(a-2)\zeta_2 + \beta(a-3)\zeta_3 + \ldots + \beta\zeta_{a-1} & \\
+\beta\zeta_{a+1} + 2\beta\zeta_{a+2} + \ldots + \beta(N-a)\zeta_N + [1+\beta(a-2)]\sum_{j=1}^{N}\vartheta_j, & l = 2, \\
(1+\beta(a-3))(\zeta_1+\zeta_2) + \beta(a-3)\zeta_3 \ldots + \beta\zeta_{a-1} & \\
+\beta\zeta_{a+1} + 2\beta\zeta_{a+2} + \ldots + \beta(N-a)\zeta_N + [1+\beta(a-3)]\sum_{j=1}^{N}\vartheta_j, & l = 3, \\
\ldots & \ldots \\
(1+\beta)(\zeta_1+\zeta_2+\ldots+\zeta_{a-2}) + \beta\zeta_{a-1} & \\
+\beta\zeta_{a+1} + 2\beta\zeta_{a+2} + \ldots + \beta(N-a)\zeta_N + (1+\beta)\sum_{j=1}^{N}\vartheta_j, & l = a-1, \\
\zeta_1 + \zeta_2 + \ldots + \zeta_{a-1} & \\
+\beta\zeta_{a+1} + 2\beta\zeta_{a+2} + \ldots + \beta(N-a)\zeta_N + \sum_{j=1}^{N}\vartheta_j, & l = a, \\
\zeta_{a+1} + (1+\beta)\zeta_{a+2} + (1+2\beta)\zeta_{a+3} + \ldots + (1+\beta(N-a-1))\zeta_N, & l = a+1, \\
\zeta_{a+2} + (1+\beta)\zeta_{a+3} + (1+2\beta)\zeta_{a+4} + \ldots + (1+\beta(N-a-2))\zeta_N, & l = a+2, \\
\ldots & \ldots \\
\zeta_{N-1} + (1+\beta)\zeta_N, & N-1, \\
\zeta_N, & N.
\end{cases} \qquad (7)$$

Let $x_{l,r} = 1$ if J_l is placed in rth position, and $x_{l,r} = 0$; otherwise. From Eq (6), the optimal sequence of $1|\widetilde{psdst}, \widetilde{con}|\sum_{l=1}^{N}(\zeta_l|L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$ can be formulated as the following assignment problem:

$$\text{Min} \sum_{l=1}^{N}\sum_{r=1}^{N}\Theta_{l,r}x_{l,r} \qquad (8)$$

$$s.t.\begin{cases} \sum_{h=1}^{N} x_{l,r} = 1, & r = 1,2,...,N, \\ \sum_{r=1}^{N} x_{l,r} = 1, & l = 1,2,...,N, \\ x_{l,r} = 0 \text{ or } 1, \end{cases} \tag{9}$$

where

$$\Theta_{l,r} = \begin{cases} \Psi_r p_l + \eta_r, & r = 1,2,...,a-1, \\ \Psi_r p_l, & r = a, \\ \Psi_r p_l + \theta_r, & r = a+1, a+2, ..., N, \end{cases} \tag{10}$$

and Ψ_r is given by Eq (7).

Based on the above analysis, to solve $1|\widetilde{psdst}, \widetilde{con}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$, Algorithm 1 was summarized as follows:

Algorithm 1

Require: β, $p_l, \zeta_l, \eta_l, \theta_l, \vartheta_l$ for $1 \le l \le N$.
Ensure: An optimal sequence ϱ^*, optimal common due date d^*.
Step 1. For each a ($a = 1, 2, \ldots, N$), calculate Ψ_r (see Eq (7)) and $\Theta_{l,r}$ (see Eq (10)), to solve the assignment problem (8)–(10), a suboptimal sequence $\varrho(a)$ and objective function value $M(a)$ can be obtained.
Step 2. The (global) optimal sequence (i.e., ϱ^*) is the one with the minimum value

$$M^* = \min\{M(a)|a = 1, 2, \ldots, N\}.$$

Step 3. Set $d^* = C_{[a]}$.

Theorem 1. *The* $1|\widetilde{psdst}, \widetilde{con}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$ *can be solved by Algorithm 1, and time complexity was* $O(N^4)$.

Proof. The correctness of **Algorithm 1** follows the above analysis. In Step 1, for each a, solving the assignment problem needs $O(N^3)$ time; Steps 2 and 3 require $O(N)$ time; $a = 1, 2, \ldots, N$. Therefore, the total time complexity was $O(N^4)$.

Lemma 4. *(Hardy et al. [23]). "The sum of products* $\sum_{l=1}^{N} a_l b_l$ *is minimized if sequence* $a_1, a_2, \ldots, a_N$ *is ordered nondecreasingly and sequence* $b_1, b_2, \ldots, b_N$ *is ordered nonincreasingly or vice versa."*

If $\eta_l = \theta_l = 0$, a can be determined by Lemma 3 (see Eqs (4) and (5)), We

$$M = \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \vartheta_l d_{[l]}) = \sum_{l=1}^{N} \Psi_l p_{[l]}, \tag{11}$$

where Ω_j is given by Eq (6).

Equation (11) can be minimized by Lemma 4 in $O(N \log N)$ time (i.e., $a_l = \Psi_l, b_l = p_l$), hence, to solve $1|\widetilde{psdst}, \widetilde{con}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \vartheta_l d_{[l]})$, the following algorithm was summarized as follows:

Algorithm 2

Require: β, $p_l, \zeta_l, \vartheta_l$ for $1 \le l \le N$.
Ensure: An optimal sequence ϱ^*, optimal common due date d^*.
Step 1. Calculate a by Lemma 3 (see Eqs (4) and (5)).
Step 2. By using Lemma 4 (let $a_l = \Psi_l, b_l = p_l$) to determine the optimal job sequence (i.e., ϱ^*), i.e., place the largest p_l at the smallest Ψ_l position, place the second largest p_l at the second smallest Ψ_l position, etc.
Step 3. Set $d^* = C_{[a]}$.

Theorem 2. *The* $1|\widetilde{psdst}, \widetilde{con}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \vartheta_l d_{[l]})$ *can be solved by Algorithm 2, and time complexity was* $O(N \log N)$.

3.2. The $1|\widetilde{psdst}, \widetilde{slk}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$

Similarly, we have

Lemma 5. *For any given sequence* ϱ *of* $1|\widetilde{psdst}, \widetilde{slk}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$*, an optimal sequence exists in which*
1) $C_{[l]} \le d_{[l]}$ *implies* $C_{[l-1]} \le d_{[l-1]}$ *and* $C_{[l]} \ge d_{[l]}$ *implies* $C_{[l+1]} \ge d_{[l+1]}$ *for all l;*
2) the optimal q is equal to the completion time of some job, i.e., $q = C_{[b-1]}$, $b = 1, 2, \ldots, N$.

Lemma 6. *For any given sequence* $\varrho = (J_{[1]}, J_{[2]}, \ldots, J_{[N]})$*, if* $\theta_l = \vartheta_l = 0$ *(*$l = 1, 2, \ldots N$*), there exists an optimal common due date* $q = C_{[b-1]}$*, where b is determined by*

$$\sum_{l=1}^{b-1} \zeta_l - \sum_{l=b}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \le 0 \tag{12}$$

and

$$\sum_{l=1}^{b} \zeta_l - \sum_{l=b+1}^{N} \zeta_l + \sum_{l=1}^{N} \vartheta_l \ge 0. \tag{13}$$

Proof. From Lemma 5, when $q = C_{[b-1]}$, we have

$$M = \sum_{l=1}^{b-1} \zeta_l (s_{[b]} + p_{[b]} + C_{[b-1]} - C_{[l]}) + \sum_{l=b+1}^{N} \zeta_l (C_{[l]} - s_{[b]} - p_{[b]} - C_{[b-1]}) + \sum_{l=1}^{N} \vartheta_l (s_{[b]} + p_{[b]} + C_{[b-1]}).$$

(i) When q reduces ε (i.e., $q = C_{[b-1]} - \varepsilon$), we have

$$M' = \sum_{l=1}^{b-1} \zeta_l (s_{[b]} + p_{[b]} + C_{[b-1]} - \varepsilon - C_{[l]}) + \sum_{l=b}^{N} \zeta_l (C_{[l]} - s_{[b]} - p_{[b]} - C_{[b-1]} + \varepsilon) + \sum_{l=1}^{N} (s_{[b]} + p_{[b]} + C_{[b-1]} - \varepsilon) \vartheta_l.$$

(ii) When q increases ε (i.e., $q = C_{[b-1]} + \varepsilon$), we have

$$M'' = \sum_{l=1}^{b} \zeta_l(s_{[b]}+p_{[b]}+C_{[b-1]}+\varepsilon-C_{[l]})+\sum_{l=b+1}^{N} \zeta_l(C_{[l]}-s_{[b]}-p_{[b]}-C_{[b-1]}-\varepsilon)+\sum_{l=1}^{N}(s_{[b]}+p_{[b]}+C_{[b-1]}+\varepsilon)\vartheta_l.$$

Hence, we have

$$M - M' = \varepsilon\left(\sum_{l=1}^{b-1}\zeta_l - \sum_{l=b}^{N}\zeta_l + \sum_{l=1}^{N}\vartheta_l\right) \leq 0$$

$$M - M'' = -\varepsilon\left(\sum_{l=1}^{b}\zeta_l - \sum_{l=b+1}^{N}\zeta_l + \sum_{l=1}^{N}\vartheta_l\right) \leq 0,$$

i.e., b is determined by $\sum_{l=1}^{b-1}\zeta_l - \sum_{l=b}^{N}\zeta_l + \sum_{l=1}^{N}\vartheta_l \leq 0$ and $\sum_{l=1}^{b}\zeta_l - \sum_{l=b+1}^{N}\zeta_l + \sum_{l=1}^{N}\vartheta_l \geq 0$.

From Lemma 5, if $q = C_{[b-1]}$ (i.e., $d_{[l]} = s_{[l]} + p_{[l]} + C_{[b-1]}$), the objective function is:

$$\begin{aligned}
M &= \sum_{l=1}^{N}(\zeta_l|L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]}) \\
&= \sum_{l=1}^{b-1}\zeta_l(s_{[l]} + p_{[l]} + C_{[b-1]} - C_{[l]}) + \sum_{l=b+1}^{N}\zeta_l(C_{[l]} - s_{[l]} - p_{[l]} - C_{[b-1]}) \\
&\quad + \sum_{l=1}^{b-1}\eta_l + \sum_{l=b+1}^{N}\theta_l + \sum_{l=1}^{N}(s_{[l]} + p_{[l]} + C_{[b-1]})\vartheta_l \\
&= \sum_{l=1}^{b-1}\zeta_l(C_{[b-1]} - C_{[l-1]}) + \sum_{l=b+1}^{N}\zeta_l(C_{[l-1]} - C_{[b-1]}) + \sum_{l=1}^{b-1}\eta_l + \sum_{l=b+1}^{N}\theta_l \\
&\quad + \sum_{l=1}^{N}(s_{[l]} + p_{[l]})\vartheta_l + \sum_{l=1}^{N}C_{[b-1]}\vartheta_l \\
&= \sum_{l=1}^{b-1}\zeta_l\left\{\sum_{j=1}^{b-1}\left[1 + \beta(b-1-j)\right]p_{[j]} - \sum_{j=1}^{l-1}\left[1 + \beta(l-1-j)\right]p_{[j]}\right\} \\
&\quad + \sum_{l=b+1}^{N}\zeta_l\left\{\sum_{j=1}^{l-1}\left[1 + \beta(l-1-j)\right]p_{[j]} - \sum_{j=1}^{b-1}\left[1 + \beta(b-1-j)\right]p_{[j]}\right\} \\
&\quad + \sum_{l=1}^{b-1}\eta_l + \sum_{l=b+1}^{N}\theta_l + \sum_{l=1}^{N}\left(\beta\sum_{j=1}^{l-1}p_{[j]} + p_{[l]}\right)\vartheta_l + \sum_{l=1}^{N}\vartheta_l\left\{\sum_{j=1}^{b-1}\left[1 + \beta(b-1-j)\right]p_{[j]}\right\} \\
&= \sum_{l=1}^{N}\Phi_l p_{[l]} + \sum_{l=1}^{b-1}\eta_l + \sum_{l=b+1}^{N}\theta_l, \qquad (14)
\end{aligned}$$

where

$$\Phi_l = \begin{cases} (1+\beta(b-2))\zeta_1 + \beta(b-2)\zeta_2 + \beta(b-3)\zeta_3 + \ldots + \beta\zeta_{b-1} & \\ +\beta\zeta_{b+1} + 2\beta\zeta_{b+2} + \ldots + \beta(N-b)\zeta_N + [1+\beta(b-2)]\sum_{j=1}^{N}\vartheta_j & \\ +\vartheta_1 + \beta\sum_{j=2}^{N}\vartheta_j, & l=1, \\ (1+\beta(b-3))(\zeta_1+\zeta_2) + \beta(b-3)\zeta_3 + \beta(b-4)\zeta_4 + \ldots + \beta\zeta_{b-1} & \\ +\beta\zeta_{b+1} + 2\beta\zeta_{b+2} + \ldots + \beta(N-b)\zeta_N + [1+\beta(b-3)]\sum_{j=1}^{N}\vartheta_j & \\ +\vartheta_2 + \beta\sum_{j=3}^{N}\vartheta_j, & l=2, \\ (1+\beta(b-4))(\zeta_1+\zeta_2+\zeta_3) + \beta(b-4)\zeta_4 + \ldots + \beta\zeta_{b-1} & \\ +\beta\zeta_{b+1} + 2\beta\zeta_{b+2} + \ldots + \beta(N-b)\zeta_N + [1+\beta(b-4)]\sum_{j=1}^{N}\vartheta_j & \\ +\vartheta_3 + \beta\sum_{j=4}^{N}\vartheta_j, & l=3, \\ \ldots & \ldots \\ (1+\beta)(\zeta_1+\zeta_2+\ldots+\zeta_{b-2}) + \beta\zeta_{b-1} & \\ +\beta\zeta_{b+1} + 2\beta\zeta_{b+2} + \ldots + \beta(N-b)\zeta_N + (1+\beta)\sum_{j=1}^{N}\vartheta_j & \\ +\vartheta_{b-2} + \beta\sum_{j=b-1}^{N}\vartheta_j, & l=b-2, \\ \zeta_1+\zeta_2+\ldots+\zeta_{b-1} & \\ +\beta\zeta_{b+1} + 2\beta\zeta_{b+2} + \ldots + \beta(N-b)\zeta_N + \sum_{j=1}^{N}\vartheta_j + \vartheta_{b-1} + \beta\sum_{j=b}^{N}\vartheta_j, & l=b-1, \\ \zeta_{b+1} + (1+\beta)\zeta_{b+2} + (1+2\beta)\zeta_{b+3} + \ldots + (1+\beta(N-b-1))\zeta_N & \\ +\vartheta_b + \beta\sum_{j=b+1}^{N}\vartheta_j, & l=b, \\ \zeta_{b+2} + (1+\beta)\zeta_{b+3} + (1+2\beta)\zeta_{b+4} + \ldots + (1+\beta(N-b-2))\zeta_N & \\ +\vartheta_{b+1} + \beta\sum_{j=b+2}^{N}\vartheta_j, & l=b+1, \\ \ldots & \ldots \\ \zeta_N + \vartheta_{N-1} + \beta\vartheta_N, & N-1, \\ \vartheta_N, & N. \end{cases} \tag{15}$$

Similarly, from Eq (14), the optimal sequence of $1|\widetilde{psdst}, \widetilde{slk}|\sum_{l=1}^{N}(\zeta_l|L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$ can be obtained as follows:

$$\text{Min} \sum_{l=1}^{N}\sum_{r=1}^{N} \Xi_{l,r} x_{l,r} \tag{16}$$

$$s.t. \begin{cases} \sum_{h=1}^{N} x_{l,r} = 1, & r = 1,2,...,N, \\ \sum_{r=1}^{N} x_{l,r} = 1, & l = 1,2,...,N, \\ x_{l,r} = 0 \text{ or } 1, & \end{cases} \tag{17}$$

where

$$\Xi_{l,r} = \begin{cases} \Phi_r p_l + \eta_r, & r = 1,2,...,b-1, \\ \Phi_r p_l, & r = b, \\ \Phi_r p_l + \theta_r, & r = b+1, b+2, ..., N, \end{cases} \tag{18}$$

and Φ_r is given by (15).

Similarly, to solve $1|\widetilde{psdst}, \widetilde{slk}|\sum_{l=1}^{N}(\zeta_l|L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$, the following algorithm can be proposed:

Algorithm 3

Require: β, $p_l, \zeta_l, \eta_l, \theta_l, \vartheta_l$ for $1 \leq l \leq N$.
Ensure: An optimal sequence ϱ^*, optimal common flow allowance q^*.
Step 1. For each b ($b = 1, 2, \dots, N$), calculate Φ_r (see Eq (15)) and $\Xi_{l,r}$ (see Eq (18)), to solve the assignment problem (16)–(18), a suboptimal sequence $\varrho(b)$ and objective function value $M(b)$ can be obtained.
Step 2. The (global) optimal sequence (i.e., ϱ^*) is the one with the minimum value

$$M^* = \min \{M(b) | b = 1, 2, \dots, N\}.$$

Step 3. Set $q^* = C_{[b-1]}$.

Theorem 3. *The* $1|\widetilde{psdst}, \widetilde{slk}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$ *can be solved by Algorithm 3, and time complexity was* $O(N^4)$.

Similarly, if $\eta_l = \theta_l = 0$, we have

Theorem 4. *The problem* $1|\widetilde{psdst}, \widetilde{slk}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \vartheta_l d_{[l]})$ *can be solved in* $O(N \log N)$ *time.*

3.3. The $1|\widetilde{psdst}, \widetilde{dif}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$

Lemma 7. *For a given sequence* π *of* $1|\widetilde{psdst}, \widetilde{dif}| \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$*, an optimal solution exists such that* $d_{[l]} \leq C_{[l]}$.

Proof. For a given sequence ϱ, the objective function for job $J_{[l]}$ was:

$$M_{[l]} = \zeta_l |C_{[l]} - d_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]}. \quad (19)$$

If $d_{[l]} > C_{[l]}$ (i.e., the job $J_{[l]}$ is an early job), it follows that

$$M_{[l]} = \zeta_l (d_{[l]} - C_{[l]}) + \eta_l U_{[l]} + \vartheta_l d_{[l]}.$$

Move $d_{[l]}$ to the left such that $d_{[l]} = C_{[l]}$, we have

$$M'_{[l]} = \vartheta_l d_{[l]} = \vartheta_l C_{[l]} < M_{[l]},$$

therefore, $d_{[l]} \leq C_{[l]}$.

Lemma 8. *For a given sequence* ϱ*, if* $\vartheta_l \geq \zeta_l$*,* $d_{[l]} = 0$*; otherwise* $d_{[l]} = C_{[l]}$ *(*$l = 1, 2, \dots, N$*).*

Proof. For a given sequence ϱ, from Lemma 7, we have $d_{[l]} \leq C_{[l]}$ and

$$M_{[l]} = \zeta_l (C_{[l]} - d_{[l]}) + \theta_l V_{[l]} + \vartheta_l d_{[l]} = \zeta_l C_{[l]} + \theta_l + (\vartheta_l - \zeta_l) d_{[l]}. \quad (20)$$

From Eq (20), when $\vartheta_l - \zeta_l \geq 0$, $d_{[l]}$ was equal to 0; otherwise, then $d_{[l]}$ was equal to $C_{[l]}$.

From Lemma 8, if $\vartheta_l \geq \zeta_l$, we have $d_{[l]} = 0$ and

$$M = \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]}) = \sum_{l=1}^{N} \zeta_l C_{[l]} + \sum_{l=1}^{N} \theta_l. \tag{21}$$

If $\vartheta_l < \zeta_l$, we have $d_{[l]} = C_{[l]}$ and

$$M = \sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]}) = \sum_{l=1}^{N} \vartheta_l C_{[l]}. \tag{22}$$

From Eqs (21) and (22), minimizing $\sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$ is equal to minimizing the expression

$$M = \sum_{l=1}^{N} \min\{\vartheta_l, \zeta_l\} C_{[l]} = \sum_{l=1}^{N} \min\{\vartheta_l, \zeta_l\} \sum_{j=1}^{l} [1 + \beta(l - j)]\, p_{[j]} = \sum_{l=1}^{N} \Upsilon_l p_{[l]}, \tag{23}$$

where

$$\Upsilon_l = \begin{cases} \min\{\vartheta_1, \zeta_1\} + (1+\beta)\min\{\vartheta_2, \zeta_2\} + \ldots + (1 + (N-1)\beta)\min\{\vartheta_N, \zeta_N\}, & l = 1, \\ \min\{\vartheta_2, \zeta_2\} + (1+\beta)\min\{\vartheta_3, \zeta_3\} + \ldots + (1 + (N-2)\beta)\min\{\vartheta_N, \zeta_N\}, & l = 2, \\ \ldots & \ldots \\ \min\{\vartheta_{N-1}, \zeta_{N-1}\} + (1+\beta)\min\{\vartheta_N, \zeta_N\}, & N-1, \\ \min\{\vartheta_N, \zeta_N\}, & N, \end{cases} \tag{24}$$

i.e.,

$$\Upsilon_l = \sum_{j=l}^{N} [1 + \beta(j - l)] \min\{\vartheta_j, \zeta_j\}, \quad l = 1, 2, \ldots, N. \tag{24'}$$

Obviously, Eq (23) can be minimized by Lemma 4.

Algorithm 4

Require: $\beta, p_l, \zeta_l, \eta_l, \theta_l, \vartheta_l$ for $1 \leq l \leq N$.
Ensure: An optimal sequence ϱ^*, optimal common due date d_l^*.
Step 1. By using Lemma 4 (let $a_l = \Upsilon_l, b_l = p_l$) to determine the optimal job sequence (i.e., ϱ^*), i.e., place the largest p_l at the smallest Υ_l position, place the second largest p_l at the second smallest Υ_l position, etc.
Step 2. If $\vartheta_l \geq \zeta_l$, $d_{[l]}^* = 0$; otherwise $d_{[l]}^* = C_{[l]}$ $(l = 1, 2, \ldots, N)$.

Theorem 5. *The* $1|\widetilde{psdst}, \widetilde{dif}|\sum_{l=1}^{N} (\zeta_l |L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]} + \vartheta_l d_{[l]})$ *can be solved by Algorithm 4, and time complexity was* $O(N \log N)$.

4. Numerical example

We present an example to illustrate the calculation steps and results of the three due date assignments.

Example 1. Consider a 6-job problem, where $\beta = 1$, $p_1 = 7$, $p_2 = 9$, $p_3 = 4$, $p_4 = 6$, $p_5 = 8$, $p_6 = 5$, $\zeta_l, \eta_l, \theta_l$ and ϑ_l are given in Table 3.

Table 3. Values of $\zeta_l, \eta_l, \theta_l$ and ϑ_l.

	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$
ζ_l	6	8	14	3	15	7
η_l	8	4	9	10	12	5
θ_l	10	8	6	5	14	17
ϑ_l	12	16	7	13	8	9

From Algorithm 1, For the $\widetilde{con}$ assignment, if $a = 1$, the values $\Psi_1 = 205, \Psi_2 = 140, \Psi_3 = 93, \Psi_4 = 54, \Psi_5 = 29, \Psi_6 = 7$, (see Eqs (7) or (7')) and $\Theta_{l,r}$ (see Eq (10)) are given in Table 4. By the assignment problems (8)–(10), the sequence is $\varrho(1) = (J_3, J_6, J_4, J_1, J_5, J_2)$ and $M(1) = 2801$. Similarly, for $a = 2, 3, 4, 5, 6$, the results are shown in Table 5. From Table 5, the optimal sequence is $\varrho^* = (J_3, J_6, J_4, J_1, J_5, J_2)$, $M^* = 2801$ and $d^* = C_{[2]} = 14$.

Table 4. Values $\Theta_{l,r}$ for $a = 1$.

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
J_1	1435	988	657	**383**	217	66
J_2	1845	1268	843	491	275	**80**
J_3	**820**	568	378	221	130	45
J_4	1230	848	**564**	329	188	59
J_5	1640	1128	750	437	**246**	73
J_6	1025	**708**	471	275	159	52

Table 5. Results for $\widetilde{con}$.

a	$\varrho(a)$	$M(a)$
1	$(J_3, J_6, J_4, J_1, J_5, J_2)$	2801
2	$(J_3, J_6, J_4, J_1, J_5, J_2)$	3017
3	$(J_3, J_6, J_4, J_1, J_5, J_2)$	3615
4	$(J_3, J_6, J_4, J_1, J_5, J_2)$	5335
5	$(J_3, J_6, J_4, J_1, J_5, J_2)$	7451
6	$(J_3, J_6, J_4, J_1, J_5, J_2)$	11,382

For the $\widetilde{slk}$ assignment, the results are shown in Table 6. From Table 6, the optimal sequence is $\varrho^* = (J_3, J_6, J_4, J_1, J_5, J_2)$, $M^* = 2832$ and $q^* = C_{[0]} = 0$.

Table 6. Results for $\widetilde{slk}$.

b	$\varrho(b)$	$M(b)$
1	$(J_3, J_6, J_4, J_1, J_5, J_2)$	2832
2	$(J_3, J_6, J_4, J_1, J_5, J_2)$	2928
3	$(J_3, J_6, J_4, J_1, J_5, J_2)$	3286
4	$(J_3, J_6, J_4, J_1, J_5, J_2)$	4310
5	$(J_3, J_6, J_4, J_1, J_5, J_2)$	5934
6	$(J_3, J_6, J_4, J_1, J_5, J_2)$	9049

For the $\widetilde{dif}$ assignment, $\Upsilon_1 = 137, \Upsilon_2 = 98, \Upsilon_3 = 65, \Upsilon_4 = 40, \Upsilon_5 = 22, \Upsilon_6 = 7$, the optimal sequence is $\varrho^* = (J_3, J_6, J_4, J_1, J_5, J_2)$, $M^* = 1987$, $d_3^* = 0$, $d_6^* = 0$, $d_4^* = C_4 = 28$, $d_1^* = 0$, $d_5^* = C_5 = 80$ and $d_2^* = 0$.

5. Conclusions

Under $\widetilde{con}$, $\widetilde{slk}$ and $\widetilde{dif}$ assignments, the single-machine scheduling problem with $\widetilde{psdst}$ and position-dependent weights had been addressed. The goal was to minimize the weighted sum of lateness, number of early and delayed jobs and due date cost. Here we showed that the problem remains polynomially solvable. If the due dates are given, from Brucker [19], the problem $1|\widetilde{psdst}|\sum_{l=1}^{N}(\zeta_l|L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]})$ is NP-dard. For future research, we suggest some interesting topics as follows:

1) Considering the problem $1|\widetilde{psdst}|\sum_{l=1}^{N}(\zeta_l|L_{[l]}| + \eta_l U_{[l]} + \theta_l V_{[l]})$;

2) Investigating the problem in a flow shop setting;

3) Studying the group technology problem with learning effects (deterioration effects) and/or resource allocation (see Wang et al. [24], Huang [25] and Liu and Xiong [26]);

4) Investigating scenario-dependent processing times (see Wu et al. [27] and Wu et al. [28]).

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. A. Allahverdi, C. T. Ng, T. C. E. Cheng, M. Y. Kovalyov, A survey of scheduling problems with setup times or costs, *Eur. J. Oper. Res.*, **187** (2008), 985–1032. https://doi.org/10.1016/j.ejor.2006.06.060

2. A. Allahverdi, The third comprehensive survey on scheduling problems with setup times/costs, *Eur. J. Oper. Res.*, **246** (2015), 345–378. https://doi.org/10.1016/j.ejor.2015.04.004

3. C. Koulamas, G. J. Kyparisis, Single-machine scheduling problems with past-sequence-dependent setup times, *Eur. J. Oper. Res.*, **187** (2008), 1045–1049. https://doi.org/10.1016/j.ejor.2006.03.066
4. C. Koulamas, G. J. Kyparisis, New results for single-machine scheduling with past-sequence-dependent setup times and due date-related objectives, *Eur. J. Oper. Res.*, **278** (2019), 149–159. https://doi.org/10.1016/j.ejor.2019.04.022
5. D. Biskup, J. Herrmann, Single-machine scheduling against due dates with past-sequence-dependent setup times, *Eur. J. Oper. Res.*, **191** (2008), 587–592. https://doi.org/10.1016/j.ejor.2007.08.028
6. J. B. Wang, Single-machine scheduling with past-sequence-dependent setup times and time-dependent learning effect, *Comput. Ind. Eng.*, **55** (2008), 584–591. https://doi.org/10.1016/j.cie.2008.01.017
7. J. B. Wang, J. X. Li, Single machine past-sequence-dependent setup times scheduling with general position-dependent and time-dependent learning effects, *Appl. Math. Modell.*, **35** (2011), 1388–1395. https://doi.org/10.1016/j.apm.2010.09.017
8. C. J. Hsu, W. H. Kuo, D. L. Yang, Unrelated parallel machine scheduling with past-sequence-dependent setup time and learning effects, *Appl. Math. Modell.*, **35** (2011), 1492–1496. https://doi.org/10.1016/j.apm.2010.09.026
9. T. C. E. Cheng, W. C. Lee, C. C. Wu, Single-machine scheduling with deteriorating jobs and past-sequence-dependent setup times, *Appl. Math. Modell.*, **35** (2011), 1861–1867. https://doi.org/10.1016/j.apm.2010.10.015
10. X. Huang, G. Li, Y. Huo, P. Ji, Single machine scheduling with general time-dependent deterioration, position-dependent learning and past sequence-dependent setup times, *Optim. Lett.*, **7** (2013), 1793–1804. https://doi.org/10.1007/s11590-012-0522-4
11. X. Y. Wang, J. J. Wang, Scheduling problems with past-sequence-dependent setup times and general effects of deterioration and learning, *Appl. Math. Modell.*, **37** (2013), 4905–4914. http://dx.doi.org/10.1016/j.apm.2012.09.044
12. J. B. Wang, J. X. Xu, F. Guo, M. Liu, Single-machine scheduling problems with job rejection, deterioration effects and past-sequence-dependent setup times, *Eng. Optim.*, **54** (2022), 471–486. https://doi.org/10.1080/0305215X.2021.1876041
13. V. S. Gordon, J. M. Proth, C. B. Chu, A survey of the state of-the-art of common due date assignment and scheduling research, *Eur. J. Oper. Res.*, **139** (2002), 1–25. https://doi.org/10.1016/S0377-2217(01)00181-3
14. V. S. Gordon, J. M. Proth, C. B. Chu, Due date assignment and scheduling: SLK, TWK and other due date assignment models, *Prod. Plan. Control*, **13** (2002), 117–132. https://doi.org/10.1080/09537280110069621
15. G. A. Rolim, M. S. Nagano, Structural properties and algorithms for earliness and tardiness scheduling against common due dates and windows: A review, *Comput. Ind. Eng.*, **149** (2020), 106803. https://doi.org/10.1016/j.cie.2020.106803
16. M. Sterna, Late and early work scheduling: A survey, *Omega*, **104** (2021), 102453. https://doi.org/10.1016/j.omega.2021.102453

17. W. Wang, Single-machine due-date assignment scheduling with generalized earliness-tardiness penalties including proportional setup times, *J. Appl. Math. Comput.*, **2021** (2021), 1–19. https://doi.org/10.1007/s12190-021-01555-4

18. L. Y. Wang, X. Huang, W. W. Liu, Y. B. Wu, J. B. Wang, Scheduling with position-dependent weights, due-date assignment and past-sequence-dependent setup times, *RAIRO Oper. Res.*, **55** (2021), S2747–S2758. https://doi.org/10.1051/ro/2020117

19. P. Brucker, *Scheduling Algorithms*, 3rd edition, Springer-Berlin, 2007. https://link.springer.com/book/10.1007/978-3-540-69516-5

20. W. Liu, X. Hu, X. Y. Wang, Single machine scheduling with slack due dates assignment, *Eng. Optim.*, **49** (2017), 709–717. https://doi.org/10.1080/0305215X.2016.1197611

21. C. Jiang, D. Zou, D. Bai, J. B. Wang, Proportionate flowshop scheduling with position-dependent weights, *Eng. Optim.*, **52** (2020), 37–52. https://doi.org/10.1080/0305215X.2019.1573898

22. M. Pinedo, *Scheduling theory, algorithms, and systems*, Prentice Hall, New Jersey, 2016. https://doi.org/10.1007/978-3-319-26580-3

23. G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1988. https://doi.org/10.1017/s0025557200143451

24. X. Y. Wang, Z. Zhou, X. Zhang, P. Ji, J. B. Wang, Several flow shop scheduling problems with truncated position-based learning effect, *Comput. Oper. Res.*, **40** (2013), 2906–2929. http://dx.doi.org/10.1016/j.cor.2013.07.001

25. X. Huang, Bicriterion scheduling with group technology and deterioration effect, *J. Appl. Math. Comput.*, **60** (2019), 455–464. https://doi.org/10.1007/s12190-018-01222-1

26. C. Liu, C. Xiong, Single machine resource allocation scheduling problems with deterioration effect and general positional effect, *Math. Biosci. Eng.*, **18** (2021), 2562–2578. https://doi.org/10.3934/mbe.2021130

27. C. C. Wu, D. Bai, X. Zhang, S. R. Cheng, J. C. Lin, Z. L. Wu, et al., A robust customer order scheduling problem along with scenario-dependent component processing times and due dates, *J. Manuf. Syst.*, **58** (2021), 291–305. https://doi.org/10.1016/j.jmsy.2020.12.013

28. C. C. Wu, D. Y. Bai, J. H. Chen, W. C. Lin, L. Xing, J. C. Lin, et al., Several variants of simulated annealing hyper-heuristic for a single-machine scheduling with two-scenario-based dependent processing times, *Swarm Evol. Comput.*, **60** (2021), 100765. https://doi.org/10.1016/j.swevo.2020.100765