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*Research article*

# A new MAGDM method with 2-tuple linguistic bipolar fuzzy Heronian mean operators

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**Abstract:** In this article, we introduce the 2-tuple linguistic bipolar fuzzy set (2TLBFS), a new strategy for dealing with uncertainty that incorporates a 2-tuple linguistic term into bipolar fuzzy set. The 2TLBFS is a better way to deal with uncertain and imprecise information in the decision-making environment. We elaborate the operational rules, based on which, the 2-tuple linguistic bipolar fuzzy weighted averaging (2TLBFWA) operator and the 2-tuple linguistic bipolar fuzzy weighted geometric (2TLBFWG) operator are presented to fuse the 2TLBF numbers (2TLBFNs). The Heronian mean (HM) operator, which can reflect the internal correlation between attributes and their influence on decision results, is integrated into the 2TLBF environment to analyze the effect of the correlation between decision factors on decision results. Initially, the generalized 2-tuple linguistic bipolar fuzzy Heronian mean (G2TLBFHM) operator and generalized 2-tuple linguistic bipolar fuzzy weighted Heronian mean (G2TLBFWHM) operator are proposed and properties are explained. Further, 2-tuple linguistic bipolar fuzzy geometric Heronian mean (2TLBFGHM) operator and 2-tuple linguistic bipolar weighted geometric Heronian mean (2TLBFWGHM) operator are proposed along with some of their desirable properties. Then, an approach to multi-attribute group decision-making (MAGDM) based on the proposed aggregation operators under the 2TLBF framework is developed. At last, a numerical illustration is provided for the selection of the best photovoltaic cell to demonstrate the use of the generated technique and exhibit its adequacy.

**Keywords:** 2-tuple linguistic bipolar fuzzy set; generalized HM operator; geometric HM operator; MAGDM; photovoltaic cells

## 1. Introduction

With the steady depletion of non-renewable resources like crude oil, natural gas, coal, and rising pollution levels, the development of solar photovoltaic is expanding, especially after the COVID-19 pandemic. Photovoltaic cells are commonly used as an economical and reliable energy source in different fields all across the world. Solar photovoltaic has become a sustainable and cost-effective energy alternative for company owners and has a significant reduction in the cost of purchasing and installing solar systems. Renewable energy has been associated with continuous development and a significant effort to select the alternative based on the supplier demands, since its beginnings. Depending on the manufacturing process, different types of photovoltaic cells are currently available. It is necessary to assess photovoltaic technologies on the market, one step in this direction was to build a MAGDM tool to assist decision-makers (DMs) in selecting the most appropriate photovoltaic technology.

Decision-making is an activity that people perform regularly in their daily lives [1, 2]. The goal of decision-making is to classify the various possibilities according to the level of reliability for DMs. The situation in which individuals collectively choose the best alternative is known as group decision-making or MAGDM. The decision is then no longer attributable to any single group member, this is because the decision is influenced by all people and social group processes. Individual decisions are mostly different from group decisions. The MAGDM is a decision-making strategy that is treated as a logical and reasonable human activity. Due to the increased complexity of the environment in decision analysis and the problem itself, DMs are periodically provided with numerical data to demonstrate decision-making information. The fuzzy set (FS) theory introduced by Zadeh [3] is the modified version of the crisp sets. In fact, FS expresses the importance of the fuzzy value and this value lies between $[0, 1]$. FS tells the membership degree (MD) of an object. Intuitionistic fuzzy set (IFS) was introduced by Atanassov [4] which is generalization of FS. IFS has two functions, MD and non-membership degree (NMD) functions, and both functions map on a closed unit interval from a non-empty set. The IFS has a condition that the sum of both degrees must belong to $[0, 1]$. Zhang [5] proposed a bipolar fuzzy set (BFS) as an extension of FS, which is represented by two components: a degree of positive membership function that belongs to $[0, 1]$ and a degree of negative membership function that belongs to $[-1, 0]$.

BFS has recently gained popularity as a useful approach for resolving uncertainty in decision-making. Akram et al. [6] developed the BFNS-TOPSIS methods for the solution of multi-attribute decision-making and MAGDM problems defined by BF N-soft information. Zhao et al. [7] established a BF interactive multi-attribute decision-making strategy known as TODIM method depending on the cumulative prospect theory TODIM model for dealing with the MAGDM issues. Certain bipolar fuzzy graphs with applications are discussed in [8]. Poulik et al. [9] proposed the Wiener index for a BF graph, and the Wiener absolute index is dependent on the total accurate connectivity between all pairs of vertices and in the entire bipolar fuzzy graph. Akram et al. [10] proposed a method for solving the LR-bipolar fuzzy linear system, the LR-complex BF linear system with complex coefficients, and the

LR-complex BF linear system with real coefficients. Further, Akram et al. [11–13] developed many decision-making methods utilizing BFSs and its extensions. Naz and Akram [14, 15] developed a new DM approach to deal with the MADM problems based on graph theory. Further, Akram et al. [16–20] introduced several DM methods under generalized fuzzy scenario.

People like to express language such as "excellent", "good", "medium", "poor", and "very poor" to evaluate some attributes of the evaluation object in MAGDM consequently, the research of information aggregation operators using language as attribute value is significantly essential. The 2-tuple linguistic (2TL) term is an effective tool, which can avoid the loss of information and get more accurate evaluation results. The 2TL representation model, firstly introduced by Herrera and Martinez [21, 22], is one of the most crucial approaches to deal with linguistic decision-making issues. Later on, several 2TL aggregation operators and decision-making approaches have been developed. Under the 2TL Pythagorean fuzzy sets, Zhang et al. [23] enhanced the standard cumulative prospect theory and TODIM method based on past study of researches. In the context of intuitionistic 2TL data, Faizi et al. [24] proposed two strategies, namely Linear best-worst method (BWM) and Euclidean BWM to the BWM, to produce superior attribute importance weights for MAGDM circumstances. To estimate correctness and generate explainable outputs, Labella et al. [25] used 2-tuple BWM to mitigate the amount of paired comparisons in MAGDM circumstances and quantify the uncertainties connected with them. For addressing MAGDM issues, Zhao et al. [26] developed the enhanced TODIM approach, which is dependent on 2TL neutrosophic sets and cumulative prospect theory. He et al. [27] introduced a combination procedure QUALIFLEX method and the Pythagorean 2TL fuzzy set to assess the effectiveness of management employees in infrastructure tasks, employing the traditional QUALIFLEX decision-making strategy and Pythagorean 2TL fuzzy numbers to convey DMs assessment on every strategy.

The Heronian mean (HM) is a powerful aggregation operator that demonstrates the aggregated parameter's interrelationships. Initially, Beliakov [28] demonstrated that the HM is an aggregation operator. Ayub et al. [29] developed a new family of cubic fuzzy HM Dombi (HMD) operators, including cubic fuzzy HMD, cubic fuzzy weighted HMD, cubic fuzzy geometric HMD, and cubic fuzzy weighted geometric HMD aggregation operators. Lin et al. [30] introduced the partitioned geometric HM operator and partitioned HM (PHM) operator, some picture fuzzy interactional PHM (PFIPHM), geometric PFIPHM operators, and its weighted forms. Deveci et al. [31] proposed a novel extension of the combined compromise solution methodology that incorporates the power Heronian function and logarithmic method. Pamucar et al. [32] utilized the weighted power Heronian and weighted geometric power Heronian functions to enhance the traditional weighted aggregated sum product assessment technique. Garg et al. [33] developed complex intuitionistic uncertain linguistic (CIUL) arithmetic HM, CIUL weighted arithmetic HM, CIUL geometric HM, and CIUL weighted geometric HM by combining the HM and the CIUL concepts. Liu et al. [34] introduced and investigated the neutrosophic cubic power Heronian and neutrosophic cubic power weighted Heronian aggregation operators.

BFS has a significant advantage when it comes to explaining decision information in MAGDM situations for DMs. Many real-world events are dependent on opposite affect, such as positive affect and negative affect, plausibility and preference consistency, neutrality and indifference, and so on. BFS plays a vital role in handling these types of situations when two opposite directional concepts are involved. When there is uncertainty due to fuzziness and vagueness, the use of fuzzy linguistic information to assess alternatives in unipolar scales might help to express DM's preferences for alternatives. However, in some situations, DMs must represent negative and positive perspectives that cannot be represented by unipolar scales. The goal of this study is to build an adaptive linguistic MAGDM model for bipolar linguistic scales in which both alternatives and attributes can change over time. People frequently use language to evaluate object attributes in actual MAGDM, and there are sometimes relationships between the attributes. As a result, the study of MAGDM with language as attributes and relationships between attributes are theoretically significant and practically beneficial. In MAGDM situations, aggregation operators are frequently used. Average and geometric aggregation operators are the two types of aggregation operators. These aggregation operators have in common that they emphasize the relevance of each attribute, but they are unable to express the interrelationships between the individual data. The HM operator is a significant operator that takes into account attribute interrelationships. This study intends to introduce the notion of the 2TLBFHM operator, because there is no published article on the 2TLBFS based on HM operator.

The following are some of the aspects of this research article that are unique:

1) We present the 2TLBFS as a novel innovation in FS theory for addressing data complexity. The 2TLBFS combines the benefits of both 2TL terms and BFS, increasing the adaptability of the BFS.

2) To cope with group decision-making situations in which the attributes have interrelationships, we propose a family of HM aggregation operators for 2TLBFS, such as G2TLBFHM, G2TLBFWHM, 2TLBFGHM, and 2TLBFWGHM operators.

3) Certain formal definitions, theorems, and properties of the suggested information aggregation operators are derived under the current conditions.

4) To rank the alternatives, a novel MAGDM approach is presented, which is based on the G2TLBFWHM and 2TLBFWGHM operators to integrate DM's evaluation preferences.

5) To demonstrate the applicability and robustness of the proposed method, an illustrative example for selecting a cost-effective solution of photovoltaic cells is presented.

To achieve this goal, the structure of this paper is arranged as follows: Section 2 introduces some initial concepts related to 2TL terms, BFSs, and HM aggregation operators. In Section 3, a new information representation form, i.e., 2TLBFS is defined, along with its basic theories, such as some basic operational rules, score function, and accuracy function of 2TLBFNs. Further, the 2TLBF weighted averaging and weighted geometric operators are developed. In Section 4, we propose a family of HM aggregation operators, including the G2TLBFHM, G2TLBFWHM, 2TLBFGHM, and 2TLBFWGHM operators along with some of its essential properties. Section 5 presents MAGDM model based on the proposed G2TLBFWHM and 2TLBFWGHM operators. Section 6 provides a numerical illustration of the approach presented in this article for selecting the optimal photovoltaic cell using 2TLBFNs. Finally, in Section 7, we summarize the paper.

## 2. Preliminaries

**Definition 2.1.** [5] Let $L$ be a fixed set. A bipolar fuzzy set (BFS) $\mathfrak{B}$ in $L$ is given as

$$\mathfrak{B} = \{\langle \ell, (\varpi^{+}_{\mathfrak{B}}(\ell), \eta^{-}_{\mathfrak{B}}(\ell))\rangle | \ell \in L\} \tag{2.1}$$

where the positive MD function $\varpi^+_{\mathfrak{B}}(\ell) : L \to [0,1]$ represents the satisfaction degree of an element $\ell$ to the property and negative MD function $\eta^-_{\mathfrak{B}}(\ell) : L \to [-1,0]$ represents the satisfaction degree of an element $\ell$ to some implicit counter property corresponding to a BFS $\mathfrak{B}$, respectively, and, for every $\ell \in L$. Let $\jmath = (\varpi^+, \eta^-)$ be a BF number (BFN).

**Definition 2.2.** Let $\jmath = (\varpi^+, \eta^-)$, $\jmath_1 = (\varpi^+_1, \eta^-_1)$ and $\jmath_2 = (\varpi^+_2, \eta^-_2)$ be three BFNs, then the basic operations are as follows:

(1) $\jmath_1 \oplus \jmath_2 = (\varpi^+_1 + \varpi^+_2 - \varpi^+_1\varpi^+_2, -|\eta^-_1||\eta^-_2|)$;
(2) $\jmath_1 \otimes \jmath_2 = (\varpi^+_1\varpi^+_2, -(|\eta^-_1| + |\eta^-_2|) + |\eta^-_1||\eta^-_2|)$;
(3) $\lambda\jmath = (1 - (1 - \varpi^+)^\lambda, -|\eta^-|^\lambda), \lambda > 0$;
(4) $(\jmath)^\lambda = ((\varpi^+)^\lambda, -1 + |1 + \eta^-|^\lambda), \lambda > 0$;
(5) $\jmath^c = (1 - \varpi^+, |\eta^-| - 1)$;
(6) $\jmath_1 \subseteq \jmath_2$ if and only if $\varpi^+_1 \leq \varpi^+_2$ and $\eta^-_1 \geq \eta^-_2$;
(7) $\jmath_1 \cup \jmath_2 = (\max\{\varpi^+_1, \varpi^+_2\}, \min\{\eta^-_1, \eta^-_2\})$;
(8) $\jmath_1 \cap \jmath_2 = (\min\{\varpi^+_1, \varpi^+_2\}, \max\{\eta^-_1, \eta^-_2\})$.

**Definition 2.3.** Let $S = \{\flat_\jmath | \jmath = 1, \ldots, \sigma\}$ be a linguistic term set (LTS) and $\varphi \in [1, \sigma]$ be a number value representing the aggregation result of linguistic symbolic [21]. Then the function $\Delta$ used to obtain the 2TL information equivalent to $\varphi$ is defined as:

$$\Delta : [1, \sigma] \to S \times [-\frac{1}{2}, \frac{1}{2}),$$

$$\Delta(\varphi) = \begin{cases} \flat_\jmath, \jmath = \text{round}(\varphi) \\ \upsilon = \varphi - \jmath, \ \upsilon \in [-\frac{1}{2}, \frac{1}{2}). \end{cases} \tag{2.2}$$

**Definition 2.4.** Let $S = \{\flat_\jmath | \jmath = 1, \ldots, \sigma\}$ be a LTS and $(\flat_\jmath, \upsilon_\jmath)$ be a 2-tuple [21], there exists a function $\Delta^{-1}$ that restore the 2-tuple to its equivalent numerical value $\varphi \in [1, \sigma] \subset R$, where

$$\Delta^{-1} : S \times [-\frac{1}{2}, \frac{1}{2}) \to [1, \sigma],$$
$$\Delta^{-1}(\flat_\jmath, \upsilon) = \jmath + \upsilon = \varphi. \tag{2.3}$$

**Definition 2.5.** Let $a_k (k = 1, 2, \ldots, \mathfrak{n})$ be a group of non-negative numbers, if [28]

$$\text{HM}(a_1, a_2, \ldots, a_n) = \frac{2}{n(n+1)} \sum_{j=1}^{n} \sum_{k=j}^{n} \sqrt{a_j a_k} \tag{2.4}$$

then HM is known as Heronian mean (HM) operator.

Based on Heronian mean, Yu [36] introduced the generalized Heronian mean (GHM) as follows:

**Definition 2.6.** Let $s, t > 0$ and $a_k (k = 1, 2, \ldots, \mathfrak{n})$ be a group of non-negative numbers [36]. If

$$\text{GHM}^{s,t}(a_1, a_2, \ldots, a_n) = \left( \frac{2}{n(n+1)} \sum_{j=1}^{n} \sum_{k=j}^{n} a_j^s a_k^t \right)^{\frac{1}{s+t}} \tag{2.5}$$

then $GHM^{s,t}$ is called GHM operator. The GHM operator decreases the HM operator when $s = t = \frac{1}{2}$. Yu [36] then introduced the geometric HM operator, which is as follows:

**Definition 2.7.** Let $s, t > 0$ and $a_k (k = 1, 2, \ldots, \mathfrak{n})$ be a group of non-negative real numbers [36], then

$$\mathrm{GHM}^{s,t}(a_1, a_2, \ldots, a_n) = \frac{1}{s+t}\left(\prod_{j=1,k=j}^{n}(sa_j + ta_k)\right)^{\frac{2}{n(n+1)}} \tag{2.6}$$

then $GHM^{s,t}$ is called the geometric HM (GHM) operator.

## 3. The 2-tuple linguistic bipolar fuzzy sets

In this section, by integrating BFS and 2TL terms, we introduce the new generalization of BFS, called 2TLBFS. Further, we develop the 2TLBFWA and 2TLBFWG operators and discuss their desirable properties.

**Definition 3.1.** Let $S = \{\mathfrak{b}_0, \mathfrak{b}_1, \mathfrak{b}_2, \ldots \mathfrak{b}_\sigma\}$ be a 2TL term set with odd cardinality $\sigma + 1$. If $((\mathfrak{b}_l, \pounds), (\mathfrak{b}_n, \aleph))$ is defined for $\mathfrak{b}_l, \mathfrak{b}_n \in S$ and $\pounds, \aleph \in [-\frac{1}{2}, \frac{1}{2})$ where $(\mathfrak{b}_l, \pounds)$ and $(\mathfrak{b}_n, \aleph)$ express independently the MD and NMD by 2TLSs, then 2TLBFS is defined as follows:

$$\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k)),$$

where $0 \le \Delta^{-1}(\mathfrak{b}_{l_k}, \pounds_k) \le \sigma, 0 \le \Delta^{-1}(\mathfrak{b}_{n_k}, \aleph_k) \le \sigma$.

**Definition 3.2.** Let $\mathfrak{F} = ((\mathfrak{b}_l, \pounds), (\mathfrak{b}_n, \aleph))$ be a 2TLBFN. Then the score and accuracy functions of $\mathfrak{F}$ are defined as:

$$\mathcal{S}(\mathfrak{F}) = \Delta\left(\frac{\sigma}{2}\left(1 + \left(\frac{\Delta^{-1}(\mathfrak{b}_l, \pounds)}{\sigma}\right) - \left(\frac{\Delta^{-1}(\mathfrak{b}_n, \aleph)}{\sigma}\right)\right)\right), \quad \mathcal{S}(\mathfrak{F}) \in [0, \sigma] \tag{3.1}$$

$$\beth(\mathfrak{F}) = \Delta\left(\sigma\left(\left(\frac{\Delta^{-1}(\mathfrak{b}_l, \pounds)}{\sigma}\right) + \left(\frac{\Delta^{-1}(\mathfrak{b}_n, \aleph)}{\sigma}\right)\right)\right), \quad \beth(\mathfrak{F}) \in [0, \sigma] \tag{3.2}$$

**Definition 3.3.** Let $\mathfrak{F}_1 = ((\mathfrak{b}_{l_1}, \pounds_1), (\mathfrak{b}_{n_1}, \aleph_1))$ and $\mathfrak{F}_2 = ((\mathfrak{b}_{l_2}, \pounds_2), (\mathfrak{b}_{n_2}, \aleph_2))$ be two 2TLBFNs, then we have

(1) $\mathcal{S}(\mathfrak{F}_1) < \mathcal{S}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 < \mathfrak{F}_2$;
(2) $\mathcal{S}(\mathfrak{F}_1) > \mathcal{S}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 > \mathfrak{F}_2$;
(3) $\mathcal{S}(\mathfrak{F}_1) = \mathcal{S}(\mathfrak{F}_2)$, $\beth(\mathfrak{F}_1) < \beth(\mathfrak{F}_2)$, then $\mathfrak{F}_1 < \mathfrak{F}_2$;
(4) $\mathcal{S}(\mathfrak{F}_1) = \mathcal{S}(\mathfrak{F}_2)$, $\beth(\mathfrak{F}_1) > \beth(\mathfrak{F}_2)$, then $\mathfrak{F}_1 > \mathfrak{F}_2$;
(5) $\mathcal{S}(\mathfrak{F}_1) = \mathcal{S}(\mathfrak{F}_2)$, $\beth(\mathfrak{F}_1) = \beth(\mathfrak{F}_2)$, then $\mathfrak{F}_1 = \mathfrak{F}_2$.

**Definition 3.4.** Let $\mathfrak{F}_1 = ((\mathfrak{b}_{l_1}, \pounds_1), (\mathfrak{b}_{n_1}, \aleph_1))$ and $\mathfrak{F}_2 = ((\mathfrak{b}_{l_2}, \pounds_2), (\mathfrak{b}_{n_2}, \aleph_2))$ be two 2TLBFNs, then

(1) $\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \left( \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{l_1}, \pounds_1)}{\sigma}\right)\right)\left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{l_2}, \pounds_2)}{\sigma}\right)\right)\right)\right), \Delta\left(\sigma\left(\frac{\Delta^{-1}(\mathfrak{b}_{n_1}, \aleph_1)}{\sigma}\right)\left(\frac{\Delta^{-1}(\mathfrak{b}_{n_2}, \aleph_2)}{\sigma}\right)\right) \right);$

(2) $\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \left( \Delta\left(\sigma\left(\frac{\Delta^{-1}(\mathfrak{b}_{l_1}, \pounds_1)}{\sigma}\right)\left(\frac{\Delta^{-1}(\mathfrak{b}_{l_2}, \pounds_2)}{\sigma}\right)\right), \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_1}, \aleph_1)}{\sigma}\right)\right)\left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_2}, \aleph_2)}{\sigma}\right)\right)\right)\right) \right);$

(3) $\lambda\mathfrak{F}_1 = \left( \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{l_1}, \pounds_1)}{\sigma}\right)\right)^{\lambda}\right)\right), \Delta\left(\sigma\left(\frac{\Delta^{-1}(\mathfrak{b}_{n_1}, \aleph_1)}{\sigma}\right)^{\lambda}\right) \right), \ \lambda > 0;$

(4) $\mathfrak{F}_1^{\lambda} = \left( \Delta\left(\sigma\left(\frac{\Delta^{-1}(\mathfrak{b}_{l_1}, \pounds_1)}{\sigma}\right)^{\lambda}\right), \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_1}, \aleph_1)}{\sigma}\right)\right)^{\lambda}\right)\right) \right), \ \lambda > 0.$

### 3.1. 2TLBFWA and 2TLBFWG operators

In this subsection, the two types of weighted aggregation operators are introduced: the 2TLBFWA operator and the 2TLBFWG operator.

**Definition 3.5.** Let $\mathfrak{F}_k = ((\flat_{l_k}, \pounds_k), (\flat_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ be a collection of 2TLBFNs. The 2TLBFWA operator is a mapping $\mathcal{P}^n \to \mathcal{P}$ such that

$$2\text{TLBFWA}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \oplus_{k=1}^{n} \varkappa_k^{\star} \mathfrak{F}_k$$

where $\varkappa^{\star} = (\varkappa_1^{\star}, \varkappa_2^{\star}, \ldots, \varkappa_n^{\star})^T$ is the weight vector of $\mathfrak{F}_k(k = 1, 2, \ldots, \mathfrak{n})$, such that $\varkappa_k^{\star} \in [0, 1]$ and $\sum\limits_{k=1}^{n} \varkappa_k^{\star} = 1$.

**Theorem 3.1.** *Let $\mathfrak{F}_k = ((\flat_{l_k}, \pounds_k), (\flat_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ be a collection of 2TLBFNs with weight vector $\varkappa^{\star} = (\varkappa_1^{\star}, \varkappa_2^{\star}, \ldots, \varkappa_n^{\star})^T$, thereby satisfying $\varkappa_k^{\star} \in [0, 1]$ and $\sum\limits_{k=1}^{n} \varkappa_k^{\star} = 1$ $(k = 1, 2, \ldots, \mathfrak{n})$. Then, their aggregation value by the 2TLBFWA operator is still a 2TLBFN, and*

$$2TLBFWA(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \left(\Delta\left(\sigma\left(1 - \prod_{k=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)\right)^{\varkappa_k^{\star}}\right)\right), \Delta\left(\sigma \prod_{k=1}^{n}\left(\frac{\Delta^{-1}(\flat_{n_k}, \aleph_k)}{\sigma}\right)^{\varkappa_k^{\star}}\right)\right). \tag{3.3}$$

*Proof.* we prove that the Eq (3.3) holds by using mathematical induction method for positive integer $n$.

**(a)** when $n = 1$, we have
$$\varkappa_1^{\star}\mathfrak{F}_1 = \left(\Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{l_1}, \pounds_1)}{\sigma}\right)\right)^{\varkappa_1^{\star}}\right)\right), \Delta\left(\sigma\left(\frac{\Delta^{-1}(\flat_{n_1}, \aleph_1)}{\sigma}\right)^{\varkappa_1^{\star}}\right)\right).$$
Thus, Eq (3.3) holds for $n = 1$.

**(b)** Suppose that Eq (3.3) holds for $n = m$,

$$2\text{TLBFWA}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m) = \left(\Delta\left(\sigma\left(1 - \prod_{k=1}^{m}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)\right)^{\varkappa_k^{\star}}\right)\right), \Delta\left(\sigma \prod_{k=1}^{m}\left(\frac{\Delta^{-1}(\flat_{n_k}, \aleph_k)}{\sigma}\right)^{\varkappa_k^{\star}}\right)\right).$$

Then, when $n = m + 1$, by inductive assumption, we have

$$\begin{aligned}
2\text{TLBFWA}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m, \mathfrak{F}_{m+1}) &= 2\text{TLBFWA}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m) \oplus \varkappa_{m+1}^{\star}\mathfrak{F}_{m+1} \\
&= \left(\Delta\left(\sigma\left(1 - \prod_{k=1}^{m}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)\right)^{\varkappa_k^{\star}}\right)\right), \Delta\left(\sigma \prod_{k=1}^{m}\left(\frac{\Delta^{-1}(\flat_{n_k}, \aleph_k)}{\sigma}\right)^{\varkappa_k^{\star}}\right)\right) \\
&\oplus \left(\Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{l_{m+1}}, \pounds_{m+1})}{\sigma}\right)\right)^{\varkappa_{m+1}^{\star}}\right)\right), \Delta\left(\sigma\left(\frac{\Delta^{-1}(\flat_{n_{m+1}}, \aleph_{m+1})}{\sigma}\right)^{\varkappa_{m+1}^{\star}}\right)\right). \\
&= \left(\Delta\left(\sigma\left(1 - \prod_{k=1}^{m+1}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)\right)^{\varkappa_k^{\star}}\right)\right), \Delta\left(\sigma \prod_{k=1}^{m+1}\left(\frac{\Delta^{-1}(\flat_{n_k}, \aleph_k)}{\sigma}\right)^{\varkappa_k^{\star}}\right)\right).
\end{aligned}$$

Therefore, Eq (3.3) holds for positive integer $n = m + 1$. Thus, by mathematical induction method, we know that Eq (3.3) holds for any $n \geq 1$. □

**Theorem 3.2.** *Let $\mathfrak{F}_k = ((\flat_{l_k}, \pounds_k), (\flat_{n_k}, \aleph_k))$, $\mathfrak{F}_k' = ((\flat_{l_k}', \pounds_k'), (\flat_{n_k}', \aleph_k'))(k = 1, 2, \ldots, \mathfrak{n})$ be two sets of 2TLBFNs, then the 2TLBFWA operator has the following properties:*

(1) *(Idempotency) If all* $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ *are equal, for all* $k = 1, 2, \ldots, \mathfrak{n}$, *then*

$$2TLBFWA(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \mathfrak{F}.$$

(2) *(Monotonicity) If* $\mathfrak{F}_k \leq \mathfrak{F}'_k$, *for all k, then*

$$2TLBFWA(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq 2TLBFWA(\mathfrak{F}'_1, \mathfrak{F}'_2, \ldots, \mathfrak{F}'_n).$$

(3) *(Boundedness) If* $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ *be a collection of 2TLBFNs, and let* $\mathfrak{F}^- = (\min_k(\mathfrak{b}_{l_k}, \pounds_k), \max_k(\mathfrak{b}_{n_k}, \aleph_k))$ *and* $\mathfrak{F}^+ = (\max_k(\mathfrak{b}_{l_k}, \pounds_k), \min_k(\mathfrak{b}_{n_k}, \aleph_k))$, *then*

$$\mathfrak{F}^- \leq 2TLBFWA(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

**Definition 3.6.** Let $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ be a collection of 2TLBFNs. The 2TLBFWG operator is a mapping $\mathcal{P}^n \to \mathcal{P}$, such that

$$\text{2TLBFWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \otimes_{k=1}^{n} \mathfrak{F}_k^{\varkappa_k^\star}$$

where $\varkappa^\star = (\varkappa_1^\star, \varkappa_2^\star, \ldots, \varkappa_n^\star)^T$ is the weight vector of $\mathfrak{F}_k(k = 1, 2, \ldots, \mathfrak{n})$, such that $\varkappa_k^\star \in [0, 1]$ and $\sum_{k=1}^{n} \varkappa_k^\star = 1$.

**Theorem 3.3.** *Let* $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ *be a collection of 2TLBFNs with weight vector* $\varkappa^\star = (\varkappa_1^\star, \varkappa_2^\star, \ldots, \varkappa_n^\star)^T$, *thereby satisfying* $\varkappa_k^\star \in [0, 1]$ *and* $\sum_{k=1}^{n} \varkappa_k^\star = 1$ $(k = 1, 2, \ldots, \mathfrak{n})$. *Then, their aggregation value by the 2TLBFWG operator is still a 2TLBFN, and*

$$2TLBFWG(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \left( \Delta\left(\sigma \prod_{k=1}^{n} \left(\frac{\Delta^{-1}(\mathfrak{b}_{l_k}, \pounds_k)}{\sigma}\right)^{\varkappa_k^\star}\right), \Delta\left(\sigma\left(1 - \prod_{k=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_k}, \aleph_k)}{\sigma}\right)\right)^{\varkappa_k^\star}\right)\right) \right). \tag{3.4}$$

**Theorem 3.4.** *Let* $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k)), \mathfrak{F}'_k = ((\mathfrak{b}'_{l_k}, \pounds'_k), (\mathfrak{b}'_{n_k}, \aleph'_k))(k = 1, 2, \ldots, \mathfrak{n})$ *be two sets of 2TLBFNs, then the 2TLBFWG operator has the following properties:*

(1) *(Idempotency) If all* $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ *are equal, for all* $k = 1, 2, \ldots, \mathfrak{n}$, *then*

$$2TLBFWG(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \mathfrak{F}.$$

(2) *(Monotonicity) If* $\mathfrak{F}_k \leq \mathfrak{F}'_k$, *for all k, then*

$$2TLBFWG(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq 2TLBFWG(\mathfrak{F}'_1, \mathfrak{F}'_2, \ldots, \mathfrak{F}'_n).$$

(3) *(Boundedness) If* $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ *be a collection of 2TLBFNs, and let* $\mathfrak{F}^- = (\min_k(\mathfrak{b}_{l_k}, \pounds_k), \max_k(\mathfrak{b}_{n_k}, \aleph_k))$ *and* $\mathfrak{F}^+ = (\max_k(\mathfrak{b}_{l_k}, \pounds_k), \min_k(\mathfrak{b}_{n_k}, \aleph_k))$, *then*

$$\mathfrak{F}^- \leq 2TLBFWG(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

## 4. The 2-tuple linguistic bipolar fuzzy Heronian mean aggregation operators

### 4.1. The G2TLBFHM operator and its weighted form

The generalized Heronian mean (GHM) operator and its weighted form in the 2TLBF environment are introduced in this subsection. We also look into some of the properties of these operators.

#### 4.1.1. The G2TLBFHM operator

In this subsection, we combine GHM with 2TLBFNs and propose the generalized 2TL bipolar fuzzy Heronian mean (G2TLBFHM) operator.

**Definition 4.1.** Let $\mathfrak{F}_k = ((\flat_{l_k}, \pounds_k), (\flat_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ be a collection of 2TLBFNs. The G2TLBFHM ia a mapping $\mathcal{P}^n \to \mathcal{P}$ such that

$$\mathrm{G2TLBFHM}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \left(\frac{2}{n(n+1)} \oplus_{j=1}^{n} \oplus_{k=j}^{n} (\mathfrak{F}_j^s \otimes \mathfrak{F}_k^t)\right)^{\frac{1}{s+t}}$$

where $s, t \geq 0$.

**Theorem 4.1.** *The aggregated value by using G2TLBFHM operator is also 2TLBFN, where*

$$\begin{aligned} &G2TLBFHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \\ &= \left( \begin{array}{c} \Delta\left(\sigma\left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right), \\ \Delta\left(\sigma\left(1 - \left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)\right) \end{array} \right) \end{aligned}$$

*where $s, t > 0$.*

*Proof.* Utilizing Def. 3.4, we have

$$\begin{aligned} (\mathfrak{F}_j)^s &= \left(\Delta\left(\sigma\left(\frac{\Delta^{-1}(\flat_{l_j}, \pounds_j)}{\sigma}\right)^s\right), \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_j}, \aleph_j)}{\sigma}\right)\right)^s\right)\right)\right), \\ (\mathfrak{F}_k)^t &= \left(\Delta\left(\sigma\left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)^t\right), \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_k}, \aleph_k)}{\sigma}\right)\right)^t\right)\right)\right). \end{aligned}$$

Thus,

$$(\mathfrak{F}_j)^s \otimes (\mathfrak{F}_k)^t = \left( \begin{array}{c} \Delta\left(\sigma\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right), \\ \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)\right) \end{array} \right).$$

Therefore,

$$\oplus_{j=1}^{n} \oplus_{k=j}^{n} (\mathfrak{F}_j^s \otimes \mathfrak{F}_k^t) = \left( \begin{array}{c} \Delta\left(\sigma\left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right)\right)\right), \\ \Delta\left(\sigma\left(\prod_{j=1,k=j}^{n}\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)\right)\right) \end{array} \right).$$

Furthermore,

$$\frac{2}{n(n+1)}\oplus_{j=1}^{n}\oplus_{k=j}^{n}(\mathfrak{F}_j^s\otimes\mathfrak{F}_k^t)$$
$$=\left(\begin{array}{l}\Delta\left(\sigma\left(1-\prod\limits_{j=1,k=j}^{n}\left(1-\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s\left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}\right)\right),\\ \Delta\left(\sigma\left(\prod\limits_{j=1,k=j}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s\left(1-\left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)^{\frac{2}{n(n+1)}}\right)\right)\end{array}\right).$$

$$\mathrm{G2TLBFHM}^{s,t}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_n)=\left(\frac{2}{n(n+1)}\oplus_{j=1}^{n}\oplus_{k=j}^{n}(\mathfrak{F}_j^s\otimes\mathfrak{F}_k^t)\right)^{\frac{1}{s+t}}$$
$$=\left(\begin{array}{l}\Delta\left(\sigma\left(1-\prod\limits_{j=1,k=j}^{n}\left(1-\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s\left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right),\\ \Delta\left(\sigma\left(1-\left(1-\prod\limits_{j=1,k=j}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s\left(1-\left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)\right)\end{array}\right).$$

□

**Example 4.1.** Let $\mathfrak{F}_1=((\flat_3,0.4),(\flat_5,-0.2))$, $\mathfrak{F}_2=((\flat_4,0.3),(\flat_2,-0.5))$, $\mathfrak{F}_3=((\flat_3,0.1),(\flat_4,-0.4))$, and $\mathfrak{F}_4=((\flat_5,0.2),(\flat_6,-0.3))$ be four 2TLBFNs, and suppose $s=2$ and $t=3$, then according to Theorem 4.1 we have

$$\mathrm{G2TLBFHM}^{s,t}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_n)=\left(\frac{2}{n(n+1)}\oplus_{j=1}^{n}\oplus_{k=j}^{n}(\mathfrak{F}_j^s\otimes\mathfrak{F}_k^t)\right)^{\frac{1}{s+t}}$$
$$=\left(\begin{array}{l}\Delta\left(6\left(1-\left(\begin{array}{c}(1-(\frac{3.4}{6})^2\times(\frac{3.4}{6})^3)\times(1-(\frac{3.4}{6})^2\times(\frac{4.3}{6})^3)\\ \times(1-(\frac{3.4}{6})^2\times(\frac{3.1}{6})^3)\times(1-(\frac{3.4}{6})^2\times(\frac{5.2}{6})^3)\\ \times(1-(\frac{4.3}{6})^2\times(\frac{4.3}{6})^3)\times(1-(\frac{4.3}{6})^2\times(\frac{3.1}{6})^3)\\ \times(1-(\frac{4.3}{6})^2\times(\frac{5.2}{6})^3)\times(1-(\frac{3.1}{6})^2\times(\frac{3.1}{6})^3)\\ \times(1-(\frac{3.1}{6})^2\times(\frac{5.2}{6})^3)\times(1-(\frac{5.2}{6})^2\times(\frac{5.2}{6})^3)\end{array}\right)^{\frac{2}{4(4+1)}}\right)^{\frac{1}{2+3}}\right),\\ \Delta\left(6\left(1-\left(1-\left(\begin{array}{c}(1-(1-\frac{4.8}{6})^2\times(1-\frac{4.8}{6})^3)\times(1-(1-\frac{4.8}{6})^2\times(1-\frac{1.5}{6})^3)\\ \times(1-(1-\frac{4.8}{6})^2\times(1-\frac{3.6}{6})^3)\times(1-(1-\frac{4.8}{6})^2\times(1-\frac{5.7}{6})^3)\\ \times(1-(1-\frac{1.5}{6})^2\times(1-\frac{1.5}{6})^3)\times(1-(1-\frac{1.5}{6})^2\times(1-\frac{3.6}{6})^3)\\ \times(1-(1-\frac{1.5}{6})^2\times(1-\frac{5.7}{6})^3)\times(1-(1-\frac{3.6}{6})^2\times(1-\frac{3.6}{6})^3)\\ \times(1-(1-\frac{3.6}{6})^2\times(1-\frac{5.7}{6})^3)\times(1-(1-\frac{5.7}{6})^2\times(1-\frac{5.7}{6})^3)\end{array}\right)^{\frac{2}{4(4+1)}}\right)^{\frac{1}{2+3}}\right)\right)\end{array}\right)$$
$$=((\flat_3,0.7626),(\flat_3,0.5081)).$$

**Property 4.1.** *(Idempotency) Let all $\mathfrak{F}_k(k=1,2,\ldots,\mathfrak{n})$ are equal, i.e., $\mathfrak{F}_k=\mathfrak{F}$ for all k, then*

$$G2TLBFHM^{s,t}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_n)=\mathfrak{F}.$$

*Proof.* Since $\mathfrak{F}_k = \mathfrak{F} = ((\flat_l, \pounds), (\flat_n, \aleph))$, then

$$\begin{aligned}
&\mathrm{G2TLBFHM}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \\
&= \left( \begin{array}{l} \Delta\left(\sigma\left(\left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_j}, \pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)\right), \\ \Delta\left(\sigma\left(1 - \left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_j}, \aleph_j)}{\sigma}\right)\right)^s \left(1 - \left(\frac{\Delta^{-1}(\flat_{n_k}, \aleph_k)}{\sigma}\right)\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)\right) \end{array} \right). \\
&= \left( \begin{array}{l} \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_l, \pounds)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_l, \pounds)}{\sigma}\right)^t\right)^{\frac{1}{s+t}}\right)\right), \\ \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_n, \aleph)}{\sigma}\right)\right)^s \left(1 - \left(\frac{\Delta^{-1}(\flat_n, \aleph)}{\sigma}\right)\right)^t\right)^{\frac{1}{s+t}}\right) \end{array} \right). \\
&= \left( \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_l, \pounds)}{\sigma}\right)\right)\right)\right), \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\flat_n, \aleph)}{\sigma}\right)\right)\right)\right) \right). \\
&= ((\flat_l, \pounds), (\flat_n, \aleph)) = \mathfrak{F}.
\end{aligned}$$

□

**Property 4.2.** *(Monotonicity) Let $\mathfrak{F}_k(k = 1, 2, \ldots, \mathfrak{n})$ and $\mathfrak{F}'_k(k = 1, 2, \ldots, \mathfrak{n})$ be two sets of 2TLBFNs. If $\mathfrak{F}_k \geq \mathfrak{F}'_k$ for all k, then*

$$G2TLBFHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \geq G2TLBFHM^{s,t}(\mathfrak{F}'_1, \mathfrak{F}'_2, \ldots, \mathfrak{F}'_n).$$

*Proof.* Let $\mathfrak{F} = \left((\flat_{l_j}, \pounds_j), (\flat_{n_j}, \aleph_j)\right)$ and $\mathfrak{F}' = \left((\flat'_{l_j}, \pounds'_j), (\flat'_{n_j}, \aleph'_j)\right)$ be two sets of 2TLBFNs. Since

$$(\flat_{l_j}, \pounds_j) \geq (\flat'_{l_j}, \pounds'_j)$$

$$\Delta^{-1}(\flat_{l_j}, \pounds_j) \geq \Delta^{-1}(\flat'_{l_j}, \pounds'_j)$$

$$\left(\frac{\Delta^{-1}(\flat_{l_j}, \pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)^t \geq \left(\frac{\Delta^{-1}(\flat'_{l_j}, \pounds'_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat'_{l_k}, \pounds'_k)}{\sigma}\right)^t$$

$$\prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_j}, \pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}} \leq \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat'_{l_j}, \pounds'_j)}{\sigma}\right)^{\prime s} \left(\frac{\Delta^{-1}(\flat'_{l_k}, \pounds'_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}.$$

$$1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_j}, \pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}} \geq 1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat'_{l_j}, \pounds'_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat'_{l_k}, \pounds'_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}.$$

$$\begin{gathered}
\Delta\left(\sigma\left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat_{l_j}, \pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat_{l_k}, \pounds_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right) \geq \\
\Delta\left(\sigma\left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\flat'_{l_j}, \pounds'_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\flat'_{l_k}, \pounds'_k)}{\sigma}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right).
\end{gathered}$$

Similarly, we can prove that $(\flat_{n_j}, \aleph_j) \leq (\flat'_{n_j}, \aleph'_j)$. □

**Property 4.3.** *(Boundedness) Let* $\mathfrak{F}_k(k = 1, 2, \ldots, \mathfrak{n})$ *be a collection of 2TLBFNs, and let* $\mathfrak{F}^- = \min_k \mathfrak{F}_k$, $\mathfrak{F}^+ = \max_k \mathfrak{F}_k$, *then*

$$\mathfrak{F}^- \leq G2TLBFHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

*Proof.* According to Property 4.1

$$G2TLBFHM^{s,t}(\mathfrak{F}_1^-, \mathfrak{F}_2^-, \ldots, \mathfrak{F}_n^-) = \mathfrak{F}^-$$

and

$$G2TLBFHM^{s,t}(\mathfrak{F}_1^+, \mathfrak{F}_2^+, \ldots, \mathfrak{F}_n^+) = \mathfrak{F}^+$$

From Property 4.2

$$\mathfrak{F}^- \leq G2TLBFHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

□

4.1.2. The G2TLBFWHM operator

**Definition 4.2.** Let $s, t > 0, \mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ be a collection of 2TLBFNs, $\varkappa^\star = (\varkappa_1^\star, \varkappa_2^\star, \ldots, \varkappa_n^\star)^T$ is the weight vector of $\mathfrak{F}_k$, satisfying $\varkappa_k^\star > 0$ and $\sum_{k=1}^n \varkappa_k^\star = 1$ $(k = 1, 2, \ldots, \mathfrak{n})$. The G2TLBFWHM operator is defined as follows:

$$\text{G2TLBFWHM}_{\varkappa^\star}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \left(\oplus_{j=1}^n \oplus_{k=j}^n \left(\varkappa_j^\star \varkappa_k^\star (\mathfrak{F}_j)^s \otimes (\mathfrak{F}_k)^t\right)\right)^{\frac{1}{s+t}}. \tag{4.1}$$

**Theorem 4.2.** *Let* $s, t > 0$, $\mathfrak{F}_k = ((\mathfrak{b}_{l_k}, \pounds_k), (\mathfrak{b}_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ *be a collection of 2TLBFNs. Then the aggregated value using by G2TLBFWHM operator is also a 2TLBFN, and*

$$\begin{aligned} & G2TLBFWHM_{\varkappa^\star}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \\ = & \begin{pmatrix} \Delta\left(\sigma\left(\left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{l_j}, \pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\mathfrak{b}_{l_k}, \pounds_k)}{\sigma}\right)^t\right)^{\varkappa_j^\star \varkappa_k^\star}\right)^{\frac{1}{s+t}}\right)\right), \\ \Delta\left(\sigma\left(1 - \left(1 - \prod_{j=1,k=j}^{n}\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_j}, \aleph_j)}{\sigma}\right)\right)^s \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_k}, \aleph_k)}{\sigma}\right)\right)^t\right)^{\varkappa_j^\star \varkappa_k^\star}\right)^{\frac{1}{s+t}}\right)\right) \end{pmatrix}. \end{aligned} \tag{4.2}$$

*Proof.* According to Definition 3.4, we can derive

$$\begin{aligned} (\mathfrak{F}_j)^s &= \left(\Delta\left(\sigma\left(\frac{\Delta^{-1}(\mathfrak{b}_{l_j}, \pounds_j)}{\sigma}\right)^s\right), \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_j}, \aleph_j)}{\sigma}\right)\right)^s\right)\right)\right), \\ (\mathfrak{F}_k)^t &= \left(\Delta\left(\sigma\left(\frac{\Delta^{-1}(\mathfrak{b}_{l_k}, \pounds_k)}{\sigma}\right)^t\right), \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_k}, \aleph_k)}{\sigma}\right)\right)^t\right)\right)\right). \end{aligned}$$

Thus,

$$(\mathfrak{F}_j)^s \otimes (\mathfrak{F}_k)^t = \begin{pmatrix} \Delta\left(\sigma\left(\frac{\Delta^{-1}(\mathfrak{b}_{l_j}, \pounds_j)}{\sigma}\right)^s \left(\frac{\Delta^{-1}(\mathfrak{b}_{l_k}, \pounds_k)}{\sigma}\right)^t\right), \\ \Delta\left(\sigma\left(1 - \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_j}, \aleph_j)}{\sigma}\right)\right)^s \left(1 - \left(\frac{\Delta^{-1}(\mathfrak{b}_{n_k}, \aleph_k)}{\sigma}\right)\right)^t\right)\right) \end{pmatrix}.$$

Therefore,

$$\varkappa_j^{\star}\varkappa_k^{\star}(\mathfrak{F}_j)^s \otimes (\mathfrak{F}_k)^t = \begin{pmatrix} \Delta\left(\sigma\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s\left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right)^{\varkappa_j^{\star}\varkappa_k^{\star}}\right)\right), \\ \Delta\left(\sigma\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s\left(1-\left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)^{\varkappa_j^{\star}\varkappa_k^{\star}}\right) \end{pmatrix}.$$

Furthermore,

$$\begin{aligned} &\oplus_{j=1}^{n}\oplus_{k=j}^{n}\left(\varkappa_j^{\star}\varkappa_k^{\star}(\mathfrak{F}_j)^s \otimes (\mathfrak{F}_k)^t\right) \\ &= \begin{pmatrix} \Delta\left(\sigma\left(1-\prod_{j=1,k=j}^{n}\left(1-\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s\left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right)^{\varkappa_j^{\star}\varkappa_k^{\star}}\right)\right), \\ \Delta\left(\sigma\left(\prod_{j=1,k=j}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s\left(1-\left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)^{\varkappa_j^{\star}\varkappa_k^{\star}}\right)\right) \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} &\mathrm{G2TLBFWHM}_{\varkappa^{\star}}^{s,t}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_n) = \left(\oplus_{j=1}^{n}\oplus_{k=j}^{n}\left(\varkappa_j^{\star}\varkappa_k^{\star}(\mathfrak{F}_j)^s \otimes (\mathfrak{F}_k)^t\right)\right)^{\frac{1}{s+t}} \\ &= \begin{pmatrix} \Delta\left(\sigma\left(1-\prod_{j=1,k=j}^{n}\left(1-\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)^s\left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)^t\right)^{\varkappa_j^{\star}\varkappa_k^{\star}}\right)^{\frac{1}{s+t}}\right), \\ \Delta\left(\sigma\left(1-\left(1-\prod_{j=1,k=j}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)\right)^s\left(1-\left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^t\right)^{\varkappa_j^{\star}\varkappa_k^{\star}}\right)^{\frac{1}{s+t}}\right)\right) \end{pmatrix}. \end{aligned}$$

□

**Example 4.2.** Let $\mathfrak{F}_1 = ((\flat_3, 0.4), (\flat_5, 0.2))$, $\mathfrak{F}_2 = ((\flat_4, 0.3), (\flat_2, -0.5))$, $\mathfrak{F}_3 = ((\flat_3, 0.1), (\flat_4, -0.4))$, and $\mathfrak{F}_4 = ((\flat_5, 0.2), (\flat_6, -0.3))$ be four 2TLBFNs, and suppose $s = 2$ and $t = 3$, $\varkappa^{\star} = (0.17, 0.32, 0.38, 0.13)$ then according to Theorem 4.2 we have

$$\begin{aligned} &\mathrm{G2TLBFWHM}_{\varkappa^{\star}}^{s,t}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_n) = \left(\oplus_{j=1}^{n}\oplus_{k=j}^{n}\left(\varkappa_j^{\star}\varkappa_k^{\star}(\mathfrak{F}_j)^s \otimes (\mathfrak{F}_k)^t\right)\right)^{\frac{1}{s+t}} \\ &= \begin{pmatrix} \Delta\left(6\left(1-\begin{pmatrix} (1-(\frac{3.4}{6})^2\times(\frac{3.4}{6})^3)^{0.17\times0.17}\times(1-(\frac{3.4}{6})^2\times(\frac{4.3}{6})^3)^{0.17\times0.32} \\ \times(1-(\frac{3.4}{6})^2\times(\frac{3.1}{6})^3)^{0.17\times0.38}\times(1-(\frac{3.4}{6})^2\times(\frac{5.2}{6})^3)^{0.17\times0.13} \\ \times(1-(\frac{4.3}{6})^2\times(\frac{4.3}{6})^3)^{0.32\times0.32}\times(1-(\frac{4.3}{6})^2\times(\frac{3.1}{6})^3)^{0.32\times0.38} \\ \times(1-(\frac{4.3}{6})^2\times(\frac{5.2}{6})^3)^{0.32\times0.13}\times(1-(\frac{3.1}{6})^2\times(\frac{3.1}{6})^3)^{0.38\times0.38} \\ \times(1-(\frac{3.1}{6})^2\times(\frac{5.2}{6})^3)^{0.38\times0.13}\times(1-(\frac{5.2}{6})^2\times(\frac{5.2}{6})^3)^{0.13\times0.13} \end{pmatrix}\right)^{\frac{1}{2+3}}\right), \\ \Delta\left(6\left(1-\left(1-\begin{pmatrix} \left(1-(1-\frac{5.2}{6})^2\times(1-\frac{5.2}{6})^3\right)^{0.17\times0.17}\times\left(1-(1-\frac{5.2}{6})^2\times(1-\frac{1.5}{6})^3\right)^{0.17\times0.32} \\ \times\left(1-(1-\frac{5.2}{6})^2\times(1-\frac{3.6}{6})^3\right)^{0.17\times0.38}\times\left(1-(1-\frac{5.2}{6})^2\times(1-\frac{5.7}{6})^3\right)^{0.17\times0.13} \\ \times\left(1-(1-\frac{1.5}{6})^2\times(1-\frac{1.5}{6})^3\right)^{0.32\times0.32}\times\left(1-(1-\frac{1.5}{6})^2\times(1-\frac{3.6}{6})^3\right)^{0.32\times0.38} \\ \times\left(1-(1-\frac{1.5}{6})^2\times(1-\frac{5.7}{6})^3\right)^{0.32\times0.13}\times\left(1-(1-\frac{3.6}{6})^2\times(1-\frac{3.6}{6})^3\right)^{0.38\times0.38} \\ \times\left(1-(1-\frac{3.6}{6})^2\times(1-\frac{5.7}{6})^3\right)^{0.38\times0.13}\times\left(1-(1-\frac{5.7}{6})^2\times(1-\frac{5.7}{6})^3\right)^{0.13\times0.13} \end{pmatrix}\right)^{\frac{1}{2+3}}\right)\right) \end{pmatrix}. \\ &= ((\flat_3, 0.2487), (\flat_2, 0.9455)). \end{aligned}$$

The G2TLBFWHM operator has the following features, which are easily shown.

**Property 4.4.** *(Idempotancy) Let all* $\mathfrak{F}_k(k = 1, 2, \ldots, \mathfrak{n})$ *are equal, i.e.,* $\mathfrak{F}_k = \mathfrak{F}$ *for all k, then*

$$G2TLBFWHM^{s,t}_{\varkappa^\star}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \mathfrak{F}.$$

**Property 4.5.** *(Monotonicity) Let* $\mathfrak{F}_k(k = 1, 2, \ldots, \mathfrak{n})$ *and* $\mathfrak{F}'_k(k = 1, 2, \ldots, \mathfrak{n})$ *be two sets of 2TLBFNs. If* $\mathfrak{F}_k \geq \mathfrak{F}'_k$ *for all k, then*

$$G2TLBFWHM^{s,t}_{\varkappa^\star}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \geq G2TLBFWHM^{s,t}_{\varkappa^\star}(\mathfrak{F}'_1, \mathfrak{F}'_2, \ldots, \mathfrak{F}'_n).$$

**Property 4.6.** *(Boundedness) Let* $\mathfrak{F}_k(k = 1, 2, \ldots, \mathfrak{n})$ *be a collection of 2TLBFNs, and let* $\mathfrak{F}^- = \min_k \mathfrak{F}_k$, $\mathfrak{F}^+ = \max_k \mathfrak{F}_k$, *then*

$$\mathfrak{F}^- \leq G2TLBFWHM^{s,t}_{\varkappa^\star}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

### *4.2. The 2TLBFGHM operator and its weighted form*

In this subsection, we introduced the geometric Heronian mean (GHM) operator and its weighted form. In addition, we investigate some properties of these operators.

#### 4.2.1. The 2TLBFGHM operator

In this subsection, we combine GHM with 2TLBFNs and propose the 2TL bipolar fuzzy geometric Heronian mean (2TLBFGHM) operator.

**Definition 4.3.** Let $\mathfrak{F}_k = ((\flat_{l_k}, \pounds_k), (\flat_{n_k}, \aleph_k))(k = 1, 2, \ldots, \mathfrak{n})$ be a collection of 2TLBFNs. The 2TLBFGHM ia a mapping $\mathcal{P}^n \to \mathcal{P}$ such that

$$\text{2TLBFGHM}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \frac{1}{s+t}\left(\otimes_{j=1}^{n} \otimes_{k=j}^{n} (s\mathfrak{F}_j \oplus t\mathfrak{F}_k)\right)^{\frac{2}{n(n+1)}} \tag{4.3}$$

where $s, t \geq 0$.

**Theorem 4.3.** *The aggregated value by using 2TLBFGHM operator is also 2TLBFN, where*

$$\begin{aligned} &2TLBFGHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \\ &= \left( \begin{array}{l} \Delta\left(\sigma\left(1-\left(1-\left(\prod_{j=1,k=j}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)\right)\right)^{s}\left(1-\left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)\right)^{t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)\right), \\ \Delta\left(\sigma\left(\left(1-\prod_{j=1,k=j}^{n}\left(1-\left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)^{s}\left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)\right)^{t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right) \end{array} \right) \end{aligned}$$

*where* $s, t > 0$.

**Example 4.3.** Let $\mathfrak{F}_1 = ((\flat_3, 0.4), (\flat_5, -0.2))$, $\mathfrak{F}_2 = ((\flat_4, 0.3), (\flat_2, -0.5))$, $\mathfrak{F}_3 = ((\flat_3, 0.1), (\flat_4, -0.4))$, and $\mathfrak{F}_4 = ((\flat_5, 0.2), (\flat_6, -0.3))$ be four 2TLBFNs, and suppose $s = 2$ and $t = 3$, then according to Theorem 4.3 we have

$$\text{2TLBFGHM}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \frac{1}{s+t}\left(\otimes_{j=1}^{n} \otimes_{k=j}^{n} (s\mathfrak{F}_j \oplus t\mathfrak{F}_k)\right)^{\frac{2}{n(n+1)}}$$

$$
= \begin{pmatrix} \Delta\left(6\left(1-\left(1-\begin{pmatrix} (1-(1-\frac{3.4}{6})^2\times(1-\frac{3.4}{6})^3)\times(1-(1-\frac{3.4}{6})^2\times(1-\frac{4.3}{6})^3) \\ \times(1-(1-\frac{3.4}{6})^2\times(1-\frac{3.1}{6})^3)\times(1-(1-\frac{3.4}{6})^2\times(1-\frac{5.2}{6})^3) \\ \times(1-(1-\frac{4.3}{6})^2\times(1-\frac{4.3}{6})^3)\times(1-(1-\frac{4.3}{6})^2\times(1-\frac{3.1}{6})^3) \\ \times(1-(1-\frac{4.3}{6})^2\times(1-\frac{5.2}{6})^3)\times(1-(1-\frac{3.1}{6})^2\times(1-\frac{3.1}{6})^3) \\ \times(1-(1-\frac{3.1}{6})^2\times(1-\frac{5.2}{6})^3)\times(1-(1-\frac{5.2}{6})^2\times(1-\frac{5.2}{6})^3) \end{pmatrix}^{\frac{2}{4(4+1)}}\right)^{\frac{1}{2+3}}\right)\right), \\ \Delta\left(6\left(1-\begin{pmatrix} (1-(\frac{4.8}{6})^2\times(\frac{4.8}{6})^3)\times(1-(\frac{4.8}{6})^2\times(\frac{1.5}{6})^3) \\ \times(1-(\frac{4.8}{6})^2\times(\frac{3.6}{6})^3)\times(1-(\frac{4.8}{6})^2\times(\frac{5.7}{6})^3) \\ \times(1-(\frac{1.5}{6})^2\times(\frac{1.5}{6})^3)\times(1-(\frac{1.5}{6})^2\times(\frac{3.6}{6})^3) \\ \times(1-(\frac{1.5}{6})^2\times(\frac{5.7}{6})^3)\times(1-(\frac{3.6}{6})^2\times(\frac{3.6}{6})^3) \\ \times(1-(\frac{3.6}{6})^2\times(\frac{5.7}{6})^3)\times(1-(\frac{5.7}{6})^2\times(\frac{5.7}{6})^3) \end{pmatrix}^{\frac{2}{4(4+1)}}\right)^{\frac{1}{2+3}}\right) \end{pmatrix}
$$

$= ((\flat_2, 0.9433), (\flat_4, 0.6685)).$

**Property 4.7.** *(Idempotency) Let all $\mathfrak{F}_k (k = 1, 2, \dots, \mathfrak{n})$ are equal, i.e., $\mathfrak{F}_k = \mathfrak{F}$ for all k, then*

$$2TLBFGHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n) = \mathfrak{F}.$$

**Property 4.8.** *(Monotonicity) Let $\mathfrak{F}_k (k = 1, 2, \dots, \mathfrak{n})$ and $\mathfrak{F}'_k (k = 1, 2, \dots, \mathfrak{n})$ be two sets of 2TLBFNs. If $\mathfrak{F}_k \geq \mathfrak{F}'_k$ for all k, then*

$$2TLBFGHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n) \geq 2TLBFGHM^{s,t}(\mathfrak{F}'_1, \mathfrak{F}'_2, \dots, \mathfrak{F}'_n).$$

**Property 4.9.** *(Boundedness) Let $\mathfrak{F}_k (k = 1, 2, \dots, \mathfrak{n})$ be a collection of 2TLBFNs, and let $\mathfrak{F}^- = \min_k \mathfrak{F}_k$, $\mathfrak{F}^+ = \max_k \mathfrak{F}_k$, then*

$$\mathfrak{F}^- \leq 2TLBFGHM^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

4.2.2. The 2TLBFWGHM operator

**Definition 4.4.** Let $s, t > 0, \mathfrak{F}_k = ((\flat_{l_k}, \pounds_k), (\flat_{n_k}, \aleph_k))(k = 1, 2, \dots, \mathfrak{n})$ be a collection of 2TLBFNs, $\varkappa^\star = (\varkappa_1^\star, \varkappa_2^\star, \dots, \varkappa_n^\star)^T$ is the weight vector of $\mathfrak{F}_k$, satisfying $\varkappa_k^\star > 0$ and $\sum_{k=1}^n \varkappa_k^\star = 1$ $(k = 1, 2, \dots, \mathfrak{n})$. The 2TLBFWGHM operator is defined as follows:

$$\text{2TLBFWGHM}_{\varkappa^\star}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n) = \frac{1}{s+t}\left(\otimes_{j=1}^n \otimes_{k=j}^n (s\mathfrak{F}_j \oplus t\mathfrak{F}_k)^{\varkappa_j^\star \varkappa_k^\star}\right). \tag{4.4}$$

**Theorem 4.4.** *Let $s, t > 0, \mathfrak{F}_k = ((\flat_{l_k}, \pounds_k), (\flat_{n_k}, \aleph_k))(k = 1, 2, \dots, \mathfrak{n})$ be a collection of 2TLBFNs. Then, the aggregated value by using the 2TLBFWGHM operator is also a 2TLBFN, and*

$$
\begin{aligned}
&2TLBFGWHM_{\varkappa^\star}^{s,t}(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n) \\
&= \begin{pmatrix} \Delta\left(\sigma\left(1-\left(1-\prod_{j=1,k=j}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(\flat_{l_j},\pounds_j)}{\sigma}\right)\right)^s\left(1-\left(\frac{\Delta^{-1}(\flat_{l_k},\pounds_k)}{\sigma}\right)\right)^t\right)^{\varkappa_j^\star\varkappa_k^\star}\right)^{\frac{1}{s+t}}\right)\right), \\ \Delta\left(\sigma\left(1-\prod_{j=1,k=j}^{n}\left(1-\left(\frac{\Delta^{-1}(\flat_{n_j},\aleph_j)}{\sigma}\right)^s\left(\frac{\Delta^{-1}(\flat_{n_k},\aleph_k)}{\sigma}\right)^t\right)^{\varkappa_j^\star\varkappa_k^\star}\right)^{\frac{1}{s+t}}\right) \end{pmatrix}.
\end{aligned} \tag{4.5}
$$

**Example 4.4.** Let $\mathfrak{F}_1 = ((\flat_3, 0.4), (\flat_5, 0.2))$, $\mathfrak{F}_2 = ((\flat_4, 0.3), (\flat_2, -0.5))$, $\mathfrak{F}_3 = ((\flat_3, 0.1), (\flat_4, -0.4))$, and $\mathfrak{F}_4 = ((\flat_5, 0.2), (\flat_6, -0.3))$ be four 2TLBFNs, and suppose $s = 2$ and $t = 3$, $\varkappa^\star = (0.17, 0.32, 0.38, 0.13)$, then according to Theorem 4.4 we have

$$\mathrm{2TLBFWGHM}^{s,t}_{\varkappa^\star}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \frac{1}{s+t}\left(\otimes_{j=1}^{n} \otimes_{k=j}^{n} (s\mathfrak{F}_j \oplus t\mathfrak{F}_k)^{\varkappa_j^\star \varkappa_k^\star}\right)$$

$$= \left(\begin{array}{l}
\Delta\left(6\left(1-\left(1-\left(\begin{array}{l}
\left(1-(1-\frac{3.4}{6})^2\times(1-\frac{3.4}{6})^3\right)^{0.17\times0.17}\times\left(1-(1-\frac{3.4}{6})^2\times(1-\frac{4.3}{6})^3\right)^{0.17\times0.32}\\
\times\left(1-(1-\frac{3.4}{6})^2\times(1-\frac{3.1}{6})^3\right)^{0.17\times0.38}\times\left(1-(1-\frac{3.4}{6})^2\times(1-\frac{5.2}{6})^3\right)^{0.17\times0.13}\\
\times\left(1-(1-\frac{4.3}{6})^2\times(1-\frac{4.3}{6})^3\right)^{0.32\times0.32}\times\left(1-(1-\frac{4.3}{6})^2\times(1-\frac{3.1}{6})^3\right)^{0.32\times0.38}\\
\times\left(1-(1-\frac{4.3}{6})^2\times(1-\frac{5.2}{6})^3\right)^{0.32\times0.13}\times\left(1-(1-\frac{3.1}{6})^2\times(1-\frac{3.1}{6})^3\right)^{0.38\times0.38}\\
\times\left(1-(1-\frac{3.1}{6})^2\times(1-\frac{5.2}{6})^3\right)^{0.38\times0.13}\times\left(1-(1-\frac{5.2}{6})^2\times(1-\frac{5.2}{6})^3\right)^{0.13\times0.13}
\end{array}\right)\right)^{\frac{1}{2+3}}\right)\right),\\
\Delta\left(6\left(1-\left(\begin{array}{l}
(1-(\frac{5.2}{6})^2\times(\frac{5.2}{6})^3)^{0.17\times0.17}\times(1-(\frac{5.2}{6})^2\times(\frac{1.5}{6})^3)^{0.17\times0.32}\\
\times(1-(\frac{5.2}{6})^2\times(\frac{3.6}{6})^3)^{0.17\times0.38}\times(1-(\frac{5.2}{6})^2\times(\frac{5.7}{6})^3)^{0.17\times0.13}\\
\times(1-(\frac{1.5}{6})^2\times(\frac{1.5}{6})^3)^{0.32\times0.32}\times(1-(\frac{1.5}{6})^2\times(\frac{3.6}{6})^3)^{0.32\times0.38}\\
\times(1-(\frac{1.5}{6})^2\times(\frac{5.7}{6})^3)^{0.32\times0.13}\times(1-(\frac{3.6}{6})^2\times(\frac{3.6}{6})^3)^{0.38\times0.38}\\
\times(1-(\frac{3.6}{6})^2\times(\frac{5.7}{6})^3)^{0.38\times0.13}\times(1-(\frac{5.7}{6})^2\times(\frac{5.7}{6})^3)^{0.13\times0.13}
\end{array}\right)\right)^{\frac{1}{2+3}}\right)
\end{array}\right).$$

$$= ((\flat_3, 0.1564), (\flat_3, 0.7362)).$$

**Property 4.10.** *(Idempotency) Let all $\mathfrak{F}_k (k = 1, 2, \ldots, \mathfrak{n})$ are equal, i.e., $\mathfrak{F}_k = \mathfrak{F}$ for all k, then*

$$2TLBFWGHM^{s,t}_{\varkappa^\star}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) = \mathfrak{F}.$$

**Property 4.11.** *(Monotonicity) Let $\mathfrak{F}_k (k = 1, 2, \ldots, \mathfrak{n})$ and $\mathfrak{F}'_k (k = 1, 2, \ldots, \mathfrak{n})$ be two sets of 2TLBFNs. If $\mathfrak{F}_k \geq \mathfrak{F}'_k$ for all k, then*

$$2TLBFWGHM^{s,t}_{\varkappa^\star}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \geq 2TLBFWGHM^{s,t}_{\varkappa^\star}(\mathfrak{F}'_1, \mathfrak{F}'_2, \ldots, \mathfrak{F}'_n).$$

**Property 4.12.** *(Boundedness) Let $\mathfrak{F}_k (k = 1, 2, \ldots, \mathfrak{n})$ be a collection of 2TLBFNs, and let $\mathfrak{F}^- = \min_k \mathfrak{F}_k$, $\mathfrak{F}^+ = \max_k \mathfrak{F}_k$, then*

$$\mathfrak{F}^- \leq 2TLBFWGHM^{s,t}_{\varkappa^\star}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

## 5. An approach to MAGDM problem with 2TLBF information

We provide a novel approach to MAGDM in this section, based on the suggested G2TLBFWHM and 2TLBFWGHM operators. Consider $F = \{F_1, F_2, \ldots, F_m\}$ be a set of alternatives, $Q = \{Q_1, Q_2, \ldots, Q_n\}$ be the set of attributes, and $D = \{D_1, D_2, \ldots, D_l\}$ be a set of experts. For attribute $Q_k (k = 1, 2, \ldots, \mathfrak{n})$ of alternative $F_j (j = 1, 2, \ldots, m)$, the DM $D_h (h = 1, 2, \ldots, l)$ expresses his assessment by $\mathfrak{F}^h_{jk} = ((\flat^h_{l_{jk}}, \pounds^h_{jk}), (\flat^h_{n_{jk}}, \aleph^h_{jk}))$, which is a 2TLBFN defined on the 2TL term set $\flat^h_{jk} \in S = (\flat_0, \flat_1, \flat_2, \ldots, \flat_\sigma)$. Thus, for each DM's an individual 2TLBF assessment matrix can be derived, which can be denoted as $F^h = (\mathfrak{F}^h_{jk})_{m\times n}$. Let $\varkappa^\star = (\varkappa^\star_1, \varkappa^\star_2, \ldots, \varkappa^\star_n)^T$ be the weight vector of attributes, such that $\varkappa^\star_k \in [0, 1]$, $\sum_{k=1}^{n} \varkappa^\star_k = 1$. Let $\varkappa^{\star\star} = (\varkappa^{\star\star}_1, \varkappa^{\star\star}_2, \ldots, \varkappa^{\star\star}_l)^T$ be the weight vector of DMs, satisfying $\varkappa^{\star\star}_h \in [0, 1]$, $\sum_{h=1}^{l} \varkappa^{\star\star}_h = 1$. The essential steps for addressing the 2TLBF-MAGDM problem are described below:

**Step 1.** Utilize the 2TLBFWA operator from Eq (3.3) and the 2TLBFWG operator from Eq (3.4) to aggregate all individual 2TLBF decision matrices $F^h = (\mathfrak{F}^h_{jk})_{m\times n}(h = 1, 2, \ldots, l)$ into a collective 2TLBF decision matrix $F = (\mathfrak{F}_{jk})_{m\times n}$.

**Step 2.** Aggregate the 2TLBF evaluation values of alternative $F_j$ on all attributes $Q_k(k = 1, 2, \ldots, n)$ into the overall evaluation value of the alternative $F_j(j = 1, 2, \ldots, m)$ by using the G2TLBFWHM operator from Eq (4.2) and the 2TLBFWGHM operator from Eq (4.5) to derive the overall preference values of the alternatives $F_j(j = 1, 2, \ldots, m)$.

**Step 3.** Determine the score $S(\mathfrak{F}_j)$ of overall assessment value $\mathfrak{F}_j(j = 1, 2, \ldots, m)$ according to Def. 3.1.

**Step 4.** By ranking the alternatives $F_j(j = 1, 2, \ldots, m)$ based on their score values, select the best alternative.

**Step 5.** End.

We give the following flowchart (Figure 1) to better explain the steps of the developed MAGDM approach in this paper.

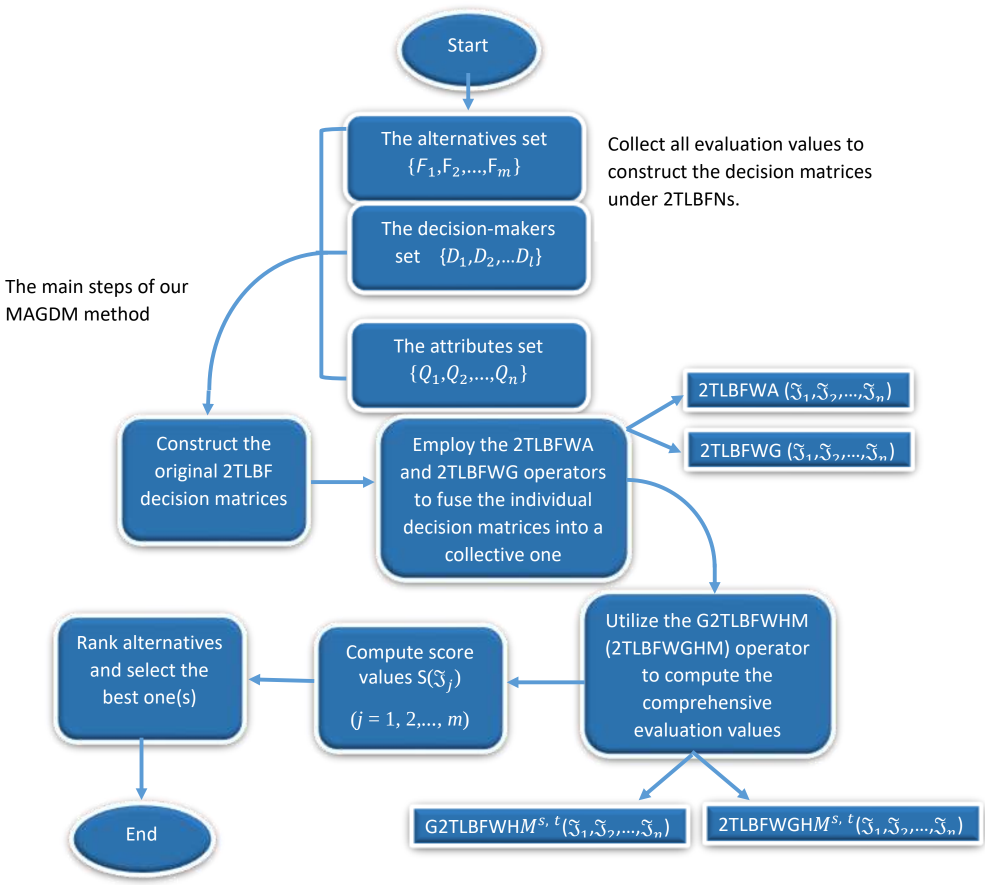


**Figure 1.** The flowchart of developed MAGDM approach.

## 6. Illustrative example and discussion

### 6.1. Evaluation process of the proposed method

Due to population growth and industrial development, the need for electrical energy is increasing rapidly. A long time ago, traditional energy sources like crude oil, natural gas, and coal were thought to be the main sources of generating electricity. Electricity prices are increasing rapidly over the last decade. Solar photovoltaic development is extending, as non-renewable resources are limited and pollution is increasing rapidly. Photovoltaic cells are commonly used as an economical and reliable energy source in different fields all across the world. As a result, academics and researchers work with government and different companies to enhance renewable energy standards and decreasing $CO_2$ emissions. Photovoltaic cells provide a cost-effective solution to this problem because it is close to demand areas and does not require extra transmission routes. Photovoltaic cells are essential for investors to conduct appropriate risk preference measures to guarantee that the project is implemented smoothly and that the expected benefits are achieved. Renewable energy sources have a rapid increase in electricity output, contributing 181 GW in 2018. Solar photovoltaic systems have the maximum capacity 55 percent, followed by wind power 28 percent, and hydropower 11 percent. The implementation of this technology provides a huge opportunity to increase system efficiency while reducing costs. A company's board of directors decided to reduce costs in order to increase profit. They observe that electricity is a major expenditure that can be reduced if solar energy is used to generate electricity. They have five alternatives of photovoltaic cells for their solar plant: Mono-crystalline photovoltaic cell ($F_1$), Poly-crystalline photovoltaic cell ($F_2$), Thin-film photovoltaic cell ($F_3$), Amorphous silicon ($F_4$), Copper indium diselenide ($F_5$). They select a photovoltaic cell based on the following attributes: Heat absorption ($Q_1$), Expenditure ($Q_2$), Efficiency and reliability ($Q_3$), Ability of charge separation ($Q_4$). Furthermore, four attributes have weight vector that is $\varkappa^\star = (0.23, 0.31, 0.27, 0.19)^T$, and $\sum_{k=1}^{n} \varkappa_k^\star = 1$. Whereas the experts believed that 2TL information is better choice for them. To select the optimal photovoltaic cell, three experts $D_h (h = 1, 2, 3)$ are invited to give their assessments using LTS S = $\{\flat_0$ = extremely poor, $\flat_1$ = very poor, $\flat_2$ = poor, $\flat_3$ = fair, $\flat_4$ = good, $\flat_5$ = very good, $\flat_6$ = extremely good$\}$. The weight vector of these experts is $\varkappa^{\star\star} = (0.3, 0.5, 0.2)$. The assessment values provided by the three experts for each attribute of each alternative are represented in the decision matrix $F^h = (\mathfrak{F}_{jk}^h)_{5\times 4} (h = 1, 2, 3)$, as in Tables 1, 2, and 3, respectively.

**Table 1.** 2TLBF decision matrix $F^1$ provided by first expert $D_1$.

| | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ |
|---|---|---|---|---|
| $F_1$ | $((\flat_3, 0), (\flat_5, 0))$ | $((\flat_1, 0), (\flat_3, 0))$ | $((\flat_2, 0), (\flat_4, 0))$ | $((\flat_3, 0), (\flat_2, 0))$ |
| $F_2$ | $((\flat_4, 0), (\flat_3, 0))$ | $((\flat_5, 0), (\flat_2, 0))$ | $((\flat_2, 0), (\flat_3, 0))$ | $((\flat_4, 0), (\flat_2, 0))$ |
| $F_3$ | $((\flat_1, 0), (\flat_4, 0))$ | $((\flat_3, 0), (\flat_3, 0))$ | $((\flat_1, 0), (\flat_3, 0))$ | $((\flat_4, 0), (\flat_3, 0))$ |
| $F_4$ | $((\flat_2, 0), (\flat_6, 0))$ | $((\flat_3, 0), (\flat_4, 0))$ | $((\flat_5, 0), (\flat_3, 0))$ | $((\flat_1, 0), (\flat_5, 0))$ |
| $F_5$ | $((\flat_4, 0), (\flat_1, 0))$ | $((\flat_1, 0), (\flat_7, 0))$ | $((\flat_6, 0), (\flat_1, 0))$ | $((\flat_2, 0), (\flat_2, 0))$ |

**Table 2.** 2TLBF decision matrix $F^2$ provided by second expert $D_2$.

| | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ |
|---|---|---|---|---|
| $F_1$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_5, 0))$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_2, 0))$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_5, 0))$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_4, 0))$ |
| $F_2$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_5, 0), (\mathfrak{b}_2, 0))$ | $((\mathfrak{b}_6, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_6, 0))$ |
| $F_3$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_2, 0))$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_5, 0))$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_3, 0))$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_4, 0))$ |
| $F_4$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_6, 0))$ | $((\mathfrak{b}_1, 0), (\mathfrak{b}_4, 0))$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_5, 0))$ | $((\mathfrak{b}_5, 0), (\mathfrak{b}_3, 0))$ |
| $F_5$ | $((\mathfrak{b}_5, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_2, 0))$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_1, 0), (\mathfrak{b}_7, 0))$ |

**Table 3.** 2TLBF decision matrix $F^3$ provided by third expert $D_3$.

| | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ |
|---|---|---|---|---|
| $F_1$ | $((\mathfrak{b}_1, 0), (\mathfrak{b}_6, 0))$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_4, 0))$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_5, 0))$ | $((\mathfrak{b}_1, 0), (\mathfrak{b}_5, 0))$ |
| $F_2$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_4, 0))$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_6, 0))$ | $((\mathfrak{b}_1, 0), (\mathfrak{b}_7, 0))$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_2, 0))$ |
| $F_3$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_3, 0))$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_3, 0))$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_3, 0))$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_1, 0))$ |
| $F_4$ | $((\mathfrak{b}_3, 0), (\mathfrak{b}_5, 0))$ | $((\mathfrak{b}_2, 0), (\mathfrak{b}_4, 0))$ | $((\mathfrak{b}_6, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_3, 0))$ |
| $F_5$ | $((\mathfrak{b}_7, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_5, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_4, 0), (\mathfrak{b}_1, 0))$ | $((\mathfrak{b}_5, 0), (\mathfrak{b}_3, 0))$ |

### 6.2. *Decision making process*

In order to choose the most desirable photovoltaic cell, the G2TLBFWHM and 2TLBFWGHM operators are used to solve the MAGDM problem with 2TLBFNs, which involves the following computing steps:

**Step 1.** Utilizing the 2TLBFWA from Eq (3.3) and the 2TLBFWG from Eq (3.4), we fuse all assessment values to get the overall 2TLBFNs $\mathfrak{F}_j (j = 1, 2, 3, 4, 5)$ of the alternatives. The fused result is shown in Table 4.

**Table 4.** Aggregated 2TLBF decision matrix by G2TLBFWA and 2TLBFWG operators.

| | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ |
|---|---|---|---|---|
| G2TLBFWA operator | | | | |
| $F_1$ | $((\mathfrak{b}_2, 0.1415), (\mathfrak{b}_5, 0.1857))$ | $((\mathfrak{b}_3, 0.0529), (\mathfrak{b}_3, -0.4054))$ | $((\mathfrak{b}_3, -0.4772), (\mathfrak{b}_5, -0.3238))$ | $((\mathfrak{b}_3, 0.2166), (\mathfrak{b}_3, 0.3973))$ |
| $F_2$ | $((\mathfrak{b}_4, 0.0000), (\mathfrak{b}_2, -0.1654))$ | $((\mathfrak{b}_5, -0.4461), (\mathfrak{b}_2, 0.4915))$ | $((\mathfrak{b}_4, 0.4274), (\mathfrak{b}_2, 0.0519))$ | $((\mathfrak{b}_3, 0.1010), (\mathfrak{b}_3, 0.4641))$ |
| $F_3$ | $((\mathfrak{b}_2, 0.2635), (\mathfrak{b}_3, -0.3297))$ | $((\mathfrak{b}_3, 0.2182), (\mathfrak{b}_4, -0.1270))$ | $((\mathfrak{b}_2, -0.0590), (\mathfrak{b}_3, 0.0000))$ | $((\mathfrak{b}_4, -0.4721), (\mathfrak{b}_3, -0.2192))$ |
| $F_4$ | $((\mathfrak{b}_2, 0.2148), (\mathfrak{b}_6, -0.2148))$ | $((\mathfrak{b}_2, -0.1358), (\mathfrak{b}_4, 0.0000))$ | $((\mathfrak{b}_4, 0.4287), (\mathfrak{b}_3, 0.1090))$ | $((\mathfrak{b}_4, -0.0973), (\mathfrak{b}_3, 0.4968))$ |
| $F_5$ | $((\mathfrak{b}_5, 0.3747), (\mathfrak{b}_1, 0.0000))$ | $((\mathfrak{b}_3, -0.4705), (\mathfrak{b}_3, -0.4646))$ | $((\mathfrak{b}_4, 0.3675), (\mathfrak{b}_1, 0.0000))$ | $((\mathfrak{b}_2, 0.3582), (\mathfrak{b}_4, 0.0577))$ |
| 2TLBFWG operator | | | | |
| $F_1$ | $((\mathfrak{b}_2, -0.0337), (\mathfrak{b}_5, 0.2337))$ | $((\mathfrak{b}_2, 0.4915), (\mathfrak{b}_3, -0.2382))$ | $((\mathfrak{b}_2, 0.4495), (\mathfrak{b}_5, -0.2704))$ | $((\mathfrak{b}_3, -0.2192), (\mathfrak{b}_4, -0.2648))$ |
| $F_2$ | $((\mathfrak{b}_4, 0.0000), (\mathfrak{b}_2, 0.3421))$ | $((\mathfrak{b}_4, 0.1628), (\mathfrak{b}_3, 0.1836))$ | $((\mathfrak{b}_3, 0.0157), (\mathfrak{b}_4, -0.2879))$ | $((\mathfrak{b}_3, -0.1716), (\mathfrak{b}_5, -0.4641))$ |
| $F_3$ | $((\mathfrak{b}_2, -0.0104), (\mathfrak{b}_3, -0.1226))$ | $((\mathfrak{b}_3, 0.1777), (\mathfrak{b}_4, 0.1270))$ | $((\mathfrak{b}_2, -0.2383), (\mathfrak{b}_3, 0.0000))$ | $((\mathfrak{b}_3, 0.4641), (\mathfrak{b}_3, 0.2166))$ |
| $F_4$ | $((\mathfrak{b}_2, 0.1689), (\mathfrak{b}_6, -0.1689))$ | $((\mathfrak{b}_2, -0.4029), (\mathfrak{b}_4, 0.0000))$ | $((\mathfrak{b}_4, 0.0168), (\mathfrak{b}_4, -0.1426))$ | $((\mathfrak{b}_3, -0.0495), (\mathfrak{b}_4, -0.2896))$ |
| $F_5$ | $((\mathfrak{b}_5, 0.0018), (\mathfrak{b}_1, 0.0000))$ | $((\mathfrak{b}_2, -0.0488), (\mathfrak{b}_4, 0.3851))$ | $((\mathfrak{b}_4, -0.0878), (\mathfrak{b}_1, 0.0000))$ | $((\mathfrak{b}_2, -0.3014), (\mathfrak{b}_6, -0.3618))$ |

**Step 2.** Using Eqs (4.2) and (4.5) to aggregate the 2TLBF assessment values $\mathfrak{F}_j$ of alternative $F_j$ on all attributes $Q_k (k = 1, 2, 3, 4)$, into the overall assessment value $\mathfrak{F}_j$ of the alternative $F_j$ $(j = 1, 2, 3, 4, 5)$ (take $s = 2$ and $t = 3$). The overall assessment values of alternatives $F_j$ $(j = 1, 2, 3, 4, 5)$ are shown in Table 5.

**Table 5.** Fused assessment values by G2TLBFWHM and 2TLBFWGHM operators.

| Alternatives | Overall assessment values by G2TLBFWHM | Overall assessment values by 2TLBFWGHM |
|---|---|---|
| $F_1$ | $((\flat_3, -0.4392), (\flat_4, -0.0151))$ | $((\flat_3, -0.1204), (\flat_4, -0.1222))$ |
| $F_2$ | $((\flat_4, -0.1760), (\flat_3, -0.1781))$ | $((\flat_4, -0.1615), (\flat_3, 0.2750))$ |
| $F_3$ | $((\flat_3, -0.3760), (\flat_4, -0.4739))$ | $((\flat_3, -0.0786), (\flat_3, 0.1544))$ |
| $F_4$ | $((\flat_3, 0.1383), (\flat_4, 0.1756))$ | $((\flat_3, -0.0911), (\flat_4, 0.0569))$ |
| $F_5$ | $((\flat_4, -0.3497), (\flat_2, 0.2376))$ | $((\flat_3, 0.1549), (\flat_4, -0.3829))$ |

**Step 3.** Determine the score function $\mathcal{S}(\mathfrak{F}_j)$ of overall assessment value $\mathfrak{F}_j$ $(j = 1, 2, 3, 4, 5)$ utilizing Eq (3.1). Score function of alternatives by G2TLBFWHM and 2TLBFWGHM operators are as follows:

$$\begin{aligned} \mathcal{S}^{HM}(F_1) &= (\flat_3, 0.2880), \mathcal{S}^{HM}(F_2) = (\flat_5, -0.4990), \mathcal{S}^{HM}(F_3) = (\flat_4, -0.4510), \\ \mathcal{S}^{HM}(F_4) &= (\flat_3, 0.4814), \mathcal{S}^{HM}(F_5) = (\flat_5, -0.2937). \end{aligned}$$

$$\begin{aligned} \mathcal{S}^{GHM}(F_1) &= (\flat_4, -0.4991), \mathcal{S}^{GHM}(F_2) = (\flat_4, 0.2817), \mathcal{S}^{GHM}(F_3) = (\flat_4, -0.1164), \\ \mathcal{S}^{GHM}(F_4) &= (\flat_3, 0.4260), \mathcal{S}^{GHM}(F_5) = (\flat_4, -0.2311). \end{aligned}$$

**Step 4.** Rank all alternatives $F_j$ $(j = 1, 2, 3, 4, 5)$ according to the score index. The ranking of all alternatives is as: $F_5 > F_2 > F_3 > F_4 > F_1$ and $F_2 > F_3 > F_5 > F_1 > F_4$ utilizing G2TLBFWHM and 2TLBFWGHM operators, respectively. Therefore, $F_5$ or $F_2$ is the best choice.

**Step 5.** End.

Different attribute weight values have a significant effect on the ranking of alternatives, as demonstrated in Tables 6 and 7. When $\varkappa_1^\star = 0.1, \varkappa_2^\star = 0.2, \varkappa_3^\star = 0.3,$ *and* $\varkappa_4^\star = 0.4$, the five alternatives are ranked $F_5 > F_2 > F_4 > F_3 > F_1$ in order of preference. In other words, $F_5$ is the best alternative. When $\varkappa_1^\star = 0.2, \varkappa_2^\star = 0.1, \varkappa_3^\star = 0.3,$ and $\varkappa_4^\star = 0.4$ , than the ranking is $F_5 > F_2 > F_4 > F_3 > F_1$, and still $F_5$ is the best alternative. When $\varkappa_1^\star = 0.3, \varkappa_2^\star = 0.1, \varkappa_3^\star = 0.2,$ and $\varkappa_4^\star = 0.4$ , than the ranking is $F_5 > F_2 > F_3 > F_4 > F_1$. Here is the slightest difference in the ranking of alternative, however $F_5$ is the best alternative. When $\varkappa_1^\star = 0.4, \varkappa_2^\star = 0.1, \varkappa_3^\star = 0.2,$ and $\varkappa_4^\star = 0.3$ , than the ranking is $F_5 > F_2 > F_3 > F_4 > F_1$ and $F_5$ is the best alternative, as shown in Table 6 based on G2TLBFWHM operator. As a consequence, in the decision-making process the attributes weight can be varied to get appropriate decision results. Similarly the different outcomes by utilizing 2TLBFWGHM operator are shown in Table 7.

**Table 6.** Influence of attribute weight $\varkappa^\star$ on alternative ranking utilizing G2TLBFWHM operator ($s = 2, t = 3$).

| Weights | Scores | Ranking |
|---|---|---|
| $\varkappa_1^\star = 0.1, \varkappa_2^\star = 0.2,$ $\varkappa_3^\star = 0.3, \varkappa_4^\star = 0.4$ | $\mathcal{S}_1^{HM} = (\flat_3, 0.3689), \mathcal{S}_2^{HM} = (\flat_4, 0.2697), \mathcal{S}_3^{HM} = (\flat_4, -0.2814),$ $\mathcal{S}_4^{HM} = (\flat_4, -0.2213), \mathcal{S}_5^{HM} = (\flat_4, 0.3436)$ | $F_5 > F_2 > F_4 > F_3 > F_1$ |
| $\varkappa_1^\star = 0.2, \varkappa_2^\star = 0.1,$ $\varkappa_3^\star = 0.3, \varkappa_4^\star = 0.4$ | $\mathcal{S}_1^{HM} = (\flat_3, 0.2348), \mathcal{S}_2^{HM} = (\flat_4, 0.2770), \mathcal{S}_3^{HM} = (\flat_4, -0.2690),$ $\mathcal{S}_4^{HM} = (\flat_4, -0.2562), \mathcal{S}_5^{HM} = (\flat_5, -0.4185)$ | $F_5 > F_2 > F_4 > F_3 > F_1$ |
| $\varkappa_1^\star = 0.3, \varkappa_2^\star = 0.1,$ $\varkappa_3^\star = 0.2, \varkappa_4^\star = 0.4,$ | $\mathcal{S}_1^{HM} = (\flat_3, 0.2133), \mathcal{S}_2^{HM} = (\flat_4, 0.2652)\mathcal{S}_3^{HM} = (\flat_4, -0.2444),$ $\mathcal{S}_4^{HM} = (\flat_4, -0.4446), \mathcal{S}_5^{HM} = (\flat_5, -0.3323)$ | $F_5 > F_2 > F_3 > F_4 > F_1$ |
| $\varkappa_1^\star = 0.4, \varkappa_2^\star = 0.1,$ $\varkappa_3^\star = 0.2, \varkappa_4^\star = 0.3,$ | $\mathcal{S}_1^{HM} = (\flat_3, 0.0671), \mathcal{S}_2^{HM} = (\flat_4, 0.4098), \mathcal{S}_3^{HM} = (\flat_4, -0.3131),$ $\mathcal{S}_4^{HM} = (\flat_3, 0.3958), \mathcal{S}_5^{HM} = (\flat_5, -0.0206)$ | $F_5 > F_2 > F_3 > F_4 > F_1$ |

**Table 7.** Influence of attribute weight $\varkappa^\star$ on alternative ranking utilizing 2TLBFWGHM operator ($s = 2, t = 3$).

| Weights | Scores | Ranking |
|---|---|---|
| $\varkappa_1^\star = 0.1, \varkappa_2^\star = 0.2,$ $\varkappa_3^\star = 0.3, \varkappa_4^\star = 0.4$ | $\mathcal{S}_1^{GHM} = (\flat_4, -0.3918), \mathcal{S}_2^{GHM} = (\flat_4, -0.0843), \mathcal{S}_3^{GHM} = (\flat_4, 0.0048),$ $\mathcal{S}_4^{GHM} = (\flat_4, -0.2370), \mathcal{S}_5^{GHM} = (\flat_3, 0.2781)$ | $F_3 > F_2 > F_4 > F_1 > F_5$ |
| $\varkappa_1^\star = 0.2, \varkappa_2^\star = 0.1,$ $\varkappa_3^\star = 0.3, \varkappa_4^\star = 0.4$ | $\mathcal{S}_1^{GHM} = (\flat_3, 0.4825), \mathcal{S}_2^{GHM} = (\flat_4, -0.0710), \mathcal{S}_3^{GHM} = (\flat_4, 0.0133),$ $\mathcal{S}_4^{GHM} = (\flat_4, -0.3175), \mathcal{S}_5^{GHM} = (\flat_3, 0.4242)$ | $F_3 > F_2 > F_4 > F_1 > F_5$ |
| $\varkappa_1^\star = 0.3, \varkappa_2^\star = 0.1,$ $\varkappa_3^\star = 0.2, \varkappa_4^\star = 0.4,$ | $\mathcal{S}_1^{GHM} = (\flat_3, 0.4177), \mathcal{S}_2^{GHM} = (\flat_4, 0.0094), \mathcal{S}_3^{GHM} = (\flat_4, 0.0358),$ $\mathcal{S}_4^{GHM} = (\flat_3, 0.4578), \mathcal{S}_5^{GHM} = (\flat_3, 0.4436)$ | $F_3 > F_2 > F_4 > F_5 > F_1$ |
| $\varkappa_1^\star = 0.4, \varkappa_2^\star = 0.1,$ $\varkappa_3^\star = 0.2, \varkappa_4^\star = 0.3,$ | $\mathcal{S}_1^{GHM} = (\flat_3, 0.2737), \mathcal{S}_2^{GHM} = (\flat_4, 0.1855), \mathcal{S}_3^{GHM} = (\flat_4, -0.0331),$ $\mathcal{S}_4^{GHM} = (\flat_3, 0.2684), \mathcal{S}_5^{GHM} = (\flat_4, -0.1974)$ | $F_2 > F_3 > F_5 > F_1 > F_4$ |

### *6.3. Parameter influence*

Surely, the parameters $s$ and $t$ have a great influence on the ranking results. The influence of parameters on score functions and ranking results based on G2TLBFWHM and 2TLBFWGHM operators are evaluated in this subsection. We fix the several values of $s$ and $t$, and evaluate the scores of the overall aggregation. Further, scores are used to rank the alternatives. In Tables 8 and 9 score values are evaluated by varying $s$ and $t$ based on G2TLBFWHM and 2TLBFGWHM operators, respectively. Then these scores are used to rank the alternatives. Ranking results from Table 10 are used to select the best alternative, $F_5$ and $F_2$ are the best alternatives based on G2TLBFWHM and 2TLBFWGHM operators, respectively.

As the values of $s$ and $t$ vary at the same time, the scores of the five alternatives change as well, resulting in an irregular change accordingly, as shown in Tables 8 and 9 based on G2TLBFWHM and 2TLBFWGHM operators. The change in values of $s$ and $t$ have a significant influence on the results of the alternatives ranking. Table 10 demonstrates that when $s$ and $t$ are changed, the ranking results are relatively stable, and the best alternative remained unchanged. The decision preference can be represented in the actual decision-making process by varying the values of $s$ and $t$ to obtain the best decision results.

**Table 8.** Score values by varying $s$ and $t$ based on the $G2TLBFWHM$ operator.

| Parameter values | Score functions | | | | |
|---|---|---|---|---|---|
| $s=t=0.5$ | $\mathcal{S}_1^{HM}=(\flat_2,\ 0.3985)$, | $\mathcal{S}_2^{HM}=(\flat_4,-0.3920)$, | $\mathcal{S}_3^{HM}=(\flat_3,-0.3138)$, | $\mathcal{S}_4^{HM}=(\flat_2,\ 0.4184)$, | $\mathcal{S}_5^{HM}=(\flat_4,-0.3323)$ |
| $s=t=1$ | $\mathcal{S}_1^{HM}=(\flat_3,-0.1830)$, | $\mathcal{S}_2^{HM}=(\flat_4,\ 0.0610)$, | $\mathcal{S}_3^{HM}=(\flat_3,\ 0.1018)$, | $\mathcal{S}_4^{HM}=(\flat_3,-0.1253)$, | $\mathcal{S}_5^{HM}=(\flat_4,\ 0.1457)$ |
| $s=t=2$ | $\mathcal{S}_1^{HM}=(\flat_3,\ 0.1731)$, | $\mathcal{S}_2^{HM}=(\flat_4,\ 0.4197)$, | $\mathcal{S}_3^{HM}=(\flat_3,\ 0.4407)$, | $\mathcal{S}_4^{HM}=(\flat_3,\ 0.3053)$, | $\mathcal{S}_5^{HM}=(\flat_5,-0.4105)$ |
| $s=3,t=4$ | $\mathcal{S}_1^{HM}=(\flat_3,\ 0.4369)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.3629)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.3229)$, | $\mathcal{S}_4^{HM}=(\flat_4,-0.3379)$, | $\mathcal{S}_5^{HM}=(\flat_5,-0.0659)$ |
| $s=4,t=3$ | $\mathcal{S}_1^{HM}=(\flat_3,\ 0.4359)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.3434)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.3369)$, | $\mathcal{S}_4^{HM}=(\flat_4,-0.3654)$, | $\mathcal{S}_5^{HM}=(\flat_5,-0.0288)$ |
| $s=5,t=6$ | $\mathcal{S}_1^{HM}=(\flat_4,-0.3663)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.2087)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.1665)$, | $\mathcal{S}_4^{HM}=(\flat_4,-0.1007)$, | $\mathcal{S}_5^{HM}=(\flat_5,\ 0.2306)$ |
| $s=6,t=5$ | $\mathcal{S}_1^{HM}=(\flat_4,-0.3639)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.2005)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.1730)$, | $\mathcal{S}_4^{HM}=(\flat_4,-0.1050)$, | $\mathcal{S}_5^{HM}=(\flat_5,\ 0.2448)$ |
| $s=7,t=8$ | $\mathcal{S}_1^{HM}=(\flat_4,-0.2402)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.1187)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.0700)$, | $\mathcal{S}_4^{HM}=(\flat_4,\ 0.0445)$, | $\mathcal{S}_5^{HM}=(\flat_5,\ 0.4174)$ |
| $s=8,t=7$ | $\mathcal{S}_1^{HM}=(\flat_4,-0.2385)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.1144)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.0732)$, | $\mathcal{S}_4^{HM}=(\flat_4,\ 0.0446)$, | $\mathcal{S}_5^{HM}=(\flat_5,\ 0.4241)$ |
| $s=1,t=9$ | $\mathcal{S}_1^{HM}=(\flat_4,-0.3257)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.2461)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.1150)$, | $\mathcal{S}_4^{HM}=(\flat_4,\ 0.0041)$, | $\mathcal{S}_5^{HM}=(\flat_5,\ 0.1821)$ |
| $s=9,t=1$ | $\mathcal{S}_1^{HM}=(\flat_4,-0.3293)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.1503)$, | $\mathcal{S}_3^{HM}=(\flat_4,-0.1849)$, | $\mathcal{S}_4^{HM}=(\flat_4,-0.1453)$, | $\mathcal{S}_5^{HM}=(\flat_5,0.3953)$ |
| $s=t=10$ | $\mathcal{S}_1^{HM}=(\flat_4,-0.1355)$, | $\mathcal{S}_2^{HM}=(\flat_5,-0.0439)$, | $\mathcal{S}_3^{HM}=(\flat_4,0.0097)$, | $\mathcal{S}_4^{HM}=(\flat_4,0.1619)$, | $\mathcal{S}_5^{HM}=(\flat_6,-0.4265)$ |

**Table 9.** Score values by varying $s$ and $t$ based on the 2TLBFWGHM operator.

| Parameter values | Score functions | | | | |
|---|---|---|---|---|---|
| $s=t=0.5$ | $\mathcal{S}_1^{GHM}=(\flat_4,0.4425)$, | $\mathcal{S}_2^{GHM}=(\flat_5,0.2342)$, | $\mathcal{S}_3^{GHM}=(\flat_5,-0.2274)$, | $\mathcal{S}_4^{GHM}=(\flat_4,0.3902)$, | $\mathcal{S}_5^{GHM}=(\flat_5,0.1811)$ |
| $s=t=1$ | $\mathcal{S}_1^{GHM}=(\flat_4,-0.0219)$, | $\mathcal{S}_2^{GHM}=(\flat_5,-0.2048)$, | $\mathcal{S}_3^{GHM}=(\flat_4,0.3335)$, | $\mathcal{S}_4^{GHM}=(\flat_4,-0.0868)$, | $\mathcal{S}_5^{GHM}=(\flat_5,-0.3859)$ |
| $s=t=2$ | $\mathcal{S}_1^{GHM}=(\flat_4,-0.4094)$, | $\mathcal{S}_2^{GHM}=(\flat_4,0.4340)$, | $\mathcal{S}_3^{GHM}=(\flat_4,-0.0223)$, | $\mathcal{S}_4^{GHM}=(\flat_4,-0.4951)$, | $\mathcal{S}_5^{GHM}=(\flat_4,0.0061)$ |
| $s=3,t=4$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.3385)$, | $\mathcal{S}_2^{GHM}=(\flat_4,0.1488)$, | $\mathcal{S}_3^{GHM}=(\flat_4,-0.2615)$, | $\mathcal{S}_4^{GHM}=(\flat_3,0.2222)$, | $\mathcal{S}_5^{GHM}=(\flat_3,0.4955)$ |
| $s=4,t=3$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.3027)$, | $\mathcal{S}_2^{GHM}=(\flat_4,0.2111)$, | $\mathcal{S}_3^{GHM}=(\flat_4,-0.2736)$, | $\mathcal{S}_4^{GHM}=(\flat_3,0.1584)$, | $\mathcal{S}_5^{GHM}=(\flat_4,-0.4688)$ |
| $s=5,t=6$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.1412)$, | $\mathcal{S}_2^{GHM}=(\flat_4,-0.0319)$, | $\mathcal{S}_3^{GHM}=(\flat_4,-0.4497)$, | $\mathcal{S}_4^{GHM}=(\flat_3,-0.0616)$, | $\mathcal{S}_5^{GHM}=(\flat_3,0.1477)$ |
| $s=6,t=5$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.1198)$, | $\mathcal{S}_2^{GHM}=(\flat_4,0.0002)$, | $\mathcal{S}_3^{GHM}=(\flat_4,-0.4552)$, | $\mathcal{S}_4^{GHM}=(\flat_3,-0.0884)$, | $\mathcal{S}_5^{GHM}=(\flat_3,0.1650)$ |
| $s=7,t=8$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.0215)$, | $\mathcal{S}_2^{GHM}=(\flat_4,-0.1589)$, | $\mathcal{S}_3^{GHM}=(\flat_3,0.4237)$, | $\mathcal{S}_4^{GHM}=(\flat_3,-0.2577)$, | $\mathcal{S}_5^{GHM}=(\flat_3,-0.0660)$ |
| $s=8,t=7$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.0074)$, | $\mathcal{S}_2^{GHM}=(\flat_4,-0.1413)$, | $\mathcal{S}_3^{GHM}=(\flat_3,0.4217)$, | $\mathcal{S}_4^{GHM}=(\flat_3,-0.2677)$, | $\mathcal{S}_5^{GHM}=(\flat_3,-0.0561)$ |
| $s=1,t=9$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.2199)$, | $\mathcal{S}_2^{GHM}=(\flat_4,-0.1981)$, | $\mathcal{S}_3^{GHM}=(\flat_4,-0.4622)$, | $\mathcal{S}_4^{GHM}=(\flat_3,0.0421)$, | $\mathcal{S}_5^{GHM}=(\flat_3,0.0067)$ |
| $s=9,t=1$ | $\mathcal{S}_1^{GHM}=(\flat_3,0.0159)$, | $\mathcal{S}_2^{GHM}=(\flat_4,0.1304)$, | $\mathcal{S}_3^{GHM}=(\flat_3,0.4816)$, | $\mathcal{S}_4^{GHM}=(\flat_3,-0.2845)$, | $\mathcal{S}_5^{GHM}=(\flat_3,0.2408)$ |
| $s=t=10$ | $\mathcal{S}_1^{GHM}=(\flat_3,-0.0821)$, | $\mathcal{S}_2^{GHM}=(\flat_4,-0.2708)$, | $\mathcal{S}_3^{GHM}=(\flat_3,0.3125)$, | $\mathcal{S}_4^{GHM}=(\flat_3,-0.4277)$, | $\mathcal{S}_5^{GHM}=(\flat_3,-0.2376)$ |

**Table 10.** Ranking by varying $s$ and $t$ based on G2TLBFWHM and 2TLBFWGHM operators.

| Parameters | G2TLBFWHM operator | 2TLBFWGHM operator |
|---|---|---|
| $s=t=0.5$ | $F_5>F_2>F_3>F_4>F_1$ | $F_2>F_5>F_3>F_1>F_4$ |
| $s=t=1$ | $F_5>F_2>F_3>F_4>F_1$ | $F_2>F_5>F_3>F_1>F_4$ |
| $s=t=2$ | $F_5>F_2>F_3>F_4>F_1$ | $F_2>F_5>F_3>F_1>F_4$ |
| $s=3,t=4$ | $F_5>F_2>F_3>F_4>F_1$ | $F_2>F_3>F_5>F_1>F_4$ |
| $s=4,t=3$ | $F_5>F_2>F_3>F_4>F_1$ | $F_2>F_3>F_5>F_1>F_4$ |
| $s=5,t=6$ | $F_5>F_2>F_4>F_3>F_1$ | $F_2>F_3>F_5>F_1>F_4$ |
| $s=6,t=5$ | $F_5>F_2>F_4>F_3>F_1$ | $F_2>F_3>F_5>F_1>F_4$ |
| $s=7,t=8$ | $F_5>F_2>F_4>F_3>F_1$ | $F_2>F_3>F_1>F_5>F_4$ |
| $s=8,t=7$ | $F_5>F_2>F_4>F_3>F_1$ | $F_2>F_3>F_1>F_5>F_4$ |
| $s=1,t=9$ | $F_5>F_2>F_4>F_3>F_1$ | $F_2>F_3>F_1>F_4>F_5$ |
| $s=9,t=1$ | $F_5>F_2>F_4>F_3>F_1$ | $F_2>F_3>F_5>F_1>F_4$ |
| $s=t=10$ | $F_5>F_2>F_4>F_3>F_1$ | $F_2>F_3>F_1>F_5>F_4$ |

Based on the G2TLBFWHM operator, the scores vary accordingly, corresponding to different values of $s$ and $t$ for alternatives $F_j(j = 1, 2, 3, 4, 5)$ as shown in Figures 2–6. In Figure 7 the value of parameter $s = 3$ is fixed, and $t$ is varied from 1 to 10. The value of parameter $t = 3$ is fixed, and $s$ is varied from 1 to 10 in Figure 8.

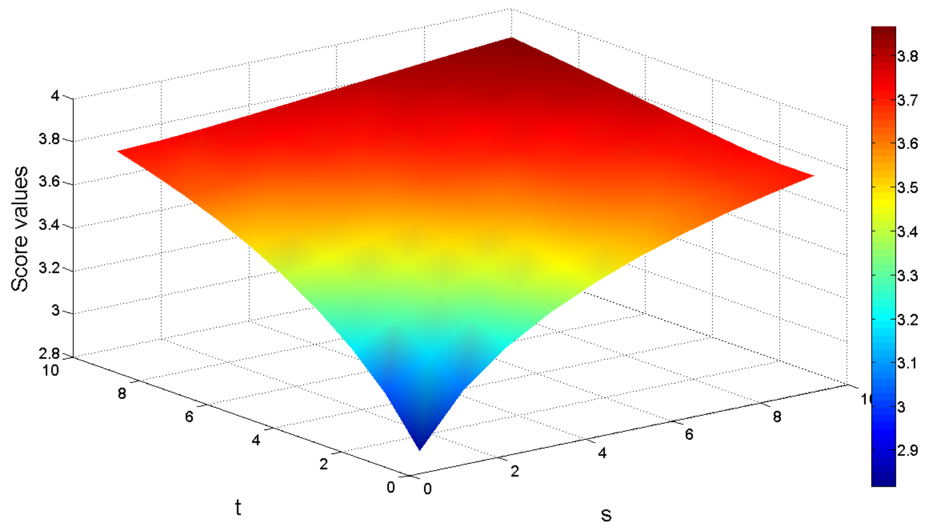


**Figure 2.** Scores of photovoltaic cell $F_1$ based on the G2TLBFWHM operator when $s, t \in (1, 10)$.

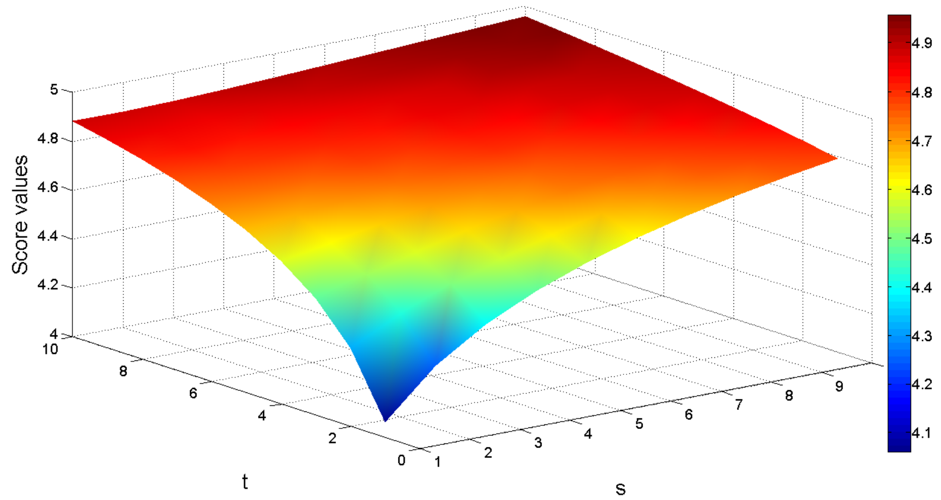


**Figure 3.** Scores of photovoltaic cell $F_2$ based on the G2TLBFWHM operator when $s, t \in (1, 10)$.

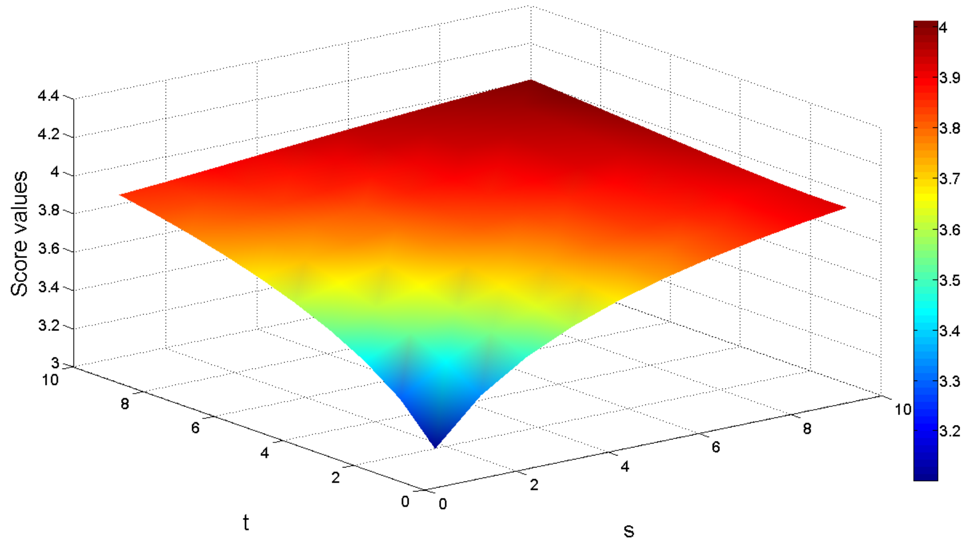


**Figure 4.** Scores of photovoltaic cell $F_3$ based on the G2TLBFWHM operator when $s, t \in (1, 10)$.

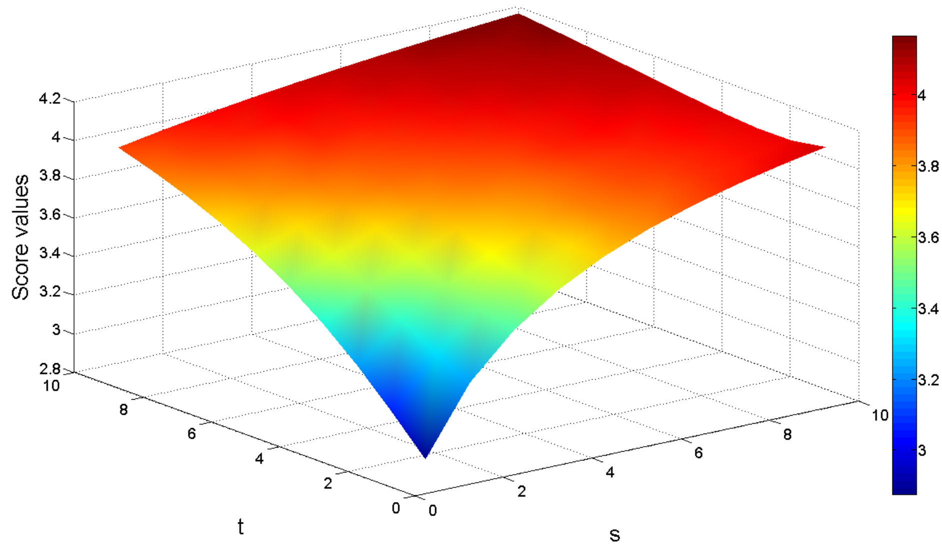


**Figure 5.** Scores of photovoltaic cell $F_4$ based on the G2TLBFWHM operator when $s, t \in (1, 10)$.

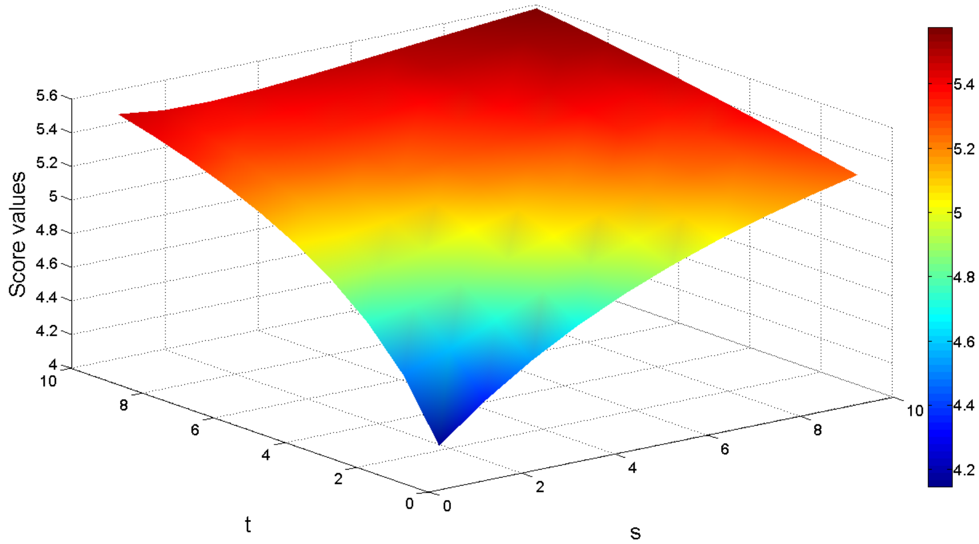


**Figure 6.** Scores of photovoltaic cell $F_5$ based on the G2TLBFWHM operator when $s, t \in (1, 10)$.

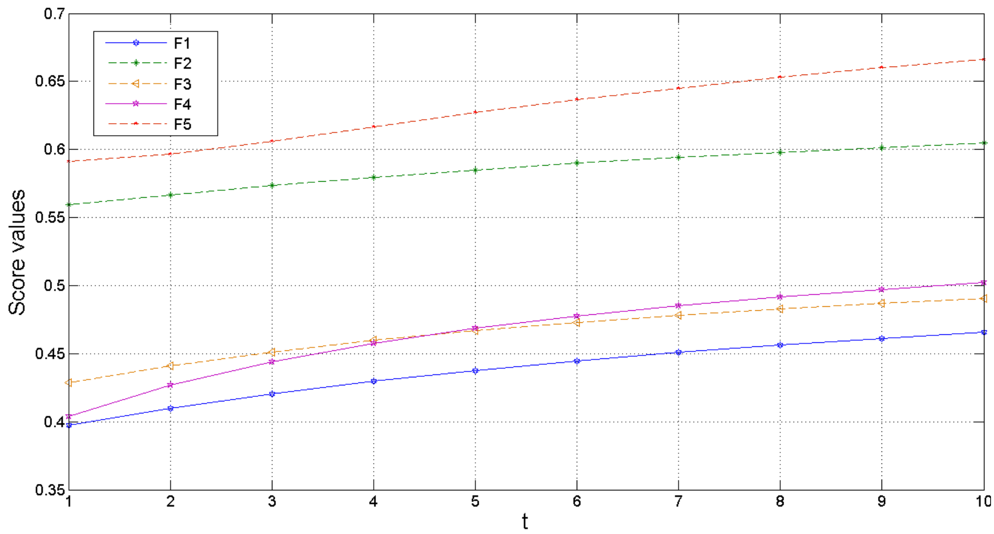


**Figure 7.** Scores of photovoltaic cells $F_j (j = 1, 2, \ldots, 5)$ based on the G2TLBFWHM operator when $s = 3, t \in (1, 10)$.

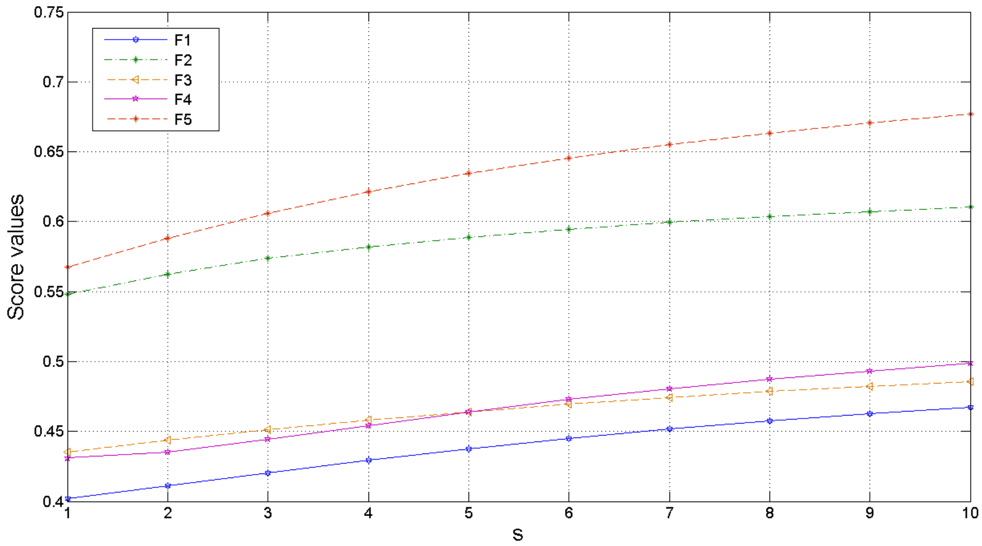


**Figure 8.** Scores of photovoltaic cells $F_j(j = 1, 2, \ldots, 5)$ based on the G2TLBFWHM operator when $t = 3$, $s \in (1, 10)$.

Based on the 2TLBFWGHM operator, the scores vary accordingly, corresponding to different values of $s$ and $t$ for alternatives $F_j(j = 1, 2, 3, 4, 5)$ as shown in Figures 9–13. In Figure 14 the value of parameter $s = 3$ is fixed, and $t$ is varied from 1 to 10. In Figure 15 the value of parameter $t = 3$ is fixed, and $s$ is varied from 1 to 10.

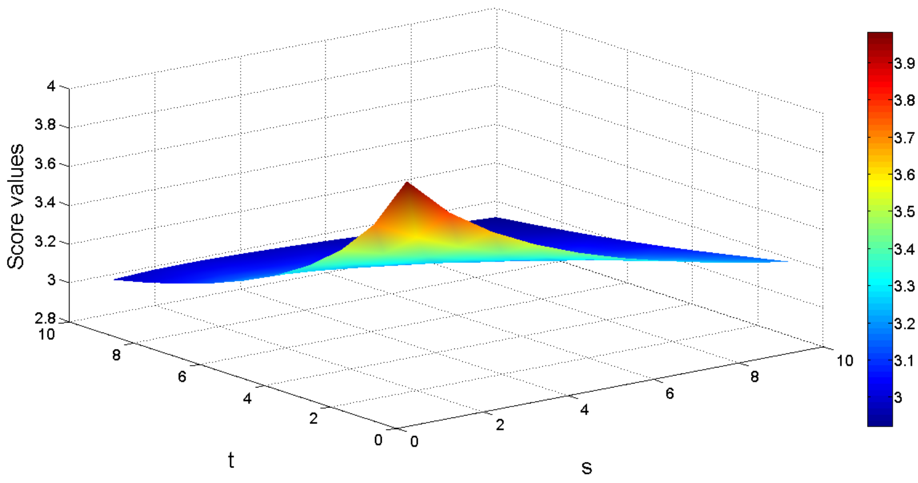


**Figure 9.** Scores of photovoltaic cell $F_1$ based on the 2TLBFWGHM operator when $s, t \in (1, 10)$.

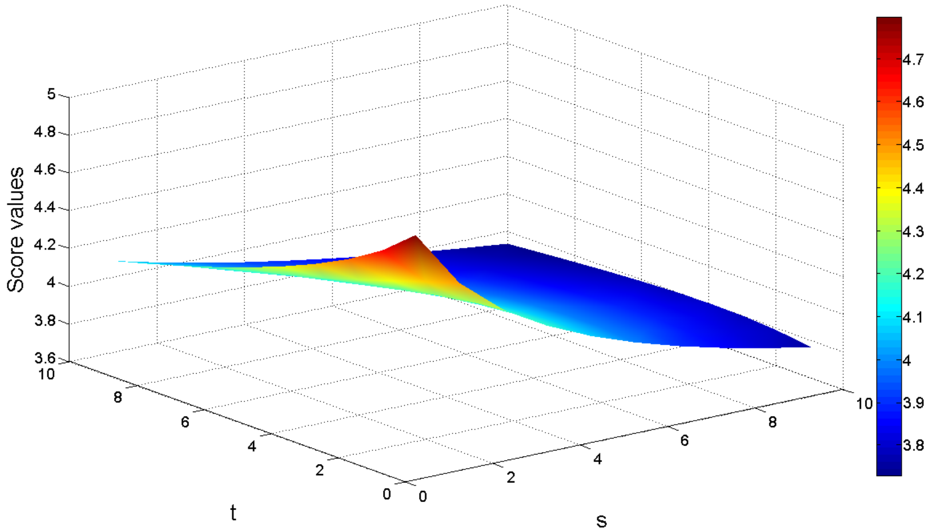


**Figure 10.** Scores of photovoltaic cell $F_2$ based on the 2TLBFWGHM operator when $s, t \in (1, 10)$.

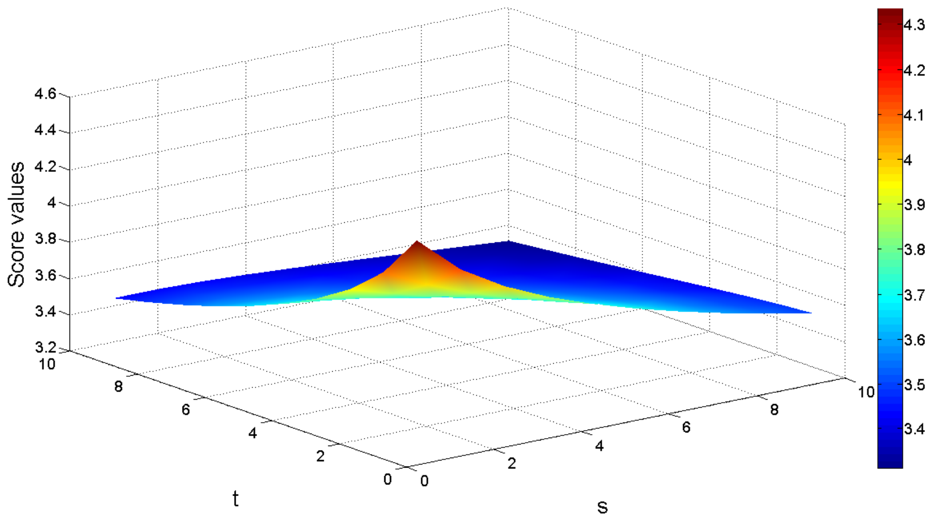


**Figure 11.** Scores of photovoltaic cell $F_3$ based on the 2TLBFWGHM operator when $s, t \in (1, 10)$.

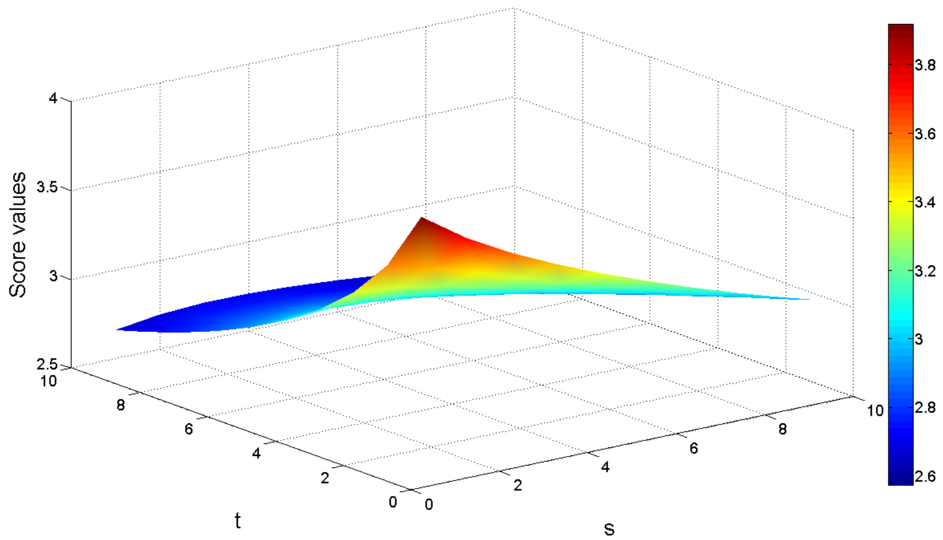


**Figure 12.** Scores of photovoltaic cell $F_4$ based on the 2TLBFWGHM operator when $s, t \in (1, 10)$.

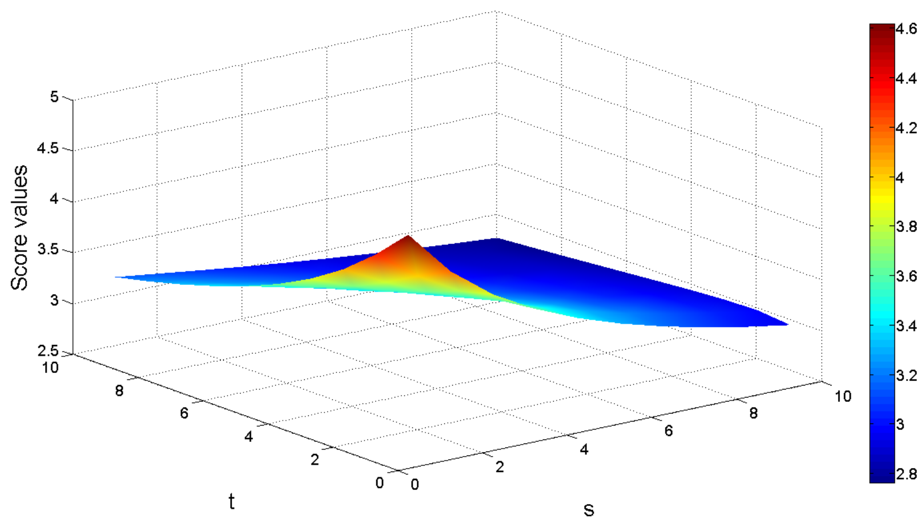


**Figure 13.** Scores of photovoltaic cell $F_5$ based on the 2TLBFWGHM operator when $s, t \in (1, 10)$.

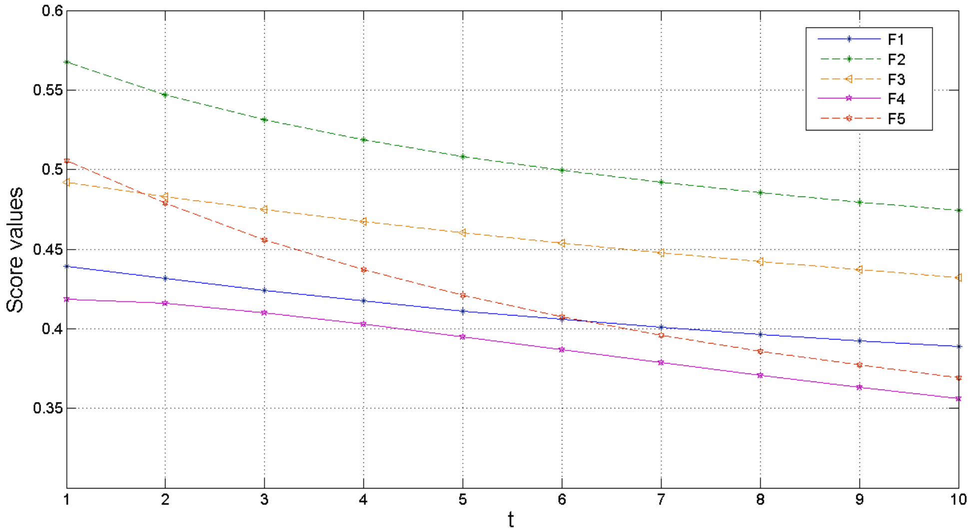


**Figure 14.** Scores of photovoltaic cells $F_j(j = 1, 2, \ldots, 5)$ based on the 2TLBFWGHM operator when $s = 3, t \in (1, 10)$.

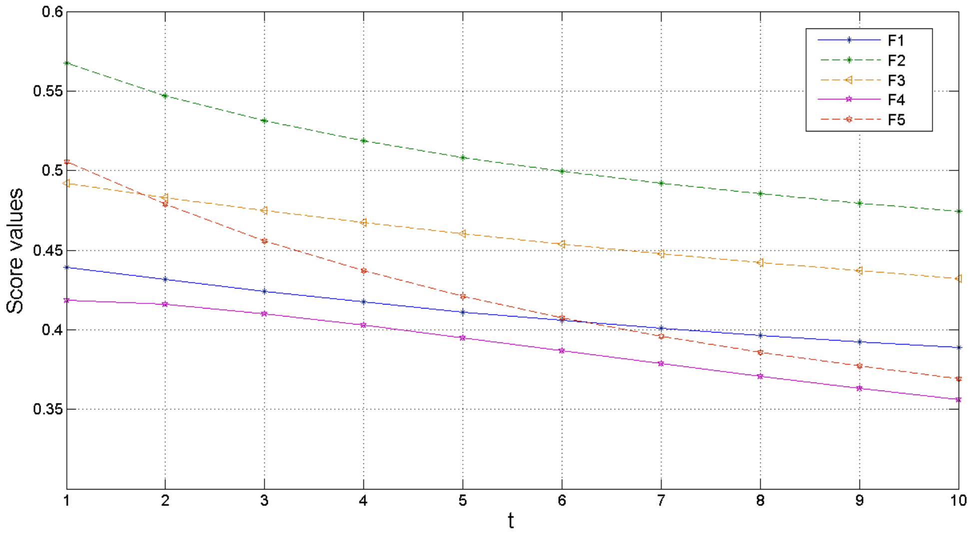


**Figure 15.** Scores of photovoltaic cells $F_j(j = 1, 2, \ldots, 5)$ based on the 2TLBFWGHM operator when $t = 3, s \in (1, 10)$.

When the parameters $s$ and $t$ are assigned the same values, the ranking results of the G2TLBFWHM and 2TLBFWGHM operators are not the same, and the score values vary differently. G2TLBFWHM and 2TLBFWGHM operators consider the relationship between two input arguments. These results indicate that the proposed methods are flexible. In actual decision-making, the parameters could be varied based on DM's preferences. The gloomy decision-maker could choose smaller parameters, whereas the optimistic decision-maker could choose larger parameters. Consequently, these results indicate that the suggested G2TLBFWHM and 2TLBFWGHM operators are more flexible and adaptable. Also the proposed strategy are most effective, realistic, and sufficient to address real-world MAGDM problems.

### 6.4. Comparative analysis

The ability to consider the interrelationship among the 2TLBFNs is a unique feature of the G2TLBFWHM and 2TLBFWGHM operators. To demonstrate the effectiveness of the suggested operators, we provide comparative analysis. To verify the validity of the developed approach, we utilize different approaches to solve the above mentioned MAGDM problem in Subsection 6.1. These methods include the 2TLBFWA operator, 2TLBFWG operator, the 2TL bipolar fuzzy weighted Hamy mean [37] (2TLBFWHM) operator, the 2TL bipolar fuzzy weighted dual Hamy mean [38] (2TLBFWDHM) operator, the 2TL bipolar fuzzy weighted Maclaurin symmetric mean [39] (2TLBFWMSM) operator, and the 2TL bipolar fuzzy weighted dual Maclaurin symmetric mean [40] (2TLBFWDMSM) operator. Detailed evaluation results gained using different MAGDM approaches are given in Tables 11–16.

**Table 11.** The outcomes utilizing 2TLBFWA operator.

| | 2TLBFWA | Scores values | The order |
|---|---|---|---|
| $F_1$ | $((\flat_3, -0.2530), (\flat_4, -0.2456))$ | $(\flat_3, 0.4963)$ | $F_5 > F_2 > F_3 > fF_4 > F_1$ |
| $F_2$ | $((\flat_4, 0.1497), (\flat_2, 0.3458))$ | $(\flat_5, -0.0981)$ | |
| $F_3$ | $((\flat_3, -0.2482), (\flat_3, 0.1161))$ | $(\flat_4, -0.1822)$ | |
| $F_4$ | $((\flat_3, 0.1559), (\flat_4, -0.0348))$ | $(\flat_4, -0.4046)$ | |
| $F_5$ | $((\flat_4, -0.1614), (\flat_2, -0.2589))$ | $(\flat_5, 0.0488)$ | |

**Table 12.** The outcomes utilizing 2TLBFWG operator.

| | 2TLBFWG | Scores values | The order |
|---|---|---|---|
| $F_1$ | $((\flat_2, 0.3982), (\flat_4, 0.1699))$ | $(\flat_3, 0.1142)$ | $F_2 > F_5 > F_3 > F_1 > F_2$ |
| $F_2$ | $((\flat_4, -0.4867), (\flat_3, 0.4502))$ | $(\flat_4, 0.0316)$ | |
| $F_3$ | $((\flat_2, 0.4733), (\flat_3, 0.3939))$ | $(\flat_4, -0.4603)$ | |
| $F_4$ | $((\flat_2, 0.4698), (\flat_4, 0.4454))$ | $(\flat_3, 0.0122)$ | |
| $F_5$ | $((\flat_3, -0.1526), (\flat_3, 0.3604))$ | $(\flat_4, -0.2565)$ | |

**Table 13.** The outcomes utilizing 2TLBFWHM operator.

| | 2TLBFWHM | Scores values | The order |
|---|---|---|---|
| $F_1$ | $((\flat_6, 0.1111), (\flat_1 0.2620))$ | $(\flat_6, 0.4246)$ | $F_2 > F_5 > F_3 > F_4 > F_1$ |
| $F_2$ | $((\flat_7, -0.2399), (\flat_1 - 0.3123))$ | $(\flat_7, 0.0362)$ | |
| $F_3$ | $((\flat_6, 0.0880), (\flat_1 - 0.0719))$ | $(\flat_7, -0.4201)$ | |
| $F_4$ | $((\flat_6, 0.2394), (\flat_1 0.3353))$ | $(\flat_6, 0.4520)$ | |
| $F_5$ | $((\flat_7, -0.4646), (\flat_1 - 0.4275))$ | $(\flat_7, -0.0185)$ | |

**Table 14.** The outcomes utilizing 2TLBFWDHM operator.

| | 2TLBFWDHM | Scores values | The order |
|---|---|---|---|
| $F_1$ | $((b_1, -0.3159), (b_7, -0.2518))$ | $(b_1, -0.0321)$ | $F_5 > F_2 > F_3 > F_1 > F_4$ |
| $F_2$ | $((b_1, 0.0870), (b_6, 0.4438))$ | $(b_1, 0.3216)$ | |
| $F_3$ | $((b_1, -0.2613), (b_6, 0.4293))$ | $(b_1, 0.1547)$ | |
| $F_4$ | $((b_1, -0.2369), (b_7, -0.1422))$ | $(b_1, -0.0473)$ | |
| $F_5$ | $((b_1, -0.0449), (b_6, -0.0374))$ | $(b_1, 0.4962)$ | |

**Table 15.** The outcomes utilizing 2TLBFWMSM operator.

| | 2TLBFWMSM | Scores values | The order |
|---|---|---|---|
| $F_1$ | $((b_6, 0.1130), (b_1, 0.2595))$ | $(b_6, 0.4267)$ | $F_2 > F_5 > F_3 > F_4 > F_1$ |
| $F_2$ | $((b_7, -0.2398), (b_1, -0.3126))$ | $(b_7, 0.0364)$ | |
| $F_3$ | $((b_6, 0.0917), (b_1, -0.0734))$ | $(b_7, -0.4175)$ | |
| $F_4$ | $((b_6, 0.2487), (b_1, 0.3320))$ | $(b_6, 0.4584)$ | |
| $F_5$ | $((b_7, -0.4588), (b_1, -0.4291))$ | $(b_7, -0.0149)$ | |

**Table 16.** The outcomes utilizing 2TLBFWDMSM operator.

| | 2TLBFWDMSM | Scores values | The order |
|---|---|---|---|
| $F_1$ | $((b_1, -0.3161), (b_7, -0.2470))$ | $(b_1, -0.0346)$ | $F_5 > F_2 > F_3 > F_1 > F_4$ |
| $F_2$ | $((b_1, 0.0852), (b_6, 0.4470))$ | $(b_1, 0.3191)$ | |
| $F_3$ | $((b_1, -0.2622), (b_6, 0.4300))$ | $(b_1, 0.1539)$ | |
| $F_4$ | $((b_1, -0.2383), (b_7, -0.1403))$ | $(b_1, -0.0490)$ | |
| $F_5$ | $((b_1, -0.0483), (b_6, -0.0121))$ | $(b_1, 0.4819)$ | |

We compare our suggested approach to other approaches such as the 2TLBFWA and 2TLBFWG operators. Tables 11,12 indicates that the 2TLBFWA and 2TLBFWG operators are unable to provide the interrelationship between the 2TLBFNs. In the above example, we should consider not only the attribute values of each photovoltaic cells, but also the correlations between these attributes while selecting the most optimal photovoltaic cell. Approaches based on 2TLBFWA and 2TLBFWG operators are ineffective in dealing with this issue. Our approach is more appropriate for dealing with this issue than, 2TLBFWA and 2TLBFWG operators as it can capture parameter correlations. The interrelationship between the 2TLBFNs is evaluated by the G2TLBFHM, G2TLBFWHM, 2TLBFGHM, and 2TLBFWGHM operators. In addition, 2TLBFWA and 2TLBFWG operators do not have any parameters, but our purposed operator has two parameters, that make our operator more flexible and adaptable. Here, we propose a novel approach for MAGDM problems based on 2TLBFS, which is an influential approach for demonstrating and indicating DM's assessments. As a result, compared with different approaches, our approach has some benefits and superiorities.

The ranking effects of the above approaches are slightly different, as shown in the above calculations, but still, the best alternative is $F_2$ or $F_5$. It shows that the G2TLBFWHM and 2TLBFWGHM operators are more effective and appropriate with 2TLBFNs for MAGDM problems. The 2TLBFS is more general and contains more information in the MAGDM process. As a result, in MAGDM, our proposed method gives more general and powerful information.

**(1)** The drawback of 2TLBFWA and 2TLBFWG is that in the process of fusion the interrelationship between arguments is not taken into account. In other words, they assume that all attributes are identical, which is to some extent inaccurate. To select the most desirable photovoltaic cell, in an illustrative example, we not only take into account the attribute values of each alternative but also their interrelationship. As a result, the 2TLBFWA and 2TLBFWG techniques are inapplicable to this situation. Our technique is more practical to tackle this problem than 2TLBFWA and 2TLBFWG techniques because it can easily capture interrelationships among multi-input arguments.

**(2)** Since both G2TLBFWHM and 2TLBFWGHM operators can consider the correlation among multi-input arguments, the optimal alternative is selected in the same way, demonstrating the rationality and validity of the proposed operators. Furthermore, 2TLBFWHM, 2TLBFWDHM, 2TLBFWMSM, and 2TLBFWDMSM operators do not reduce the effects of extremely strong or weak ranking outcomes, whereas our suggested method does. This shows our proposed technique is more straightforward and superior.

## 7. Conclusions

Choosing the optimal photovoltaic cell is essential for a company's board of directors. The selection of an appropriate photovoltaic cell not only increases profit but also reduces expenditures. As a result, choosing the best photovoltaic cell is an important MAGDM problem. In comparison to the classical and FSs, a BFS gives better precision, flexibility, and compatibility for the system. In this article, we investigated the MAGDM problem by combining the concepts of 2TL terms and HM operators with BF numbers. As a result of being inspired by the generalized HM operator and the geometric HM operator, we introduced some G2TLBFHM and 2TLBFGHM aggregation operators, as well as their weighted forms. The G2TLBFWHM and 2TLBFWGHM operators were different from other operators not only by accommodating the 2TLBFNs as well as by considering the interdependent phenomena among the arguments. Providing our operators to have a broader range of practical application potentials, a numerical example has been provided to evaluate the developed approach and demonstrate the feasibility and usefulness of the suggested method. By using the illustrated example, we can see that the parameters of the aggregation operators influence the ranking of alternatives. The significance of the parameters $s$ and $t$ on decision-making results and comparative analysis have been demonstrated.

On the other hand, there are still some limitations in this research study. Firstly, this work only deals with the aggregation of 2TLBFNs. Secondly, this research is only applied to the evaluation of photovoltaic cells with 2TLBFS. In fact, it has broad implications for assessment methods and related fields such as cognitive computation, engineering, natural and artificial cognitive systems, and management applications.s To overcome discussed limitations, in the future work, we will integrate this approach with other generalized FSs, including dual hesitant $q$-ROF 2-tuple linguistic sets, probability hesitant FSs, cloud approaches, and so on, to enlarge the area for expression of assessment results, adapt to a wider range of assessment environments, and increase the versatility of the strategy. Furthermore, we will investigate more aggregation operators for fusing 2TLBFNs, such as the 2TLBF Bonferroni operators, the 2TLBF power Bonferroni operators, and the 2TLBF Muirhead mean operators, which can capture the interrelationships among given input attributes.

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**Conflict of interest**

The authors declare there is no conflict of interest.

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