Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 19(12): 13988–14028
DOI: 10.3934/mbe.2022652
Received: 01 August 2022
Revised: 05 September 2022
Accepted: 07 September 2022
Published: 22 September 2022

Research article

Asymptotic stability of spiky steady states for a singular chemotaxis model with signal-suppressed motility

Xu Song and Jingyu Li*

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

* **Correspondence:** Email: lijy645@nenu.edu.cn.

Abstract: We study the nonlinear stability of spiky solutions to a chemotaxis model of consumption type with singular signal-suppressed motility in the half space. We show that, when the no-flux boundary condition for the bacteria density and the nonhomogeneous Dirichlet boundary condition for the nutrient are prescribed, this chemotaxis model admits a unique smooth spiky steady state, and it is nonlinearly stable under appropriate perturbations. The challenge of the problem is that there are two types of singularities involved in the model: one is the logarithmic singularity of the sensitive function; and the other is the inverse square singularity of the motility. We employ a Cole-Hopf transformation to relegate the former singularity to a nonlocality that can be resolved by the method of anti-derivative. To deal with the latter singularity, we construct an approximate system that retains a key structure of the original singular system in the local theory, and develop a new strategy, which combines a weighted elliptic estimate and the weighted energy estimate, to establish a priori estimate in the global theory.

Keywords: chemotaxis; singular sensitivity; signal-suppressed motility; spike; asymptotic stability

1. Introduction

In a series of works [1,2], Kim and his collaborators introduced the following chemotaxis model

$$\begin{cases} u_t = \left(\frac{1}{w}\left(\frac{u}{w}\right)_x\right)_x, \\ w_t = dw_{xx} - \kappa(w)u, \end{cases} \tag{1.1}$$

where u is the bacterial density and w is the concentration of nutrient. $d \geq 0$ is the diffusion rate of nutrient. $\kappa(w) \geq 0$ is the consumption rate. Typical examples of $\kappa(w)$ include $\kappa(w) = w^m$ with $m \geq 0$.

System (1.1) is an alternative model to describe the propagation of traveling band of bacteria observed in the experiment of Adler [3]. Compared with the classical Keller-Segel system [4], this model is rigorously derived from the notion of "metric of food", and brings the theory of Riemannian geometry to the field of chemotaxis. Choi and Kim [1] have proved that system (1.1) with $d = 0$ can generate traveling bands and traveling fronts under various assumptions on $\kappa(w)$. They also generalized their results of [1] to the models with porous medium diffusion for the bacterial density, and showed that there exist compactly supported traveling waves for chemotaxis. Ahn, Choi and Yoo [5] proved the global existence of strong solutions of Cauchy problem of system (1.1) if the initial value of w has positive lower bound. Very recently, they [6] have generalized this result to the case where w, with infinite initial mass, can be zero at spatial infinity.

In this paper, we are interested in the existence and stability of spiky patterns to system (1.1). We assume that the consumption rate is linear for simplicity, and write system (1.1) as

$$\begin{cases} u_t = \left(\dfrac{1}{w^2}\left(u_x - \dfrac{uw_x}{w}\right)\right)_x, \\ w_t = dw_{xx} - wu. \end{cases} \tag{1.2}$$

We shall consider the system in the half-space $\mathbb{R}_+ = [0, \infty)$, with the following initial value

$$(u, w)(x, 0) = (u_0(x), w_0(x)), \tag{1.3}$$

and boundary conditions

$$\begin{cases} (u_x - \frac{uw_x}{w})(0, t) = 0, \ w(0, t) = b, \\ (u, w)(+\infty, t) = (0, 0). \end{cases} \tag{1.4}$$

where $b > 0$ is a constant. That means we prescribe no-flux boundary condition for the bacterial density and saturated boundary condition for the oxygen. This kind of boundary conditions have also been used in a chemotaxis-fluid model to describe the formation of concentration patterns for aerobic bacteria observed in the experiment of [7].

System (1.2) is actually a chemotaxis model with signal-suppressed motility. In other words, the diffusion rate of the bacterial density is monotonically decreasing as the concentration of the signal increases. There are several analytical works for the chemotaxis model of self-aggregation type with signal-suppressed motility in bounded domains. See [8, 9] for the global existence of classical solutions if the motility function satisfies the power law, [10,11] for the existence of critical mass generating blowup if the motility is an exponential function, [12] for the formation of spiky patterns. In contrast, system (1.2) is of consumption type. That is the chemical signal w is consumed by the bacteria u. It turns out that such two types of chemotaxis model may exhibit different dynamics. Indeed, one can easily verify (following the argument of Proposition 2.1 of [13]) that if $b = 0$ or w satisfies homogeneous Neumann boundary in the half space, then system (1.2) only has constant steady states, and no pattern exists. In other words, it is the nonhomogenous boundary condition that generates spiky patterns. Such phenomenon is quite different from the solution structures of chemotaxis model of self-aggregation type for which the intrinsic mechanics of chemotactic interaction generates spiky patterns (see [14, 15]). Furthermore, Tao [16] showed that, under homogeneous Neumann boundary conditions in bounded domains, the multidimensional classical chemotaxis model of consumption type has a unique global bounded solutions under suitable assumptions on the initial data w_0 and the chemotactic coefficient. In particular, the global existence or blow-up of solutions is independent of the initial data u_0. This study indicates that the chemotaxis model of consumption type posses another different property from the one of self-aggregation type since the latter has the well-known critical mass on u_0 for blow-up in dimension 2. This work subsequently led to various generalizations. Baghaei and Khelghati [17] improved the results of [16] to a larger set of w_0 and chemotactic coefficient. Frassu and Viglialoro [18] further generalized the works of [16] to the models with indirect signal consumption. Recently, Li and Zhao [19] and Wang [20] proved that under homogeneous Neumann boundary conditions, the chemotaxis-consumption system with regular signal-dependent motility also has global bounded solutions under some assumptions on w_0. It is worth mentioning that for the chemotaxis-consumption system with logarithmic sensitivity, Winkler [21, 22] introduced the notion of renormalized solutions to handle the singularity in the study of global existence of large solutions.

There are some studies on the dynamics of classical chemotaxis model of consumption type with nonhomogeneous boundary conditions. In the one dimensional case, Hong and Wang [23] studied the stability of steady state to the minimal model with Dirichlet boundary condition for the nutrient in bounded domains; Carrillo, Li and Wang [13] obtained the stability of steady state to the model with constant motility and logarithmic singular sensitivity in the half space. In the multidimensional case, Braukhoff and Lankeit [24] proved the existence and uniqueness of steady state to the minimal model with nonhomogeneous Robin boundary condition, while Lee, Wang and Yang [25] obtained similar results for the minimal model with Dirichlet boundary condition, and they further analyzed the boundary layer phenomena. Recently, Fuest, Lankeit and Mizukami [26] further showed the stability of steady state for the minimal parabolic-elliptic model on the base of the works on the steady state obtained in [24].

One can observe from the boundary condition $w(+\infty, t) = 0$ that, in contrast with the models studied in the above mentioned works, system (1.2) is actually a chemotaxis model with singular sensitivity and singular motility. In this paper, we shall develop some new strategies to overcome analytical difficulties caused by the coupling of nonhomogenous boundary condition and singularities. And we obtain the following results:

(1) system (1.2)–(1.4) admits a unique steady state (U, W), and $U \to \lambda\delta(x)$ as $d \to 0$, where $\delta(x)$ is the Dirac function and λ is the initial bacterial density, i.e., $\lambda = \int_0^\infty u_0(x)dx$;

(2) this spiky steady state (U, W) is asymptotically stable in the sense that if the initial data (u_0, w_0) is a small perturbation of (U, W) in some topology, then the solution (u, w) will converge to (U, W) time asymptotically.

Following the argument of [13], one can easily show that result (1) holds. The aim of this paper is to show the nonlinear stability of steady state. The main difficulty of the problem is the presence of two types of singularities in the model: one is the logarithmic singularity of the sensitivity function, the other is the inverse square singularity of the signal-dependent motility. As in the arguments of [13, 27–29], we relegate the former singularity by using the Cole-Hopf transformation to a nonlinear nonlocal term. However, this transformation is not powerful enough to settle the latter singularity. We shall develop new ideas to deal with the challenge of inverse square singularity of motility. Indeed, we first reformulate the problem in the perturbation variables using the method of anti-derivative, to classify the strength of singularity. Then we construct an appropriately approximate system, which retains some key structures of the original system, to establish the local well-posedness of the perturbation equations. In this step we will first prove that the approximate system is locally well-posed in a time interval $[0, T]$, where T is independent of the artificial parameter ε; and then pass to the limit $\varepsilon \to 0^+$ by using the Aubin-Lions compactness lemma and a diagonal argument. Finally, to close the a priori estimate that is necessary to obtain the global well-posedness of the perturbation equations (or the asymptotic stability of steady state), we establish a new weighted elliptic estimate upon the weighted energy estimates where the weights are artfully chosen according to the nice structures of the equations.

The paper is organized as follows. In Section 2, we present some elementary calculations and state the main results of this paper. In Section 3, we derive the perturbation equations, and establish the local well-posedness theory. Section 4 is devoted to the proof of nonlinear stability of the spiky steady state.

2. Preliminaries and main results

In this section, we first show the existence of spiky steady state to the system (1.2)–(1.4). Then we present some elementary calculations and state the main results on the asymptotic stability of such spike profile.

Owing to the zero-flux boundary condition for u, the mass of bacterial should be conserved. In other words,

$$\lambda := \int_0^\infty u(x,t)dx = \int_0^\infty u_0(x)dx. \tag{2.1}$$

Thus, the steady state of (1.2) satisfies

$$\begin{cases} \left(\frac{1}{W^2}\left(U_x - \frac{UW_x}{W}\right)\right)_x = 0, \\ dW_{xx} - WU = 0, \\ \int_0^\infty U(x)dx = \lambda > 0. \end{cases} \tag{2.2}$$

with boundary conditions

$$\left(U_x - \frac{UW_x}{W}\right)(0) = 0,\ W(0) = b,\ (U,W)(+\infty) = (0,0). \tag{2.3}$$

Observe that when $W > 0$, the steady state equations (2.2) and (2.3) is actually the $m = \chi = 1$ case of the chemotaxis model studied in [13]. Thus, according to Proposition 2.1 and Theorem 2.1 of [13], we have the following result.

Proposition 2.1. *(1) The system* (2.2) *and* (2.3) *has a unique smooth solution* (U,W) *satisfying* $U' < 0$, $W' < 0$*, and*

$$U(x) = \frac{\lambda^2}{6d}(1 + \frac{\lambda}{6d}x)^{-2},\ W(x) = b(1 + \frac{\lambda}{6d}x)^{-2}. \tag{2.4}$$

(2) U concentrates as a spike at $x = 0$ as $d \to 0^+$, i.e.,

$$U(x) \to \lambda\delta(x) \textit{ in the sense of distribution as } d \to 0^+.$$

We next pay attention to the asymptotic stability of (U,W) to system (1.2)–(1.4). Because the chemical concentration $w(x,t)$ has vacuum end state at $x = +\infty$, there are two types of singularities in system (1.2): one is the singular sensitivity $\frac{w_x}{w}$, the other is the singular motility w^{-2}. To handle the former singularity, motivated by the works of [13, 27–29], we employ the following Cole-Hopf type transformation

$$v := -\frac{w_x}{w},\ \text{that is,}\ (\ln w)_x = -v, \tag{2.5}$$

which along with the boundary condition $w(0,t) = b$ gives

$$w(x,t) = be^{-\int_0^x v(y,t)dy}. \tag{2.6}$$

Hence we transform system (1.2) into a nonlocal system of viscous conservation laws as follows

$$\begin{cases} u_t = (w^{-2}(u_x + uv))_x, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ v_t = dv_{xx} - (dv^2 - u)_x, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ w(x,t) = be^{-\int_0^x v(y,t)}dy, \\ (u,v)(x,0) = (u_0(x), v_0(x)), \end{cases} \tag{2.7}$$

where $v_0 = -\frac{w_{0x}}{w_0}$. One may observe that the new system (2.7) still has singular motility near $x = +\infty$ for the bacterial mass u. In this paper, we shall develop some novel ideas to solve this challenging problem.

We next determine the boundary conditions of (2.7). The second equation of (1.2) gives $(\ln w)_t = -dv_x + dv^2 - u$. Because b is a constant, for smooth solutions $(\ln w)_t = 0$ at $x = 0$, it then follows that

$$dv_x - (dv^2 - u) = 0 \text{ at } x = 0.$$

Denote by $(U, V)(x)$ the steady state of (2.7), where $U(x)$ is given in (2.4). Then we have

$$V(x) = -\frac{W_x}{W} = \frac{\lambda}{3d}(1 + \frac{\lambda}{6d}x)^{-1}.$$

It is easy to see that

$$V(x) \to 0 \text{ as } x \to \infty.$$

Because it is expected that $v(x,t) \to V(x)$ as $t \to \infty$, it is natural to impose $v(+\infty, t) = 0$. Therefore, the boundary conditions of (2.7) are

$$\begin{cases} u_x + uv = 0, & x = 0 \\ dv_x - (dv^2 - u) = 0, & x = 0 \\ (u,v)(x,t) \to (0,0), & x \to \infty. \end{cases} \tag{2.8}$$

We also need some notation. H^k denotes the usual Sobolev space whose norm is abbreviated as $\|f\|_k^2 := \sum_{j=0}^k \|\partial_x^j f\|^2$ with $\|f\| := \|f\|_{L^2(\mathbb{R}_+)}$, and H_ω^k is the weighted Sobolev space of measurable function f such that $\sqrt{\omega}\partial_x^j f \in L^2(\mathbb{R}_+)$ with norm $\|f\|_\omega := \|\sqrt{\omega} f\|_{L^2(\mathbb{R}_+)}$ and $\|f\|_{k,\omega}^2 := \sum_{j=0}^k \|\sqrt{\omega}\partial_x^j f\|^2$ for $0 \le j \le k$.

We are now ready to state the main results.

Theorem 2.1 (Local well-posedness). *Let* (U, V) *be the steady state of* (2.7) *and* (2.8). *Assume that the initial perturbation around* (U, V) *satisfies* $\phi_0(\infty) = \psi_0(\infty) = 0$, *where*

$$(\phi_0, \psi_0)(x) = \int_0^x (u_0(y) - U(y), v_0(y) - V(y))dy.$$

Suppose that

$$\phi_0 \in H^1(\mathbb{R}_+),\ \psi_0 \in L^2(\mathbb{R}_+),\ \frac{\psi_{0x}}{\sqrt{W}} \in L^2(\mathbb{R}_+).$$

Then there is a time $T > 0$*, such that the system* (2.7) *and* (2.8) *has a unique strong solution* (u, v) *on* $\mathbb{R}_+ \times (0, T)$*, satisfying*

$$u - U \in C([0, T]; L^2_{\omega_1}) \cap L^2((0, T); H^1_{\omega_2}),\ v - V \in C([0, T]; L^2_{\omega_2}) \cap L^2((0, T); H^1_{\omega_2}),$$

where $\omega_1 = U$ *and* $\omega_2 = \frac{1}{U}$.

Theorem 2.2 (Global well-posedness)**.** *Assume that the conditions of Theorem 2.1 hold, and that there exists a constant* $\delta_0 > 0$ *such that,*

$$\|\psi_0\|^2 + \|\phi_0\|^2 + \|\psi_{0x}\|^2_{1,w_2} + \|\phi_{0x}\|^2_{\omega_3} + \|\phi_{0xx}\|^2_{\omega_4} \le \delta_0,$$

where $\omega_3 = \frac{1}{U^2}$ *and* $\omega_4 = \frac{1}{U^3}$. *Then the system* (2.7) *and* (2.8) *has a unique global solution* $(u, v)(x, t)$ *satisfying*

$$\begin{cases} u - U \in C([0, \infty); H^1) \cap L^2((0, \infty); H^2), \\ v - V \in C([0, \infty); H^1) \cap L^2((0, \infty); H^2). \end{cases} \tag{2.9}$$

Moreover, the following asymptotic convergence hold:

$$\sup_{x \in \mathbb{R}_+} |(u, v)(x, t) - (U, V)(x)| \to 0 \text{ as } t \to +\infty, \tag{2.10}$$

and

$$\|u(\cdot, t) - U(\cdot)\|_{L^1(\mathbb{R}_+)} \to 0 \text{ as } t \to +\infty. \tag{2.11}$$

Using the Cole-Hopf transformation (2.5), we transfer Theorem 2.2 to the original system (1.2)–(1.4).

Theorem 2.3. *Let* (U, W) *be the unique steady state of* (1.2)–(1.4)*. Assume that the initial perturbation satisfies* $\phi_0(\infty) = \psi_0(\infty) = 0$*, where*

$$\phi_0(x) = \int_0^x (u_0(y) - U(y))dy,\ \psi_0(x) = -\ln w_0(x) + \ln W(x).$$

Suppose that there is a constant $\delta_0 > 0$ *such that*

$$\|\psi_0\|^2 + \|\phi_0\|^2 + \|\psi_{0x}\|^2_{1,w_2} + \|\phi_{0x}\|^2_{\omega_3} + \|\phi_{0xx}\|^2_{\omega_4} \le \delta_0.$$

Then the system (1.2)–(1.4) *has a unique global solution* $(u, w)(x, t)$ *satisfying*

$$\begin{cases} u - U \in C([0, \infty); H^1) \cap L^2((0, \infty); H^2), \\ w - W \in C([0, \infty); H^1) \cap L^2((0, \infty); H^2). \end{cases}$$

Moreover, we have the following asymptotic convergence:

$$\sup_{x \in \mathbb{R}_+} |(u, w)(x, t) - (U, W)(x)| \to 0 \text{ as } t \to +\infty,$$

and

$$\|(u, w)(\cdot, t) - (U, W)(\cdot)\|_{L^1(\mathbb{R}_+)} \to 0 \text{ as } t \to +\infty.$$

Remark 2.1. *We provide both the pointwise convergence and L^1 convergence for the solution. In contrast with the result of [6] where it is required infinite initial mass for w, our Theorem 2.3 implies that the chemical concentration w carries finite mass for all time.*

Remark 2.2. *In view of its biological background, it is also interesting to study the stability of traveling waves to system* (1.1). *However, when we apply our argument to that problem, the perturbation equation involves several unfavorable terms which are sophisticated to estimate. We leave this problem for the future study.*

Remark 2.3. *We shall remark that the steady state* (U, W) *obtained in Proposition 2.1 is a smooth solution of system* (2.2) *and* (2.3), *and it satisfies* $U(x) > 0$ *and* $W(x) > 0$ *for any* $x \in [0, +\infty)$. *In other words,* $U(x)$ *only vanishes at the far field, and the singularity only happens at* $x = +\infty$. *This fact enables us to take* $\frac{1}{U}$ *as the key weight function, and derive the stability of steady state in specific weighted space.*

3. Local well-posedness

This section is devoted to proving Theorem 2.1, i.e., the local well-posedness of system (2.7) and (2.8). We first reformulate the problem in the perturbation variables using the method of anti-derivative. Because the perturbation system still has a singularity, we have to construct an appropriately approximate system. Then we prove that the approximate system is locally well-posed in a time interval $[0, T]$ where T is independent of the artificial parameter ε. After establishing the uniqueness of solutions in weighted Sobolev space, we finally derive the local well-posedness of system (2.7) and (2.8) by the Aubin-Lions compactness lemma and a diagonal argument.

3.1. Reformulation of the problem

The steady state (U, V) of system (2.7) and (2.8) satisfies

$$\begin{cases} \left(W^{-2}(U_x + UV)\right)_x = 0, \\ dV_{xx} - (dV^2 - U)_x = 0, \end{cases} \tag{3.1}$$

where the boundary conditions are given by

$$(U_x + UV)(0) = (dV_x - (dV^2 - U))(0) = 0, \ (U, V)(+\infty) = (0, 0). \tag{3.2}$$

Integrating (3.1) in x, we have

$$\begin{cases} U_x + UV = 0, \\ dV_x - dV^2 + U = 0. \end{cases} \tag{3.3}$$

Observing that (u, v) satisfies the zero-flux boundary condition, the perturbation around (U, V) should have the conservation of mass. That is

$$\int_0^\infty (u(x,t) - U(x), v(x,t) - V(x))dx = \int_0^\infty (u_0(x) - U(x), v_0(x) - V(x))dx = (0, 0). \tag{3.4}$$

Then we could adopt the method of anti-derivative to decompose the solution (u, v) as

$$(\phi, \psi)(x, t) = \int_0^x (u(y, t) - U(y), v(y, t) - V(y))dy,$$

which implies

$$\phi_x = u - U,\ \psi_x = v - V. \tag{3.5}$$

Substituting (3.5) into (2.7), integrating the equations in x, noting $w = e^{-\psi}W$, and using (3.1), we have

$$\begin{cases} \phi_t = W^{-2}e^{2\psi}(\phi_{xx} + \phi_x\psi_x + U\psi_x + V\phi_x), \\ \psi_t = d\psi_{xx} - 2dV\psi_x - d\psi_x^2 + \phi_x, \end{cases}$$

which is equivalent to

$$\begin{cases} W^2\phi_t = e^{2\psi}(\phi_{xx} + \phi_x\psi_x + U\psi_x + V\phi_x), \\ \psi_t = d\psi_{xx} - 2dV\psi_x - d\psi_x^2 + \phi_x. \end{cases} \tag{3.6}$$

The initial value of (ϕ, ψ) is given by

$$(\phi_0, \psi_0)(x) := (\phi, \psi)(x, 0) = \int_0^x (u_0(y) - U(y), v_0(y) - V(y))dy, \tag{3.7}$$

with

$$(\phi_0, \psi_0)(+\infty) = (0, 0), \tag{3.8}$$

and the boundary condition satisfies

$$(\phi, \psi)(0, t) = (0, 0),\ (\phi, \psi)(+\infty, t) = (0, 0). \tag{3.9}$$

We shall remark that the anti-derivative for v could remove the nonlocality of the problem, but it can not handle the singularity of the motility. Indeed, to overcome the difficulties caused by the singular motility (or degenerate relaxation), we construct an approximate system of (3.6) as

$$\begin{cases} W_\varepsilon^2\phi_t = e^{2\psi}(\phi_{xx} + \phi_x\psi_x + U\psi_x + V_\varepsilon\phi_x), \\ \psi_t = d\psi_{xx} - 2dV_\varepsilon\psi_x - d\psi_x^2 + \phi_x, \end{cases} \tag{3.10}$$

where $\varepsilon > 0$ is a constant, $W_\varepsilon = W + \varepsilon$ and $V_\varepsilon = \frac{W}{W_\varepsilon} \cdot V$. Here we also approximate V by V_ε so that system (3.10) retains the key structure of system (3.6):

$$\frac{V_\varepsilon}{W_\varepsilon} - (\frac{1}{W_\varepsilon})_x = 0. \tag{3.11}$$

Indeed, recalling that $V = -\frac{W_x}{W}$, a direct calculation leads to

$$\frac{V_\varepsilon}{W_\varepsilon} - (\frac{1}{W_\varepsilon})_x = \frac{V_\varepsilon W_\varepsilon + W_{\varepsilon x}}{W_\varepsilon^2} = \frac{VW + W_x}{W_\varepsilon^2} = 0.$$

3.2. Local well-posedness for the approximate system

Employing the principle of contraction mapping (e.g., see [30]), one could easily get the local well-posedness for the approximate system on a time interval that may depend on ε.

Proposition 3.1. *Assume that the initial data* (ϕ_0, ψ_0) *satisfies*

$$\phi_0 \in H^1(\mathbb{R}_+),\ \psi_0 \in H^1(\mathbb{R}_+).$$

Then, there exists a constant $T > 0$ *depending on* ε, $\|\phi_0\|_{H^1}$ *and* $\|\psi_0\|_{H^1}$ *such that the approximate system* (3.10) *with* (3.7)–(3.9) *has a unique local strong solution on* $\mathbb{R}_+ \times [0, T]$ *satisfying*

$$(\phi, \psi) \in C([0, T]; H^1) \cap L^2((0, T); H^2).$$

Proof.

By Proposition 3.1, there exists a time $T_1 > 0$ such that the system (3.10) with (3.7)–(3.9) has a unique solution (ϕ, ψ) on $(0, T_1)$ that satisfies $\phi \in C([0, T]; H^1) \cap L^2((0, T); H^2)$. Starting at T_1, applying Proposition 3.1 again, we can extend the solution (ϕ, ψ) to another time $T_2 = T_1 + t_1$, where $t_1 > 0$ depends on ε, $\|\phi(T_1)\|_{H^1}$ and $\|\psi(T_1)\|_{H^1}$. Continuing this procedure, we get two sequences $\{t_j\}_{j=1}^{\infty}$ and $\{T_j\}_{j=1}^{\infty}$, where t_j depends on ε, $\|\phi(T_j)\|_{H^1}$ and $\|\psi(T_j)\|_{H^1}$, such that the solution (ϕ, ψ) exists on the time interval $(0, T_j)$, and satisfies

$$(\phi, \psi) \in C([0, T_j]; H^1) \cap L^2((0, T_j); H^2).$$

Take the maximal existing time T^* as $T^* = T_1 + \sum_{j=1}^{\infty} t_j$. Then the solution can be extended to $(0, T^*)$ and satisfies

$$(\phi, \psi) \in C([0, T]; H^1) \cap L^2((0, T); H^2),$$

for any $T \in (0, T^*)$. Clearly, if $T^* < \infty$, then

$$\overline{\lim_{t \to T^*}} (\|\phi(t)\|_{H^1} + \|\psi(t)\|_{H^1}) = \infty. \tag{3.12}$$

However, one can not use Proposition 3.1 to derive the local well-posedness of system (3.6) by directly passing to the limit $\varepsilon \to 0$, since the time interval $[0, T]$ obtained in Proposition 3.1 depends on ε. In the following, we have to establish appropriate a priori estimates that are independent of ε.

Proposition 3.2. *Assume that* (ϕ_0, ψ_0) *satisfies*

$$\phi_0 \in H^1(\mathbb{R}_+),\ \psi_0 \in L^2(\mathbb{R}_+),\ \frac{\psi_{0x}}{\sqrt{W}} \in L^2(\mathbb{R}_+). \tag{3.13}$$

Then there exists a constant $T_0 > 0$ *independent of* ε, *such that the approximate system* (3.10) *with* (3.7)–(3.9) *has a unique solution on* $\mathbb{R}_+ \times [0, T_0]$, *which satisfies*

$$\sup_{t \in [0, T_0]} \int_0^{\infty} \left(W_\varepsilon \phi^2 + \psi^2 + W_\varepsilon \phi_x^2 + \frac{\psi_x^2}{W_\varepsilon} \right) dx \leq 2 \int_0^{\infty} \left(W_\varepsilon \phi_0^2 + \psi_0^2 + W_\varepsilon \phi_{0x}^2 + \frac{\psi_{0x}^2}{W_\varepsilon} \right) dx, \tag{3.14}$$

and

$$\int_0^{T_0} \int_0^{\infty} \left(\frac{\phi_x^2}{W_\varepsilon} + \frac{\phi_{xx}^2}{W_\varepsilon} + \psi_x^2 + \frac{\psi_{xx}^2}{W_\varepsilon} \right) \leq C(T_0). \tag{3.15}$$

Proof. Thanks to (3.12), it suffices to establish a priori estimate in the weighted Sobolev space that is independent of ε. To achieve this, we multiply the first equation of (3.10) by $\frac{\phi}{W_\varepsilon}$, and integrate the resultant equation over $(0,t)\times(0,+\infty)$ to get

$$\begin{aligned}\frac{1}{2}\int_0^\infty W_\varepsilon\phi^2 &= \int_0^t\int_0^\infty e^{2\psi}\left(\frac{V_\varepsilon}{W_\varepsilon}\phi\phi_x+\frac{\phi\phi_{xx}}{W_\varepsilon}+\frac{\phi\phi_x\psi_x}{W_\varepsilon}+\frac{U}{W_\varepsilon}\phi\psi_x\right)+\frac{1}{2}\int_0^\infty W_\varepsilon\phi_0^2\\ &=\int_0^t\int_0^\infty\left[\frac{V_\varepsilon}{W_\varepsilon}-(\frac{1}{W_\varepsilon})_x\right]e^{2\psi}\phi\phi_x-\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_x^2}{W_\varepsilon}-\int_0^t\int_0^\infty e^{2\psi}\frac{\phi\phi_x\psi_x}{W_\varepsilon}\\ &\quad+\int_0^t\int_0^\infty\frac{U}{W_\varepsilon}e^{2\psi}\phi\psi_x+\frac{1}{2}\int_0^\infty W_\varepsilon\phi_0^2.\end{aligned}\tag{3.16}$$

By (3.11) we have

$$\frac{1}{2}\int_0^\infty W_\varepsilon\phi^2+\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_x^2}{W_\varepsilon}=\int_0^t\int_0^\infty\frac{U}{W_\varepsilon}e^{2\psi}\phi\psi_x-\int_0^t\int_0^\infty e^{2\psi}\frac{\phi\phi_x\psi_x}{W_\varepsilon}+\frac{1}{2}\int_0^\infty W_\varepsilon\phi_0^2.$$

It follows from Young's inequality that

$$\Big|\int_0^\infty\frac{U}{W_\varepsilon}e^{2\psi}\phi\psi_x dx\Big|\le Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty W_\varepsilon\phi^2dx+Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty\frac{\psi_x^2}{W_\varepsilon}dx.\tag{3.17}$$

Using the inequality

$$\phi^2(x,t)=-2\int_x^\infty\phi\phi_x(y,t)dy\le 2\left(\int_0^\infty W_\varepsilon\phi^2\right)^{\frac{1}{2}}\left(\int_0^\infty\frac{\phi_x^2}{W_\varepsilon}\right)^{\frac{1}{2}},\tag{3.18}$$

we get

$$\begin{aligned}&\Big|\int_0^\infty e^{2\psi}\frac{\phi\phi_x\psi_x}{W_\varepsilon}dx\Big|\\ \le&Ce^{\|\psi\|_{L^\infty}}\|\phi\|_{L^\infty(\mathbb{R}_+)}\Big|\int_0^\infty e^{\psi}\frac{\phi_x\psi_x}{W_\varepsilon}dx\Big|\\ \le&Ce^{\|\psi\|_{L^\infty}}\|\sqrt{W_\varepsilon}\phi\|^{\frac{1}{2}}_{L^2(\mathbb{R}_+)}\|\frac{\phi_x}{\sqrt{W_\varepsilon}}\|^{\frac{1}{2}}_{L^2(\mathbb{R}_+)}\|\frac{e^{\psi}\phi_x}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}\|\frac{\psi_x}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}\\ \le&Ce^{\frac{3}{2}\|\psi\|_{L^\infty}}\|\sqrt{W_\varepsilon}\phi\|^{\frac{1}{2}}_{L^2(\mathbb{R}_+)}\|\frac{\psi_x}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}\|\frac{e^{\psi}\phi_x}{\sqrt{W_\varepsilon}}\|^{\frac{3}{2}}_{L^2(\mathbb{R}_+)}\\ \le&Ce^{6\|\psi\|_{L^\infty}}\|\sqrt{W_\varepsilon}\phi\|^2_{L^2(\mathbb{R}_+)}\|\frac{\psi_x}{\sqrt{W_\varepsilon}}\|^4_{L^2(\mathbb{R}_+)}+\frac{1}{8}\|\frac{e^{\psi}\phi_x}{\sqrt{W_\varepsilon}}\|^2_{L^2(\mathbb{R}_+)}\\ \le&Ce^{6\|\psi\|_{L^\infty}}\|\sqrt{W_\varepsilon}\phi\|^6_{L^2(\mathbb{R}_+)}+Ce^{6\|\psi\|_{L^\infty}}\|\frac{\psi_x}{\sqrt{W_\varepsilon}}\|^6_{L^2(\mathbb{R}_+)}+\frac{1}{8}\|\frac{e^{\psi}\phi_x}{\sqrt{W_\varepsilon}}\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}\tag{3.19}$$

Combining (3.17) and (3.19), one obtains

$$\begin{aligned}&\frac{1}{2}\int_0^\infty W_\varepsilon\phi^2+\frac{7}{8}\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_x^2}{W_\varepsilon}\\ \le&C\int_0^t e^{2\|\psi\|_{L^\infty}}\int_0^\infty\left(W_\varepsilon\phi^2+\frac{\psi_x^2}{W_\varepsilon}\right)\\ &+C\int_0^t e^{6\|\psi\|_{L^\infty}}\left(\int_0^\infty W_\varepsilon\phi^2dx+\int_0^\infty\frac{\psi_x^2}{W_\varepsilon}dx\right)^3+\frac{1}{2}\int_0^\infty W_\varepsilon\phi_0^2.\end{aligned}\tag{3.20}$$

Multiplying the second equation of (3.10) by ψ, we get

$$\frac{1}{2}\int_0^\infty \psi^2 + d\int_0^t\int_0^\infty \psi_x^2 = -2d\int_0^t\int_0^\infty V_\varepsilon\psi\psi_x + \int_0^t\int_0^\infty \phi_x\psi - d\int_0^t\int_0^\infty \psi\psi_x^2 + \frac{1}{2}\int_0^\infty \psi_0^2. \quad (3.21)$$

By Young's inequality, we have

$$\left|2d\int_0^\infty V_\varepsilon\psi\psi_x dx\right| \le \frac{d}{4}\int_0^\infty \psi_x^2 dx + C\int_0^\infty \psi^2 dx, \quad (3.22)$$

and

$$\left|\int_0^\infty \phi_x\psi dx\right| \le Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty \psi^2 dx + \frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_x^2}{W_\varepsilon}dx.$$

Moreover, using (3.18) yields

$$\begin{aligned}\left|d\int_0^\infty \psi\psi_x^2 dx\right| \le& C\|\psi\|_{L^\infty(\mathbb{R}_+)}\int_0^\infty \psi_x^2 dx\\ \le& C\|\psi\|_{L^2(\mathbb{R}_+)}^{\frac{1}{2}}\|\psi_x\|_{L^2}^{\frac{5}{2}}\\ \le& C\|\psi\|_{L^2(\mathbb{R}_+)}^2 + C\|\psi_x\|_{L^2(\mathbb{R}_+)}^{\frac{10}{3}}.\end{aligned} \quad (3.23)$$

Now substituting (3.22) and (3.23) into (3.21), we have

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \psi^2 + \frac{d}{4}\int_0^t\int_0^\infty \psi_x^2\\ \le& C\int_0^t\left(1+e^{2\|\psi\|_{L^\infty}}\right)\int_0^\infty \psi^2 + \frac{1}{8}\int_0^t\int_0^\infty \frac{e^{2\psi}\phi_x^2}{W_\varepsilon} + C\int_0^t\left(\int_0^\infty \psi_x^2 dx\right)^{\frac{5}{3}} + \frac{1}{2}\int_0^\infty \psi_0^2.\end{aligned} \quad (3.24)$$

Combining (3.20) with (3.24) yields

$$\begin{aligned}&\frac{1}{2}\int_0^\infty (W_\varepsilon\phi^2+\psi^2)dx + \frac{3}{4}\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_x^2}{W_\varepsilon}dxd\tau + \frac{d}{4}\int_0^t\int_0^\infty \psi_x^2 dxd\tau\\ \le& C\int_0^t e^{2\|\psi\|_{L^\infty}}\int_0^\infty W_\varepsilon\phi^2 dxd\tau + C\int_0^t e^{6\|\psi\|_{L^\infty}}\Big(\int_0^\infty W_\varepsilon\phi^2 dx\Big)^3 d\tau\\ &+ C\int_0^t\left(1+e^{2\|\psi\|_{L^\infty}}\right)\int_0^\infty \psi^2 dxd\tau + C\int_0^t e^{2\|\psi\|_{L^\infty}}\int_0^\infty \psi_x^2 dxd\tau\\ &+ C\int_0^t e^{6\|\psi\|_{L^\infty}}\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^3 d\tau + C\int_0^t\Big(\int_0^\infty \psi_x^2 dx\Big)^{\frac{5}{3}} d\tau + \frac{1}{2}\int_0^\infty (W_\varepsilon\phi_0^2+\psi_0^2)dx.\end{aligned} \quad (3.25)$$

Multiplying the first equation of (3.10) by $\frac{\phi_{xx}}{W_\varepsilon}$, one gets

$$\begin{aligned}&\frac{1}{2}\int_0^\infty W_\varepsilon\phi_x^2 + \int_0^t\int_0^\infty e^{2\psi}\frac{\phi_{xx}^2}{W_\varepsilon}\\ =& -\int_0^t\int_0^\infty\left(e^{2\psi}\frac{V_\varepsilon}{W_\varepsilon}\phi_x\phi_{xx} + e^{2\psi}\frac{\phi_x\phi_{xx}\psi_x}{W_\varepsilon} + e^{2\psi}\frac{U}{W_\varepsilon}\phi_{xx}\psi_x + W_x\phi_t\phi_x\right) + \frac{1}{2}\int_0^\infty W_\varepsilon\phi_{0x}^2\\ =& -\int_0^t\int_0^\infty\left[e^{2\psi}\left(\frac{V_\varepsilon}{W_\varepsilon}+\frac{W_x}{W_\varepsilon^2}\right)\phi_x\phi_{xx} + e^{2\psi}\frac{\phi_x\phi_{xx}\psi_x}{W_\varepsilon} + e^{2\psi}\frac{U}{W_\varepsilon}\phi_{xx}\psi_x\right]\\ &-\int_0^t\int_0^\infty e^{2\psi}\frac{W_x}{W_\varepsilon^2}(U\phi_x\psi_x + \phi_x^2\psi_x + V_\varepsilon\phi_x^2) + \frac{1}{2}\int_0^\infty W_\varepsilon\phi_{0x}^2,\end{aligned} \quad (3.26)$$

where we have used the first equation of (3.10) in the second equality. As in (3.19),

$$\begin{aligned}&\Big|\int_0^\infty e^{2\psi}\frac{\phi_x\phi_{xx}\psi_x}{W_\varepsilon}dx\Big|\\ \leq&\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_{xx}^2}{W_\varepsilon}dx+Ce^{6\|\psi\|_{L^\infty}}\Big(\int_0^\infty W_\varepsilon\phi_x^2dx\Big)^3+Ce^{6\|\psi\|_{L^\infty}}\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^3.\end{aligned}\tag{3.27}$$

From Young's inequality, it follows that

$$\Big|\int_0^\infty e^{2\psi}\frac{U}{W_\varepsilon}\phi_{xx}\psi_x dx\Big|\leq Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty \psi_x^2dx+\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_{xx}^2}{W_\varepsilon}dx,\tag{3.28}$$

$$\Big|\int_0^\infty e^{2\psi}\frac{W_x}{W_\varepsilon^2}U\phi_x\psi_x dx\Big|\leq Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty W_\varepsilon\phi_x^2dx+Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx,\tag{3.29}$$

and

$$\begin{aligned}&\Big|\int_0^\infty \frac{W_x}{W_\varepsilon^2}e^{2\psi}\phi_x^2\psi_x dx\Big|\\ \leq&Ce^{\|\psi\|_{L^\infty}}\|\phi_x\|_{L^\infty(\mathbb{R}_+)}\int_0^\infty \frac{e^\psi|\phi_x\psi_x|}{\sqrt{W_\varepsilon}}dx\\ \leq&Ce^{\frac{3}{2}\|\psi\|_{L^\infty}}\|\sqrt{W_\varepsilon}\phi_x\|_{L^2(\mathbb{R}_+)}^{\frac12}\|e^\psi\frac{\phi_{xx}}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}^{\frac12}\|e^\psi\frac{\phi_x}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}\|\psi_x\|_{L^2(\mathbb{R}_+)}\\ \leq&\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_x^2}{W_\varepsilon}dx+Ce^{3\|\psi\|_{L^\infty}}\|\sqrt{W_\varepsilon}\phi_x\|_{L^2(\mathbb{R}_+)}\|e^\psi\frac{\phi_{xx}}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}\|\psi_x\|_{L^2(\mathbb{R}_+)}^2\\ \leq&\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_x^2}{W_\varepsilon}dx+\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_{xx}^2}{W_\varepsilon}dx+Ce^{6\|\psi\|_{L^\infty}}\|\sqrt{W_\varepsilon}\phi_x\|_{L^2(\mathbb{R}_+)}^2\|\psi_x\|_{L^2(\mathbb{R}_+)}^4\\ \leq&\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_x^2}{W_\varepsilon}dx+\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_{xx}^2}{W_\varepsilon}dx+Ce^{6\|\psi\|_{L^\infty}}(\|\sqrt{W_\varepsilon}\phi_x\|_{L^2}^6+\|\psi_x\|_{L^2}^6).\end{aligned}\tag{3.30}$$

Substituting (3.27)–(3.30) into (3.26) leads to

$$\begin{aligned}&\frac{1}{2}\int_0^\infty W_\varepsilon\phi_x^2dx+\frac{1}{2}\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_{xx}^2}{W_\varepsilon}dxd\tau\\ \leq&C\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_x^2}{W_\varepsilon}dxd\tau+C\int_0^t e^{2\|\psi\|_{L^\infty}}\left(\int_0^\infty \psi_x^2dx+\int_0^\infty W_\varepsilon\phi_x^2dx\right)d\tau\\ &+C\int_0^t e^{6\|\psi\|_{L^\infty}}\left(\int_0^\infty W_\varepsilon\phi_x^2dx+\int_0^\infty \psi_x^2dx\right)^3d\tau+\frac{1}{2}\int_0^\infty W_\varepsilon\phi_{0x}^2dx.\end{aligned}\tag{3.31}$$

Multiplying the second equation of (3.10) by $\frac{\psi_{xx}}{W_\varepsilon}$, one gets

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}+\int_0^t\int_0^\infty d\frac{\psi_{xx}^2}{W_\varepsilon}\\ =&\int_0^t\int_0^\infty\left(2d\frac{V_\varepsilon}{W_\varepsilon}\psi_x\psi_{xx}-\frac{\phi_x\psi_{xx}}{W_\varepsilon}+d\frac{\psi_x^2\psi_{xx}}{W_\varepsilon}-(\frac{1}{W_\varepsilon})_x\psi_t\psi_x\right)+\frac{1}{2}\int_0^\infty \frac{\psi_{0x}^2}{W_\varepsilon}.\end{aligned}\tag{3.32}$$

By Young's inequality,

$$\Big|\int_0^\infty 2d\frac{V_\varepsilon}{W_\varepsilon}\psi_x\psi_{xx}dx\Big| \le C\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx + \frac{d}{8}\int_0^\infty \frac{\psi_{xx}^2}{W_\varepsilon}dx.$$

Moreover, integration by parts leads to

$$\begin{aligned}
&\Big|\int_0^\infty \frac{\phi_x\psi_{xx}}{W_\varepsilon}dx\Big| \\
=&\Big|-\int_0^\infty \frac{\phi_{xx}\psi_x}{W_\varepsilon}dx - \int_0^\infty (\frac{1}{W_\varepsilon})_x\phi_x\psi_x dx + \frac{\phi_x\psi_x}{b+\varepsilon}\big|_{x=0}\Big| \\
\le&\frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_x^2}{W_\varepsilon}dx + \frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_{xx}^2}{W_\varepsilon}dx + Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx + \Big|\frac{\phi_x\psi_x}{b+\varepsilon}\big|_{x=0}\Big|.
\end{aligned}$$

The boundary term can be estimated as

$$\begin{aligned}
\Big|\frac{\phi_x\psi_x}{b+\varepsilon}\big|_{x=0}\Big| \le&\frac{1}{2(b+\varepsilon)}\phi_x^2\big|_{x=0} + \frac{1}{2(b+\varepsilon)}\psi_x^2\big|_{x=0} \\
=&-\frac{1}{b+\varepsilon}\int_0^\infty \phi_x\phi_{xx}dx - \frac{1}{b+\varepsilon}\int_0^\infty \psi_x\psi_{xx}dx \\
\le&\delta\int_0^\infty e^{2\psi}\frac{\phi_{xx}^2}{W_\varepsilon}dx + Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty W_\varepsilon\phi_x^2 \\
&+\delta\int_0^\infty \frac{\psi_{xx}^2}{W_\varepsilon}dx + C\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx,
\end{aligned}$$

where δ is a small constant. It follows from (3.18) that

$$\begin{aligned}
\Big|\int_0^\infty d\frac{\psi_x^2\psi_{xx}}{W_\varepsilon}dx\Big| \le&C\|\psi_x\|_{L^\infty}\int_0^\infty \frac{\psi_x\psi_{xx}}{W_\varepsilon}dx \\
\le&C\|\psi_x\|_{L^2(\mathbb{R}_+)}^{\frac{1}{2}}\|\psi_{xx}\|_{L^2(\mathbb{R}_+)}^{\frac{1}{2}}\|\frac{\psi_x}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}\|\frac{\psi_{xx}}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)} \\
\le&C\|\frac{\psi_{xx}}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}^{\frac{3}{2}}\|\frac{\psi_x}{\sqrt{W_\varepsilon}}\|_{L^2(\mathbb{R}_+)}^{\frac{3}{2}} \\
\le&\frac{d}{8}\int_0^\infty \frac{\psi_{xx}^2}{W_\varepsilon}dx + C\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^3.
\end{aligned}$$

In view of the second equation of (3.10), it holds:

$$\begin{aligned}
-\int_0^\infty (\frac{1}{W_\varepsilon})_x\psi_t\psi_x dx =& \int_0^\infty 2dV_\varepsilon(\frac{1}{W_\varepsilon})_x\psi_x^2 dx - \int_0^\infty d(\frac{1}{W_\varepsilon})_x\psi_{xx}\psi_x dx \\
&-\int_0^\infty (\frac{1}{W_\varepsilon})_x\phi_x\psi_x dx + \int_0^\infty d(\frac{1}{W_\varepsilon})_x\psi_x^3 dx,
\end{aligned}$$

where

$$\Big|\int_0^\infty 2dV_\varepsilon(\frac{1}{W_\varepsilon})_x\psi_x^2 dx\Big| \le C\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx,$$

$$\Big|\int_0^\infty d(\frac{1}{W_\varepsilon})_x\psi_{xx}\psi_x dx\Big| \leq C\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx + \frac{d}{8}\int_0^\infty \frac{\psi_{xx}^2}{W_\varepsilon}dx,$$

$$\Big|\int_0^\infty (\frac{1}{W_\varepsilon})_x\phi_x\psi_x dx\Big| \leq \frac{1}{8}\int_0^\infty \frac{e^{2\psi}\phi_x^2}{W_\varepsilon}dx + Ce^{2\|\psi\|_{L^\infty}}\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx,$$

and by (3.18),

$$\begin{aligned}\Big|\int_0^\infty d(\frac{1}{W_\varepsilon})_x\psi_x^3 dx\Big| \leq & C\|\psi_x\|^{\frac{1}{2}}_{L^2(\mathbb{R}_+)}\|\psi_{xx}\|^{\frac{1}{2}}_{L^2(\mathbb{R}_+)}\|\frac{\psi_x}{\sqrt{W_\varepsilon}}\|^2_{L^2(\mathbb{R}_+)} \\ \leq & \frac{d}{8}\int_0^\infty \frac{\psi_{xx}^2}{W_\varepsilon}dx + C\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^{\frac{5}{3}}.\end{aligned}$$

Then choosing $\delta \ll 1$, by (3.31), we get

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \frac{\psi_x^2}{W_\varepsilon} + \frac{d}{4}\int_0^t\int_0^\infty \frac{\psi_{xx}^2}{W_\varepsilon} \\ \leq & C\int_0^t\Big(1+e^{2\|\psi\|_{L^\infty}}\Big)\int_0^\infty \frac{\psi_x^2}{W_\varepsilon} + \frac{1}{4}\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_{xx}^2}{W_\varepsilon} + C\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_x^2}{W_\varepsilon} \\ &+ C\int_0^t\Big[\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^{\frac{5}{3}} + \Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^3\Big] + C\int_0^t e^{2\|\psi\|_{L^\infty}}\int_0^\infty W_\varepsilon\phi_x^2 + \frac{1}{2}\int_0^\infty \frac{\psi_{0x}^2}{W_\varepsilon}.\end{aligned} \tag{3.33}$$

Combining (3.31) with (3.33), we have

$$\begin{aligned}&\frac{1}{2}\int_0^\infty (W_\varepsilon\phi_x^2 + \frac{\psi_x^2}{W_\varepsilon}) + \frac{1}{4}\int_0^t\int_0^\infty \Big(e^{2\psi}\frac{\phi_{xx}^2}{W_\varepsilon} + d\frac{\psi_{xx}^2}{W_\varepsilon}\Big) \\ \leq & C\int_0^t\int_0^\infty e^{2\psi}\frac{\phi_x^2}{W_\varepsilon} + C\int_0^t e^{6\|\psi\|_{L^\infty}}\Big(\int_0^\infty W_\varepsilon\phi_x^2 dx\Big)^3 \\ &+ C\int_0^t e^{2\|\psi\|_{L^\infty}}\int_0^\infty W_\varepsilon\phi_x^2 + C\int_0^t(1+e^{6\|\psi\|_{L^\infty}})\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^3 \\ &+ C\int_0^t(1+e^{2\|\psi\|_{L^\infty}})\int_0^\infty \frac{\psi_x^2}{W_\varepsilon} + C\int_0^t\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^{\frac{5}{3}} + \frac{1}{2}\int_0^\infty (W_\varepsilon\phi_{0x}^2 + \frac{\psi_{0x}^2}{W_\varepsilon}).\end{aligned} \tag{3.34}$$

Multiplying (3.25) by $K \gg 1$ and combing the resultant inequality with (3.34), we have

$$\begin{aligned}&\int_0^\infty \Big(W_\varepsilon\phi^2 + \psi^2 + W_\varepsilon\phi_x^2 + \frac{\psi_x^2}{W_\varepsilon}\Big) + \int_0^t\int_0^\infty \Big(e^{2\psi}\frac{\phi_x^2}{W_\varepsilon} + e^{2\psi}\frac{\phi_{xx}^2}{W_\varepsilon} + \psi_x^2 + \frac{\psi_{xx}^2}{W_\varepsilon}\Big) \\ \leq & C\int_0^t e^{2\|\psi\|_{L^\infty}}\int_0^\infty (W_\varepsilon\phi^2 + W_\varepsilon\phi_x^2) + C\int_0^t e^{6\|\psi\|_{L^\infty}}\Big(\Big(\int_0^\infty W_\varepsilon\phi^2\Big)^3 + \Big(\int_0^\infty W_\varepsilon\phi_x^2\Big)^3\Big) \\ &+ C\int_0^t\Big(1+e^{2\|\psi\|_{L^\infty}}\Big)\int_0^\infty \Big(\psi^2 + \frac{\psi_x^2}{W_\varepsilon}\Big) + C\int_0^t(1+e^{6\|\psi\|_{L^\infty}})\Big(\int_0^\infty \frac{\psi_x^2}{W_\varepsilon}dx\Big)^3 \\ &+ \int_0^\infty \Big(W_\varepsilon\phi_0^2 + \psi_0^2 + W_\varepsilon\phi_{0x}^2 + \frac{\psi_{0x}^2}{W_\varepsilon}\Big),\end{aligned} \tag{3.35}$$

which further gives

$$\begin{aligned}&\int_0^\infty \Big(W_\varepsilon\phi^2+\psi^2+W_\varepsilon\phi_x^2+\frac{\psi_x^2}{W_\varepsilon}\Big)dx\\ &\le C\int_0^t\left(1+\left[\int_0^\infty \Big(W_\varepsilon\phi^2+\psi^2+W_\varepsilon\phi_x^2+\frac{\psi_x^2}{W_\varepsilon}\Big)dx\right]^3 e^{6\|\psi\|_{L^\infty}}\right)\\ &\quad+\int_0^\infty\left(W_\varepsilon\phi_0^2+\psi_0^2+W_\varepsilon\phi_{0x}^2+\frac{\psi_{0x}^2}{W_\varepsilon}\right).\end{aligned}$$

Set $H(t):=\int_0^\infty(W_\varepsilon\phi^2+\psi^2+W_\varepsilon\phi_x^2+\frac{\psi_x^2}{W_\varepsilon})dx$. Noting

$$e^z>1 \text{ for } z>0 \ \text{ and } \|\psi\|_{L^\infty}^2\le\int_0^\infty(\psi^2+\psi_x^2)dx,$$

we are led to

$$H(t)\le C\int_0^t(H+1)^3e^{6\sqrt{H}}+H_0,$$

where $H_0=\int_0^\infty(W_\varepsilon\phi_0^2+\psi_0^2+W_\varepsilon\phi_{0x}^2+\frac{\psi_{0x}^2}{W_\varepsilon})dx$. It is easy to verify that when $\overline{T_0}$ satisfies

$$2Ce^{12\sqrt{H_0}}(H_0+1)^2\overline{T_0}\le\min\{\frac{1}{2},\frac{H_0}{2}\}, \tag{3.36}$$

then

$$H(t)\le 2H_0 \text{ for } t\in(0,\overline{T_0}). \tag{3.37}$$

Indeed, consider

$$(H(t)+1)\le M\int_0^t(H+1)^3+(H_0+1),$$

where $M=Ce^{12\sqrt{H_0}}$, then

$$H(t)\le(H_0+1)(1-2M(H_0+1)^2t)^{-\frac{1}{2}}-1 \text{ for } t \text{ small}.$$

Since $(1-x)^{-\frac{1}{2}}<1+x$ for $x\in(0,\frac{1}{2})$, it holds

$$H(t)\le(H_0+1)(1+2M(H_0+1)^2t)-1=H_0+2M(H_0+1)^2tH_0+2M(H_0+1)^2t$$

for t small. Thus, when we take $\overline{T_0}$ satisfying (3.36), we have (3.37).

If we take

$$\begin{aligned}H_1&=\int_0^\infty W(\phi_0^2+\phi_{0x}^2)dx+\int_0^\infty(\phi_0^2+\phi_{0x}^2)dx+\int_0^\infty(\psi^2+\frac{\psi_{0x}^2}{W})dx,\\ H_2&=\int_0^\infty W(\phi_0^2+\phi_{0x}^2)dx+\int_0^\infty(\psi^2+\frac{\psi_{0x}^2}{W})dx,\end{aligned}$$

then $H_1 > H_0$ and $H_2 < H_0$. Now we take T_0 satisfying

$$2Ce^{12\sqrt{H_1}}(H_1+1)^2T_0 = \min\{\frac{1}{2}, \frac{H_2}{2}\}.$$

Clearly, T_0 is independent of ε, and

$$H(t) \le 2H_0 \text{ for } t \in (0, T_0). \tag{3.38}$$

Thanks to Proposition 3.1, (3.13) and (3.38), for any $0 < \varepsilon < 1$, system (3.10) with (3.7)–(3.9) has a unique solution (ϕ, ψ) on $\mathbb{R}_+ \times (0, T_0)$ satisfying (3.14). The other desired estimate (3.15) follows from (3.38) and an integration of (3.35) in t. □

3.3. Local well-posedness for the singular system

Let us now study the local well-posedness of (3.6)–(3.9). We start with the uniqueness of the solutions.

Proposition 3.3. *Let* (ϕ_1, ψ_1) *and* (ϕ_2, ψ_2) *be two solutions of system* (3.6)–(3.9) *satisfying*

$$\sqrt{W}\phi_i \in L^\infty((0,T);H^1), \frac{\phi_{ix}}{\sqrt{W}} \in L^2((0,T);H^1),$$
$$\psi_i \in L^\infty((0,T);L^2), \frac{\psi_{ix}}{\sqrt{W}} \in L^\infty((0,T);L^2), \frac{\psi_{ixx}}{\sqrt{W}} \in L^2((0,T);L^2),$$

for $i = 1, 2$. *Then* $(\phi_1, \psi_1) \equiv (\phi_2, \psi_2)$ *on* $\mathbb{R}_+ \times [0, T]$.

Proof. Define (ϕ, ψ) by

$$\phi = \phi_1 - \phi_2,\ \psi = \psi_1 - \psi_2.$$

Then (ϕ, ψ) satisfies

$$\begin{cases} W^2\phi_t = e^{2\psi_1}\phi_{xx} + (\psi_{1x}+V)e^{2\psi_1}\phi_x + (e^{2\psi_1} - e^{2\psi_2})\phi_{2xx} \\ \qquad\quad + (e^{2\psi_1} - e^{2\psi_2})\phi_{2x}\psi_{1x} + e^{2\psi_2}\phi_{2x}\psi_x + V(e^{2\psi_1} - e^{2\psi_2})\phi_{2x} \\ \qquad\quad + Ue^{2\psi_1}\psi_x + U(e^{2\psi_1} - e^{2\psi_2})\psi_{2x}, \\ \psi_t = d\psi_{xx} - (2dV + d\psi_{1x} + d\psi_{2x})\psi_x + \phi_x. \end{cases} \tag{3.39}$$

Multiplying the first equation of (3.39) by $\frac{\phi}{W}$, and the second one by ψ, summing the resultant equations up, one gets after an integration by parts that

$$\begin{aligned} &\frac{1}{2}\int_0^\infty \Big(W\phi^2 + \psi^2\Big) + \int_0^t\int_0^\infty \Big(\frac{e^{2\psi_1}\phi_x^2}{W} + d\psi_x^2\Big) \\ =& \int_0^t\int_0^\infty (e^{2\psi_1} - e^{2\psi_2})\frac{\phi_{2x}\psi_{1x}\phi}{W} + \int_0^t\int_0^\infty e^{2\psi_2}\frac{\phi_{2x}\phi\psi_x}{W} + \int_0^t\int_0^\infty \frac{U}{W}e^{2\psi_1}\phi\psi_x \\ &+ \int_0^t\int_0^\infty \frac{U}{W}(e^{2\psi_1} - e^{2\psi_2})\psi_{2x}\phi + \int_0^t\int_0^\infty \frac{V}{W}(e^{2\psi_1} - e^{2\psi_2})\phi_{2x}\phi + \int_0^t\int_0^\infty \phi_x\psi \\ &- \int_0^t\int_0^\infty (2dV + d\psi_{1x} + d\psi_{2x})\psi_x\psi - \int_0^t\int_0^\infty \Big(\frac{\psi_{1x}}{W} - \frac{V}{W} + (\frac{1}{W})_x\Big)e^{2\psi_1}\phi\phi_x \\ &+ \int_0^t\int_0^\infty (e^{2\psi_1} - e^{2\psi_2})\frac{\phi_{2xx}\phi}{W} + \frac{1}{2}\int_0^\infty \Big(W\phi_0^2 + \psi_0^2\Big). \end{aligned} \tag{3.40}$$

By Young's inequality, we have

$$\begin{aligned}
&\Big|\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})\frac{\phi_{2x}\psi_{1x}\phi}{W}dx\Big|\\
&\le C\int_0^\infty \frac{|\psi\phi\phi_{2x}\psi_{1x}|}{W}dx\\
&\le C\|\phi\|_{L^\infty}\|\psi\|_{L^\infty}\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}\|\frac{\psi_{1x}}{\sqrt{W}}\|_{L^2} \qquad (3.41)\\
&\le C\|\sqrt{W}\phi\|_{L^2}\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}+C\|\psi\|_{L^2}\|\psi_x\|_{L^2}\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}\\
&\le C(\|\sqrt{W}\phi\|_{L^2}^2+\|\psi\|_{L^2}^2)\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2+\delta(\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2+\|\psi_x\|_{L^2}^2),
\end{aligned}$$

$$\begin{aligned}
\Big|\int_0^\infty e^{2\psi_2}\frac{\phi_{2x}\phi\psi_x}{W}dx\Big| &\le C\|\phi\|_{L^\infty}\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}\|\frac{\psi_x}{\sqrt{W}}\|_{L^2}\\
&\le C\|\sqrt{W}\phi\|_{L^2}\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}+\|\frac{\psi_x}{\sqrt{W}}\|_{L^2}^2\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2 \qquad (3.42)\\
&\le \delta\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2+C\|\sqrt{W}\phi\|_{L^2}^2+C\|\frac{\psi_x}{\sqrt{W}}\|_{L^2}^2\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2,
\end{aligned}$$

$$\Big|\int_0^\infty \frac{U}{W}e^{2\psi_1}\phi\psi_x dx\Big| \le C\|\frac{\psi_x}{\sqrt{W}}\|_{L^2}^2+C\|\sqrt{W}\phi\|_{L^2}^2, \qquad (3.43)$$

$$\begin{aligned}
\Big|\int_0^\infty \frac{U}{W}(e^{2\psi_1}-e^{2\psi_2})\psi_{2x}\phi dx\Big| &\le C\int_0^\infty |\psi\psi_{2x}\phi|dx\\
&\le C\|\phi\|_{L^\infty}\|\psi\|_{L^2}\|\psi_{2x}\|_{L^2}\\
&\le C\|\sqrt{W}\phi\|_{L^2}^{\frac{1}{2}}\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^{\frac{1}{2}}\|\psi\|_{L^2} \qquad (3.44)\\
&\le C(\|\psi\|_{L^2}^2+\|\sqrt{W}\phi\|_{L^2}^2)+\delta\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
\Big|\int_0^\infty \frac{V}{W}(e^{2\psi_1}-e^{2\psi_2})\phi_{2x}\phi dx\Big| &\le C\int_0^\infty \frac{|\psi\phi_{2x}\phi|}{\sqrt{W}}dx \qquad (3.45)\\
&\le \delta\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2+C\|\sqrt{W}\phi\|_{L^2}^2+C\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2\|\psi\|_{L^2}^2,
\end{aligned}$$

and

$$\Big|\int_0^\infty \phi_x\psi dx-\int_0^\infty (2dV+d\psi_{1x}+d\psi_{2x})\psi_x\psi dx\Big| \le \delta\|\psi_x\|_{L^2}^2+\delta\|\phi_x\|_{L^2}^2+C\|\psi\|_{L^2}^2,$$

where $\delta>0$ is a small constant. A direct calculation yields $-\frac{V}{W}+(\frac{1}{W})_x=0$, and then we get

$$\Big|\int_0^\infty \big(\frac{\psi_{1x}}{W}-\frac{V}{W}+(\frac{1}{W})_x\big)e^{2\psi_1}\phi\phi_x\Big| = \Big|\int_0^\infty e^{2\psi_1}\frac{\psi_{1x}\phi\phi_x}{W}\Big| \le \delta\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2+C\|\sqrt{W}\phi\|_{L^2}^2. \qquad (3.46)$$

Integration by parts leads to

$$
\begin{aligned}
\Big|\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})\frac{\phi_{2xx}\phi}{W}\Big| =&\Big|\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})_x\frac{\phi\phi_{2x}}{W}+\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})\frac{\phi_x\phi_{2x}}{W}\\
&+\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})(\frac{1}{W})_x\phi\phi_{2x}\Big|\\
=&\Big|\int_0^\infty 2e^{2\psi_1}\frac{\psi_x\phi\phi_{2x}}{W}+2\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})\frac{\psi_{2x}\phi\phi_{2x}}{W}\\
&+\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})\frac{\phi_x\phi_{2x}}{W}+\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})(\frac{1}{W})_x\phi\phi_{2x}\Big|.
\end{aligned}
$$

As in (3.42),

$$
\Big|\int_0^\infty 2e^{2\psi_1}\frac{\psi_x\phi\phi_{2x}}{W}\Big| \le \delta\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2 + C\|\sqrt{W}\phi\|_{L^2}^2 + C\|\frac{\psi_x}{\sqrt{W}}\|_{L^2}^2\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2. \tag{3.47}
$$

As in (3.41),

$$
\Big|\int_0^\infty 2(e^{2\psi_1}-e^{2\psi_2})\frac{\psi_{2x}\phi\phi_{2x}}{W}\Big| \le C(\|\sqrt{W}\phi\|_{L^2}^2+\|\psi\|_{L^2}^2)\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2+\delta(\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2+\|\psi_x\|_{L^2}^2),
$$

$$
\begin{aligned}
\Big|\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})\frac{\phi_x\phi_{2x}}{W}\Big| &\le C\|\psi\|_{L^\infty}\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}\\
&\le C(\|\psi\|_{L^2}^2+\|\psi_x\|_{L^2}^2)\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2+\delta\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\int_0^\infty (e^{2\psi_1}-e^{2\psi_2})(\frac{1}{W})_x\phi\phi_{2x}\Big| \le& C\|\phi\|_{L^\infty}\|\psi\|_{L^2}\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}\\
\le&\delta\|\frac{\phi_x}{\sqrt{W}}\|_{L^2}^2+C\|\sqrt{W}\phi\|_{L^2}^2+C\|\psi\|_{L^2}^2\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2.
\end{aligned} \tag{3.48}
$$

Now substituting (3.41)–(3.48) into (3.40), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty (W\phi^2+\psi^2)dx+\int_0^t\int_0^\infty \Big[(e^{2\psi_1}-C\delta)\frac{\phi_x^2}{W}+(d-C\delta)\psi_x^2\Big]dx\\
\le& C\int_0^t\Big(\|\sqrt{W}\phi\|_{L^2}^2+\|\psi\|_{L^2}^2+\|\frac{\psi_x}{\sqrt{W}}\|_{L^2}^2\Big)\Big(1+\|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2\Big).
\end{aligned} \tag{3.49}
$$

We next present the estimate for $\int_0^\infty \frac{\psi_x^2}{W}dx$. Multiplying the second equation of (3.39) by $\frac{\psi_{xx}}{W}$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty \frac{\psi_x^2}{W}+\int_0^t\int_0^\infty d\frac{\psi_{xx}^2}{W}\\
=&-\int_0^t\int_0^\infty \Big(\frac{1}{W}\Big)_x\psi_x\psi_t+\int_0^t\int_0^\infty \frac{\psi_{xx}}{W}(2dV+d\psi_{1x}+d\psi_{2x})\psi_x+\int_0^t\int_0^\infty \frac{\psi_{xx}\phi_x}{W}.
\end{aligned} \tag{3.50}
$$

Using the second equation of (3.39), we get

$$\int_0^\infty \psi_t^2 \le C\int_0^\infty (\psi_{xx}^2 + \psi_x^2 + (\psi_{1x}^2 + \psi_{2x}^2)\psi_x^2 + \phi_x^2),$$

and

$$\int_0^\infty (\psi_{1x}^2 + \psi_{2x}^2)\psi_x^2 \le C\|\psi_x\|_{L^\infty}^2 \le C\|\psi_x\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2.$$

Thus,

$$\left|\int_0^\infty \left(\frac{1}{W}\right)_x \psi_x\psi_t\right| \le C\int_0^\infty \frac{\psi_x^2}{W} + \delta\int_0^\infty \psi_t^2 \le C\int_0^\infty \frac{\psi_x^2}{W} + \delta\int_0^\infty (\psi_{xx}^2 + \phi_x^2).$$

Similarly,

$$\left|\int_0^\infty \frac{\psi_{xx}}{W}(2dV + d\psi_{1x} + d\psi_{2x})\psi_x\right| \le C\int_0^\infty \frac{\psi_x^2}{W} + \delta\int_0^\infty \frac{\psi_{xx}^2}{W},$$

and

$$\int_0^\infty \frac{|\psi_{xx}\phi_x|}{W} \le \frac{d}{2}\int_0^\infty \frac{\psi_{xx}^2}{W} + C\int_0^\infty \frac{\phi_x^2}{W}.$$

Substituting these inequalities into (3.50), we get

$$\frac{1}{2}\int_0^\infty \frac{\psi_x^2}{W} + \int_0^t\int_0^\infty (\frac{d}{2} - C\delta)\frac{\psi_{xx}^2}{W} \le C\int_0^t\int_0^\infty \left(\frac{\psi_x^2}{W} + \frac{\phi_x^2}{W}\right). \tag{3.51}$$

Multiplying (3.49) by $K \gg 1$ and combing the resultant inequality with (3.51), we have

$$\int_0^\infty (W\phi^2 + \psi^2 + \frac{\psi_x^2}{W})dx \le C\int_0^t \left(\|\sqrt{W}\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|\frac{\psi_x}{\sqrt{W}}\|_{L^2}^2\right)\left(1 + \|\frac{\phi_{2x}}{\sqrt{W}}\|_{L^2}^2\right).$$

It then follows from the Gronwall's inequality that

$$\int_0^\infty \left(W\phi^2 + \frac{\psi^2}{W} + \frac{\psi_x^2}{W}\right)dx = 0.$$

Therefore, $\phi \equiv 0$ and $\psi \equiv 0$. We complete the proof.

We are now ready to prove the local existence of solutions to system (3.6)–(3.9).

Proposition 3.4. *Assume that* (ϕ_0, ψ_0) *satisfies*

$$\phi_0 \in H^1(\mathbb{R}_+),\ \psi_0 \in L^2(\mathbb{R}_+),\ \frac{\psi_{0x}}{\sqrt{W}} \in L^2(\mathbb{R}_+).$$

Then there exists a constant $T > 0$*, such that the system* (3.6)–(3.9) *has a unique solution* (ϕ, ψ) *on* $\mathbb{R}_+ \times (0, T)$*, which satisfies*

$$\sup_{t\in[0,T]}\int_0^\infty \left(W\phi^2 + \psi^2 + W\phi_x^2 + \frac{\psi_x^2}{W}\right)dx \le C\int_0^\infty \left(W\phi_0^2 + \psi_0^2 + W\phi_{0x}^2 + \frac{\psi_{0x}^2}{W}\right)dx, \tag{3.52}$$

where C is a constant independent of T.

Proof. Owing to Proposition 3.2, there exists a constant $T > 0$ independent of $\varepsilon > 0$ such that the approximate system (3.10), subject to (3.7)–(3.9), has a unique solution $(\phi_\epsilon, \psi_\epsilon)$ satisfying

$$\begin{aligned}\sup_{t\in[0,T]}\int_0^\infty \left(W_\varepsilon\phi_\varepsilon^2 + \psi_\varepsilon^2 + W_\varepsilon\phi_{\varepsilon x}^2 + \frac{\psi_{\varepsilon x}^2}{W_\varepsilon}\right)dx &\le 2\int_0^\infty \left(W_\varepsilon\phi_0^2 + \psi_0^2 + W_\varepsilon\phi_{0x}^2 + \frac{\psi_{0x}^2}{W_\varepsilon}\right)dx \\ &\le C\int_0^\infty (\phi_0^2 + \phi_{0x}^2 + \psi_0^2 + \frac{\psi_{0x}^2}{W}),\end{aligned} \tag{3.53}$$

where C is a constant independent of ε. Owing to (3.53) and (3.15), passing to the limit $\varepsilon \to 0^+$, applying the Banach-Alaoglu theorem and the diagonal argument, we know that there is a subsequence, still denoted by $(\phi_\varepsilon, \psi_\varepsilon)$, such that for any $r \in (0, \infty)$

$$\begin{aligned}\psi_{\varepsilon t} \to \psi_t &\ \text{ weakly in } L^2((0,T);L^2(0,r)),\\ \phi_{\varepsilon t} \to \phi_t &\ \text{ weakly in } L^2((0,T);L^2(0,r)),\\ \psi_\varepsilon \to \psi &\ \text{ weakly in } L^2((0,T);H^2(0,r)),\\ \phi_\varepsilon \to \phi &\ \text{ weakly in } L^2((0,T);H^2(0,r)).\end{aligned}$$

Noting $H^2(0,r)$ and $H^1(0,r)$ compactly embed into $H^1(0,r)$ and $L^\infty(0,r)$, respectively, for any $r > 0$, we obtain from the Aubin-Lions compactness lemma that

$$\begin{aligned}\psi_\varepsilon \to \psi &\ \text{ strongly in } L^2((0,T);H^1(0,r)) \cap C([0,T];L^\infty(0,r)),\\ \phi_\varepsilon \to \phi &\ \text{ strongly in } L^2((0,T);H^1(0,r)).\end{aligned}$$

Observing that $W_\varepsilon \to W$ and $V_\varepsilon \to V$ in $C[0,r]$, one can see that the nonlinear terms in (3.10), $e^{2\psi_\varepsilon}\phi_{\varepsilon x}\psi_{\varepsilon x}$ and $d\psi_{\varepsilon x}^2$ converge strongly in $L^2((0,T);L^2(0,r))$ to $e^{2\psi}\phi_x\psi_x$ and $d\psi_x^2$, respectively. Then one can take the limit as $\varepsilon \to 0$ in (3.10) to derive that (ϕ, ψ) satisfies (3.6) in the sense of distribution. Moreover, it follows from the weakly lower semi-continuity of the norms, the first inequality of (3.53) that

$$\begin{aligned}&\|\sqrt{W}\phi\|^2_{L^\infty((0,T);H^1(0,r))} + \|\psi\|^2_{L^\infty((0,T);L^2(0,r))} + \|\frac{\psi_x}{\sqrt{W}}\|^2_{L^\infty((0,T);L^2(0,r))}\\ &\le \varliminf_{\varepsilon\to 0}(\|\sqrt{W_\varepsilon}\phi_\varepsilon\|^2_{L^\infty((0,T);H^1(0,r))} + \|\psi_\varepsilon\|^2_{L^\infty((0,T);L^2(0,r))} + \|\frac{\psi_{\varepsilon x}}{\sqrt{W_\varepsilon}}\|^2_{L^\infty((0,T);L^2(0,r))})\\ &\le 2\varliminf_{\varepsilon\to 0}(\|\sqrt{W_\varepsilon}\phi_0\|^2_{L^\infty((0,T);H^1(0,r))} + \|\psi_0\|^2_{L^\infty((0,T);L^2(0,r))} + \|\frac{\psi_{0x}}{\sqrt{W_\varepsilon}}\|^2_{L^\infty((0,T);L^2(0,r))})\\ &= 2(\|\sqrt{W}\phi_0\|^2_{L^\infty((0,T);H^1(0,r))} + \|\psi_0\|^2_{L^\infty((0,T);L^2(0,r))} + \|\frac{\psi_{0x}}{\sqrt{W}}\|^2_{L^\infty((0,T);L^2(0,r))}).\end{aligned}$$

Therefore, (3.52) holds, and the proof is complete.

Proof. [Proof of Theorem 2.1] It is a consequence of Propositions 3.3 and 3.4.

4. Nonlinear stability

In this section, we prove the global well-posedness of strong solutions to the system (3.6)–(3.9), which also implies the nonlinear stability of spiky steady state to the original chemotaxis system (1.2)–

(1.4). We construct global solutions of system (3.6)–(3.9) in the more regular space:

$$X(0,T) := \{(\phi,\psi)(x,t)|\phi \in C([0,T];H^2), \phi_x \in C([0,T];L^2_{\omega_3}) \cap L^2((0,T);H^2_{\omega_2}),$$
$$\phi_{xx} \in C([0,T];L^2_{\omega_4}), \psi \in C([0,T];H^2), \psi_x \in C([0,T];H^1_{\omega_2}) \cap L^2((0,T);H^2_{\omega_2}).$$

for $T \in (0,+\infty]$, where $\omega_2 = \frac{1}{U}$, $\omega_3 = \frac{1}{U^2}$ and $\omega_4 = \frac{1}{U^3}$. Set

$$N^2(t) := \sup_{\tau\in[0,t]} (\|\phi(\cdot,\tau)\|^2 + \|\psi(\cdot,\tau)\|^2 + \|\phi_x(\cdot,\tau)\|^2_{\omega_3} + \|\psi_x(\cdot,\tau)\|^2_{1,\omega_2} + \|\phi_{xx}(\cdot,\tau)\|^2_{\omega_4}).$$

Since $U(x) \le \frac{\lambda^2}{6d}$, the Sobolev embedding theorem implies

$$\sup_{\tau\in[0,t]} \{\|\phi(\cdot,\tau)\|_{L^\infty}, \|\psi(\cdot,\tau)\|_{L^\infty}\} \le N(t).$$

Moreover, noting

$$\begin{aligned}\frac{\psi_x^2}{U}(x,t) = -\int_x^\infty (\frac{\psi_x^2}{U})_x dx = -\int_x^\infty 2\frac{\psi_x\psi_{xx}}{U}dx - \int_x^\infty \left(\frac{1}{U}\right)_x \psi_x^2 dx \\ \le C\int_0^\infty \frac{\psi_x^2}{U}dx + C\int_0^\infty \frac{\psi_{xx}^2}{U}dx,\end{aligned} \tag{4.1}$$

we have

$$\left\|\frac{\psi_x(\cdot,t)}{\sqrt{U}}\right\|_{L^\infty} \le CN(t). \tag{4.2}$$

Similarly,

$$\begin{aligned}\frac{\phi_x^2}{U}(x,t) = -\int_x^\infty (\frac{\phi_x^2}{U})_x dx = -\int_x^\infty 2\frac{\phi_x\phi_{xx}}{U}dx - \int_x^\infty (\frac{1}{U})_x \phi_x^2 dx \\ \le C\int_0^\infty \frac{\phi_x^2}{U}dx + C\int_0^\infty \frac{\phi_{xx}^2}{U}dx,\end{aligned} \tag{4.3}$$

which implies

$$\left\|\frac{\phi_x(\cdot,t)}{\sqrt{U}}\right\|_{L^\infty} \le CN(t). \tag{4.4}$$

For system (3.6)–(3.9), we have the following results.

Proposition 4.1. *There exists a constant $\delta_1 > 0$, such that if $N(0) \le \delta_1$, then the system* (3.6)–(3.9) *has a unique global solution $(\phi,\psi) \in X(0,\infty)$ satisfying*

$$\begin{aligned}&\|\phi(\cdot,\tau)\|^2 + \|\psi(\cdot,\tau)\|^2 + \|\phi_x(\cdot,\tau)\|^2_{\omega_3} + \|\psi_x(\cdot,\tau)\|^2_{1,\omega_2} + \|\phi_{xx}(\cdot,\tau)\|^2_{\omega_4} \\ &+ \int_0^t (\|\phi_x(\tau)\|^2_{2,\omega_2} + \|\psi_x(\tau)\|^2_{2,\omega_2} + \|\phi_{xx}(\tau)\|^2_{\omega_3})d\tau \le CN^2(0)\end{aligned} \tag{4.5}$$

for any $t \in [0,\infty)$.

Thanks to the local well-posedness established in Propositions 3.3 and 3.4, we only need to derive the following a priori estimates to prove Proposition 4.1.

Proposition 4.2. *Assume that the conditions of Proposition* 4.1 *hold, and that* $(\phi, \psi) \in X(0, T)$ *is a solution of system* (3.6)–(3.9) *for some constant* $T > 0$*. Then there is a constant* $\varepsilon > 0$*, independent of* T*, such that if* $N(t) \leq \varepsilon$ *for any* $0 < t \leq T$*, then* (ϕ, ψ) *satisfies* (4.5) *for any* $0 \leq t \leq T$*.*

To establish the a priori estimate, we need the following Hardy inequality (see Lemma 3.4 of [13] for the proof).

Lemma 4.1. **(Hardy inequality)** *If* $f \in H_0^1(0, \infty)$*, then for* $j \neq -1$*, it holds that*

$$\int_0^\infty (1 + kx)^j f^2(x) dx \leq \frac{4}{(j + 1)^2 k^2} \int_0^\infty (1 + kx)^{j+2} f_x^2(x) dx. \tag{4.6}$$

where $k > 0$ *is a constant.*

We start with the L^2 estimate.

Lemma 4.2. *If* $N(t) \ll 1$*, then there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\int_0^\infty (U\phi^2 + \psi^2) dx + \int_0^t \int_0^\infty (\phi^2 + U\psi^2) dx d\tau + \int_0^t \int_0^\infty \left(\frac{\phi_x^2}{U} + \psi_x^2\right) dx d\tau \\ &\leq C \int_0^\infty (U\phi_0^2 + \psi_0^2) dx. \end{aligned} \tag{4.7}$$

Proof. We rewrite (3.6) as

$$\begin{cases} W^2\phi_t = \phi_{xx} + V\phi_x + U\psi_x + \phi_x\psi_x + (e^{2\psi} - 1)(\phi_{xx} + V\phi_x + U\psi_x + \phi_x\psi_x), \\ \psi_t = d\psi_{xx} - 2dV\psi_x - d\psi_x^2 + \phi_x. \end{cases} \tag{4.8}$$

Multiplying the first equation of (4.8) by $\frac{\phi}{U}$, the second one by ψ, and integrating the resulting equations on $(0, t) \times (0, +\infty)$, we have

$$\begin{aligned} &\frac{1}{2} \int_0^\infty \left(\frac{W^2}{U}\phi^2 + \psi^2\right) dx + \int_0^t \int_0^\infty \frac{\phi_x^2}{U} dx d\tau + d \int_0^t \int_0^\infty \psi_x^2 dx d\tau + d \int_0^\infty |V_x| \psi_x^2 dx d\tau \\ &= \int_0^t \int_0^\infty \frac{1}{2} \left[\left(\frac{1}{U}\right)_{xx} - \left(\frac{V}{U}\right)_x\right] \phi^2 dx d\tau + \int_0^t \int_0^\infty \frac{\phi\phi_x\psi_x}{U} dx d\tau - \int_0^t \int_0^\infty d\psi\psi_x^2 dx d\tau \\ &+ \int_0^t \int_0^\infty (e^{2\psi} - 1) \frac{\phi}{U} (\phi_{xx} + \phi_x\psi_x + U\psi_x + V\phi_x) dx d\tau + \frac{1}{2} \int_0^\infty \left(\frac{W^2}{U}\phi_0^2 + \psi_0^2\right) dx. \end{aligned} \tag{4.9}$$

By (2.4) and Hardy inequality, we get

$$\frac{1}{2} \int_0^\infty \frac{\phi_x^2}{U} dx = \int_0^\infty \frac{3d}{\lambda^2} (1 + \frac{\lambda}{6d} x)^2 \phi_x^2 dx \geq \frac{1}{48d} \int_0^\infty \phi^2 dx. \tag{4.10}$$

Owing to (3.3), it is easy to compute that

$$\left(\frac{1}{U}\right)_{xx} - \left(\frac{V}{U}\right)_x = 0, \tag{4.11}$$

which gives

$$\int_0^\infty \frac{1}{2}\left[\left(\frac{1}{U}\right)_{xx} - \left(\frac{V}{U}\right)_x\right]\phi^2 dx = 0.$$

By (4.2) and Young's inequality, we derive that

$$\Big|\int_0^\infty \frac{\phi\phi_x\psi_x}{U}dx\Big| \le CN(t)\int_0^\infty \frac{|\phi\phi_x|}{\sqrt{U}}dx \le CN(t)\int_0^\infty \frac{\phi_x^2}{U}dx + CN(t)\int_0^\infty \phi^2 dx. \tag{4.12}$$

Similarly, since $\|\psi(\cdot,t)\|_{L^\infty} \le N(t)$ and

$$V = \sqrt{\frac{2U}{3d}}, \tag{4.13}$$

we have

$$\Big|\int_0^\infty d\psi\psi_x^2 dx\Big| \le dN(t)\int_0^\infty \psi_x^2 dx, \tag{4.14}$$

and

$$\begin{aligned}
&\int_0^\infty (e^{2\psi}-1)\frac{\phi}{U}(\phi_{xx} + \phi_x\psi_x + U\psi_x + V\phi_x)\\
&= -2\int_0^\infty e^{2\psi}\psi_x\frac{\phi\phi_x}{U} - \int_0^\infty (e^{2\psi}-1)\left(\frac{\phi}{U}\right)_x\phi_x + \int_0^\infty (e^{2\psi}-1)\frac{\phi}{U}(\phi_x\psi_x + U\psi_x + V\phi_x)\\
&\le C\left\|\frac{\psi_x(\cdot,t)}{\sqrt{U}}\right\|_{L^\infty}\int_0^\infty \frac{|\phi\phi_x|}{\sqrt{U}} + C\int_0^\infty |\psi|\left(\frac{\phi_x^2}{U} + \frac{|U_x\phi\phi_x|}{U^2}\right)\\
&\quad + C\int_0^\infty \frac{1}{U}\,|\phi\psi(\phi_x\psi_x + U\psi_x + V\phi_x)|\\
&\le CN(t)\int_0^\infty (\phi^2 + \frac{\phi_x^2}{U} + \psi_x^2),
\end{aligned} \tag{4.15}$$

where we have used the Taylor expansion

$$\begin{aligned}
\left|e^{2\psi}-1\right| &= \left|2\psi + \sum_{n=2}^\infty \frac{2^n\psi^n}{n!}\right|\\
&\le 2|\psi| + 2\left|\psi\sum_{n=2}^\infty \frac{2^{n-1}\psi^{n-1}}{n!}\right|\\
&\le 2|\psi| + 2|\psi|\sum_{n=2}^\infty 2^{n-1}(\frac{1}{2})^{n-1}\\
&\le C|\psi|.
\end{aligned} \tag{4.16}$$

Now substituting (4.10)–(4.15) into (4.9), noting $V_x < 0$, and using Hardy inequality, we get

$$\begin{aligned}
&\int_0^\infty (U\phi^2 + \psi^2)dx + \int_0^t\int_0^\infty \left(\frac{\phi_x^2}{U} + \psi_x^2\right)dxd\tau + \int_0^t\int_0^\infty (\phi^2 + U\psi^2)dxd\tau\\
&\le CN(t)\int_0^t\int_0^\infty \left(\phi^2 + \frac{\phi_x^2}{U} + \psi_x^2\right)dxd\tau + \int_0^\infty (U\phi_0^2 + \psi_0^2)dx.
\end{aligned}$$

Thus, we obtain (4.7) provided $N(t) \ll 1$.

We next establish the H^1 estimate.

Lemma 4.3. *If* $N(t) \ll 1$, *then the solution of* (3.6)–(3.9) *satisfies*

$$\begin{aligned}&\int_0^\infty \left(\phi_x^2 + \frac{\psi_x^2}{U}\right) dx + \int_0^t \int_0^\infty \left(\frac{\phi_{xx}^2}{U^2} + \frac{\psi_{xx}^2}{U}\right) dx d\tau \\ &\leq C \int_0^\infty \left(\phi_{0x}^2 + \frac{\psi_{0x}^2}{U} + U\phi_0^2 + \psi_0^2\right) dx.\end{aligned} \tag{4.17}$$

Proof. Multiplying the first equation of (4.8) by $\frac{\phi_{xx}}{W^2}$, we get

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \phi_x^2 + \int_0^t \int_0^\infty \frac{\phi_{xx}^2}{W^2} \\ =& -\int_0^t \int_0^\infty \frac{U}{W^2}\phi_{xx}\psi_x - \int_0^t \int_0^\infty \frac{V}{W^2}\phi_x\phi_{xx} - \int_0^t \int_0^\infty \frac{\phi_x\phi_{xx}\psi_x}{W^2} \\ &-\int_0^t \int_0^\infty (e^{2\psi} - 1)\frac{\phi_{xx}}{W^2}(\phi_{xx} + \phi_x\psi_x + U\psi_x + V\phi_x) + \frac{1}{2}\int_0^\infty \phi_{0x}^2.\end{aligned} \tag{4.18}$$

In view of (2.4), it holds

$$U(x) = \frac{\lambda^2}{6db}W(x). \tag{4.19}$$

It then follows from Young's inequality that

$$\int_0^\infty \frac{U}{W^2}|\phi_{xx}\psi_x| dx \leq C \int_0^\infty \frac{|\phi_{xx}\psi_x|}{W} dx \leq \frac{1}{4}\int_0^\infty \frac{\phi_{xx}^2}{W^2} dx + C\int_0^\infty \psi_x^2 dx. \tag{4.20}$$

Moreover, by (4.13),

$$\begin{aligned}\Big|\int_0^\infty \frac{V}{W^2}\phi_x\phi_{xx} dx\Big| &\leq \frac{1}{2}\int_0^\infty \frac{\phi_{xx}^2}{W^2} dx + \frac{1}{2}\int_0^\infty \frac{V^2}{W^2}\phi_x^2 dx \\ &\leq \frac{1}{2}\int_0^\infty \frac{\phi_{xx}^2}{W^2} dx + C\int_0^\infty \frac{\phi_x^2}{W} dx.\end{aligned} \tag{4.21}$$

Using (4.2), it is easy to see that

$$\begin{aligned}\Big|\int_0^\infty \frac{\phi_x\phi_{xx}\psi_x}{W^2} dx| &\leq CN(t)\int_0^\infty \frac{|\phi_x\phi_{xx}|}{W^{\frac{3}{2}}} dx \\ &\leq CN(t)\int_0^\infty \frac{\phi_x^2}{W} dx + CN(t)\int_0^\infty \frac{\phi_{xx}^2}{W^2} dx.\end{aligned} \tag{4.22}$$

By (4.16), the fact that $\|\psi(\cdot,t)\|_{L^\infty} \leq N(t)$ and (4.2) again, one has

$$\int_0^\infty \Big|(e^{2\psi} - 1)\frac{\phi_{xx}}{W^2}(\phi_{xx} + \phi_x\psi_x + U\psi_x + V\phi_x)\Big| dx \leq CN(t)\int_0^\infty \left(\frac{\phi_x^2}{W} + \frac{\phi_{xx}^2}{W^2} + \psi_x^2\right) dx. \tag{4.23}$$

Substituting (4.20)–(4.23) into (4.18), we get

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \phi_x^2 + \left(\frac{1}{4} - CN(t)\right)\int_0^t \int_0^\infty \frac{\phi_{xx}^2}{W^2} \\ &\leq (C + CN(t))\int_0^t \int_0^\infty \frac{\phi_x^2}{W} + (C + CN(t))\int_0^t \int_0^\infty \psi_x^2 + \frac{1}{2}\int_0^\infty \phi_{0x}^2.\end{aligned}$$

Thus, by (4.19) and Lemma 4.2, when $N(t) \ll 1$, we arrive at

$$\int_0^\infty \phi_x^2 dx + \int_0^t \int_0^\infty \frac{\phi_{xx}^2}{U^2} dx d\tau \le \int_0^\infty \phi_{0x}^2 dx + C \int_0^\infty (U\phi_0^2 + \psi_0^2) dx. \tag{4.24}$$

Multiplying the second equation of (4.8) by $\frac{\psi_{xx}}{U}$, we have

$$\begin{aligned}\frac{1}{2}\int_0^\infty \frac{\psi_x^2}{U} + d\int_0^t \int_0^\infty \frac{\psi_{xx}^2}{U} =& -\int_0^t \int_0^\infty \frac{\phi_x \psi_{xx}}{U} + 2d \int_0^t \int_0^\infty \frac{V}{U}\psi_x\psi_{xx} \\ & - \int_0^t \int_0^\infty \left(\frac{1}{U}\right)_x \psi_t \psi_x + \int_0^t \int_0^\infty \frac{d}{U}\psi_x^2 \psi_{xx} + \frac{1}{2}\int_0^\infty \frac{\psi_{0x}^2}{U}.\end{aligned} \tag{4.25}$$

By Young's inequality,

$$\left| \int_0^\infty \frac{\phi_x \psi_{xx}}{U} dx \right| \le \frac{1}{2d} \int_0^\infty \frac{\phi_x^2}{U} dx + \frac{d}{2} \int_0^\infty \frac{\psi_{xx}^2}{U} dx. \tag{4.26}$$

Moreover, (4.13) gives

$$\begin{aligned}\left| 2d \int_0^\infty \frac{V}{U} \psi_x \psi_{xx} dx \right| &\le C \int_0^\infty \frac{1}{\sqrt{U}} |\psi_x \psi_{xx}| dx \\ &\le C \int_0^\infty \psi_x^2 dx + \frac{d}{4} \int_0^\infty \frac{\psi_{xx}^2}{U} dx.\end{aligned} \tag{4.27}$$

By (3.3) and (4.13),

$$\left| \left(\frac{1}{U}\right)_x \right| = \frac{|U_x|}{U^2} = \frac{V}{U} = \sqrt{\frac{2}{3d}} \cdot \frac{1}{\sqrt{U}}, \tag{4.28}$$

which in combination with (4.2) leads to

$$\begin{aligned}&\left| \int_0^\infty \left(\frac{1}{U}\right)_x \psi_t \psi_x dx \right| \\ \le& \int_0^\infty \left| \left(\frac{1}{U}\right)_x (d\psi_{xx} - 2dV\psi_x - d\psi_x^2 + \phi_x)\psi_x \right| dx \\ \le& C \int_0^\infty \left| \frac{\psi_x \psi_{xx}}{\sqrt{U}} \right| dx + C \int_0^\infty \frac{V}{\sqrt{U}} \psi_x^2 dx + C \int_0^\infty \frac{|\psi_x^3|}{\sqrt{U}} dx + C \int_0^\infty \left| \frac{\phi_x \psi_x}{\sqrt{U}} \right| dx \\ \le& C \int_0^\infty \frac{\phi_x^2}{U} dx + \big(C + CN(t)\big) \int_0^\infty \psi_x^2 dx + \frac{d}{8} \int_0^\infty \frac{\psi_{xx}^2}{U} dx,\end{aligned} \tag{4.29}$$

and

$$\left| \int_0^\infty \frac{d}{U} \psi_x^2 \psi_{xx} dx \right| \le CN(t) \int_0^\infty \left| \frac{\psi_{xx}\psi_x}{\sqrt{U}} \right| dx \le CN(t) \int_0^\infty \frac{\psi_{xx}^2}{U} dx + CN(t) \int_0^\infty \psi_x^2 dx. \tag{4.30}$$

Now substituting (4.26)–(4.30) into (4.25), we derive that

$$\begin{aligned}\frac{1}{2}\int_0^\infty \frac{\psi_x^2}{U} + \int_0^t \int_0^\infty \left(\frac{d}{8} - CN(t)\right)\frac{\psi_{xx}^2}{U} \le& \big(C + CN(t)\big) \int_0^t \int_0^\infty \psi_x^2 + C \int_0^t \int_0^\infty \frac{\phi_x^2}{U} \\ &+ \frac{1}{2}\int_0^\infty \frac{\psi_{0x}^2}{U}.\end{aligned}$$

Then by Lemma 4.2, when $N(t) \ll 1$, we have

$$\int_0^\infty \frac{\psi_x^2}{U}dx + \int_0^t \int_0^\infty \frac{\psi_{xx}^2}{U}dxd\tau \leq \int_0^\infty \frac{\psi_{0x}^2}{U}dx + C\int_0^\infty (U\phi_0^2 + \psi_0^2)dx. \tag{4.31}$$

Combining (4.31) and (4.24), we get (4.17).

The H^2 estimate is as follows.

Lemma 4.4. *If $N(t) \ll 1$, then it holds*

$$\begin{aligned}&\int_0^\infty \left(U\phi_t^2 + \frac{\psi_t^2}{U} + \frac{\phi_{xx}^2}{U} + \frac{\psi_{xx}^2}{U}\right) + \int_0^t \int_0^\infty \left(\frac{\phi_{tx}^2}{U} + \frac{\psi_{tx}^2}{U} + \frac{\phi_{xxx}^2}{U} + \frac{\psi_{xxx}^2}{U}\right)\\ &\leq C\int_0^\infty \left(U\phi_0^2 + \psi_0^2 + \frac{\phi_{0x}^2}{U^2} + \frac{\psi_{0x}^2}{U} + \frac{\phi_{0xx}^2}{U^3} + \frac{\psi_{0xx}^2}{U}\right).\end{aligned} \tag{4.32}$$

Proof. Differentiating the first equation of (4.8) with respect to t leads to

$$\begin{aligned}W^2\phi_{tt} =&\phi_{txx} + \phi_{tx}\psi_x + \phi_x\psi_{tx} + U\psi_{tx} + V\phi_{tx}\\ &+ (e^{2\psi} - 1)(\phi_{txx} + \phi_{tx}\psi_x + \phi_x\psi_{tx} + U\psi_{tx} + V\phi_{tx})\\ &+ 2e^{2\psi}\psi_t(\phi_{xx} + \phi_x\psi_x + U\psi_x + V\phi_x).\end{aligned} \tag{4.33}$$

Multiplying (4.33) by $\frac{\phi_t}{U}$ and integrating it in x and t, we get

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \frac{W^2}{U}\phi_t^2 + \int_0^t\int_0^\infty \frac{\phi_{tx}^2}{U} - \frac{1}{2}\int_0^t\int_0^\infty \left[\left(\frac{1}{U}\right)_{xx} - \left(\frac{V}{U}\right)_x\right]\phi_t^2\\ &\leq \int_0^t\int_0^\infty \frac{\phi_t\phi_{tx}\psi_x}{U} + \int_0^t\int_0^\infty \frac{\phi_t\phi_x\psi_{tx}}{U} + \int_0^t\int_0^\infty \phi_t\psi_{tx}\\ &\quad + \int_0^t\int_0^\infty (e^{2\psi}-1)\phi_{txx}\frac{\phi_t}{U} + \int_0^t\int_0^\infty (e^{2\psi}-1)(\phi_{tx}\psi_x + \phi_x\psi_{tx} + U\psi_{tx} + V\phi_{tx})\frac{\phi_t}{U}\\ &\quad + \int_0^t\int_0^\infty 2e^{2\psi}\frac{\psi_t\phi_t}{U}\phi_{xx} + \int_0^t\int_0^\infty 2e^{2\psi}\frac{\psi_t\phi_t}{U}(\phi_x\psi_x + U\psi_x + V\phi_x)\\ &\quad + C\int_0^\infty \left(\frac{\phi_{0xx}^2}{U^3} + \frac{\phi_{0x}^2}{U^2} + \frac{\psi_{0x}^2}{U}\right),\end{aligned} \tag{4.34}$$

where we have used

$$U\phi_t^2\big|_{t=0} \leq C\left(\frac{\phi_{0xx}^2}{U^3} + \frac{\phi_{0x}^2}{U^2} + \frac{\psi_{0x}^2}{U}\right).$$

By Young's inequality, Hardy's inequality, (4.2) and (4.4), we get

$$\begin{aligned}\Big|\int_0^\infty \frac{\phi_t\phi_{tx}\psi_x}{U}dx\Big| &\leq \frac{N(t)}{2}\int_0^\infty \frac{\phi_{tx}^2}{U}dx + \frac{N(t)}{2}\int_0^\infty \phi_t^2 dx\\ &\leq CN(t)\int_0^\infty \frac{\phi_{tx}^2}{U}dx,\end{aligned} \tag{4.35}$$

and

$$\Big|\int_0^\infty \frac{\phi_t\phi_x\psi_{tx}}{U}dx\Big| \le CN(t)\int_0^\infty \frac{\psi_{tx}^2}{U}dx + CN(t)\int_0^\infty \phi_t^2 dx \\ \le CN(t)\int_0^\infty \frac{\psi_{tx}^2}{U}dx + CN(t)\int_0^\infty \frac{\phi_{tx}^2}{U}dx. \tag{4.36}$$

Moreover, integration by parts leads to

$$\Big|\int_0^\infty \phi_t\psi_{tx}dx\Big| = \Big|\int_0^\infty \psi_t\phi_{tx}dx\Big| \le \frac{1}{4}\int_0^\infty \frac{\phi_{tx}^2}{U}dx + C\int_0^\infty \frac{\psi_t^2}{U}dx. \tag{4.37}$$

Using (4.16), (4.2) and (4.4) again, a simple calculation yields

$$\Big|\int_0^\infty (e^{2\psi}-1)\phi_{txx}\frac{\phi_t}{U}dx\Big| = \Big|\int_0^\infty \Big[(e^{2\psi}-1)\Big(\frac{\phi_{tx}^2}{U}+\phi_{tx}\phi_t(\frac{1}{U})_x\Big)+2e^{2\psi}\psi_x\phi_{tx}\frac{\phi_t}{U}\Big]dx\Big| \\ \le CN(t)\int_0^\infty \Big(\frac{\phi_{tx}^2}{U}+\phi_t^2\Big)dx \\ \le CN(t)\int_0^\infty \frac{\phi_{tx}^2}{U}dx, \tag{4.38}$$

and

$$\Big|\int_0^\infty (e^{2\psi}-1)(\phi_{tx}\psi_x+\phi_x\psi_{tx}+U\psi_{tx}+V\phi_{tx})\frac{\phi_t}{U}dx\Big| \\ \le CN(t)\int_0^\infty \Big|\frac{\phi_t}{U}(\phi_{tx}\psi_x+\phi_x\psi_{tx}+U\psi_{tx}+V\phi_{tx})\Big|dx \\ \le CN(t)\int_0^\infty \frac{\phi_{tx}^2}{U}dx + CN(t)\int_0^\infty \frac{\psi_{tx}^2}{U}dx + CN(t)\int_0^\infty \frac{\psi_t^2}{U}dx. \tag{4.39}$$

Similarly,

$$\int_0^\infty 2e^{2\psi}\frac{\phi_t\psi_t}{U}\phi_{xx}dx = -\int_0^\infty 2\Big(e^{2\psi}\frac{\phi_t\psi_t}{U}\Big)_x\phi_x dx \\ \le CN(t)\int_0^\infty \Big(\frac{\phi_{tx}^2}{U}+\frac{\psi_{tx}^2}{U}+\phi_t^2+\psi_t^2\Big)dx \\ \le CN(t)\int_0^\infty \Big(\frac{\phi_{tx}^2}{U}+\frac{\psi_{tx}^2}{U}\Big)dx, \tag{4.40}$$

and

$$\Big|\int_0^\infty 2e^{2\psi}\frac{\phi_t\psi_t}{U}(\phi_x\psi_x+U\psi_x+V\phi_x)dx\Big| \le CN(t)\int_0^\infty (\frac{\phi_{tx}^2}{U}+\frac{\psi_{tx}^2}{U})dx. \tag{4.41}$$

Now substituting (4.35)–(4.41) into (4.34), we arrive at

$$\frac{1}{2}\int_0^\infty U\phi_t^2 + \Big(\frac{3}{4}-CN(t)\Big)\int_0^t\int_0^\infty \frac{\phi_{tx}^2}{U} \le CN(t)\int_0^t\int_0^\infty \frac{\psi_{tx}^2}{U} + C\int_0^t\int_0^\infty \frac{\psi_t^2}{U} \\ + C\int_0^\infty \Big(\frac{\phi_{0xx}^2}{U^3}+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}\Big). \tag{4.42}$$

By (4.2), (3.6), and Lemmas 4.2 and 4.3, we estimate the last term of (4.42) as

$$\begin{aligned}\int_0^t\int_0^\infty \frac{\psi_t^2}{U}dxd\tau \leq& C\int_0^t\int_0^\infty\left(\frac{\psi_{xx}^2}{U}+\frac{V^2}{U}\psi_x^2+\frac{1}{U}\psi_x^4+\frac{\phi_x^2}{U}\right)dxd\tau\\ \leq& C\int_0^t\int_0^\infty\left(\frac{\psi_{xx}^2}{U}+(1+N^2(t))\psi_x^2+\frac{\phi_x^2}{U}\right)dxd\tau\\ \leq& C\int_0^\infty(\phi_{0x}^2+\frac{\psi_{0x}^2}{U}+U\phi_0^2+\psi_0^2)dx.\end{aligned}\tag{4.43}$$

We next estimate $\int_0^t\int_0^\infty \frac{\psi_{tx}^2}{U}$. Differentiating the second equation of (3.6) with respect to t leads to

$$\psi_{tt}=d\psi_{txx}-2dV\psi_{tx}-2d\psi_x\psi_{tx}+\phi_{tx}.\tag{4.44}$$

Multiplying (4.44) by $\frac{\psi_t}{U}$, we have

$$\begin{aligned}\frac{1}{2}\int_0^\infty\frac{\psi_t^2}{U}+\int_0^t\int_0^\infty d\frac{\psi_{tx}^2}{U}=&\int_0^t\int_0^\infty\frac{\phi_{tx}\psi_t}{U}-\int_0^t\int_0^\infty d(\frac{1}{U})_x\psi_{tx}\psi_t\\ &-\int_0^t\int_0^\infty 2d\frac{1}{U}(V\psi_t\psi_{tx}+\psi_x\psi_t\psi_{tx})+\frac{1}{2}\int_0^\infty\frac{\psi_t^2}{U}\Big|_{t=0}.\end{aligned}\tag{4.45}$$

Owing to Young's inequality, we have

$$\Big|\int_0^\infty\frac{\phi_{tx}\psi_t}{U}dx\Big|\leq\frac{1}{2}\int_0^\infty\frac{\phi_{tx}^2}{U}dx+\frac{1}{2}\int_0^\infty\frac{\psi_t^2}{U}dx\tag{4.46}$$

and

$$\begin{aligned}\Big|\int_0^\infty d(\frac{1}{U})_x\psi_{tx}\psi_t dx\Big|\leq& C\Big|\int_0^\infty\frac{\psi_{tx}\psi_t}{\sqrt{U}}dx\Big|\\ \leq&\frac{d}{4}\int_0^\infty\frac{\psi_{tx}^2}{U}dx+C\int_0^\infty\frac{\psi_t^2}{U}dx.\end{aligned}\tag{4.47}$$

By the boundedness of $U(x)$ and $V(x)$, we arrive that

$$\Big|\int_0^\infty 2d\frac{V}{U}\psi_t\psi_{tx}dx\Big|\leq C\int_0^\infty\frac{1}{U}|\psi_t\psi_{tx}|dx\leq C\int_0^\infty\frac{\psi_t^2}{U}dx+\frac{d}{4}\int_0^\infty\frac{\psi_{tx}^2}{U}dx.\tag{4.48}$$

Moreover, the fact that $\|\psi_x(\cdot,t)\|_{L^\infty}\leq CN(t)$ leads to

$$\Big|\int_0^\infty 2d\frac{\psi_x\psi_t\psi_{tx}}{U}dx\Big|\leq CN(t)\int_0^\infty\frac{\psi_{tx}^2}{U}dx+CN(t)\int_0^\infty\frac{\psi_t^2}{U}dx.\tag{4.49}$$

Substituting (4.46)–(4.49) into (4.45) gives

$$\begin{aligned}&\frac{1}{2}\int_0^\infty\frac{\psi_t^2}{U}+\Big(\frac{d}{2}-CN(t)\Big)\int_0^t\int_0^\infty\frac{\psi_{tx}^2}{U}\\ \leq&(C+CN(t))\int_0^t\int_0^\infty\frac{\psi_t^2}{U}+\frac{1}{2}\int_0^t\int_0^\infty\frac{\phi_{tx}^2}{U}+\frac{1}{2}\int_0^\infty\frac{\psi_t^2}{U}\Big|_{t=0}.\end{aligned}\tag{4.50}$$

Combing (4.42) and (4.50), by (4.43), we get

$$\begin{aligned}&\int_0^\infty U\phi_t^2dx+\int_0^\infty \frac{\psi_t^2}{U}dx+\int_0^t\int_0^\infty \frac{\phi_{tx}^2}{U}dxd\tau+\int_0^t\int_0^\infty \frac{\psi_{tx}^2}{U}dxd\tau\\ \le{}& C\int_0^\infty\left(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U}\right)dx.\end{aligned}\tag{4.51}$$

Squaring (3.6) and multiplying the resultant equations by $\frac{1}{U}$, owing to (4.51) and Lemma 4.3, we obtain

$$\begin{aligned}\int_0^\infty \frac{\phi_{xx}^2}{U}dx \le& C\int_0^\infty\left(\frac{W^4}{U}e^{-4\psi}\phi_t^2+\frac{\phi_x^2\psi_x^2}{U}+U\psi_x^2+\frac{V^2}{U}\phi_x^2\right)dx\\ \le& C\int_0^\infty\left(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U}\right)dx,\end{aligned}\tag{4.52}$$

and

$$\begin{aligned}\int_0^\infty \frac{\psi_{xx}^2}{U}dx \le& C\int_0^\infty\left(\frac{\psi_t^2}{U}+\frac{V^2}{U}\psi_x^2+\frac{\psi_x^4}{U}+\frac{\phi_x^2}{U}\right)dx\\ \le& C\int_0^\infty\left(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U}\right)dx.\end{aligned}\tag{4.53}$$

Differentiating the first equation of (3.6) in x yields

$$\begin{aligned}\phi_{xxx}=&-\phi_{xx}\psi_x-\phi_x\psi_{xx}-U_x\psi_x-U\psi_{xx}-V_x\phi_x-V\phi_{xx}+W^2\phi_{tx}+2WW_x\phi_t\\ &-[(e^{2\psi}-1)(\phi_{xx}+\phi_x\psi_x+U\psi_x+V\phi_x)]_x.\end{aligned}\tag{4.54}$$

Squaring (4.54) and multiplying the resultant equations by $\frac{1}{U}$ lead to

$$\begin{aligned}&\int_0^t\int_0^\infty \frac{\phi_{xxx}^2}{U}dxd\tau\\ \le{}& C\int_0^t\int_0^\infty\left(\frac{\phi_{xx}^2\psi_x^2}{U}+\frac{\phi_x^2\psi_{xx}^2}{U}+\frac{U_x^2}{U}\psi_x^2+U\psi_{xx}^2+\frac{V_x^2}{U}\phi_x^2+\frac{V^2}{U}\phi_{xx}^2+\frac{W^4}{U}\phi_{tx}^2\right)\\ &+C\int_0^t\int_0^\infty \frac{W^2W_x^2}{U}\phi_t^2+C\int_0^t\int_0^\infty \frac{1}{U}\left|\Big((e^{2\psi}-1)(\phi_{xx}+\phi_x\psi_x+U\psi_x+V\phi_x)\Big)_x\right|^2.\end{aligned}\tag{4.55}$$

By (2.4), (4.51), the boundedness of $U(x)$, and Hardy inequality, we have

$$\begin{aligned}\left|\int_0^t\int_0^\infty 4\frac{W^2W_x^2}{U}\phi_t^2dxd\tau\right| \le& C\int_0^t\int_0^\infty \frac{\phi_{tx}^2}{U}dxd\tau\\ \le& C\int_0^\infty(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U})dx,\end{aligned}\tag{4.56}$$

and

$$\begin{aligned}\int_0^t\int_0^\infty \frac{W^4}{U}\phi_{tx}^2dxd\tau \le& C\int_0^t\int_0^\infty \frac{\phi_{tx}^2}{U}dxd\tau\\ \le& C\int_0^\infty(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U})dx.\end{aligned}\tag{4.57}$$

Using (4.16) and Lemmas 4.2 and 4.3, with the fact that $\|\psi(\cdot,t)\|_{L^\infty} \le N(t)$, we get

$$\begin{aligned}
&\int_0^t\int_0^\infty \frac{1}{U}\left|((e^{2\psi}-1)(\phi_{xx}+\phi_x\psi_x+U\psi_x+V\phi_x))_x\right|^2\\
\le& C\int_0^t\int_0^\infty \frac{\psi^2}{U}|\phi_{xxx}+\phi_{xx}\psi_x+\phi_x\psi_{xx}+U_x\psi_x+U\psi_{xx}+V_x\phi_x+V\phi_{xx}|^2\\
&+C\int_0^t\int_0^\infty \frac{e^{4\psi}}{U}[\psi_x(\phi_{xx}+\phi_x\psi_x+U\psi_x+V\phi_x)]^2\\
\le& CN(t)\int_0^\infty\left(\phi_{0x}^2+\frac{\psi_{0x}^2}{U}+U\phi_0^2+\psi_0^2\right)+CN(t)\int_0^t\int_0^\infty\frac{\phi_{xxx}^2}{U}.
\end{aligned} \tag{4.58}$$

Substituting (4.56)–(4.58) into (4.55), and using (4.52) and (4.53), when $N(t) \ll 1$, one gets

$$\int_0^t\int_0^\infty \frac{\phi_{xxx}^2}{U}dxd\tau \le C\int_0^\infty\left(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U}\right)dx. \tag{4.59}$$

Similarly, differentiating the second equation of (3.6) in x yields

$$d\psi_{xxx} = \psi_{tx}+2dV_x\psi_x+2dV\psi_{xx}+2d\psi_x\psi_{xx}-\phi_{xx}. \tag{4.60}$$

Then squaring (4.60), multiplying the resultant equations by $\frac{1}{U}$, using (4.51) and Lemmas 4.2 and 4.3, we arrive at

$$\begin{aligned}
\int_0^t\int_0^\infty \frac{\psi_{xxx}^2}{U}dxd\tau \le& C\int_0^t\int_0^\infty\left(\frac{\psi_{tx}^2}{U}+\frac{V_x^2}{U}\psi_x^2+\frac{V^2}{U}\psi_{xx}^2+\frac{\psi_x^2\psi_{xx}^2}{U}+\frac{\phi_{xx}^2}{U}\right)dxd\tau\\
\le& C\int_0^\infty\left(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U}\right)dx.
\end{aligned} \tag{4.61}$$

From (4.59), (4.61) and (4.51)–(4.53), we get the desired (4.32).

Notice that the estimate (4.32) requires that the initial data satisfies $\frac{\phi_{0x}^2}{U^2}+\frac{\phi_{0xx}^2}{U^3} < \infty$. Hence, to guarantee the extension procedure works, we further need the following weighted elliptic estimate.

Lemma 4.5. *If $N(t) \ll 1$, we have*

$$\begin{aligned}
&\left\|\frac{\phi_x(\cdot,t)}{\sqrt{U}}\right\|_{L^\infty}^2+\left\|\frac{\psi_x(\cdot,t)}{\sqrt{U}}\right\|_{L^\infty}^2+\int_0^\infty\phi^2dx+\int_0^\infty\frac{\phi_x^2}{U^2}dx+\int_0^\infty\frac{\phi_{xx}^2}{U^3}dx\\
&\le C\int_0^\infty\left(\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U}\right)dx.
\end{aligned} \tag{4.62}$$

Proof. By (4.1), (4.17) and (4.53), we have

$$\|\frac{\psi_x(\cdot,t)}{\sqrt{U}}\|_{L^\infty}^2 \le C\int_0^\infty(U\phi_0^2+\psi_0^2+\frac{\phi_{0x}^2}{U^2}+\frac{\psi_{0x}^2}{U}+\frac{\phi_{0xx}^2}{U^3}+\frac{\psi_{0xx}^2}{U})dx. \tag{4.63}$$

We write the first equation of (4.8) as

$$\phi_{xx}+V\phi_x = W^2\phi_t-U\psi_x-\phi_x\psi_x-(e^{2\psi}-1)(\phi_{xx}+\phi_x\psi_x+U\psi_x+V\phi_x). \tag{4.64}$$

Multiplying (4.64) by $-\frac{\phi_x}{U^2V}$, noting

$$\int_0^\infty \frac{\phi_x\phi_{xx}}{U^2V}dx = -U^{-2}V^{-1}\phi_x^2\Big|_{x=0} - \int_0^\infty (U^{-2}V^{-1})_x\phi_x^2 dx,$$

$$\int_0^\infty \frac{(e^{2\psi}-1)}{U^2V}\phi_{xx}\phi_x dx = -\int_0^\infty \phi_x^2\frac{e^{2\psi}\psi_x}{U^2V}dx - \int_0^\infty \frac{\phi_x^2}{2}(e^{2\psi}-1)\left(\frac{1}{U^2V}\right)_x dx - \frac{(e^{2\psi}-1)}{2U^2V}\phi_x^2\Big|_{x=0},$$

we get

$$\begin{aligned}
&\frac{(e^{2\psi}+1)}{2U^2V}\phi_x^2\Big|_{x=0} + \int_0^\infty \left(-U^{-2}+(U^{-2}V^{-1})_x\right)\phi_x^2 dx \\
=& -\int_0^\infty \frac{W^2}{U^2V}\phi_x\phi_t dx + \int_0^\infty \frac{\phi_x\psi_x}{UV}dx + \int_0^\infty \frac{\phi_x^2\psi_x}{U^2V}dx - \int_0^\infty \phi_x^2\frac{e^{2\psi}\psi_x}{U^2V}dx \\
&- \int_0^\infty \frac{\phi_x^2}{2}(e^{2\psi}-1)\left(\frac{1}{U^2V}\right)_x dx + \int_0^\infty \frac{(e^{2\psi}-1)}{U^2V}\phi_x(\phi_x\psi_x + U\psi_x + V\phi_x)dx.
\end{aligned} \tag{4.65}$$

A direct calculation by (3.3) gives

$$-U^{-2} + (U^{-2}V^{-1})_x = \frac{1}{U^2} - \frac{V_x}{U^2V^2} > \frac{1}{U^2}.$$

Thus,

$$\text{LHS of (4.65)} \geq \int_0^\infty \frac{\phi_x^2}{U^2}dx. \tag{4.66}$$

We next estimate the RHS of (4.65). By (4.2),

$$\left|\int_0^\infty \frac{\phi_x^2}{2}\frac{e^{2\psi}\psi_x}{U^2V}dx\right| + \left|\int_0^\infty \frac{\phi_x^2}{2}(e^{2\psi}-1)\left(\frac{1}{U^2V}\right)_x dx\right| \leq CN(t)\int_0^\infty \frac{\phi_x^2}{U^2}dx.$$

By (4.16),

$$\int_0^\infty \frac{(e^{2\psi}-1)}{U^2V}\phi_x(\phi_x\psi_x + U\psi_x + V\phi_x)dx \leq CN(t)\int_0^\infty \left(\frac{\phi_x^2}{U^2} + \frac{|\phi_x\psi_x|}{UV}\right)dx.$$

Then we get from (4.13) and Young's inequality that

$$\begin{aligned}
\text{RHS of (4.65)} \leq& C\int_0^\infty \frac{|\phi_t\phi_x|}{\sqrt{U}} + C\int_0^\infty \frac{|\phi_x\psi_x|}{UV} + C\int_0^\infty \frac{|\phi_x^2\psi_x|}{U^2V} + CN(t)\int_0^\infty \frac{\phi_x^2}{U^2}dx \\
\leq& \left(\frac{1}{2} + CN(t)\right)\int_0^\infty \frac{\phi_x^2}{U^2}dx + C\int_0^\infty U\phi_t^2 dx + C\int_0^\infty \frac{\psi_x^2}{U}dx.
\end{aligned} \tag{4.67}$$

Now substituting (4.66) and (4.67) into (4.65), by Lemmas 4.3 and 4.4, we have

$$\int_0^\infty \frac{\phi_x^2}{U^2}dx \leq C\int_0^\infty (U\phi_0^2 + \psi_0^2 + \frac{\phi_{0x}^2}{U^2} + \frac{\psi_{0x}^2}{U} + \frac{\phi_{0xx}^2}{U^3} + \frac{\psi_{0xx}^2}{U})dx, \tag{4.68}$$

which along with Hardy inequality gives

$$\int_0^\infty \phi^2 dx \le C \int_0^\infty (U\phi_0^2 + \psi_0^2 + \frac{\phi_{0x}^2}{U^2} + \frac{\psi_{0x}^2}{U} + \frac{\phi_{0xx}^2}{U^3} + \frac{\psi_{0xx}^2}{U})dx. \tag{4.69}$$

We next write the first equation of (3.6) as

$$\phi_{xx} = W^2 e^{-2\psi}\phi_t - (V\phi_x + U\psi_x + \phi_x\psi_x).$$

Squaring this equation and multiplying the resultant equation by $\frac{1}{U^3}$, owing to (4.2) again, we obtain

$$\begin{aligned}\int_0^\infty \frac{\phi_{xx}^2}{U^3}dx \le& C\int_0^\infty \Big(U\phi_t^2 + \frac{V^2}{U^3}\phi_x^2 + \frac{\psi_x^2}{U} + \frac{\phi_x^2\psi_x^2}{U^3}\Big)dx\\ \le& C\Big(1+N(t)\Big)\int_0^\infty \frac{\phi_x^2}{U^2}dx + C\int_0^\infty U\phi_t^2 dx + C\int_0^\infty \frac{\psi_x^2}{U}dx.\end{aligned}$$

Thus, by (4.68) and Lemma 4.4,

$$\int_0^\infty \frac{\phi_{xx}^2}{U^3}dx \le C\int_0^\infty (U\phi_0^2 + \psi_0^2 + \frac{\phi_{0x}^2}{U^2} + \frac{\psi_{0x}^2}{U} + \frac{\phi_{0xx}^2}{U^3} + \frac{\psi_{0xx}^2}{U})dx. \tag{4.70}$$

Using (4.3), (4.68) and (4.70), we get

$$\begin{aligned}\left\|\frac{\phi_x(\cdot,t)}{\sqrt{U}}\right\|_{L^\infty}^2 \le& \int_0^\infty \frac{\phi_x^2}{U}dx + \int_0^\infty \frac{\phi_{xx}^2}{U}dx\\ \le& C\int_0^\infty \left(U\phi_0^2 + \psi_0^2 + \frac{\phi_{0x}^2}{U^2} + \frac{\psi_{0x}^2}{U} + \frac{\phi_{0xx}^2}{U^3} + \frac{\psi_{0xx}^2}{U}\right)dx.\end{aligned} \tag{4.71}$$

Therefore, (4.62) follows from (4.63) and (4.68)–(4.71). We complete the proof.

Proof. [Proof of Proposition 4.2] It is a direct consequence of Lemmas 4.2–4.5.

Proof. [Proof of Theorem 2.2] The a priori estimate (4.5) guarantees that if $N(0)$ is small, then $N(t)$ is small for all $t > 0$. Thus, applying the standard extension argument, we obtain the global well-posedness of system (3.6)–(3.9) in $X(0,\infty)$. Owing to the transformation (3.5), system (2.7) and (2.8) has a unique global solution $(u,v)(x,t)$ satisfying (2.9).

We next prove the convergence (2.10). We first show that

$$\|\phi_x(\cdot,t)\| + \|\psi_x(\cdot,t)\| \to 0 \text{ as } t\to\infty. \tag{4.72}$$

It suffices to prove that $\|\phi_x(\cdot,t)\|^2 \in W^{1,1}(0,\infty)$ and $\|\psi_x(\cdot,t)\|^2 \in W^{1,1}(0,\infty)$. By Lemma 4.2, we get

$$\int_0^\infty\int_0^\infty \phi_x^2 dxdt \le C\int_0^\infty\int_0^\infty \frac{\phi_x^2}{U}dxdt < \infty. \tag{4.73}$$

By Lemma 4.5, we have $\|\psi(\cdot,t)\|_{L^\infty} \leq C$ and $\|\frac{\psi_x(\cdot,t)}{\sqrt{W}}\|_{L^\infty} \leq C$. In view of the first equation of (3.6), there exists a constant C such that

$$\begin{aligned}\left|\frac{d}{dt}\int_0^\infty \phi_x^2 dx\right| =&2\left|\int_0^\infty \phi_{tx}\phi_x dx\right|\\ =&2\left|\int_0^\infty \phi_{xx}\phi_t dx\right|\\ =&2\left|\int_0^\infty W^{-2}\phi_{xx}e^{2\psi}(\phi_{xx}+\phi_x\psi_x+U\psi_x+V\phi_x)dx\right|\\ \leq& C\|e^\psi\|^2_{L^\infty}\left(\int_0^\infty \frac{\phi_{xx}^2}{W^2}dx + C(1+\|\frac{\psi_x}{\sqrt{W}}\|^2_{L^\infty})\int_0^\infty \frac{\phi_x^2}{W}dx + \int_0^\infty \psi_x^2 dx\right),\end{aligned} \tag{4.74}$$

where we have used (4.13) and (4.19). Then integrating (4.74) with respect to t and using (4.5), we get

$$\int_0^\infty \left|\frac{d}{dt}\int_0^\infty \phi_x^2 dx\right| dt < \infty,$$

which, along with (4.73) leads to $\|\phi_x(\cdot,t)\|^2 \in W^{1,1}(0,\infty)$. Thus

$$\|\phi_x(\cdot,t)\| \to 0 \text{ as } t \to \infty. \tag{4.75}$$

Similarly, one has

$$\int_0^\infty \int_0^\infty \psi_x^2 dxdt < \infty. \tag{4.76}$$

Using the second equation of (3.6), there is a constant $C > 0$ such that

$$\begin{aligned}\left|\frac{d}{dt}\int_0^\infty \psi_x^2 dx\right| =&2\left|\int_0^\infty \psi_{xx}\psi_t dx\right|\\ =&2\left|\int_0^\infty \psi_{xx}(d\psi_{xx}-2dV\psi_x-d\psi_x^2+\phi_x)dx\right|\\ \leq& C(1+\|\psi_x\|^2_{L^\infty})\int_0^\infty \psi_{xx}^2 dx + C\int_0^\infty \psi_x^2 dx + C\int_0^\infty \phi_x^2 dx.\end{aligned} \tag{4.77}$$

Then we integrate (4.77) with respect to t and make use of (4.5) to get

$$\int_0^\infty \left|\frac{d}{dt}\int_0^\infty \psi_x^2 dx\right| dt < \infty,$$

which, along with (4.76) implies $\|\psi_x(\cdot,t)\|^2 \in W^{1,1}(0,\infty)$. Thus

$$\|\psi_x(\cdot,t)\| \to 0 \text{ as } t \to \infty. \tag{4.78}$$

(4.72) then follows from (4.75) and (4.78). By Cauchy-Schwarz inequality and (4.5), we get

$$\begin{aligned}\phi_x^2(x,t) = -2\int_x^\infty \phi_x\phi_{xx}(y,t)dy \leq& 2\Big(\int_0^\infty \phi_x^2 dy\Big)^{\frac{1}{2}}\Big(\int_0^\infty \phi_{xx}^2 dy\Big)^{\frac{1}{2}}\\ \leq& C\|\phi_x(\cdot,t)\|\\ \to& 0 \text{ as } t \to +\infty.\end{aligned}$$

This implies

$$\sup_{x\in\mathbb{R}_+} |\phi_x(x,t)| \to 0 \text{ as } t \to +\infty.$$

Similarly, we have

$$\sup_{x\in\mathbb{R}_+} |\psi_x(x,t)| \to 0 \text{ as } t \to +\infty.$$

Thus, (2.10) holds.

Finally, we prove the L^1 convergence. By Lemmas 4.2 and 4.4, we get

$$\int_0^\infty \int_0^\infty \frac{\phi_x^2}{U} \le C \text{ and } \int_0^\infty \int_0^\infty \frac{\phi_{tx}^2}{U} \le C. \tag{4.79}$$

A simple calculation gives

$$\left|\frac{d}{dt}\int_0^\infty \frac{\phi_x^2}{U}dx\right| dt = 2\left|\int_0^\infty \frac{\phi_{tx}\phi_x}{U}dx\right| \le \int_0^\infty \frac{\phi_x^2}{U}dx + \int_0^\infty \frac{\phi_{tx}^2}{U}dx. \tag{4.80}$$

Integrating (4.80) with respect to t and using (4.79), we obtain

$$\int_0^\infty \left|\frac{d}{dt}\int_0^\infty \frac{\phi_x^2}{U}dx\right| dt < \infty,$$

which, along with the first inequality (4.79) yields $\int_0^\infty \frac{\phi_x^2(x,t)}{U}dx \in W^{1,1}(0,\infty)$. And then

$$\int_0^\infty \frac{\phi_x^2(x,t)}{U}dx \to 0 \text{ as } t \to \infty.$$

Thus, from Hölder inequality and the fact that $\int_0^\infty U dx < \infty$, it follows that

$$\int_0^\infty |\phi_x(x,t)|dx \le \left(\int_0^\infty \frac{\phi_x^2(x,t)}{U}dx\right)^{\frac{1}{2}} \left(\int_0^\infty U dx\right)^{\frac{1}{2}} \to 0 \text{ as } t \to \infty.$$

This yields the convergence (2.11).

Proof. [Proof of Theorem 2.3.] We just need to pass the results from v to w to complete the proof of Theorem 2.3. The transformation (3.5) and Theorem 2.2 give the regularity of $\frac{w_x}{w} - \frac{W_x}{W}$.

Next, we derive the results of $w - W$. Let $\xi := w - W$. Owing to (2.5) and (3.5), it is easy to calculate that

$$w(x,t) = be^{-\int_0^x v(y,t)dy} = be^{-\int_0^x (\psi_x + V)dy} = e^{-\psi}W.$$

Thus, $\xi = W(e^{-\psi} - 1)$ and $\xi_x = W_x(e^{-\psi} - 1) - We^{-\psi}\psi_x$, which gives the regularity of $w - W$.

It is left to show the convergence. By Cauchy-Schwarz inequality and (4.5), we get

$$\psi^2(x,t) = 2\int_0^x \psi\psi_x(y,t)dy \le 2\Big(\int_0^\infty \psi^2 dy\Big)^{\frac{1}{2}}\Big(\int_0^\infty \psi_x^2 dy\Big)^{\frac{1}{2}} \le C\|\psi_x(\cdot,t)\|.$$

Noting $\|\psi(\cdot,t)\|_{L^\infty} \le N(t) \ll 1$, the Taylor expansion gives

$$|1 - e^{-\psi}| = \left|\psi - \sum_{n=2}^\infty \frac{(-1)^n\psi^n}{n!}\right| \le C|\psi|.$$

Therefore, by (4.78), we get

$$\sup_{x\in\mathbb{R}_+}|\xi(x,t)| \le C\sup_{x\in\mathbb{R}_+}|\psi(x,t)| \le C\|\psi_x(\cdot,t)\|^{\frac{1}{2}} \to 0 \text{ as } t\to\infty.$$

For the L^1 convergence, noting $\int_0^\infty W(x)dx < \infty$, it follows from Hölder inequality and Hardy inequality that

$$\begin{aligned}\int_0^\infty |\xi(x,t)|dx \le C\int_0^\infty W|\psi(x,t)|dx &\le C\left(\int_0^\infty W^2dx\right)^{\frac{1}{2}}\left(\int_0^\infty W^2\psi^2(x,t)dx\right)^{\frac{1}{2}}\\ &\le C\|\psi_x(\cdot,t)\|\\ &\to 0 \text{ as } t\to\infty.\end{aligned}$$

We complete the proof of Theorem 2.3.

5. Summary and discussion

We are concerned with the existence and stability of spiky patterns to the chemotaxis model (1.1) proposed by Kim and his collaborators [1, 2]. This model was derived from the notion of "metric of food"which measures the amount of food. It avoids the mysterious assumption that the microscopic scale bacteria sense the macroscopic scale gradient of food. Moreover, this model also admits two types of traveling waves: traveling band and traveling front, under suitable assumptions on the consumption rates. Hence, it can be viewed as an alternative model to describe the propagation of traveling bands of bacteria observed in the experiment of Adler [3]. However, since the traveling wave of oxygen W vanishes at far field, one has to encounter the challenge of presence of two types of singularities in the study of stability of traveling waves. As the first step we investigate instead the stability of stationary waves to the model in the half space. In this case the model remains singular at the far field. We successfully find an effective strategy to handle the two types of singularities. In the following studies, we will apply the strategy of this paper to study the stability of traveling waves of the model by modifying some estimates.

The potential biological application of our results is the explanation of formation of a plume pattern for aerobic bacteria observed in the experiment of [7], where the bacteria consume oxygen in a water drop. We conjecture that this plume pattern is a superposition of series of one dimensional spikes. However, owing to the lack of effective mathematical tools to handle the stability of biological patterns to a chemotaxis-fluid model, we consider a simplified fluid free chemotaxis model. And we expect that our argument is effective for more general chemotaxis models and even some chemotaxis-fluid models.

Acknowledgments

The authors are grateful to the three referees for their insightful comments and suggestions, which lead to great improvements of our original manuscript. This work is supported by the Natural Science Foundation of Jilin Province (20210101144JC).

Conflict of interest

The authors declare there is no conflict of interest.

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Appendix

Proof. [Proof of Proposition 3.1] The local existence can be proved using the principle of contraction mapping. Set

$$\mathcal{Y}_T := \{(f,g) | (f,g) \in L^\infty((0,T);H^1), (f_x,g_x) \in L^2((0,T);H^1)\}$$

equipped with norm

$$\|(f,g)\|_{\mathcal{Y}_T} := \|(f,g)\|_{L^\infty((0,T);H^1)} + \|(f_x,g_x)\|_{L^2((0,T);H^1)}.$$

Define a mapping $\mathcal{Z}$: $(\hat{\phi},\hat{\psi}) \in \mathcal{Y}_T \mapsto \mathcal{Z}(\hat{\phi},\hat{\psi})$ such that $(\phi,\psi) = \mathcal{Z}(\hat{\phi},\hat{\psi})$ is a solution of

$$\begin{cases} W_\varepsilon^2 \phi_t = e^{2\psi}(\phi_{xx} + \hat{\phi}_x\hat{\psi}_x + U\hat{\psi}_x + V_\varepsilon\hat{\phi}_x), \\ \psi_t = d\psi_{xx} - 2dV_\varepsilon\hat{\psi}_x - d\hat{\psi}_x^2 + \hat{\phi}_x, \end{cases} \tag{A1}$$

with the initial and boundary conditions (3.7)–(3.9). Taking a ball

$$B_{M,T} := \{(\hat{\phi},\hat{\psi}) : \|(\hat{\phi},\hat{\psi})\|_{\mathcal{Y}_T} \le M\},$$

where M is a constant to be determined later. We shall show that there are M and T such that (i) $\mathcal{Z}$ maps $B_{M,T}$ into itself; (ii) $\mathcal{Z}$ is a contraction in $B_{M,T}$.

We first show (i). According to the standard linear parabolic theory, for any $(\hat{\phi},\hat{\psi}) \in \mathcal{Y}_T$, the second equation of $(A1)$ has a unique strong solution ψ. Substituting ψ into the first equation, we obtain the existence of strong solution ϕ. Hence the mapping $\mathcal{Z}$ is well-defined.

We next derive the estimates for (ϕ,ψ). Multiplying the second equation of $(A1)$ by ψ gives

$$\frac{1}{2}\int_0^\infty \psi^2 + d\int_0^t\int_0^\infty \psi_x^2 \le \frac{1}{2}\int_0^t\int_0^\infty \psi^2 + C\int_0^t\int_0^\infty (\hat{\psi}_x^2 + \hat{\phi}_x^2) + d\int_0^t\int_0^\infty \hat{\psi}_x^2|\psi| + \frac{1}{2}\int_0^\infty \psi_0^2,$$

where

$$d\int_0^\infty \hat{\psi}_x^2|\psi| \le d\|\psi\|_{L^\infty}\int_0^\infty \hat{\psi}_x^2 \le dM^2\|\psi\|_{L^2}^{\frac{1}{2}}\|\psi_x\|_{L^2}^{\frac{1}{2}} \le \frac{d}{2}\int_0^\infty \psi_x^2 + \int_0^\infty \psi^2 + CM^4.$$

Then choosing $T \le \frac{1}{2}$, we get

$$\sup_{0\le t\le T}\int_0^\infty \psi^2 dx + \int_0^T\int_0^\infty \psi_x^2 dxdt \le C(M^2+M^4)T + 2\int_0^\infty \psi_0^2 dx. \tag{A2}$$

Similarly, multiplying the second equation of (A1) by ψ_{xx} leads to

$$\begin{aligned} \sup_{0\le t\le T}\int_0^\infty \psi_x^2 dx + \int_0^T\int_0^\infty \psi_{xx}^2 dxdt &\le CM^2T + CM^3T^{\frac{1}{2}}\left(\int_0^T\int_0^\infty \hat{\psi}_{xx}^2\right)^{\frac{1}{2}} + \int_0^\infty \psi_{0x}^2 dx \\ &\le CM^2T + CM^4T^{\frac{1}{2}} + \int_0^\infty \psi_{0x}^2 dx, \end{aligned} \tag{A3}$$

where we have used

$$\int_0^\infty \hat{\psi}_x^4 \leq M^2\|\hat{\psi}_x\|_{L^\infty}^2 \leq M^2\|\hat{\psi}_x\|_{L^2}\|\hat{\psi}_{xx}\|_{L^2} \leq M^3\|\hat{\psi}_{xx}\|_{L^2}.$$

By (A2) and (A3), if we take $M^2 \geq 2\int_0^\infty(\psi_0^2+\psi_{0x}^2)$ and chose T small enough, then

$$\|\psi\|_{L^\infty((0,T);H^1)}^2 \leq M^2. \tag{A4}$$

Multiplying the first equation of (A1) by ϕ gives

$$\begin{aligned}&\frac{1}{2}\int_0^\infty W_\varepsilon^2\phi^2 + \int_0^t\int_0^\infty e^{2\psi}\phi_x^2\\ &= -2\int_0^t\int_0^\infty e^{2\psi}\psi_x\phi_x\phi + \int_0^t\int_0^\infty e^{2\psi}(\hat{\phi}_x\hat{\psi}_x + U\hat{\psi}_x + V_\varepsilon\hat{\phi}_x)\phi + \frac{1}{2}\int_0^\infty W_\varepsilon^2\phi_0^2,\end{aligned} \tag{A5}$$

where

$$\begin{aligned}2\int_0^\infty e^{2\psi}|\psi_x\phi_x\phi| &\leq 2e^{2M}\|e^{2\psi}\phi_x\|_{L^2}^{\frac{3}{2}}\|\phi\|_{L^2}^{\frac{1}{2}}\|\psi_x\|_{L^2}\\ &\leq \frac{1}{2}\int_0^\infty e^{2\psi}\phi_x^2 + e^{4M}M^4\int_0^\infty\phi^2,\end{aligned}$$

$$\begin{aligned}\int_0^\infty e^{2\psi}|\hat{\phi}_x\hat{\psi}_x\phi| &\leq \int_0^\infty \phi^2 + e^{4M}\|\hat{\phi}_x\|_{L^\infty}^2\|\hat{\psi}_x\|_{L^2}^2\\ &\leq \int_0^\infty\phi^2 + e^{4M}M^3\|\hat{\psi}_{xx}\|_{L^2},\end{aligned}$$

and

$$\int_0^\infty e^{2\psi}|(U\hat{\psi}_x + V_\varepsilon\hat{\phi}_x)\phi| \leq \int_0^\infty\phi^2 + e^{4M}M^2.$$

If $e^{4M}M^4T \leq \frac{\varepsilon^2}{2}$, we get from (A5) that

$$\sup_{0\leq t\leq T}\int_0^\infty\phi^2 + \int_0^T\int_0^\infty e^{2\psi}\phi_x^2 \leq Ce^{4M}M^4T^{\frac{1}{2}}\varepsilon^{-2} + C\varepsilon^{-2}\int_0^\infty\phi_0^2. \tag{A6}$$

Similarly, multiplying the first equation of (A1) by ϕ_{xx} gives

$$\begin{aligned}&\frac{1}{2}\int_0^\infty W_\varepsilon^2\phi_x^2 + \frac{1}{2}\int_0^t\int_0^\infty e^{2\psi}\phi_{xx}^2\\ &\leq \int_0^t\int_0^\infty e^{2\psi}|(\hat{\phi}_x\hat{\psi}_x + U\hat{\psi}_x + V_\varepsilon\hat{\phi}_x)\phi_{xx}| + 2\int_0^t\int_0^\infty W_\varepsilon|W_{\varepsilon x}\phi_x\phi_t| + \frac{1}{2}\int_0^\infty W_\varepsilon^2\phi_{0x}^2\\ &\triangleq \int_0^t(I+II) + \frac{1}{2}\int_0^\infty W_\varepsilon^2\phi_{0x}^2,\end{aligned}$$

where

$$I \leq e^{2M}M^3\|\hat{\psi}_{xx}\|_{L^2} + e^{2M}M^2,\ II \leq \frac{1}{4}\int_0^\infty e^{2\psi}\phi_{xx}^2 + Ce^{2M}M^4 + \int_0^\infty\phi_x^2.$$

Then choosing $T \leq \frac{\varepsilon^2}{2}$, we have

$$\sup_{0\leq t\leq T}\int_0^\infty \phi_x^2 + \int_0^T\int_0^\infty e^{2\psi}\phi_{xx}^2 \leq C\varepsilon^{-2}e^{2M}M^4T^{\frac{1}{2}} + C\varepsilon^{-2}\int_0^\infty \phi_{0x}^2. \tag{A7}$$

In view of (A2), (A3), (A6) and (A7), we choose M and T satisfying

$$M = 4\int_0^\infty(\psi_0^2+\psi_{0x}^2) + 4C\varepsilon^{-2}\int_0^\infty(\phi_0^2+\phi_{0x}^2) + 1,\ 8C(M+M^3)T^{\frac{1}{2}} + 8Ce^{4M}M^4\varepsilon^{-2}T^{\frac{1}{2}} \leq 1,$$

then $\|(\phi,\psi)\|_{\mathcal{Y}_T} \leq M$, which verifies (i).

We proceed to show (ii). For any $(\hat{\phi}_1,\hat{\psi}_1)$, $(\hat{\phi}_2,\hat{\psi}_2) \in B_{M,T}$, set $(\phi_1,\psi_1) = \mathcal{Z}(\hat{\phi}_1,\hat{\psi}_1)$, $(\phi_2,\psi_2) = \mathcal{Z}(\hat{\phi}_2,\hat{\psi}_2)$ and $(\bar{\phi},\bar{\psi}) := (\phi_1,\psi_1) - (\phi_2,\psi_2)$. Then $(\bar{\phi},\bar{\psi})$ satisfies

$$\begin{cases} W_\varepsilon^2\bar{\phi}_t = e^{2\psi_1}\bar{\phi}_{xx} + (e^{2\psi_1}-e^{2\psi_2})\phi_{2xx} + (e^{2\hat{\psi}_1}-e^{2\hat{\psi}_2})\hat{\phi}_{1x}\hat{\psi}_{1x} \\ \qquad\quad + e^{2\hat{\psi}_2}(\hat{\phi}_{1x}-\hat{\phi}_{2x})\hat{\psi}_{1x} + e^{2\hat{\psi}_2}\hat{\phi}_{2x}(\hat{\psi}_{1x}-\hat{\psi}_{2x}) \\ \qquad\quad + U(e^{2\hat{\psi}_1}-e^{2\hat{\psi}_2})\hat{\psi}_{1x} + Ue^{2\hat{\psi}_2}(\hat{\psi}_{1x}-\hat{\psi}_{2x}) \\ \qquad\quad + V_\varepsilon(e^{2\hat{\psi}_1}-e^{2\hat{\psi}_2})\hat{\phi}_{1x} + V_\varepsilon e^{2\hat{\psi}_2}(\hat{\phi}_{1x}-\hat{\phi}_{2x}), \\ \bar{\psi}_t = d\bar{\psi}_{xx} - 2dV_\varepsilon(\hat{\psi}_1-\hat{\psi}_2)_x - d(\hat{\psi}_1-\hat{\psi}_2)_x(\hat{\psi}_{1x}+\hat{\psi}_{2x}) + (\hat{\phi}_1-\hat{\phi}_2)_x \end{cases} \tag{A8}$$

with zero initial-boundary conditions. Multiplying the second equation of (A8) by $\bar{\psi}$ gives

$$\int_0^\infty \bar{\psi}^2 + d\int_0^T\int_0^\infty \bar{\psi}_x^2 \leq (C+M^2)\int_0^T\int_0^\infty |(\hat{\psi}_1-\hat{\psi}_2)_x|^2 + \int_0^T\int_0^\infty(\bar{\psi}^2 + C|(\hat{\phi}_1-\hat{\phi}_2)_x|^2).$$

Thus, choosing $T \leq \frac{1}{2}$, we get

$$\begin{aligned} \|\bar{\psi}\|^2_{L^\infty((0,T);L^2)} + \|\bar{\psi}_x\|^2_{L^2((0,T);L^2)} \leq & C(1+M^2)T\|(\hat{\psi}_1-\hat{\psi}_2)_x\|^2_{L^\infty((0,T);L^2)} \\ & + CT\|(\hat{\phi}_1-\hat{\phi}_2)_x\|^2_{L^\infty((0,T);L^2)}. \end{aligned}$$

Multiplying the second equation of (A8) by $\bar{\psi}_{xx}$, and noting

$$\begin{aligned} \int_0^T\int_0^\infty |(\hat{\psi}_1+\hat{\psi}_2)_x|^2|(\hat{\psi}_1-\hat{\psi}_2)_x|^2 &\leq M\|(\hat{\psi}_1-\hat{\psi}_2)_x\|^2_{L^\infty((0,T);L^2)}\int_0^T\|(\hat{\psi}_1+\hat{\psi}_2)_{xx}\|_{L^2} \\ &\leq M^2T^{\frac{1}{2}}\|(\hat{\psi}_1-\hat{\psi}_2)_x\|^2_{L^\infty((0,T);L^2)}, \end{aligned}$$

we get after choosing $T \leq \frac{1}{2}$ that

$$\begin{aligned} &\|\bar{\psi}_x\|^2_{L^\infty((0,T);L^2)} + \|\bar{\psi}_{xx}\|^2_{L^2((0,T);L^2)} \\ &\leq CM^2T^{\frac{1}{2}}\|(\hat{\psi}_1-\hat{\psi}_2)_x\|^2_{L^\infty((0,T);L^2)} + CT\|(\hat{\phi}_1-\hat{\phi}_2)_x\|^2_{L^\infty((0,T);L^2)}. \end{aligned} \tag{A9}$$

This also implies

$$\|\bar{\psi}\|^2_{L^\infty((0,T);L^\infty)} \leq CM^2T^{\frac{1}{2}}\|(\hat{\psi}_1-\hat{\psi}_2)_x\|^2_{L^\infty((0,T);L^2)} + CT\|(\hat{\phi}_1-\hat{\phi}_2)_x\|^2_{L^\infty((0,T);L^2)}. \tag{A10}$$

Multiplying the first equation of (A8) by $\bar{\phi}$, noting

$$\int_0^T \int_0^\infty |(e^{2\psi_1} - e^{2\psi_2})\phi_{2xx}\bar{\phi}| \le \int_0^T \|\bar{\psi}\|_{L^\infty}\|\phi_{2xx}\|_{L^2}\|\bar{\phi}\|_{L^2} \le T\|\bar{\phi}\|^2_{L^\infty((0,T);L^2)} + M^2\|\bar{\psi}\|^2_{L^\infty((0,T);L^\infty)},$$

we have

$$\begin{aligned}&\varepsilon^2\|\bar{\phi}\|^2_{L^\infty((0,T);L^2)} + \|\bar{\phi}_x\|^2_{L^2((0,T);L^2)}\\ &\le C(M)T\left(\|\bar{\phi}\|^2_{L^\infty((0,T);L^2)} + \|\bar{\psi}\|^2_{L^\infty((0,T);L^\infty)}\right)\\ &\quad + C(M)T\left(\|(\hat{\psi}_1 - \hat{\psi}_2)_x\|^2_{L^\infty((0,T);L^2)} + \|(\hat{\phi}_1 - \hat{\phi}_2)_x\|^2_{L^\infty((0,T);L^2)}\right).\end{aligned}$$

Multiplying the first equation of (A8) by $\bar{\phi}_{xx}$, noting

$$\begin{aligned}\int_0^T \int_0^\infty |(e^{2\psi_1} - e^{2\psi_2})\phi_{2xx}\bar{\phi}_{xx}| &\le \frac{1}{2}\int_0^T \int_0^\infty \bar{\phi}^2_{xx} + C\int_0^T e^{6M}\|\bar{\psi}\|_{L^2}\|\bar{\psi}_x\|_{L^2}\|\phi_{2xx}\|^2_{L^2}\\ &\le \frac{1}{2}\int_0^T \int_0^\infty \bar{\phi}^2_{xx} + C(M)\|\bar{\psi}\|_{L^\infty((0,T);L^2)}\|\bar{\psi}_x\|_{L^\infty((0,T);L^2)},\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \int_0^\infty e^{2\psi_2}(\hat{\phi}_1 - \hat{\phi}_2)_x\hat{\psi}_{1x}\bar{\phi}_{xx}\\ &\le \frac{1}{2}\int_0^T \int_0^\infty \bar{\phi}^2_{xx} + C(M)\|(\hat{\phi}_1 - \hat{\phi}_2)_x\|^2_{L^\infty((0,T);L^2)}\int_0^T \|\hat{\psi}_{1xx}(\cdot,t)\|_{L^2}\\ &\le \frac{1}{2}\int_0^T \int_0^\infty \bar{\phi}^2_{xx} + C(M)T^{\frac{1}{2}}\|(\hat{\phi}_1 - \hat{\phi}_2)_x\|^2_{L^\infty((0,T);L^2)},\end{aligned}$$

we have

$$\begin{aligned}&\varepsilon^2\|\bar{\phi}_x\|^2_{L^\infty((0,T);L^2)} + \|\bar{\phi}_{xx}\|^2_{L^2((0,T);L^2)}\\ &\le C(M)\|\bar{\psi}\|_{L^\infty((0,T);L^2)}\|\bar{\psi}_x\|_{L^\infty((0,T);L^2)}\\ &\quad + C(M)T\left(\|(\hat{\psi}_1 - \hat{\psi}_2)_x\|^2_{L^\infty((0,T);L^2)} + \|(\hat{\phi}_1 - \hat{\phi}_2)_x\|^2_{L^\infty((0,T);L^2)}\right).\end{aligned}$$

Therefore, owing to (A9)–(A10), we can take T small enough to derive

$$\|(\bar{\phi}, \bar{\psi})\|_{\mathcal{Y}_T} \le \frac{1}{2}\|(\hat{\phi}_1 - \hat{\phi}_2, \hat{\psi}_1 - \hat{\psi}_2)\|_{\mathcal{Y}_T}, \tag{A11}$$

which verifies (ii).

Now we apply the contraction mapping principle to obtain that system (3.10) has a solution. The uniqueness follows from a similar argument as (A11) and the Gronwall's inequality.

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