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Research article

New classifications of monotonicity investigation for discrete operators with Mittag-Leffler kernel

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Abstract: This paper deals with studying monotonicity analysis for discrete fractional operators with Mittag-Leffler in kernel. The $\nu-$monotonicity definitions, namely $\nu-$(strictly) increasing and $\nu-$(strictly) decreasing, are presented as well. By examining the basic properties of the proposed discrete fractional operators together with $\nu-$monotonicity definitions, we find that the investigated discrete fractional operators will be ν^2-(strictly) increasing or ν^2-(strictly) decreasing in certain domains of the time scale $\mathbb{N}_a := \{a, a+1, \dots\}$. Finally, the correctness of developed theories is verified by deriving mean value theorem in discrete fractional calculus.

Keywords: discrete fractional calculus; nabla AB-fractional operators; monotonicity investigation

1. Introduction

During the last years an important effort was done to discrete fractional calculus (DFC) due to their wide range applications into real world phenomena (see [1]). Various attempts have been made in order to present these phenomena in a superior way and to explore new discrete nabla or delta fractional differences with different approaches such as Riemann-Liouville (RL), Caputo, Caputo-Fabrizio (CF), and Atangana-Baleanu (AB) (see [2–9]). Actually, fractional difference operators are nonlocal in mathematical nature, and they describe many nonlinear phenomena very precisely and they have a huge impact on different disciplines of science like cancer, tumor growth, tumor modeling, control theory, hydrodynamics, image processing, and signal processing, see [10–14], and more applications in these multidisciplinary sciences can be traced therein.

It has been verified that fractional and discrete fractional calculus models are a powerful tool for estimating the impacts of molecular mechanisms on discrete analysis. In the past two decades, plenty of theoretical models have been developed to characterize fractional operators including RL, Caputo, CF, and AB, see [15–21].

One of the more challenging and mathematically rich areas is to study the relationship between fractional differences and the qualitative properties of the functions on which they act. For example, it is a simple observation that if $(\nabla u)(z) := u(z) - u(z-1) > 0$, then u is increasing at $a+1$. But for fractional differences, which are inherently nonlocal, things are not so simple. Dahal and Goodrich [22] initiated work in this area in 2014 in the setting of fractional delta differences of Riemann-Liouville type. Since then this topic has attracted great attention among the researchers for most of types of nabla and delta discrete fractional differences such as Riemann-Liouville, Caputo-Fabrizio-Riemann (CFR), and Atangana-Baleanu-Riemann (ABR) fractional differences (see [23–29]). In addition to the already mentioned papers, another remarkable area which has recently received considerable attention as the central topics in DFC are positivity and convexity analyses of discrete *sequential* fractional differences, by which we mean the analysis of compositions of two or more fractional differences. Some recent papers in this direction are [30–35], and these papers contain positivity, monotonicity, and convexity results for a variety of types of discrete fractional sequential differences. Also, the results in these papers have demonstrated both similarities and dissimilarities between fractional differences with a variety of kernels, the providing a comprehensive analysis of the positivity, monotonicity, and convexity properties of discrete fractional operators, including numerical analysis of these properties (e.g., [31, 33]).

Motivated and inspired by those works, we aim to establish some new monotonicity investigations for the discrete ABC fractional difference operators. We organize our results in this article as follows: Section 2 is dedicated to recall discrete Mittag-Leffler (ML) functions, their properties and calculating some of their initial values, and the basic concept of discrete ABC fractional operators and some related properties. The Section 3 deals with the ν^2-monotonicity investigations of the results for the discrete nabla AB fractional operators. Section 4 is the discussion of results by means of the discrete mean value theorem. Finally, the concluding remarks and future directions are given in Section 5 by briefly emphasizing the relevance of the obtained investigation results.

2. Basic results

At first, let us indicate the definitions of discrete ML functions and left discrete nabla AB fractional operators on $\mathbb{N}_a$ that we will consider in this article. For any $\eta \in \mathsf{R}$ (the set of real numbers) and $\nu, \beta, z \in \mathbb{C}$ with $\mathrm{Re}(\nu) > 0$, the nabla discrete two parameters ML function is defined by (see [4]):

$$\mathsf{E}_{\overline{\nu,\beta}}(\eta, \mathsf{z}) := \sum_{k=0}^{\infty} \eta^k \frac{\mathsf{z}^{\overline{k\nu+\beta-1}}}{\Gamma(k\nu+\beta)} \quad (\text{for } |\eta| < 1), \tag{2.1}$$

where $\mathsf{z}^{\overline{\nu}}$ is the rising function, defined by

$$\mathsf{z}^{\overline{\nu}} = \frac{\Gamma(\mathsf{z}+\nu)}{\Gamma(\mathsf{z})} \quad (\nu \in \mathsf{R}), \tag{2.2}$$

for z in $\mathsf{R} \setminus \{\ldots, -2, -1, 0\}$.

Particularly, the nabla discrete one parameter ML function can be obtained for $\beta = \gamma = 1$:

$$\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}) := \sum_{k=0}^{\infty} \eta^k \frac{\mathsf{z}^{\overline{k\nu}}}{\Gamma(k\nu+1)} \quad (\text{for } |\eta| < 1). \tag{2.3}$$

Remark 2.1. *From [23, Remark 1], we have some initial values of* $\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z})$ *at* $\mathsf{z} = 0, 1, 2, 3$ *are as follows:*

- $\mathsf{E}_{\overline{\nu}}(\eta, 0) = 1,$
- $\mathsf{E}_{\overline{\nu}}(\eta, 1) = 1 - \nu,$
- $\mathsf{E}_{\overline{\nu}}(\eta, 2) = (1+\nu)(1-\nu)^2,$
- $\mathsf{E}_{\overline{\nu}}(\eta, 3) = \frac{1-\nu}{2}\left[\nu^3(2\nu-1) - 3\nu^2 + 2\right]$

for $\eta = -\frac{\nu}{1-\nu}$ *and* $0 < \nu < \frac{1}{2}$. *In general, one can see that* $0 < \mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}) < 1$ *for each* $\mathsf{z} = 1, 2, 3, \ldots$.

On the other hand, we have that $\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z})$ *is monotonically decreasing for each* $\mathsf{z} = 0, 1, 2, \ldots$.

Now, we recall the left discrete nabla ABC and ABR fractional differences with discrete ML function kernels.

Definition 2.1. Let $\Delta \mathsf{u}(\mathsf{z}) := \mathsf{u}(\mathsf{z}+1) - \mathsf{u}(\mathsf{z})$ be the delta (forward) difference operator and $\nabla \mathsf{u}(\mathsf{z}) := \mathsf{u}(\mathsf{z}) - \mathsf{u}(\mathsf{z}-1)$ be the nabla (backward) difference operator, $\eta = -\frac{\nu}{1-\nu}$, $\nu \in [0, 0.5)$ and $a \in \mathsf{R}$ (see [2, 5, 6]). Then, for any function u defined on $\mathbb{N}_a$, the left discrete nabla ABC and ABR fractional differences are defined by

$$\left({}^{ABC}_{\;\;a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) = \frac{\mathcal{B}(\nu)}{1-\nu} \sum_{\jmath=a+1}^{\mathsf{z}} (\nabla \mathsf{u})(\jmath) \mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-\jmath+1) \quad (\text{for each } \mathsf{z} \in \mathbb{N}_{a+1}), \tag{2.4}$$

and

$$\left({}^{ABR}_{\;\;a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) = \frac{\mathcal{B}(\nu)}{1-\nu} \nabla_{\mathsf{z}} \sum_{\jmath=a+1}^{\mathsf{z}} \mathsf{u}(\jmath) \mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-\jmath+1) \quad (\text{for each } \mathsf{z} \in \mathbb{N}_{a+1}), \tag{2.5}$$

respectively, where $\mathcal{B}(\nu) > 0$ with $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

The corresponding fractional sum to the ABR fractional difference Eq (2.5) is given by

$$\left({}^{AB}_{\;a}\nabla^{-\nu}\mathsf{u}\right)(\mathsf{z}) = \frac{1-\nu}{\mathcal{B}(\nu)} \mathsf{u}(\mathsf{z}) + \frac{\nu}{\mathcal{B}(\nu)} \left({}^{RL}_{\;a}\nabla^{-\nu}\mathsf{u}\right)(\mathsf{z}) \quad (\text{for each } \mathsf{z} \in \mathbb{N}_a), \tag{2.6}$$

where ${}^{RL}_{a}\nabla^{-\nu}$ is the RL fractional difference, defined by

$$\left({}^{RL}_{a}\nabla^{-\nu}\mathsf{u}\right)(\mathsf{z}) = \frac{1}{\Gamma(\nu)}\sum_{\jmath=a+1}^{\mathsf{z}} \mathsf{u}(\jmath)(\mathsf{z}-\jmath+1)^{\overline{\nu-1}} \quad (\text{for each } \mathsf{z}\in\mathbb{N}_a), \tag{2.7}$$

where when $\mathsf{z} = a$ we use the standard convention that $\sum_{\jmath=a+1}^{a}(\cdot) := 0$. Its major property which is used in this article is

$${}^{RL}_{a}\nabla^{-\nu}\mathsf{E}_{\overline{\mu}}(\eta, \mathsf{z}-a) = \mathsf{E}_{\overline{\mu,\nu+1}}(\eta, \mathsf{z}-a). \tag{2.8}$$

Lemma 2.1 (see [6, Relationship between ABC and ABR]). *Let* u *be a function defined on* $\mathbb{N}_a$*, then the following relation may be held:*

$$\left({}^{ABC}_{a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) = \left({}^{ABR}_{a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) - \frac{\mathcal{B}(\nu)}{1-\nu}\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a)\mathsf{u}(a) \quad (\textit{for each } \mathsf{z}\in\mathbb{N}_a).$$

3. $\nu-$monotonicity investigations

In this section, we focus on implementing $\nu-$monotonicity investigation for the discrete nabla AB fractional operators discussed in the previous section. First, we recall the $\nu-$monotone definitions for each $0 < \nu \leq 1$ and a function $\mathsf{u} : \mathbb{N}_a \to \mathsf{R}$ satisfying $\mathsf{u}(a) \geq 0$ that stated in [24, 25]: The function u is called $\nu-$monotone increasing (or decreasing) function on $\mathbb{N}_a$, if:

$$\mathsf{u}(\mathsf{z}+1) \geq \nu\,\mathsf{u}(\mathsf{z}) \quad (\text{or } \mathsf{u}(\mathsf{z}+1) \leq \nu\,\mathsf{u}(\mathsf{z})) \quad (\text{for each } \mathsf{z}\in\mathbb{N}_a).$$

The function u is called $\nu-$monotone strictly increasing (or strictly decreasing) function on $\mathbb{N}_a$, if:

$$\mathsf{u}(\mathsf{z}+1) > \nu\,\mathsf{u}(\mathsf{z}) \quad (\text{or } \mathsf{u}(\mathsf{z}+1) < \nu\,\mathsf{u}(\mathsf{z})) \quad (\text{for each } \mathsf{z}\in\mathbb{N}_a).$$

Remark 3.1. *It is clear that if* $\mathsf{u}(\mathsf{z})$ *is increasing (or decreasing) on* $\mathbb{N}_a$*, then we have*

$$\mathsf{u}(\mathsf{z}+1) \geq \mathsf{u}(\mathsf{z}) \geq \nu\,\mathsf{u}(\mathsf{z}) \quad (\textit{or } \mathsf{u}(\mathsf{z}+1) \leq \mathsf{u}(\mathsf{z}) \leq \nu\,\mathsf{u}(\mathsf{z})),$$

for all $\mathsf{z}\in\mathbb{N}_a$ *and* $\nu \in (0, 1]$*. This means that* $\mathsf{u}(\mathsf{z})$ *is* ν*-monotone decreasing (or* $\nu-$*monotone decreasing) on* $\mathbb{N}_a$*.*

We start this section by proving a useful result, the ν^2-monotone investigation for ABC fractional difference.

Theorem 3.1. *Let* $\nu \in \left(0, \frac{1}{2}\right)$*. If a function* $\mathsf{u} : \mathbb{N}_a \to \mathsf{R}$ *satisfies* $\mathsf{u}(a) \geq 0$ *and*

$$\left({}^{ABC}_{a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) \geq 0 \quad (\textit{for each } \mathsf{z}\in\mathbb{N}_{a+1}),$$

then $\mathsf{u}(\mathsf{z}) > 0$*. Moreover,* u *is* ν^2-*monotone increasing on* $\mathbb{N}_a$*.*

Proof. From Definition 2.1 and Remark 2.1, we can deduce for each $z \in \mathbb{N}_{a+1}$:

$$\begin{aligned}
&\left({}^{ABC}_{\ a}\nabla^{\nu}u\right)(z)\\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\sum_{\jmath=a+1}^{z}(\nabla u)(\jmath)E_{\overline{\nu}}(\eta, z-\jmath+1)\\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\sum_{\jmath=a+1}^{z}(u(\jmath)-u(\jmath-1))E_{\overline{\nu}}(\eta, z-\jmath+1)\\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\left[\sum_{\jmath=a+1}^{z}u(\jmath)E_{\overline{\nu}}(\eta, z-\jmath+1)-\sum_{\jmath=a+1}^{z}u(\jmath-1)E_{\overline{\nu}}(\eta, z-\jmath+1)\right]\\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\left[E_{\overline{\nu}}(\eta,1)u(z)-E_{\overline{\nu}}(\eta, z-a)u(a)+\sum_{\jmath=a+1}^{z-1}u(\jmath)E_{\overline{\nu}}(\eta, z-\jmath+1)-\sum_{\jmath=a+1}^{z-1}u(\jmath)E_{\overline{\nu}}(\eta, z-\jmath)\right]\\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\left((1-\nu)u(z)-E_{\overline{\nu}}(\eta, z-a)u(a)+\sum_{\jmath=a+1}^{z-1}u(\jmath)\left[E_{\overline{\nu}}(\eta, z-\jmath+1)-E_{\overline{\nu}}(\eta, z-\jmath)\right]\right). \qquad (3.1)
\end{aligned}$$

By using the fact that $\frac{\mathcal{B}(\nu)}{1-\nu} > 0$ and $\left({}^{ABC}_{\ a}\nabla^{\nu}u\right)(z) \geq 0$ for each $z \in \mathbb{N}_{a+1}$ into Eq (3.1), we get

$$u(z) \geq \frac{1}{1-\nu}E_{\overline{\nu}}(\eta, z-a)u(a)+\frac{1}{1-\nu}\sum_{\jmath=a+1}^{z-1}u(\jmath)\left[E_{\overline{\nu}}(\eta, z-\jmath)-E_{\overline{\nu}}(\eta, z-\jmath+1)\right] \quad (z \in \mathbb{N}_{a+1}). \qquad (3.2)$$

To prove that $u(z) > 0$ for each $z \in \mathbb{N}_a$ we use the principle of strong induction. Recalling that, by assumption, $u(a) > 0$, if we assume that $u(z) > 0$ for each $z \in \mathbb{N}_a^j := \{a, a+1, \ldots, j\}$, for some $j \in \mathbb{N}_a$, then as a consequence of Eq (3.2) we conclude that

$$u(j+1) \geq \frac{1}{1-\nu}E_{\overline{\nu}}(\eta, z-a)u(a)+\frac{1}{1-\nu}\sum_{\jmath=a+1}^{j}\underbrace{u(\jmath)}_{>0}\underbrace{\left[E_{\overline{\nu}}(\eta, z-\jmath)-E_{\overline{\nu}}(\eta, z-\jmath+1)\right]}_{>0} > 0,$$

where we recall from Remark 2.1 that

$$E_{\overline{\nu}}(\eta, z-\jmath)-E_{\overline{\nu}}(\eta, z-\jmath+1) > 0.$$

Thus, we conclude that $u(z) > 0$ for $z \in \mathbb{N}_a$, as desired.

To prove the ν^2–monotonicity of $u(z)$, we rewrite Eq (3.2) as follows:

$$\begin{aligned}
u(z) &\geq \frac{1}{1-\nu}E_{\overline{\nu}}(\eta, z-a)u(a)+\frac{1}{1-\nu}\left[E_{\overline{\nu}}(\eta,1)-E_{\overline{\nu}}(\eta,2)\right]u(z-1)\\
&\qquad +\frac{1}{1-\nu}\sum_{\jmath=a+1}^{z-2}u(\jmath)\left[E_{\overline{\nu}}(\eta, z-\jmath)-E_{\overline{\nu}}(\eta, z-\jmath+1)\right]\\
&= \frac{1}{1-\nu}E_{\overline{\nu}}(\eta, z-a)u(a)+\nu^2u(z-1)+\frac{1}{1-\nu}\sum_{\jmath=a+1}^{z-2}u(\jmath)\left[E_{\overline{\nu}}(\eta, z-\jmath)-E_{\overline{\nu}}(\eta, z-\jmath+1)\right], \qquad (3.3)
\end{aligned}$$

for each $z \in \mathbb{N}_{a+1}$. It is shown that $u(z) \geq 0$ for all $z \in \mathbb{N}_a$ and we know from Remark 2.1 that $E_{\overline{\nu}}(\eta, z)$ is monotonically decreasing for each $z = 0, 1, \ldots$. Then, it follows from Eq (3.3) that

$$u(z) \geq \nu^2 u(z-1) \quad (\forall\, z \in \mathbb{N}_{a+1}).$$

Thus, ν^2–monotone increasing of $u(z)$ on $\mathbb{N}_a$ is proved. □

Corollary 3.1. *Changing* ABC *to* ABR *fractional difference monotone investigation in Theorem 3.1 will need to replace the condition*

$$\left({}^{ABC}_{\ a}\nabla^{\nu} u\right)(z) \geq 0 \quad (\textit{for each } z \in \mathbb{N}_{a+1}),$$

with the condition:

$$\left({}^{ABR}_{\ a}\nabla^{\nu} u\right)(z) \geq \frac{\mathcal{B}(\nu)}{1-\nu} E_{\overline{\nu}}(\eta, z-a) u(a) \quad (\textit{for each } z \in \mathbb{N}_{a+1}),$$

where u *is assumed to be a function satisfying all remaining assumptions of Theorem 3.1. Therefore, by using the Lemma 2.1, we see that* u *is* ν^2*–monotone increasing on* $\mathbb{N}_a$.

Remark 3.2. *By restricting the conditions in Theorem 3.1 and Corollary 3.1 to* $u(a) > 0$, $\left({}^{ABC}_{\ a}\nabla^{\nu} u\right)(z) > 0$, *and* $u(a) > 0$, $\left({}^{ABR}_{\ a}\nabla^{\nu} u\right)(z) > \frac{\mathcal{B}(\nu)}{1-\nu} E_{\overline{\nu}}(\eta, z-a) u(a)$ *for each* $z \in \mathbb{N}_{a+1}$, *respectively, we can deduce that* u *is* ν^2*–monotone strictly increasing on* $\mathbb{N}_a$.

Theorem 3.2. *Suppose that* u *is defined on* $\mathbb{N}_a$ *and increasing on* $\mathbb{N}_{a+1}$ *with* $u(a) \geq 0$. *Then for* $\nu \in \left(0, \frac{1}{2}\right)$ *we have*

$$\left({}^{ABC}_{\ a}\nabla^{\nu} u\right)(z) \geq 0 \quad (\textit{for each } z \in \mathbb{N}_{a+1}).$$

Proof. From Eq (3.1) in Theorem 3.1, we have

$$\left({}^{ABC}_{\ a}\nabla^{\nu} u\right)(z) = \frac{\mathcal{B}(\nu)}{1-\nu}\left\{(1-\nu)u(z) - E_{\overline{\nu}}(\eta, z-a)u(a) + \sum_{j=a+1}^{z-1} u(j)\Big[E_{\overline{\nu}}(\eta, z-j+1) - E_{\overline{\nu}}(\eta, z-j)\Big]\right\}$$

Then by using Remark 2.1 and increasing of u on $\mathbb{N}_{a+1}$, it follows that

$$\begin{aligned}
\left({}^{ABC}_{\ a}\nabla^{\nu} u\right)(z) &= \frac{\mathcal{B}(\nu)}{1-\nu}\Bigg\{(1-\nu)u(z) - E_{\overline{\nu}}(\eta, z-a)u(a) - \nu^2(1-\nu)u(z-1) \\
&\qquad + \sum_{j=a+1}^{z-2} u(j)\Big[E_{\overline{\nu}}(\eta, z-j+1) - E_{\overline{\nu}}(\eta, z-j)\Big]\Bigg\} \\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\Bigg\{(1-\nu)u(z) - E_{\overline{\nu}}(\eta, z-a)u(a) - \nu^2(1-\nu)u(z-1) \\
&\qquad + \sum_{j=a+1}^{z-2} \underbrace{\underbrace{\big[u(j) - u(z-1)\big]}_{\leq 0}\underbrace{\Big[E_{\overline{\nu}}(\eta, z-j+1) - E_{\overline{\nu}}(\eta, z-j)\Big]}_{<0}}_{\geq 0}
\end{aligned}$$

$$+\sum_{\jmath=a+1}^{z-2} u(z-1)\Big[E_{\overline{\nu}}(\eta, z-\jmath+1)-E_{\overline{\nu}}(\eta, z-\jmath)\Big]\Big\}$$

$$\geq \frac{\mathcal{B}(\nu)}{1-\nu}\Big\{(1-\nu)u(z)-E_{\overline{\nu}}(\eta, z-a)u(a)+\sum_{\jmath=a+1}^{z-1} u(z-1)\Big[E_{\overline{\nu}}(\eta, z-\jmath+1)-E_{\overline{\nu}}(\eta, z-\jmath)\Big]\Big\}.$$

Considering, u is increasing on $\mathbb{N}_{a+1}$ and $u(a) \geq 0$, we can deduce

$$u(z) \geq u(z-1) \geq u(a) \geq 0 \quad (\text{for each } z \in \mathbb{N}_{a+1}). \tag{3.4}$$

It follows from Eq (3.4) that

$$\begin{aligned}
&\left({}^{ABC}_{\ a}\nabla^{\nu}u\right)(z)\\
&\geq \frac{\mathcal{B}(\nu)}{1-\nu}\Big\{(1-\nu)u(z)-E_{\overline{\nu}}(\eta, z-a)u(a)-(1-\nu)u(z-1)\\
&\qquad +(1-\nu)u(z-1)+u(z-1)\sum_{\jmath=a+1}^{z-1}\Big[E_{\overline{\nu}}(\eta, z-\jmath+1)-E_{\overline{\nu}}(\eta, z-\jmath)\Big]\Big\}\\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\Big\{\underbrace{\underbrace{(1-\nu)}_{>0}\underbrace{[u(z)-u(z-1)]}_{\geq 0}}_{\geq 0}-E_{\overline{\nu}}(\eta, z-a)u(a)\\
&\qquad +(1-\nu)u(z-1)+u(z-1)\sum_{\jmath=a+1}^{z-1}\Big[E_{\overline{\nu}}(\eta, z-\jmath+1)-E_{\overline{\nu}}(\eta, z-\jmath)\Big]\Big\}\\
&\geq \frac{\mathcal{B}(\nu)}{1-\nu}\Big\{(1-\nu)u(z-1)-E_{\overline{\nu}}(\eta, z-a)u(a)+u(z-1)\sum_{\jmath=a+1}^{z-1}\Big[E_{\overline{\nu}}(\eta, z-\jmath+1)-E_{\overline{\nu}}(\eta, z-\jmath)\Big]\Big\}\\
&= \frac{\mathcal{B}(\nu)}{1-\nu}\Big\{E_{\overline{\nu}}(\eta, 1)u(z-1)-E_{\overline{\nu}}(\eta, z-a)u(a)+u(z-1)\Big[E_{\overline{\nu}}(\eta, z-a)-E_{\overline{\nu}}(\eta, 1)\Big]\Big\}\\
&= \underbrace{\underbrace{\frac{\mathcal{B}(\nu)}{1-\nu}}_{>0}\,\underbrace{E_{\overline{\nu}}(\eta, z-a)}_{>0}\,\underbrace{\Big\{u(z-1)-u(a)\Big\}}_{\geq 0}}_{\geq 0}.
\end{aligned}$$

Thus, the result is proved. □

Corollary 3.2. *Again, if* u *is a function satisfying all assumptions of Theorem 3.2, then by using the relationship 2.1, we can change* ABC *to* ABR *fractional difference monotone investigation in Theorem 3.2 as follows:*

$$\left({}^{ABR}_{\ a}\nabla^{\nu}u\right)(z) \geq \frac{\mathcal{B}(\nu)}{1-\nu}E_{\overline{\nu}}(\eta, z-a)u(a) \quad (\textit{for each } z \in \mathbb{N}_{a+1}).$$

Remark 3.3. *By restricting the conditions in Theorem 3.1 and Corollary 3.2 to* $u(a) > 0$ *and* u *is strictly increasing on* $\mathbb{N}_{a+1}$*, we can obtain*

$$\left({}^{ABC}_{\ a}\nabla^{\nu}u\right)(z) > 0 \quad (\textit{for each } z \in \mathbb{N}_{a+1}),$$

and

$$\left({}^{ABR}_{\quad a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) > \frac{\mathcal{B}(\nu)}{1-\nu}\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a)\mathsf{u}(a) \quad (\textit{for each } \mathsf{z} \in \mathbb{N}_{a+1}),$$

respectively.

Remark 3.4. *By the same method as above, all the above results can be obtained for decreasing (or strictly decreasing) functions by matching conditions with their corresponding difference operators.*

4. Discrete mean value theorem

In this section, we collect a series of directions in which the discrete mean value theorem can be established.

Lemma 4.1. *For $\eta = -\frac{\nu}{1-\nu}$ with $\nu \in \left(0, \frac{1}{2}\right)$, we have for $\mathsf{z} \in \mathbb{N}_{a+1}$:*

$$\left({}^{AB}_{a+1}\nabla^{-\nu}\,\mathsf{E}_{\overline{\nu}}\right)(\eta, \mathsf{z}-a) = \frac{1-\nu}{\mathcal{B}(\nu)}\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a) + \frac{\nu}{\mathcal{B}(\nu)}\mathsf{E}_{\overline{\nu,\nu+1}}(\eta, \mathsf{z}-a) - \frac{\nu(1-\nu)}{\mathcal{B}(\nu)}\frac{(\mathsf{z}-a)^{\overline{\nu-1}}}{\Gamma(\nu)}.$$

Proof. From the definition Eq (2.6) of fractional sum with $\mathsf{u}(\mathsf{z}) := \mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a)$, we have for each $\mathsf{z} \in \mathbb{N}_a$:

$$\left({}^{AB}_{a+1}\nabla^{-\nu}\mathsf{E}_{\overline{\nu}}\right)(\eta, \mathsf{z}-a) = \frac{1-\nu}{\mathcal{B}(\nu)}\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a) + \frac{\nu}{\mathcal{B}(\nu)}\left({}^{RL}_{a+1}\nabla^{-\nu}\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a)\right)(\mathsf{z}).$$

Considering the identity Eq (2.7):

$$\begin{aligned}\left({}^{RL}_{a+1}\nabla^{-\nu}\mathsf{E}_{\overline{\nu}}\right)(\eta, \mathsf{z}-a) &= \left({}^{RL}_{\quad a}\nabla^{-\nu}\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a)\right)(\mathsf{z}) - \frac{(\mathsf{z}-a)^{\overline{\nu-1}}}{\Gamma(\nu)}\mathsf{E}_{\overline{\nu}}(\eta, 1) \\ &= \underbrace{\mathsf{E}_{\overline{\nu,\nu+1}}(\eta, \mathsf{z}-a)}_{\text{by using (2.8)}} - \frac{(1-\nu)(\mathsf{z}-a)^{\overline{\nu-1}}}{\Gamma(\nu)},\end{aligned}$$

it follows that

$$\left({}^{AB}_{a+1}\nabla^{-\nu}\mathsf{E}_{\overline{\nu}}\right)(\eta, \mathsf{z}-a) = \frac{1-\nu}{\mathcal{B}(\nu)}\mathsf{E}_{\overline{\nu}}(\eta, \mathsf{z}-a) + \frac{\nu}{\mathcal{B}(\nu)}\mathsf{E}_{\overline{\nu,\nu+1}}(\eta, \mathsf{z}-a) - \frac{\nu}{\mathcal{B}(\nu)}\frac{(1-\nu)(\mathsf{z}-a)^{\overline{\nu-1}}}{\Gamma(\nu)},$$

which is the result, as required. □

Theorem 4.1. *If u is defined on $\mathbb{N}_a$, $\eta = -\frac{\nu}{1-\nu}$ and $\nu \in \left(0, \frac{1}{2}\right)$, then for any $\mathsf{z} \in \mathbb{N}_{a+1}$, we have*

$$\left({}^{AB}_{a+1}\nabla^{-\nu}\ {}^{ABC}_{\quad a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) = \mathsf{u}(\mathsf{z}) - \mathsf{u}(a) - \frac{\nu\,(\mathsf{z}-a)^{\overline{\nu-1}}}{\Gamma(\nu)}(\nabla\,\mathsf{u})(a+1).$$

Proof. Following the definition Eq (2.4) and the identity (see [6]):

$$\left({}^{AB}_{\ a}\nabla^{-\nu}\ {}^{ABC}_{\quad a}\nabla^{\nu}\mathsf{u}\right)(\mathsf{z}) = \mathsf{u}(\mathsf{z}) - \mathsf{u}(a),$$

we see that

$$\begin{aligned}\left({}^{AB}_{a+1}\nabla^{-\nu}\ {}^{ABC}_{\ a}\nabla^{\nu}\mathtt{u}\right)(\mathtt{z}) &= \frac{\mathcal{B}(\nu)}{1-\nu}\ {}^{AB}_{a+1}\nabla^{-\nu}\left[\sum_{\jmath=a+1}^{\mathtt{z}}(\nabla\mathtt{u})(\jmath)\mathsf{E}_{\overline{\nu}}(\eta,\mathtt{z}-\jmath+1)\right]\\ &= \frac{\mathcal{B}(\nu)}{1-\nu}\ {}^{AB}_{a+1}\nabla^{-\nu}\left[\sum_{\jmath=a+2}^{\mathtt{z}}(\nabla\mathtt{u})(\jmath)\mathsf{E}_{\overline{\nu}}(\eta,\mathtt{z}-\jmath+1)+\mathsf{E}_{\overline{\nu}}(\eta,\mathtt{z}-a)(\nabla\mathtt{u})(a+1)\right]\\ &= \left({}^{AB}_{a+1}\nabla^{-\nu}\ {}^{ABC}_{a+1}\nabla^{\nu}\mathtt{u}\right)(\mathtt{z})+\frac{\mathcal{B}(\nu)}{1-\nu}\left({}^{AB}_{a+1}\nabla^{-\nu}\,\mathsf{E}_{\overline{\nu}}(\eta,\mathtt{z}-a)\right)(\mathtt{z})(\nabla\mathtt{u})(a+1)\\ &= \mathtt{u}(\mathtt{z})-\mathtt{u}(a+1)+\frac{\mathcal{B}(\nu)}{1-\nu}\left({}^{AB}_{a+1}\nabla^{-\nu}\,\mathsf{E}_{\overline{\nu}}(\eta,\mathtt{z}-a)\right)(\nabla\mathtt{u})(a+1). \qquad (4.1)\end{aligned}$$

By using Lemma 4.1 in Eq (4.1) and the fact that $\mathsf{E}_{\overline{\nu}}(\eta,\mathtt{z}-a)-\eta\,\mathsf{E}_{\overline{\nu,\nu+1}}(\eta,\mathtt{z}-a)\equiv 1$, we get the desired result. □

Remark 4.1. *Denote by $\mathcal{R}(\nu,\mathtt{z},a)$ the function $\mathcal{R}(\nu,\mathtt{z},a):=\frac{\nu\,(\mathtt{z}-a)^{\overline{\nu-1}}}{\Gamma(\nu)}$. If $\mathtt{v}$ is strictly increasing function, then by using Remark 3.3 we find that*

$$\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{v}\right)(\mathtt{z})>0.$$

Taking ${}^{AB}_{a+1}\nabla^{-\nu}$ to both sides, we get

$$\left({}^{AB}_{a+1}\nabla^{-\nu}\ {}^{ABC}_{\ a}\nabla^{\nu}\mathtt{v}\right)(\mathtt{z})>\left({}^{AB}_{a+1}\nabla^{-\nu}\,0\right)(\mathtt{z})=0. \qquad (4.2)$$

Considering Theorem 4.1, it follows that

$$\mathtt{v}(\mathtt{z})-\mathtt{v}(a)-\mathcal{R}(\nu,\mathtt{z},a)(\nabla\,\mathtt{v})(a+1)>0.$$

We are now ready to prove the discrete mean value theorem on the set $\mathrm{J}:=\mathbb{N}_{a+1}^{b}=\{a+1,a+2,\ldots,b\}$, where $b=a+k$ for some $k\in\mathbb{N}_1$. We remark that Theorem 4.2 can be compared to a related result by Atici and Uyanik [36, Theorem 4.11].

Theorem 4.2. *If $\mathtt{u}$ and $\mathtt{v}$ are two functions defined on J, $\mathtt{v}$ is strictly increasing and $0<\nu<\frac{1}{2}$, then there exist $\mathtt{t}_1,\mathtt{t}_2\in\mathrm{J}$ with*

$$\frac{\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{u}\right)(\mathtt{t}_1)}{\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{v}\right)(\mathtt{t}_1)}\le\frac{\mathtt{u}(b)-\mathtt{u}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{u})(a+1)}{\mathtt{v}(b)-\mathtt{v}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{v})(a+1)}\le\frac{\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{u}\right)(\mathtt{t}_2)}{\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{v}\right)(\mathtt{t}_2)}. \qquad (4.3)$$

Proof. On contrary, we assume that inequality (4.3) does not hold. Therefore, either

$$\frac{\mathtt{u}(b)-\mathtt{u}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{u})(a+1)}{\mathtt{v}(b)-\mathtt{v}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{v})(a+1)}>\frac{\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{u}\right)(\mathtt{z})}{\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{v}\right)(\mathtt{z})}\quad(\text{for each } \mathtt{z}\in\mathrm{J}), \qquad (4.4)$$

or

$$\frac{\mathtt{u}(b)-\mathtt{u}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{u})(a+1)}{\mathtt{v}(b)-\mathtt{v}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{v})(a+1)}<\frac{\left({}^{ABC}_{\ a}\nabla^{\nu}\mathtt{u}\right)(\mathtt{z})}{\left({}^{ABR}_{\ a}\nabla^{\nu}\mathtt{v}\right)(\mathtt{z})}\quad(\text{for each } \mathtt{z}\in\mathrm{J}). \qquad (4.5)$$

According to Eqs (4.1) and (4.2), the denominators in inequality (4.3) are all positive. Without loss of generality, the inequality (4.4) can be expressed as follows:

$$\frac{\mathtt{u}(b)-\mathtt{u}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{u})(a+1)}{\mathtt{v}(b)-\mathtt{v}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{v})(a+1)}\left({}^{ABC}_{\ \ a}\nabla^{\nu}\mathtt{v}\right)(\mathtt{z}) > \left({}^{ABC}_{\ \ a}\nabla^{\nu}\mathtt{u}\right)(\mathtt{z}). \tag{4.6}$$

Taking ${}^{AB}_{a+1}\nabla^{-\nu}$ to both sides of inequality (4.6) and making use of Theorem 4.1 we obtain

$$\frac{\mathtt{u}(b)-\mathtt{u}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{u})(a+1)}{\mathtt{v}(b)-\mathtt{v}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{v})(a+1)}\Big[\mathtt{v}(\mathtt{z})-\mathtt{v}(a)-\mathcal{R}(\nu,\mathtt{z},a)(\nabla\,\mathtt{v})(a+1)\Big]$$
$$> \mathtt{u}(\mathtt{z})-\mathtt{u}(a)-\mathcal{R}(\nu,\mathtt{z},a)(\nabla\,\mathtt{u})(a+1). \tag{4.7}$$

By substituting $\mathtt{z}=b$ into inequality (4.7), we get

$$\mathtt{u}(b)-\mathtt{u}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{u})(a+1) > \mathtt{u}(b)-\mathtt{u}(a)-\mathcal{R}(\nu,b,a)(\nabla\,\mathtt{u})(a+1),$$

which is a contradiction. Thus, inequality (4.4) cannot be true. A completely symmetric argument shows that inequality (4.5) must also be false. Thus, we are led to conclude that inequality (4.3) must be true, as desired. And this completes the proof. □

5. Concluding remarks

It is important to notice the great development in the study of monotonicity analyses of the discrete fractional operators in the last few years because it appears to be more effective for modeling discrete fractional calculus processes in theoretical physics and applied mathematics. This point has led to the thought that developing new discrete fractional operators to analyse these monotonicity results has been one of the most significant concerns for researchers of discrete fractional calculus for decades. In this article, we aimed to retrieve some novel monotonicity analyses for discrete ABC fractional difference operators are considered. The results showed that these operators could be ν^2-(strictly) increasing or ν^2-(strictly) decreasing in certain domains of the time scale $\mathbb{N}_a$. In addition, in Remark 4.1, we have found that the denominators in mean value theorem, which contains the reminder $\mathcal{R}(\nu,b,a)$, are all positive. This helped us to apply one of our strict monotonicity results to mean value theorem in the context of discrete fractional calculus as a final result of our article.

As a future extension of our work, it is possible for the readers to extend our results here by considering the discrete generalized fractional operators, introduced in [4].

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Conflict of interest

The authors declare there is no conflict of interest.

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