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Research article

Existence of solutions of an impulsive integro-differential equation with a general boundary value condition

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Abstract: In this paper, we discuss the existence of solutions for a first-order nonlinear impulsive integro-differential equation with a general boundary value condition. New comparison principles are developed, and existence results for extremal solutions are obtained using the established principles and the monotone iterative technique. The results are more general than those of the periodic boundary problems, which may be widely applied in this field.

Keywords: impulsive integro-differential equation; boundary value condition; comparison result; minimal and maximal solutions

1. Introduction

In recent years, impulsive integro-differential equations with boundary conditions have attracted much attentions and been studied extensively [1–3]. We notice that the periodic and antiperiodic boundary value problems are very common, and they have a wide range of applications [4–11]. The monotone iterative technique is a common method to prove the existence of extremal solutions for impulsive integro-differential equations [4]. The monotone sequences of a linear system is developed from the upper and lower solutions, and this method can prove monotone sequences converge monotonically to the extremal solutions of the original system [12, 13]. Luo et al. [5] developed some new comparison principles and existence results of solutions for an impulsive integro-differential equation with the periodic boundary conditions. Recently, Kumar et al. [9] discussed the stability and existence of a fractional integro differential equation with the periodic boundary condition. Gou et al. [14] explored the existence of mild solutions for the periodic boundary conditions in a semilinear fractional evolution system. The existence result for the periodic boundary conditions in the phi-Laplacian impulsive differential equation can refer to [15]. Ibnelazyz et al. [11] studied the existence results for a fractional integro-differential equation with the antiperiodic boundary conditions. Ding et al. used the monotone iterative technique to discuss the existence of solutions for a class of impulsive functional differential equations with the anti-periodic boundary value condition [16]. Zuo et al. [6] studied the existence and uniqueness of solutions of an antiperiodic boundary problem in a mixed impulsive fractional integro-differential equation. However, we note that the periodic and anti-periodic boundary values are both two special conditions. For the impulsive integro-differential equation with a more general boundary condition, such as "$w(0) = \chi w(T), \chi \in R$", have not been involved by now.

Inspired by Luo and Nieto [5],in this paper, we consider the follow two-point boundary value problem (TP-BVP) for a first-order impulsive integro-differential system:

$$\begin{cases} w'(\theta) = f(\theta, w(\theta), [\Gamma w](\theta), [\delta w](\theta)) \quad \theta \in \xi' = \xi - \theta_1, \theta_2, \dots, \theta_m, \\ \triangle w(\theta_i) = I_k(w(\theta_i)) \quad i = 1, 2, \cdots, m \\ w(0) = \chi w(T) \end{cases} \tag{1.1}$$

where $\xi = [0, T], f \in C(\xi \times R^3), I_k \in C(R, R), 0 = \theta_0 < \theta_1 < \cdots \theta_m < \theta_{m+1} = T, \triangle w(\theta_k) = w(\theta_k^+) - w(\theta_k^-)$, $w(\theta_k^-)$ and $w(\theta_k^+)$ are the left and right limits of $w(\theta)$ at $\theta = \theta_k$,

$$[\Gamma w](\theta) = \int_0^\theta \Phi(\theta, s) w(s) ds, \qquad [\delta w](\theta) = \int_0^T \Psi(\theta, s) w(s) ds,$$

$\Phi \in C(D, R^+), D = \{(\theta, s) \in \xi \times \xi : \theta \geq s\}, \Psi \in C(\xi \times \xi, R^+), R^+ = [0, \infty), \chi \in R$. It should be noted that χ is an arbitrary real number and the boundary condition in Eq (1.1) is more general than the periodic or antiperiodic boundary value. Therefore, the existence result of the solution of Eq (1.1) will have a wider range of application than previous studies.

In Section 2, we establish new comparison principles. In Section 3, we discuss the existence and uniqueness of the solutions for a linear BVP. Finally, we obtain the extremal solutions for TP-BVP Eq (1.1) in Section 4.

2. Preliminaries and some basic results

Similar to previous studies [2,5,17,18], we give the follow spaces to define the solution of Eq (1.1): $LC(\xi) = \{w : \xi \to R : w\big|_{(\theta_k, \theta_{k+1}]} \in C((\theta_k, \theta_{k+1}], R),\ k = 0, 1, \cdots m;\ w(\theta_k^+)$ and $w(\theta_k^-)$ exist for $k = 1, 2, \cdots, m$ with $w(\theta_k^-) = w(\theta_k)\}$; $LC'(\xi) = \{w \in LC(\xi); w\big|_{(\theta_k, \theta_{k+1}]} \in C'((\theta_k, \theta_{k+1}], R),\ k = 0, 1, \cdots, m$; Limits $w'(\theta_k^-)$, $w'(\theta_k^+)$, $w'(0^+)$ and $w'(T^-)$ exist when $k = 1, 2, \cdots, m\}$. It is not difficult to verify that $LC(\xi)$ and $LC'(\xi)$ are both Bananch spaces with the following norms [5]:

$$||w||_{LC} = sup\{|w(\theta)|; \theta \in \xi\}, \qquad ||w||_{LC'} = ||w||_{LC} + ||w'||_{LC}.$$

Then, a function $w \in LC'(\xi)$ is a solution of Eq (1.1) when it satisfies Eq (1.1).

Now, we prove the follow key comparison lemmas.

Lemma 2.1. (New comparison principles) Suppose that $\Lambda_k > -1 (k = 1, 2, \cdots, m), \rho_1, \rho_2 \geq 0, \chi > e^{-\varepsilon T}$ and $\varepsilon > 0$, such as

$$\begin{cases} w'(\theta) - \varepsilon w(\theta) - \rho_1[\Gamma w](\theta) - \rho_2[\delta w](\theta) \geq 0 \qquad \theta \in \xi' \\ \triangle w(\theta_k) \geq \Lambda_k w(\theta_k), \qquad k = 1, 2, \cdots, m \\ w(0) \geq \chi w(T) \end{cases} \tag{2.1}$$

or

$$\begin{cases} w'(\theta) - \varepsilon w(\theta) - \rho_1[\Gamma w](\theta) - \rho_2[\delta w](\theta) - b_w(\theta) \geq 0 & \theta \in \xi' \\ \triangle w(\theta_k) \geq \Lambda_k w(\theta_k) + l_{wk}, & k = 1, 2, \cdots, m \\ w(0) < \chi w(T) \end{cases} \tag{2.2}$$

$l_{wk} = \Lambda_k g(\theta_k) - \Delta g(\theta_k)$ and $b_w(\theta) = -g'(\theta) + \varepsilon g(\theta) + \rho_1[\Gamma g](\theta) + \rho_2[\delta g](\theta)$. Where $g \geq 0$ is a function in space $LC'(\xi)$, which satisfies $g(0) - \chi g(T) \geq \chi w(T) - w(0) > 0$.

We define $\overline{\overline{\Lambda_k}} = \min\{\Lambda_k, 0\}$ for $k = 1, 2, ..., m$, and

$$\overline{\pi}(\theta) = \rho_1 \int_0^\theta \Phi(\theta, s) e^{-\varepsilon(\theta - s)} \prod_{s<\theta_k<T} (1 + \Lambda_k) ds + \rho_2 \int_0^T \Psi(\theta, s) e^{-\varepsilon(\theta - s)} \prod_{s<\theta_k<T} (1 + \Lambda_k) ds$$

the following inequality is assumed to be true:

$$\chi e^{\varepsilon T} \int_0^T \overline{\pi}(s) ds \leq \prod_{j=1}^m \left(1 + \overline{\overline{\Lambda_j}}\right) \tag{2.3}$$

Then, we can draw a conclusion that $w(\theta) \leq 0$ for $\theta \in \xi$.

Proof. For $\Lambda_k > -1$, then we define $c_k = 1 + \Lambda_k > 0$. If boundary conditions satisfy $w(0) \geq \chi w(T)$, we let $\zeta(\theta) = \left(\prod_{\theta<\theta_k<T} c_k^{-1}\right) w(\theta) e^{-\varepsilon\theta}$, then the signs of w and ζ are same and it can be obtained that

$$\begin{cases} \zeta'(\theta) \geq \left(\prod_{\theta<\theta_k<T} c_k^{-1}\right)\left(\rho_1 \int_0^\theta \Phi(\theta, s) e^{-\varepsilon(\theta-s)} \zeta(s) \prod_{s<\theta_k<T} (c_k) ds \right. \\ \left. +\rho_2 \int_0^T \Psi(\theta, s) e^{-\varepsilon(\theta-s)} \zeta(s) \prod_{s<\theta_k<T} (c_k) ds\right), \qquad \theta \in \xi' \\ \zeta(\theta_k^+) \geq c_k \zeta(\theta_k), \qquad k = 1, 2, \cdots, m \\ \zeta(0) \geq \chi \zeta(T) \left(\prod_{k=1}^m c_k^{-1}\right) e^{\varepsilon T} \end{cases}$$

Now, we complete the proof by two cases:

(i): If $\zeta \geq 0$ and $\zeta \not\equiv 0$; then clearly $\zeta'(\theta) \geq 0$ for $\theta \in \xi'$ and $\zeta(T) \geq \zeta(0) \prod_{i=1}^m c_j \geq \chi \zeta(T) e^{\varepsilon T}$. If $\zeta(T) = 0$, it is easy to know that $\zeta(\theta) \leq 0$ by two conditions $c_k > 0$ and $\zeta'(\theta) \geq 0$, i.e., $\zeta \equiv 0$, which is inconsistent with the assumption. Moreover, if $\zeta(T) > 0$, one can obtain that $\chi e^{\varepsilon T} \leq 1$, which is a wrong conclusion.

(ii): Denote $r_1 \in [0, T]$, which satisfies $\zeta(r_1) > 0$. Suppose that $\zeta(r_2) = \min_{\theta \in [0,T]} \zeta(\theta) = n$, then clearly $n < 0$. It can be obtained that

$$\zeta'(\theta) \geq n \left(\prod_{\theta<\theta_k<T} c_k^{-1}\right) \pi(\theta)$$

If $r_1 > r_2$, we have

$$n = \zeta(r_2) \geq \zeta(r_1) \prod_{r_1<\theta_k<r_2} c_k + n \int_{r_1}^{r_2} \left(\prod_{\theta<\theta_k<r_2} c_k\right)\left(\prod_{\theta<\theta_k<T} c_k^{-1}\right) \pi(\theta) d\theta$$

$$> n\int_{r_1}^{r_2}\left(\prod_{r_2<\theta_k<T} c_k^{-1}\right)\pi(\theta)\,d\theta$$

then

$$\chi e^{\varepsilon T}\int_0^T \pi(s)\,ds \geq \int_{r_1}^{r_2}\pi(\theta)\,d\theta > \prod_{r_2<\theta_k<T} c_k \geq \prod_{j=1}^{m}\overline{\overline{c_j}},$$

$\overline{\overline{c_j}} = 1 + \overline{\overline{\Lambda_j}}$, $j = 1, 2, ...m$, it is a contradiction with the condition Eq (2.3). If $r_1 < r_2$, we have

$$\begin{aligned} n = \zeta(r_2) &\geq \zeta(0)\prod_{0<\theta_\chi<r_2} c_\chi + n\int_0^{r_2}\left(\prod_{\theta<\theta_k<r_2} c_k\right)\left(\prod_{\theta<\theta_k<T} c_k^{-1}\right)\pi(\theta)\,d\theta \\ &= \zeta(0)\prod_{0<\theta_k<r_2} c_k + n\int_0^{r_2}\left(\prod_{r_2\leq\theta_k<T} c_k^{-1}\right)\pi(\theta)\,d\theta \end{aligned}$$

and

$$\begin{aligned} \zeta(T) &\geq \zeta(r_1)\prod_{r_1<\theta_k<T} c_k + n\int_{r_1}^{T}\left(\prod_{\theta<\theta_k<T} c_k\right)\left(\prod_{\theta<\theta_k<T} c_k^{-1}\right)\pi(\theta)\,d\theta \\ &> n\int_{r_1}^{T}\pi(\theta)d\theta \end{aligned}$$

For $\zeta(0) \geq \chi\zeta(T)\left(\prod_{k=1}^{m} c_k^{-1}\right)e^{\varepsilon T}$, it is easy to obtain that

$$\begin{aligned} n &\geq \chi\zeta(T)\left(\prod_{k=1}^{m} c_k^{-1}\right)e^{\varepsilon T}\prod_{0<\theta_k<r_2} c_k + n\int_0^{r_2}\left(\prod_{r_2\leq\theta_k<T} c_k^{-1}\right)\pi(\theta)\,d\theta \\ &> \chi n e^{\varepsilon T}\left(\prod_{r_2\leq\theta_k<T} c_k^{-1}\right)\int_{r_1}^{T}\pi(\theta)\,d\theta + n\int_0^{r_2}\left(\prod_{r_2\leq\theta_k<T} c_k^{-1}\right)\pi(\theta)\,d\theta. \end{aligned}$$

Then, the follow inequality can be obtained with conditions $r_2 < r_1, \chi e^{\varepsilon T} > 1$ and $n < 0$:

$$\begin{aligned} 1 &< \chi e^{\varepsilon T}\left(\prod_{r_2\leq\theta_k<T} c_k^{-1}\right)\int_{r_1}^{T}\pi(\theta)\,d\theta + \int_0^{r_2}\left(\prod_{r_2\leq\theta_k<T} c_k^{-1}\right)\pi(\theta)\,d\theta \\ &\leq \chi e^{\varepsilon T}\left(\prod_{r_2\leq\theta_k<T} c_k^{-1}\right)\int_0^{T}\pi(\theta)\,d\theta. \end{aligned}$$

i.e.,

$$\chi e^{\varepsilon T}\int_0^T \pi(s)\,ds > \prod_{r_2\leq\theta_k<T} c_k \geq \prod_{j=1}^{m}\overline{\overline{c_j}},$$

It is a contradiction with Eq (2.3).

On the other hand, if boundary conditions satisfy $w(0) < \chi w(T)$, we let $b(\theta) = w(\theta) + g(\theta)$. It is easy to get that

$$\begin{cases} b'(\theta) - \varepsilon b(\theta) - \rho_1[\Gamma b](\theta) - \rho_2[\delta b](\theta) \geq 0, & \theta \in \xi' \\ \Delta b(\theta_k) \geq \Lambda_k b(\theta_k), \quad k = 1, 2, \cdots, m \\ b(0) \geq \chi b(T), \end{cases}$$

Clearly, $b \leq 0$ from the above proof, and $w(\theta) \leq 0$.

Remark 2.1. Lemma 2.1 is a key comparison result to obtain extremal solutions of Eq (1.1). Expression and proof of Lemma 2.1 are similar to previous studies [5]. Moreover, the boundary condition "$w(0) = \chi w(T)$" with $\chi > e^{-\varepsilon T}$ is more general than the periodic condition "$\chi = 1$". Our method can also generalize some known results, such as Corollary 2.1 [5, 19], Corollary 2.2 [5] and Corollary 2.3 [5, 20].

Corollary 2.1. Let $\varepsilon > 0, \rho_1, \rho_2 \geq 0, \chi > e^{-\varepsilon T}$, $\Lambda_k \geq 0, k = 1, 2, \cdots, m$, $w \in LC'(\xi)$ satisfies Eq (2.1) or (2.2), and define

$$\pi_1(\theta) = \rho_1 \int_0^\theta \Phi(\theta, s) e^{-\varepsilon(\theta - s)} ds + \rho_2 \int_0^T \Psi(\theta, s) e^{-\varepsilon(\theta - s)} ds$$

if the follow inequality Eq (2.4) holds

$$\chi e^{\varepsilon T} \int_0^T \left(\prod_{0<\theta_k \leq s} (1 + \Lambda_k)^{-1} \right) \pi_1(s)\, ds \leq \left(\prod_{j=1}^m \left(1 + \Lambda_j\right)^{-1} \right)^2 \tag{2.4}$$

then, we have $w(\theta) \leq 0$ for $\theta \in \xi$.

Proof. We prove that Eq (2.3) holds as follows:

$$\begin{aligned} \chi e^{\varepsilon T} \int_0^T \overline{\pi}(s) ds &\leq \chi e^{\varepsilon T} \left(\prod_{j=1}^m \left(1 + \Lambda_j\right) \right) \int_0^T \left(\rho_1 \int_0^\theta \Phi(\theta, s) e^{-\varepsilon(\theta - s)} ds \right. \\ &\quad \left. + \rho_2 \int_0^T \Psi(\theta, s) e^{-\varepsilon(\theta - s)} ds \right) dt \\ &= \chi e^{\varepsilon T} \left(\prod_{j=1}^m \left(1 + \Lambda_j\right) \right) \int_0^T \left(\prod_{0<\theta_k \leq s} (1 + \Lambda_k) \right) \left(\prod_{0<\theta_k \leq s} (1 + \Lambda_k)^{-1} \right) \pi_1(\theta)\, d\theta \\ &\leq \chi e^{\varepsilon T} \left(\prod_{j=1}^m \left(1 + \Lambda_j\right) \right)^2 \int_0^T \left(\prod_{0<\theta_k \leq s} (1 + \Lambda_k)^{-1} \right) \pi_1(\theta)\, d\theta \\ &\leq \left(\prod_{j=1}^m \left(1 + \Lambda_j\right) \right)^2 \left(\prod_{j=1}^m \left(1 + \Lambda_j\right)^{-1} \right)^2 = 1 \leq \prod_{j=1}^m \left(1 + \overline{\overline{\Lambda_j}}\right). \end{aligned}$$

We know that $w(\theta) \leq 0$ by Lemma 2.1.

Corollary 2.2. Suppose that $\varepsilon > 0, \rho_1, \rho_2 \geq 0, \chi > e^{-\varepsilon T}$, $\Lambda_k \geq 0, k = 1, 2, \cdots, m$, $w \in LC'(\xi)$ satisfies Eq (2.1) or (2.2), and let

$$\chi \frac{(\rho_1 k_0 + \rho_2 h_0)\left(e^{\varepsilon T} - 1\right) e^{\varepsilon T}}{\varepsilon} \leq \frac{\left(\prod_{j=1}^m \left(1 + \Lambda_j\right)^{-1} \right)^2}{\int_0^T \left(\prod_{0<\theta_k \leq s} (1 + \Lambda_k)^{-1} \right) ds}, \tag{2.5}$$

then, we can obtain $w(\theta) \leq 0$ on ξ.

Proof. The condition Eq (2.3) is derived as follows:

$$\begin{aligned}
\chi e^{\varepsilon T}\int_0^T \overline{\pi}(\theta)d\theta &\leq \chi e^{\varepsilon T}(\rho_1 k_0+\rho_2 h_0)\int_0^T\left(\int_0^T\left(\prod_{s<\theta_k<T}(1+\Lambda_k)\right)e^{-\varepsilon(\theta-s)}ds\right)d\theta\\
&\leq \chi(\rho_1 k_0+\rho_2 h_0)\frac{e^{\varepsilon T}-1}{\varepsilon}e^{\varepsilon T}\int_0^T\left(\prod_{s<\theta_k<T}(1+\Lambda_k)\right)ds\\
&= \chi(\rho_1 k_0+\rho_2 h_0)\frac{e^{\varepsilon T}-1}{\varepsilon}e^{\varepsilon T}\left(\prod_{j=1}^m\left(1+\Lambda_j\right)\right)\int_0^T\left(\prod_{0<\theta_k\leq s}(1+\Lambda_k)^{-1}\right)ds\\
&\leq\left(\prod_{j=1}^m\left(1+\Lambda_j\right)\right)\left(\prod_{j=1}^m\left(1+\Lambda_j\right)^{-1}\right)^2=\left(\prod_{j=1}^m\left(1+\Lambda_j\right)^{-1}\right)\leq 1\leq\prod_{j=1}^m\left(1+\overline{\overline{\Lambda_j}}\right)
\end{aligned}$$

From Lemma 2.1, we have $w(\theta) \leq 0$ on ξ.

Corollary 2.3. Let $\Lambda_k \geq 0, k = 1, 2, \cdots, m, \rho_1, \rho_2 \geq 0, \varepsilon > 0, \chi > e^{-\varepsilon T}$, $w \in LC'(\xi)$ satisfies Eq (2.1) or (2.2), and suppose that

$$(\varepsilon+\chi\rho_1 Tk_0+\chi\rho_2 Th_0)\,\tau\left(1+(m+1)\prod_{j=1}^m\left(1+\Lambda_j\right)\right)\leq 1, \tag{2.6}$$

where $\tau = \max\{\theta_k - \theta_{k-1} : k = 1, 2, ...m + 1\}$. Then, we have $w(\theta) \leq 0$ on ξ.

Proof. We prove that the inequality Eq (2.3) holds as follows:

$$\begin{aligned}
\chi e^{\varepsilon T}\int_0^T \overline{\pi}(\theta)d\theta &\leq \chi e^{\varepsilon T}(\rho_1 k_0+\rho_2 h_0)\int_0^T\left(\int_0^T\left(\prod_{k=1}^m(1+\Lambda_k)\right)e^{-\varepsilon(\theta-s)}ds\right)d\theta\\
&= \chi e^{\varepsilon T}(\rho_1 k_0+\rho_2 h_0)\frac{\left(1-e^{-\varepsilon T}\right)\left(e^{\varepsilon T}-1\right)}{\varepsilon^2}\prod_{j=1}^m\left(1+\Lambda_j\right)\\
&\leq \chi(\rho_1 Tk_0+\rho_2 Th_0)\frac{(e^{\varepsilon T}-1)^2}{\varepsilon^2 T^2}\tau(m+1)\prod_{j=1}^m\left(1+\Lambda_j\right)\\
&\leq \chi\frac{\left(e^{\varepsilon T}-1\right)^2}{\varepsilon^2 T^2}(\varepsilon+\rho_1 Tk_0+\rho_2 Th_0-\varepsilon)\,\tau\left(1+(m+1)\prod_{j=1}^m\left(1+\Lambda_j\right)\right)\\
&\leq \frac{\left(e^{\varepsilon T}-1\right)^2}{\varepsilon^2 T^2}\left(1-\varepsilon\tau\left(1+(m+1)\prod_{j=1}^m\left(1+\Lambda_j\right)\right)\right)\\
&\leq \frac{\left(e^{\varepsilon T}-1\right)^2}{\varepsilon^2 T^2}(1-\varepsilon T)\leq 1\leq\prod_{j=1}^m\left(1+\overline{\overline{\Lambda_j}}\right).
\end{aligned}$$

3. A linear system

Now, we study the solution of a linear system (LS) with the general boundary condition:

$$w'(\theta)+\varepsilon w(\theta)=\varsigma(\theta), \qquad \theta\in\xi', \tag{3.1}$$

$$w\left(\theta_j^+\right) = w\left(\theta_j^-\right) + I_j\left(w\left(\theta_j\right)\right), \qquad j = 1, ..., m, \tag{3.2}$$

$$w(0) = \chi w(T), \tag{3.3}$$

where $I_j \in C(R, R)$ and $\varsigma \in LC(\xi)$.

Lemma 3.1. The solution of (LS) can be described as follows:

$$w(\theta) = \int_0^T U(\theta, s)\varsigma(s)ds + \sum_{j=1}^m U\left(\theta, \theta_j\right)I_j\left(w\left(\theta_j\right)\right), \qquad \theta \in \xi', \tag{3.4}$$

where

$$U(\theta, s) = \frac{1}{e^{\varepsilon T} - \chi}\begin{cases} e^{\varepsilon(T-\theta+s)}, & 0 \le s \le \theta \le T \\ \chi e^{-\varepsilon(\theta-s)}, & 0 \le \theta < s \le T. \end{cases} \tag{3.5}$$

Proof. Set $z(\theta) = e^{\varepsilon\theta}w(\theta), \theta \in \xi$. Then

$$z'(\theta) = \varsigma^*(\theta), \theta \in \xi'; \qquad z(0) = \chi z(T)e^{-\varepsilon T};$$

$$z\left(\theta_j^+\right) = z\left(\theta_j^-\right) + I_j^*\left(z\left(\theta_j\right)\right),$$

where $I_j^*(x) = e^{\varepsilon\theta_j}I_j\left(e^{-\varepsilon\theta_j}x\right)$, $\varsigma^*(\theta) = e^{\varepsilon\theta}\varsigma(\theta)$.

If $\theta \in \left(\theta_j, \theta_{j+1}\right]$, $j = 1, ..., m$, we obtain

$$z(\theta) = z\left(\theta_j^+\right) + \int_{\theta_j}^{\theta} \varsigma^*(s)ds.$$

Since

$$z\left(\theta_j^-\right) = z\left(\theta_{j-1}^+\right) + \int_{\theta_{j-1}}^{\theta_j} \varsigma^*(s)ds.$$

So, when $\theta \in (\theta_j, \theta_{j+1}]$, we have

$$z(\theta) = z\left(\theta_{j-1}^+\right) + \int_{\theta_{j-1}}^{\theta} \varsigma^*(s)ds + I_j^*\left(z\left(\theta_j\right)\right).$$

Therefore,

$$z(\theta) = z(0) + \int_0^{\theta} \varsigma^*(s)ds + \sum_{j:\theta_j \in (0,\theta)} I_j^*\left(z\left(\theta_j\right)\right), \theta \in \xi. \tag{3.6}$$

In Eq (3.6), we let $\theta = T$, then we have

$$z(0) = \frac{\chi}{e^{\varepsilon T} - \chi}\int_0^T \varsigma^*(s)ds + \frac{\chi}{e^{\varepsilon T} - \chi}\sum_{j=1}^m I_j^*\left(z\left(\theta_j\right)\right). \qquad \theta \in \xi.$$

Finally, substitute z(0) into Eq (3.6), we get that

$$w(\theta) = \frac{e^{-\varepsilon\theta}\chi}{e^{\varepsilon T} - \chi}\int_0^T \varsigma^*(s)ds + \frac{e^{-\varepsilon\theta}\chi}{e^{\varepsilon T} - \chi}\sum_{j=1}^m I_j^*\left(z\left(\theta_j\right)\right)$$

$$+ e^{-\varepsilon\theta}\int_0^{\theta}\varsigma^*(s)ds + e^{-\varepsilon\theta}\sum_{j:\theta_j\in(0,\theta)} I_j^*\left(z\left(\theta_j\right)\right)$$

$$= \frac{e^{-\varepsilon\theta}\chi}{e^{\varepsilon T}-\chi}\int_0^T e^{\varepsilon s}\varsigma(s)ds + \frac{e^{-\varepsilon\theta}\chi}{e^{\varepsilon T}-\chi}\sum_{j=1}^m e^{\varepsilon\theta_j} I_j\left(w\left(\theta_j\right)\right)$$

$$+ e^{-\varepsilon\theta}\int_0^{\theta} e^{\varepsilon s}\varsigma(s)ds + e^{-\varepsilon\theta}\sum_{j:\theta_j\in(0,\theta)} e^{\varepsilon\theta_j} I_j\left(w\left(\theta_j\right)\right)$$

$$= \int_0^T U(\theta,s)\,\varsigma(s)ds + \sum_{j=1}^m U\left(\theta,\theta_j\right) I_j\left(w\left(\theta_j\right)\right).$$

4. Main results

Lemma 4.1 is given without proof since it is similar to Lemma 3.1.

Lemma 4.1. Let $\Lambda_k > -1, k = 1,2,\cdots,m, \rho_1,\rho_2 \geq 0, \varepsilon > 0, \chi > e^{-\varepsilon T}$, $\vartheta \in LC'(\xi)$ and $\varsigma \in LC(\xi)$, then the solution ($w \in LC'(\xi)$) of the follow Eq (4.1) can be expressed as Eq (4.2):

$$\begin{cases} w'(\theta) - \varepsilon w(\theta) - \rho_1[\Gamma w](\theta) - \rho_2[\delta w](\theta) = \varsigma(\theta), & \theta \in \xi' \\ \triangle w(\theta_k) = \Lambda_k w(\theta_k) - I_k(\vartheta(\theta_k)) - \Lambda_k(\vartheta(\theta_k)), & k = 1,2,\cdots,m \\ w(0) = \chi w(T) \end{cases} \tag{4.1}$$

$$\begin{aligned} w(\theta) = & -\int_0^T U(\theta,s)\{\rho_1[\Gamma w](s) + \rho_2[\delta w](s) + \varsigma(s)\}ds \\ & - \sum_{0<\theta_k<T} U(\theta,\theta_k)(\Lambda_k w(\theta_k) - I_k(\vartheta(\theta_k)) - \Lambda_k(\vartheta(\theta_k))) \qquad \theta \in \xi \end{aligned} \tag{4.2}$$

where

$$U(\theta,s) = \frac{1}{1-\chi e^{\varepsilon T}}\begin{cases} e^{\varepsilon(\theta-s)}, 0 \leq s \leq \theta \leq T \\ e^{\varepsilon(T+\theta-s)}, 0 \leq \theta < s \leq T \end{cases}$$

Lemma 4.2. Let $\Lambda_k > -1, k = 1,2,\cdots,m, \rho_1,\rho_2 \geq 0, \varepsilon > 0, \chi > e^{-\varepsilon T}$, $I_k \in C(R,R)$, $\vartheta \in LC'(\xi)$ and $\varsigma \in LC(\xi)$, if the follow inequality holds:

$$\sup_{\theta\in\xi}\int_0^T U(\theta,s)\left\{\rho_1\int_0^s \Phi(s,r)\,dr + \rho_2\int_0^T \Psi(s,r)dr\right\}ds + \frac{1}{e^{-\varepsilon T}-\chi}\sum_{j=1}^m |\Lambda_k| < 1 \tag{4.3}$$

then the solution of Eq (4.1) is unique.

Proof. Define the operator $F : LC(\xi) \to LC(\xi)$, where Fw is given by the right-hand term in Eq (4.2). Clearly, the solution of Eq (4.1) is also the fixed point of the operator equation w=Fw. Since,

$$\|Fw - F\zeta\| = \sup_{\theta\in\xi}\left| -\int_0^T U(\theta,s)\{\rho_1\{[\Gamma w](s) - [\Gamma\zeta](s)\} + \rho_2\{[\delta w](s) - [\delta\zeta](s)\}ds\}\right.$$

$$- \sum_{0<\theta_k<T} U(\theta,\theta_k)\Lambda_k (w(\theta_k) - \zeta(\theta_k))\Bigg|$$

$$\leq \sup_{\theta\in\xi}\left\{\int_0^T U(\theta,s)\left[\rho_1\int_0^s \Phi(s,r)|w(r)-\zeta(r)|\,dr\right.\right.$$

$$\left.\left. + \rho_2\int_0^T \Psi(s,r)|w(r)-\zeta(r)|\,dr\right]ds + \sum_{0<\theta_k<T} U(\theta,\theta_k)|\Lambda_k||(w(\theta_k)-\zeta(\theta_k))|\right\}$$

$$\leq \|w-\zeta\|\left(\sup_{\theta\in\xi}\int_0^T U(\theta,s)\left[\rho_1\int_0^s \Phi(s,r)\,dr + \rho_2\int_0^T \Psi(s,r)dr\right]ds\right.$$

$$\left. + \frac{1}{e^{-\varepsilon T}-\chi}\sum_{k=1}^{p}|\Lambda_k|\right)$$

Then, we know that F is a contractive mapping by condition Eq (4.3). So, according to Banach's fixed point theorem, the solution of Eq (4.1) is unique.

Finally, we can obtain the following existence theorem of extremal solutions by using Lemmas 2.1 and 4.2. The arguments of Theorem 4.1 are similar to that in [4] and [5], the proof process is omitted.

Theorem 4.1. Suppose that $\Lambda_k > -1, k = 1,2,\cdots,m$, $\rho_1,\rho_2 \geq 0, \varepsilon > 0, \chi > e^{-\varepsilon T}$, and the follow four conditions satisfy:

(i)The conditions Eqs (2.3) and (4.3) hold.

(ii)There exist two functions $\nu,\mu \in LC'(\xi)$ such as $\mu(\theta) \leq \nu(\theta)$ and

$$\begin{cases} \mu'(\theta) \geq f(\theta,\mu(\theta),[\Gamma\mu](\theta),[\delta\mu](\theta)) & \theta\in\xi' = \xi-\theta_1,\theta_2,\ldots,\theta_m, \\ \triangle\mu(\theta_k) \geq I_k(\mu(\theta_k)) & k=1,2,\cdots,m \\ \mu(0) \geq \chi\mu(T) \end{cases}$$

or

$$\begin{cases} \mu'(\theta) \geq f(\theta,\mu(\theta),[\Gamma\mu](\theta),[\delta\mu](\theta)) + b_\mu(\theta) & \theta\in\xi' = \xi-\theta_1,\theta_2,\ldots,\theta_m, \\ \triangle\mu(\theta_k) \geq I_k(\mu(\theta_k)) + l_{\mu k} & k=1,2,\cdots,m \\ \mu(0) < \chi\mu(T) \end{cases}$$

$b_\mu(\theta) = -g_2'(\theta) + \varepsilon g_2(\theta) + \rho_1[\Gamma g_2](\theta) + \rho_2[\delta g_2](\theta)$,$l_{\mu k} = \Lambda_k g_2(\theta_k) - \Delta g_2(\theta_k)$, where $g_2 \in LC'(\xi)$ with $g_2 \geq 0, g_2(0) - \chi g_2(T) \geq \chi\mu(T) - \mu(0) > 0$.
and

$$\begin{cases} \nu'(\theta) \leq f(\theta,\nu(\theta),[\Gamma\nu](\theta),[\delta\nu](\theta)) & \theta\in\xi' = \xi-\theta_1,\theta_2,\ldots,\theta_m, \\ \triangle\nu(\theta_k) \leq I_k(\nu(\theta_k)) & k=1,2,\cdots,m \\ \nu(0) \leq \chi\nu(T) \end{cases}$$

or

$$\begin{cases} \nu'(\theta) \leq f(\theta,\nu(\theta),[\Gamma\nu](\theta),[\delta\nu](\theta)) - b_\nu(\theta) & \theta\in\xi' = \xi-\theta_1,\theta_2,\ldots,\theta_m, \\ \triangle\nu(\theta_k) \leq I_k(\nu(\theta_k)) - l_{\nu k} & k=1,2,\cdots,m \\ \nu(0) > \chi\nu(T) \end{cases}$$

$b_\nu(\theta) = -g_1'(\theta) + \varepsilon g_1(\theta) + \rho_1[\Gamma g_1](\theta) + \rho_2[\delta g_1](\theta)$,$l_{\nu k} = \Lambda_k g_1(\theta_k) - \Delta g_1(\theta_k)$, where $g_1 \in LC'(\xi)$ with $g_1 \geq 0, g_1(0) - \chi g_1(T) \geq \nu(0) - \chi\nu(T) > 0$.

(iii) When $\mu(\theta_k) \leq y \leq x \leq \nu(\theta_k), I_k(k = 1, 2, \cdots, m)$ meet

$$I_k(x) - I_k(y) \leq \Lambda_k(x - y).$$

(iv) When $\theta \in \xi, \mu \leq \overline{w} \leq w \leq \nu, [\Gamma\mu](\theta) \leq \overline{\zeta}(\theta) \leq \zeta(\theta) \leq [\Gamma\nu](\theta), [\delta\mu](\theta) \leq \overline{z}(\theta) \leq z(\theta) \leq [\delta\nu](\theta)$, and f meets

$$f(\theta, w, \zeta, z) - f(\theta, \overline{w}, \overline{\zeta}, \overline{z}) \leq \varepsilon(w - \overline{w}) + \rho_1(\zeta - \overline{\zeta}) + \rho_2(z - \overline{z})$$

Then, two monotone sequences ν_n, μ_n can be found such as $\nu = \nu_0 \geq \nu_n \geq ... \geq \mu_n \geq \mu_0 = \mu$, which converge uniformly to the maximal and minimal solutions of Eq (1.1) in

$$[\mu, \nu] = \{w \in LC(\xi) : \mu(\theta) \leq w(\theta) \leq \nu(\theta), \theta \in \xi\}.$$

Remark 4.1. If $\chi = 1$, then the Eq (1.1) is a periodic BVP. Therefore, the condition "$\chi > e^{-\varepsilon T}$" is more general than the periodic boundary condition. Existence result of solution (Theorem 4.1) obtained in this paper is more applicable than that in the periodic BVP.

5. Conclusions

In this paper, we discuss the existence of solutions for a first-order nonlinear impulsive integro-differential equation with a general boundary value condition "$w(0) = \chi w(T)$". Firstly, new comparison principles are developed in Section 2, which are key comparison results to obtain extremal solutions of Eq (1.1). We note that the boundary condition "$w(0) = \chi w(T)$" with $\chi > e^{-\varepsilon T}$ is more general than the periodic condition "$\chi = 1$". Then, the expression of solution for a linear system is given in Section 3. Finally, we obtain the existence results of extremal solutions for Eq (1.1) by using the monotone iterative technique, as shown in Theorem 4.1. Previous studies mainly focused on the periodic and antiperiodic boundary value conditions, therefore, the condition "$\chi > e^{-\varepsilon T}$" is more general. The main results in Section 4 are more general than previous studies, which may be widely applied in this field.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. K. Zhao, L. Suo, Y. Liao, Boundary value problem for a class of fractional integro-differential coupled systems with Hadamard fractional calculus and impulses, *Boundary Value Probl.*, **2019** (2019), 1–18. https://doi.org/10.1186/s13661-019-1219-8

2. M. J. Mardanov, Y. A. Sharifov, F. M. Zeynalli, Existence and uniqueness of the solutions to impulsive nonlinear integro-differential equations with nonlocal boundary conditions, in *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan.*, **45** (2019), 222–233. https://doi.org/10.29228/proc.6

3. S. Asawasamrit, S. K. Ntouyas, P. Thiramanus, J. Tariboon, Periodic boundary value problems for impulsive conformable fractional integro-differential equations, *Boundary Value Probl.*, **2016** (2016), 1–18. https://doi.org/10.1186/s13661-016-0629-0

4. Z. He, X. He, Monotone iterative technique for impulsive integro-differential equations with periodic boundary conditions, *Comput. Math. Appl.*, **48** (2004), 73–84. https://doi.org/10.1016/j.camwa.2004.01.005

5. Z. Luo, J. J. Nieto, New results for the periodic boundary value problem for impulsive integro-differential equations, *Nonlinear Anal.: Theory, Methods Appl.*, **70** (2009), 2248–2260. https://doi.org/10.1016/j.na.2008.03.004

6. M. Zuo, X. Hao, L. Liu, Y. Cui, Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions, *Boundary Value Probl.*, **2017** (2017), 1–15. https://doi.org/10.1186/s13661-017-0892-8

7. B. Zhu, L. Liu, Periodic boundary value problems for fractional semilinear integro-differential equations with non-instantaneous impulses, *Boundary Value Probl.*, **2018** (2018), 1–14. https://doi.org/10.1186/s13661-018-1048-1

8. L. Zhang, A. Liu, L. Xiao, Anti-periodic boundary value problem for second-order impulsive integro-differential equation with delay, *J. Cent. China Norm. Univ.*, **52** (2018), 298–302.

9. V. Kumar, M. Malik, Existence and stability of fractional integro differential equation with non-instantaneous integrable impulses and periodic boundary condition on time scales, *J. King Saud Univ.-Sci.*, **31** (2019), 1311–1317. https://doi.org/10.1016/j.jksus.2018.10.011

10. A. Anguraj, P. Karthikeyan, Anti-periodic boundary value problem for impulsive fractional integro differential equations, *Fractional Calculus Appl. Anal.*, **13** (2010), 281–294. http://hdl.handle.net/10525/1653

11. L. Ibnelazyz, K. Guida, K. Hilal, S. Melliani, Existence results for nonlinear fractional integro-differential equations with integral and antiperiodic boundary conditions, *Comput. Appl. Math.*, **40** (2021), 1–12. https://doi.org/10.1007/s40314-021-01419-4

12. R. Agarwal, D. O'Regan, S. Hristova, Monotone iterative technique for the initial value problem for differential equations with non-instantaneous impulses, *Appl. Math. Comput.*, **298** (2017), 45–56. https://doi.org/10.1016/j.amc.2016.10.009

13. R. Chaudhary, D. N. Pandey, Monotone iterative technique for impulsive Riemann-Liouville fractional differential equations, *Filomat*, **32** (2018), 3381–3395. https://doi.org/10.2298/FIL1809381C

14. H. Gou, Y. Li, Existence of mild solutions for impulsive fractional evolution equations with periodic boundary conditions, *J. Pseudo-Differ. Oper. Appl.*, **11** (2020), 425–445. https://doi.org/10.1007/s11868-019-00278-2

15. J. Henderson, A. Ouahab, S. Youcefi, Existence results for phi-Laplacian impulsive differential equations with periodic conditions, *Aims Math.*, **4** (2019), 1640–1633. https://doi.org/10.3934/math.2019.6.1610
16. W. Ding, Y. Xing, M. Han, Anti-periodic boundary value problems for first order impulsive functional differential equations, *Appl. Math. Comput.*, **186** (2007), 45–53. https://doi.org/10.1016/j.amc.2006.07.087
17. B. Ahmad, A. Alsaedi, Existence of solutions for anti-periodic boundary value problems of nonlinear impulsive functional integro-differential equations of mixed type, *Nonlinear Anal. Hybrid Syts.*, **3** (2009), 501–509. https://doi.org/10.1016/j.nahs.2009.03.007
18. Y. Hou, L. Zhang, G. Wang, A new comparison principle and its application to nonlinear impulsive functional integro-differential equations, *Adv. Differ. Equations*, **2018** (2018), 380. https://doi.org/10.1186/s13662-018-1849-7
19. J. J. Nieto, R. Rodriguez-Lopez, New comparison results for impulsive integro-differential equations and applications, *J. Math. Anal. Appl.*, **328** (2007), 1343–1368. https://doi.org/10.1016/j.jmaa.2006.06.029
20. L. Chen, J. Sun, Nonlinear boundary problem of first order impulsive integro-differential equations, *J. Comput. Appl. Math.*, **202** (2007), 392–401. https://doi.org/10.1016/j.cam.2005.10.041