Mathematical Biosciences and Engineering

MBE, 19(12): 13628–13659.
DOI: 10.3934/mbe.2022636
Received: 21 June 2022
Revised: 28 August 2022
Accepted: 30 August 2022
Published: 16 September 2022

http://www.aimspress.com/journal/MBE

Research article

Acceptance sampling plans for the three-parameter inverted Topp–Leone model

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Abstract: The quadratic rank transmutation map is used in this article to suggest a novel extension of the power inverted Topp–Leone distribution. The newly generated distribution is known as the transmuted power inverted Topp–Leone (TPITL) distribution. The power inverted Topp–Leone and the inverted Topp–Leone are included in the recommended distribution as specific models. Aspects of the offered model, including the quantile function, moments and incomplete moments, stochastic ordering, and various uncertainty measures, are all discussed. Plans for acceptance sampling are created for the TPITL model with the assumption that the life test will end at a specific time. The median lifetime of the TPITL distribution with the chosen variables is the truncation time. The smallest sample size is required to obtain the stated life test under a certain consumer's risk. Five conventional estimation techniques, including maximum likelihood, least squares, weighted least squares, maximum product of spacing, and Cramer-von Mises, are used to assess the characteristics of TPITL distribution. A rigorous Monte Carlo simulation study is used to evaluate the effectiveness of these estimators. To determine how well the most recent model handled data modeling, we tested it on a range of datasets. The simulation results demonstrated that, in most cases, the maximum likelihood estimates had the smallest mean squared errors among all other estimates. In some cases, the Cramer-von Mises estimates performed better than others. Finally, we observed that precision measures decrease for all estimation techniques when the sample size increases, indicating that all estimation approaches are consistent. Through two real data analyses, the suggested model's validity and adaptability are contrasted with those of other models, including the power inverted Topp–Leone, log-normal, Weibull, generalized exponential, generalized inverse exponential, inverse Weibull, inverse gamma, and extended inverse exponential distributions.

Keywords: power inverted Topp–Leone distribution; transmuted family; stochastic ordering; maximum product spacing

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Symbols: $h(t;\,\Theta)$: PDF; $H(t;\,\Theta)$: CDF; $S(t;\,\Theta)$: SF; $\lambda(t;\,\Theta)$: HRF; $\psi(\tau)$: AE; $\gamma(\tau)$: H-CE; $\mathfrak{M}(\tau)$: RE; $\rho(\tau)$: Ts; $L(p)$: AP; $Q(u)$: QF; $GM(\Theta)$: GM; $-2\log L$: $-2\log$ likelihood; p^*: Consumer's risk

Abbreviations: AIC: Akaike information criterion; ASP: Acceptance sampling plan; AE: Arimoto entropy; AP: Acceptance probability of the lot as function of the failure probability; BIC: Bayesian information criterion; EIE: Extended inverse exponential; Ge-Ex: Generalized exponential; GIE: Generalized inverse exponential; in-Ga: inverse gamma; in-We: Inverse Weibull; K–S: Kolmogorov–Smirnov; CAIC: Corrected AIC; CvMEs: Cramer-von Mises estimators; CDF: Cumulative distribution function; HRF: Hazard rate function; H-CE: Havrda and Charvat entropy; ITL: Inverted Topp–Leone; Ku: Kurtosis;; LSEs: Least squares estimators; MPSEs: Maximum product spacing estimators; PITL: Power ITL; PDF: Probability density function; PP: Probability-probability; QF: Quantile function; QRTM: Quadratic rank transmutation map; RE: Rényi entropy; Sw: Skewness; SE: Standard error; MSE: Mean squared error; med: Median lifetime; SO: Stochastic ordering

1. Introduction

The validity of statistical investigations of real-world occurrences is determined by the appropriateness of the distributions used as models. Although well-known distribution families are frequently employed to simulate many events, their modeling capabilities may not always meet expectations. To address this issue, several scholars have presented new distribution families for ideally representing real-world events by employing generalizations of existing families or novel distribution generating procedures in the last two decades. In addition, several strategies for creating a new distribution have been described in the literature. For example, reference [1] devised a mechanism called the quadratic rank transmutation map (QRTM). This technique generates a transmuted distribution, as the name implies. In addition to having the same characteristics as baseline distributions, transmuted distributions are always more flexible.

Reference [1] defined the transmuted class with the following cumulative distribution function (CDF) and probability density function (PDF):

$$F_{TC}(t) = G(t)[1 + \upsilon - \upsilon G(t)]\,, |\upsilon| \leq 1, t \in R, \tag{1}$$

$$f_{TC}(t) = g(t)[1 + \upsilon - 2\upsilon G(t)], \tag{2}$$

where υ is a transmuted parameter known as the transmutation parameter. The transmuted density (2) is a mixture of the baseline density and the exponentiated-G density with power parameter equal two for $\upsilon = 0$, the transmuted CDF $F_{TC}(t)$ and the baseline CDF, G(t) are identical. For more information about the generalization of the probability distributions via the QRTM method, the reader can refer to references [2–8]. Other generalization methods can be found in [9–15].

Reference [16] recently presented the inverted Topp–Leone (ITL) distribution as a probability distribution with only one shape parameter. The PDF and CDF of the ITL distribution are defined as follows:

$$G(t) = 1 - (1 + 2t)^{\vartheta}(1 + t)^{-2\vartheta}; t, \vartheta > 0,$$

$$g(t) = 2\vartheta t(1 + t)^{-2\vartheta-1}(1 + 2t)^{\vartheta-1}; t, \vartheta > 0.$$

Many researchers have investigated the ITL distribution extensions and generalizations to increase flexibility in modeling a wide range of data. Reference [17] presented an "alpha power ITL distribution," a new form of ITL model with an extra parameter based on the alpha power-G family. Reference [18] created a three-parameter ITL distribution for a particular case from the Kumaraswamy-G family. Reference [19] presented the two-parameter half-logistic ITL distribution, and parameter estimators were investigated under ranked samples. Reference [20] investigated parameter estimators using various estimating methods for the modified Kies ITL distribution. Reference [21] proposed a new two-parameter ITL model via the odd Weibull–G family. Our focus here is on a power ITL (PITL) distribution prepared in [22] with an extra shape parameter. The PDF and CDF of the PITL distribution are defined as follows:

$$G_1(t) = 1 - \left(1 + 2t^{\delta}\right)^{\vartheta}\left(1 + t^{\delta}\right)^{-2\vartheta}; t, \vartheta, \delta > 0, \quad (3)$$

$$g_1(t) = 2\vartheta\delta t^{2\delta-1}\left(1 + t^{\delta}\right)^{-2\vartheta-1}\left(1 + 2t^{\delta}\right)^{\vartheta-1}; t, \vartheta, \delta > 0.$$

Acceptance sampling is used by industries worldwide to ensure the quality of incoming and outgoing goods, using statistical principles to create a plan for accepting or rejecting these goods. This process is called an acceptance sampling plan (ASP). It is one of the oldest quality assurance procedures, and it involves inspecting and deciding on quantities of goods to be accepted. An example of ASP in action is shown below:

• Required: A corporation gets a product shipment from a vendor. This product is frequently a component or raw material utilized during manufacturing.

• Sampling: A sample of the lot is obtained, and the relevant quality characteristics of the units in the sample are examined.

• Decision: Based on the information provided by the presented sample, a decision is made about whether to accept or reject the lot.

1) **For accepted samples**: Accepted lots are placed into production.
2) **For rejected samples**: Rejected lots may be transferred from the seller or subjected to further lot disposition actions.

Parameter estimation is essential to the study of any probability distribution. Maximum likelihood (ML) estimation is frequently used to estimate any model's parameters due to its desirable qualities. ML estimators are asymptotically consistent, unbiased, and impartial [23]. Over time, other methods for estimating distributions have emerged, including least squares (LS) and weighted LS (WLS) [24], maximum product of spacing (MPS) [25], and Cramer-von Mises (CvM) [26].

In this study, we established a new generalization method of PITL distribution to enhance the overall flexibility of the PITL model by employing QRTM. The newly created distribution is termed "transmuted PITL (TPITL) distribution". We are motivated to introduce the TPITL distribution due to the following reasons:

(i) This modification significantly impacts important distributional features, such as skewness, kurtosis, mean, and variance.
(ii) The related density and hazard rate functions have a large panel of monotonic and nonmonotonic forms, making them appropriate for data fitting and other applications.
(iii) It includes the PITL and ITL distributions as sub-models.
(iv) Important aspects of the presented model are explored, including quantile function, moments and incomplete moments, stochastic ordering, and some uncertainty measures.
(v) We created a sampling strategy, extracted its operational characteristic function, and gave it a decision rule.
(vi) The TPITL distribution parameters are evaluated using five traditional estimation methodologies.
(vii) An inclusive simulation study was conducted to determine if estimators based on accuracy criteria were successful.
(viii) The proposed model's validity and flexibility are compared with those of existing models, including the PITL, log-normal, Weibull, generalized inverse exponential, inverse Weibull, inverse gamma, generalized exponential, and extended inverse exponential distributions via two real data analyses.

The remaining paper is laid out as follows: Section 2 describes the CDF, survival function, and hazard rate function (HRF) of the TPITL distribution. In Section 3, the statistical features of the TPITL distribution are examined, and various helpful representations, measurements, and functions are derived. Section 4 discusses the suggested ASP's design under a truncated life test. In Section 5, the parametric estimate of the TPITL distribution is examined using some classical techniques. A simulation analysis ensuring its numerical performance is presented in Section 6. Section 7 discusses two real-world datasets to demonstrate how beneficial the TPITL model may be. Lastly, Section 8 contains a few findings.

2. Model description

We constructed the TPITL distribution using the CDF (3) of the PITL model as a baseline distribution in (1). The CDF of the TPITL distribution is given by:

$$H(t;\Theta)=\left[1-\left\{\frac{\left(1+2t^{\delta}\right)^{\vartheta}}{\left(1+t^{\delta}\right)^{2\vartheta}}\right\}\right]\left[1+\upsilon\left\{\frac{\left(1+2t^{\delta}\right)^{\vartheta}}{\left(1+t^{\delta}\right)^{2\vartheta}}\right\}\right],|\upsilon|\leq 1,t>0, \tag{4}$$

where $\vartheta>0$ and $\delta>0$ are two shape parameters, υ is the transmuted parameter, and $\Theta\equiv(\upsilon,\vartheta,\delta)$ is the set of parameters. A random variable with CDF (4) will be denoted by $T\sim TPITL(\Theta)$. The PDF associated with (4) is given as follows:

$$h(t;\Theta)=2\vartheta\delta t^{2\delta-1}\left(1+t^{\delta}\right)^{-2\vartheta-1}\left(1+2t^{\delta}\right)^{\vartheta-1}\left[1-\upsilon+2\upsilon\left\{\frac{\left(1+2t^{\delta}\right)^{\vartheta}}{\left(1+t^{\delta}\right)^{2\vartheta}}\right\}\right],t>0. \tag{5}$$

For $\upsilon = 0$, the PDF (5) reduces to PITL distribution (Abushal et al. [22]).
For $\upsilon = 0,\ \vartheta = 1$, the PDF (5) gives the ITL distribution (Hassen et al. [16]).

The form of the TPITL distribution is controlled by two shape parameters υ, and ϑ, similar to the PITL distribution. The parameter υ adds to the distribution's flexibility in addition to its role in influencing the behavior of the distribution. The survival function and HRF of the TPITL distribution are derived, respectively, as follows:

$$S(t;\Theta) = 1 - \{1 - \mathrm{K}(\delta,\vartheta)\}[1 + \upsilon\{\mathrm{K}(\delta,\vartheta)\}],$$

$$\lambda(t;\Theta) = \frac{2\vartheta\delta t^{2\delta-1}\left(1+t^{\delta}\right)^{-2\vartheta-1}\left(1+2t^{\delta}\right)^{\vartheta-1}[1-\upsilon+2\upsilon\{\mathrm{K}(\delta,\vartheta)\}]}{1-\{1-\mathrm{K}(\delta,\vartheta)\}[1+\upsilon\{\mathrm{K}(\delta,\vartheta)\}]},$$

where $\mathrm{K}(\delta,\vartheta) = \left(1 + 2t^{\delta}\right)^{\vartheta}\left(1 + t^{\delta}\right)^{-2\vartheta}$. For the given parameter values, the PDF and HRF plots are illustrated in Figures 1 and 2, respectively.

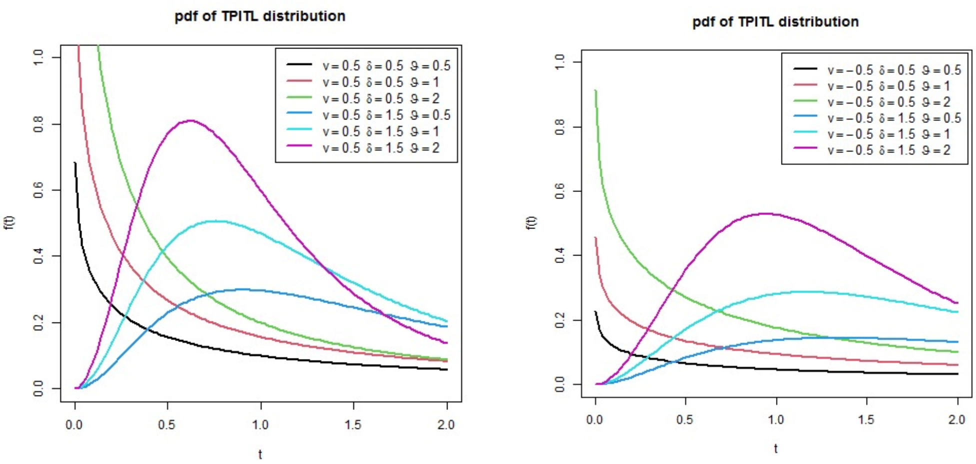

Figure 1. The PDF plots for the TPITL distribution.

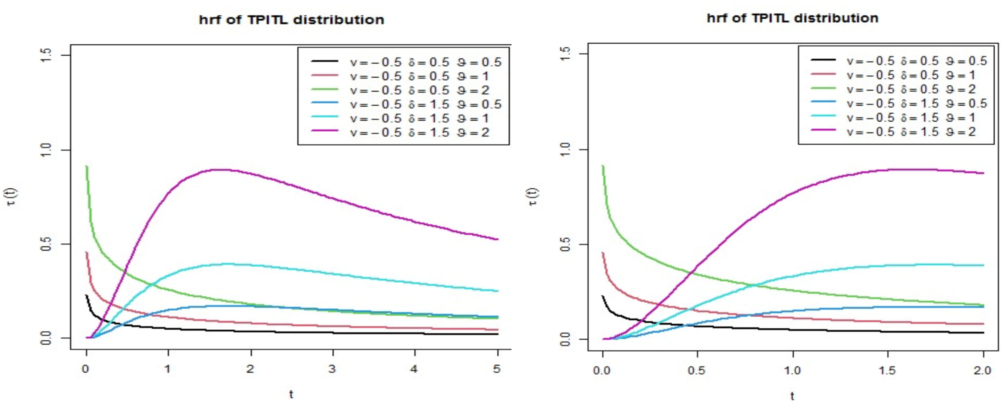

Figure 2. The HRF plots for the TPITL distribution.

Reversed J-shaped, unimodal, nearly symmetrical, and positively skewed are all possible shapes for the PDF of the TPITL model. Additionally, the HRF of the TPITL model has a growing, decreasing, inverted J-shaped, upside-down form. These figures indicate the flexibility of the TPITL distribution to model right-skewed data as well as data with decreasing and upside-down bathtub shapes.

Furthermore, for estimation and simulation, quantiles are necessary. Inverting Equation (4) yields the quantile function, say, $Q(u) = H^{-1}(u)$, where $u \in (0,1)$, as follows:

$$Q(u) = \left\{-1 + (1-A)^{-1/\vartheta} + \sqrt{(1-(1-A)^{-1/\vartheta})^2 - 1 + (1-A)^{-1/\vartheta}}\right\}^{1/\delta},\ A = \left(\frac{(1+\upsilon) - \sqrt{(1+\upsilon)^2 - 4u\upsilon}}{2\upsilon}\right). \quad (6)$$

Setting $u = 0.5$ in (6), we obtain the median.

3. Principal characteristics

Herein, we present ordinary and incomplete moments, stochastic ordering (SO), and some uncertainty measures of the TPITL distribution.

3.1. Moments measures

Here, we obtain the m^{th} moment, incomplete moments of the TPITL distribution. The m^{th} moment for the TPITL model is derived as follows:

$$\mu_m' = \int_0^\infty 2\vartheta\delta t^{m+2\delta-1}\left(1+t^\delta\right)^{-2\vartheta-1}\left(1+2t^\delta\right)^{\vartheta-1}\left[1-\upsilon+2\upsilon\left\{\frac{\left(1+2t^\delta\right)^\vartheta}{\left(1+t^\delta\right)^{2\vartheta}}\right\}\right] dt = (1-\upsilon)I_1 + \upsilon I_2,$$

where

$$I_1 = \int_0^\infty 2\vartheta\delta t^{m+2\delta-1}\left(1+t^\delta\right)^{-2\vartheta-1}\left(1+2t^\delta\right)^{\vartheta-1} dt,$$

$$I_2 = \int_0^\infty 4\vartheta\delta t^{m+2\delta-1}\left(1+t^\delta\right)^{-4\vartheta-1}\left(1+2t^\delta\right)^{2\vartheta-1} dt.$$

Using the following generalized binomial expansion, for $\vartheta > 0$ is a real non-integer and $|z| < 1$,

$$(1+z)^{\vartheta-1} = \sum_{j=0}^\infty \binom{\vartheta-1}{j} z^j,$$

in I_1 and I_2, then, we have

$$I_1 = \sum_{j=0}^\infty 2\vartheta \binom{\vartheta-1}{j} \mathrm{B}\left(\frac{m}{\delta}+j+2, \vartheta-\frac{m}{\delta}\right),\ I_2 = \sum_{i=0}^\infty 4\vartheta \binom{2\vartheta-1}{i} \mathrm{B}\left(\frac{m}{\delta}+i+2, 2\vartheta-\frac{m}{\delta}\right).$$

where $\mathrm{B}(.,.)$ is the beta function. Hence, the m^{th} moment of the TPITL distribution can be written as follows:

$$\mu_m' = (1-\upsilon)\sum_{j=0}^\infty 2\vartheta \binom{\vartheta-1}{j} \mathrm{B}\left(\frac{m}{\delta}+j+2, \vartheta-\frac{m}{\delta}\right) + \upsilon\sum_{i=0}^\infty 4\vartheta \binom{2\vartheta-1}{i} \mathrm{B}\left(\frac{m}{\delta}+i+2, 2\vartheta-\frac{m}{\delta}\right). \quad (7)$$

The first four moments about zero are obtained after putting $m = 1, 2, 3$, and 4 in (7). The m^{th} central moment (μ_m) of the TPITL distribution is given by:

$$\mu_m = E(T - \acute{\mu}_1)^m = \sum_{u=0}^{m}(-1)^u \binom{m}{u} (\acute{\mu}_1)^u \, \acute{\mu}_{m-u}.$$

Numerical values of the first four moments, variance (σ^2), skewness (Sw), and kurtosis (Ku) of the TPITL distribution for $\vartheta = 0.5$, and at some choice values of parameters are displayed in Table 1.

Table 1. Moments measures for selected parameters values.

$\acute{\mu}_m$	$\delta = 7$ $\upsilon = -0.3$	$\delta = 7$ $\upsilon = 0.3$	$\delta = 9$ $\upsilon = 0.5$	$\delta = 9$ $\upsilon = -0.5$	$\delta = 7$ $\upsilon = -0.5$	$\delta = 7$ $\upsilon = 0.5$
$\acute{\mu}_1$	1.567	1.389	1.236	1.439	1.626	1.33
$\acute{\mu}_2$	3.056	2.327	1.657	2.318	3.299	2.084
$\acute{\mu}_3$	11.183	6.932	2.603	4.584	12.6	5.515
$\acute{\mu}_4$	392.029	212.752	6.807	16.324	451.896	152.346
σ^2	0.601	0.397	0.131	0.247	0.655	0.315
Sw	9.669	10.388	4.904	4.377	9.618	10.8
Ku	964.521	1206	124.816	22.875	934.164	1371

According to Table 1, all moments' values, variance, and skewness measures of the TPITL model are less for negative values of transmuted parameters than for positive ones. While the TPITL's kurtosis values was the highest for positive values of υ. The TPITL distribution is right-skewed according to the values of the skewness measure (Figure 1). The TPITL distribution is leptokurtic according to the values of the kurtosis measure (Figure 2).

Moreover, the m^{th} incomplete moment, say $\zeta_m(z)$ of the TPITL distribution, is obtained as follows:

$$\zeta_m(z) = 2\vartheta\delta(1-\upsilon)\int_0^z t^{m+2\delta-1}(1+t^\delta)^{-2\vartheta-1}(1+2t^\delta)^{\vartheta-1}\,dt + 4\upsilon\vartheta\delta\int_0^z t^{m+2\delta-1}(1+t^\delta)^{-4\vartheta-1}(1+2t^\delta)^{2\vartheta-1}\,dt.$$

After simplification, the m^{th} incomplete moment of the TPITL distribution is obtained as:

$$\zeta_m(z) = 2\vartheta(1-\upsilon)\sum_{j=0}^{\infty}\binom{\vartheta-1}{j} \mathrm{B}\left(\frac{m}{\delta}+j+2, \vartheta-\frac{m}{\delta}, \frac{z^\delta}{1+z^\delta}\right) + 4\vartheta\upsilon\sum_{i=0}^{\infty}\binom{2\vartheta-1}{i} \mathrm{B}\left(\frac{m}{\delta}+i+2, 2\vartheta-\frac{m}{\delta}, \frac{z^\delta}{1+z^\delta}\right),$$

where $\mathrm{B}(.,.,x)$ is the incomplete beta function. Inequality measures, including the Bonferroni and Lorenz curves, are commonly utilized in various fields. These are the first incomplete moments' principal applications. The Lorenz and Bonferroni curves are formed, respectively, as follows:

$$\mathbb{Q} = \frac{(1-\upsilon)\sum_{j=0}^{\infty}\binom{\vartheta-1}{j}\mathrm{B}\left(\frac{1}{\delta}+j+2,\ \vartheta-\frac{1}{\delta}\right)+2\upsilon\sum_{i=0}^{\infty}\binom{2\vartheta-1}{i}\mathrm{B}\left(\frac{1}{\delta}+i+2,\ 2\vartheta-\frac{1}{\delta}\right)}{(1-\upsilon)\sum_{j=0}^{\infty}\binom{\vartheta-1}{j}\mathrm{B}\left(\frac{1}{\delta}+j+2,\ \vartheta-\frac{1}{\delta},\ \frac{z^\delta}{1+z^\delta}\right)+2\upsilon\sum_{i=0}^{\infty}\binom{2\vartheta-1}{i}\mathrm{B}\left(\frac{1}{\delta}+i+2,\ 2\vartheta-\frac{1}{\delta},\ \frac{z^\delta}{1+z^\delta}\right)},$$

$$\mathbb{C} = \mathbb{Q}\left\{\left[1-\left(\frac{(1+2z^\delta)^\vartheta}{(1+z^\delta)^{2\vartheta}}\right)\right]\left[1+\upsilon\left(\frac{(1+2z^\delta)^\vartheta}{(1+z^\delta)^{2\vartheta}}\right)\right]\right\}^{-1}.$$

3.2. Stochastic ordering

In reliability theory and other fields, SO is a well-studied notion in probability distributions, and it is used to assess the performance of random variables. Let T_i has the TPITL distribution with parameters $\Theta_i = (\vartheta_i, \delta_i, \upsilon_i)$ $i = 1,2$. Let $H_i(t;\ \Theta_i)$ indicates T_i's CDF and $h_i(t;\ \Theta_i)$ represents T_i's PDF.

If $h_1(t;\ \Theta_1)/h_2(t;\ \Theta_2)$, is a decreasing function $\forall\ t$, then T_1 is said to be stochastically less than T_2 (represented by $T_1 \leq_{sr} T_2$) in terms of likelihood ratio order.
Let $T_1 \sim TPITL(\vartheta_1, \delta_1, \upsilon_1)$ and $T_2 \sim ITL(\vartheta_2, \delta_2, \upsilon_2)$ then the likelihood ratio ordering is

$$\frac{h_1(t;\Theta_1)}{h_2(t;\Theta_2)} = \frac{\vartheta_1\delta_1 t^{2\delta_1-1}\left(1+t^{\delta_1}\right)^{-2\vartheta_1-1}\left(1+2t^{\delta_1}\right)^{\vartheta_1-1}[1-\upsilon_1+2\upsilon_1\mathrm{K}(\delta_1,\vartheta_1)]}{\vartheta_2\delta_2 t^{2\delta_2-1}\left(1+t^{\delta_2}\right)^{-2\vartheta_2-1}\left(1+2t^{\delta_2}\right)^{\vartheta_2-1}[1-\upsilon_2+2\upsilon_2\mathrm{K}(\delta_2,\vartheta_2)]},$$

$$\frac{d}{dt}\log\frac{h_1(t;\Theta_1)}{h_2(t;\Theta_2)} = \frac{2\delta_1-2\delta_2}{t} - \frac{(2\vartheta_1+1)\delta_1 t^{\delta_1-1}}{1+t^{\delta_1}} + \frac{(2\vartheta_2+1)\delta_2 t^{\delta_2-1}}{1+t^{\delta_2}} + \frac{2(\vartheta_1-1)\delta_1 t^{\delta_1-1}}{1+2t^{\delta_1}}$$

$$- \frac{2(\vartheta_2-1)\delta_2 t^{\delta_2-1}}{1+2t^{\delta_2}} + \frac{2\upsilon_1[d\mathrm{K}(\delta_1,\vartheta_1)/dt]}{[1-\upsilon_1+2\upsilon_1\mathrm{K}(\delta_1,\vartheta_1)]} - \frac{2\upsilon_2[d\mathrm{K}(\delta_2,\vartheta_2)/dt]}{[1-\upsilon_2+2\upsilon_2\mathrm{K}(\delta_2,\vartheta_2)]},$$

where, $\mathrm{K}(\delta_i,\vartheta_i) = \frac{\left(1+2t^{\delta_i}\right)^{\vartheta_i}}{\left(1+t^{\delta_i}\right)^{2\vartheta_i}}$, $i = 1,2$ $\frac{d\mathrm{K}(\delta_i,\vartheta_i)}{dt} = \frac{-2\delta_i\vartheta_i t^{2\delta_i-1}\left(1+2t^{\delta_i}\right)^{\vartheta_i-1}}{\left(1+t^{\delta_i}\right)^{2\vartheta_i+1}}$, $i = 1,2$

At $\vartheta_1 < \vartheta_2$, $\delta_1 < \delta_2$, $\upsilon_1 < \upsilon_2$, we get ${}^{d}/_{dt}\,[\log h_1(t;\Theta_1) - \log h_2(t;\Theta_2)] < 0$, for all $t \geq 0$, hence, $\frac{h_1(t;\Theta_1)}{h_2(t;\Theta_2)}$ is decreasing in t, and hence, $T_1 \leq_{lr} T_2$. Moreover, T_1 is said to be smaller than T_2 in other different orderings, such as SO (denoted by $T_1 \leq_{st} T_2$), hazard rate order (denoted by $T_1 \leq_{hr} T_2$), and reversed hazard rate order (denoted by $T_1 \leq_{rhr} T_2$).

3.3. Information measures

The entropy of a random variable with PDF (5) is a measure of the uncertainty's fluctuation. A high entropy number indicates that the data is more unpredictable. The Rényi entropy (RE) [27] is defined by:

$$\mathfrak{M}(\tau) = (1-\tau)^{-1}\log\left(\int_{-\infty}^{\infty}\left(h(t)\right)^{\tau}\,dt\right),\ \tau \neq 1,\ \tau > 0. \tag{8}$$

To obtain $\mathfrak{M}(\tau)$ of the TPITL model, we must obtain $\left(h(t;\ \Theta)\right)^{\tau}$, as follows:

$$\left(h(t;\Theta)\right)^{\tau} = \Xi_1\Xi_2,$$

where, $\Xi_1 = (1-\upsilon)^{\tau}(2\vartheta\delta)^{\tau}t^{\tau(2\delta-1)}\left(1+t^{\delta}\right)^{-\tau(2\vartheta+1)}\left(1+2t^{\delta}\right)^{\tau(\vartheta-1)}$ and $\Xi_2 = \left[1+\frac{2\upsilon}{1-\upsilon}\mathrm{K}(\delta,\vartheta)\right]^{\tau}$.

Using the binomial expansion, then Ξ_1 and Ξ_2 noticing that $\left[\frac{2\upsilon}{1-\upsilon}\mathrm{K}(\delta,\vartheta)\right]<1$, we have

$$\Xi_1=\sum_{j=0}^{\infty}(1-\upsilon)^{\tau}\binom{\tau(\vartheta-1)}{j}(2\vartheta\delta)^{\tau}t^{\tau(2\delta-1)+\delta j}\left(1+t^{\delta}\right)^{-\tau\vartheta-2\tau-j},$$

$$\Xi_2=\sum_{i=0}^{\infty}\binom{\tau}{i}\left(\frac{2\upsilon}{1-\upsilon}\right)^{i}\left(1+t^{\delta}\right)^{-\vartheta i}\left(1+\frac{t^{\delta}}{1+t^{\delta}}\right)^{\vartheta i}=\sum_{i,k=0}^{\infty}\binom{\tau}{i}\binom{\vartheta i}{k}\left(\frac{2\upsilon}{1-\upsilon}\right)^{i}t^{\delta k}\left(1+t^{\delta}\right)^{-\vartheta i-k}.$$

Hence, $\left(h(t;\Theta)\right)^{\tau}$, is formed as

$$\left(h(t;\Theta)\right)^{\tau}=\left[\Upsilon_j\xi_{i,k}t^{\tau(2\delta-1)+\delta j+\delta k}\left(1+t^{\delta}\right)^{-\tau\vartheta-2\tau-j-\vartheta i-k}\right], \tag{9}$$

$$\Upsilon_j=\sum_{j=0}^{\infty}(1-\upsilon)^{\tau}\binom{\tau(\vartheta-1)}{j}(2\vartheta\delta)^{\tau}\ ,\ \xi_{i,k}=\sum_{i,k=0}^{\infty}\binom{\tau}{i}\binom{\vartheta i}{k}\left(\frac{2\upsilon}{1-\upsilon}\right)^{i}.$$

Insert (9) in (8), then, we get the RE of the TPITL distribution as follows:

$$\mathfrak{M}(\tau)=(1-\tau)^{-1}\log\left\{\left[\frac{\xi_{i,k}\Upsilon_j}{\delta}\ \mathrm{B}\left(2\tau-\frac{\tau}{\delta}+j+k+\frac{1}{\delta},\tau\vartheta+\vartheta i+\frac{\tau}{\delta}-\frac{1}{\delta}\right)\right]\right\}.$$

The Tsallis entropy (TE) measure [28] is defined by:

$$\rho(\tau)=\frac{1}{\tau-1}\left[1-\int_{-\infty}^{\infty}\left(h(t)\right)^{\tau}\,dt\right]\ ,\tau\neq1,\tau>0.$$

The TE of the TPITL distribution is obtained as follows:

$$\rho(\tau)=\frac{1}{\tau-1}\left[1-\left\{\left[\frac{\xi_{i,k}\Upsilon_j}{\delta}\ \mathrm{B}\left(2\tau-\frac{\tau}{\delta}+j+k+\frac{1}{\delta},\tau\vartheta+\vartheta i+\frac{\tau}{\delta}-\frac{1}{\delta}\right)\right]\right\}\right].$$

The Arimoto entropy (**AE**) measure ([29]) is given by:

$$\psi(\tau)=\frac{\tau}{1-\tau}\left[\left(\int_{-\infty}^{\infty}\left(h(t)\right)^{\tau}\,dt\right)^{\frac{1}{\tau}}-1\right]\ ,\tau\neq1,\tau>0.$$

Hence, the AE of the TPITL distribution is given by:

$$\psi(\tau)=\frac{\tau}{1-\tau}\left\{\left[\frac{\xi_{i,k}\Upsilon_j}{\delta}\ \mathrm{B}\left(2\tau-\frac{\tau}{\delta}+j+k+\frac{1}{\delta},\tau\vartheta+\vartheta i+\frac{\tau}{\delta}-\frac{1}{\delta}\right)\right]^{\frac{1}{\tau}}-1\right\}.$$

The Havrda and Charvat entropy (H–CE) measure ([30]) is defined by:

$$\gamma(\tau)=\frac{1}{2^{1-\tau}-1}\left[\left(\int_{-\infty}^{\infty}h^{\tau}(t)\,dt\right)^{\frac{1}{\tau}}-1\right]\ ,\tau\neq1,\tau>0.$$

Hence, the H–CE of the TPITL distribution is given by:

$$\gamma(\tau)=\frac{1}{2^{1-\tau}-1}\left\{\left[\frac{\xi_{i,k}\,\Upsilon_j}{\delta}\,\mathrm{B}\left(2\tau-\frac{\tau}{\delta}+j+k+\frac{1}{\delta},\tau\vartheta+\vartheta i+\frac{\tau}{\delta}-\frac{1}{\delta}\right)\right]^{\frac{1}{\tau}}-1\right\}.$$

Table 2 shows the numerical entropy values for some of the elected parameter values.

Table 2. Entropy values for the TPITL model.

τ	$(\vartheta,\delta,\upsilon)$	RE	TE	AE	H-CE
0.5	(0.5,3,0.3)	3.28	8.308	25.563	61.716
	(1,3,0.3)	1.535	2.308	3.64	8.789
	(1.5,3,0.3)	1.028	1.345	1.797	4.337
	(2,3,0.3)	0.76	0.924	1.138	2.746
	(0.5,5,–0.3)	1.944	3.286	5.985	14.448
	(1,5,–0.3)	0.876	1.1	1.402	3.384
	(1.5,5,–0.3)	0.483	0.546	0.62	1.497
	(2,5,–0.3)	0.263	0.281	0.301	0.726
2	(0.5,3,0.3)	1.163	0.687	0.882	0.882
	(1,3,0.3)	0.615	0.46	0.53	0.53
	(1.5,3,0.3)	0.353	0.297	0.323	0.323
	(2,3,0.3)	0.188	0.172	0.18	0.18
	(0.5,5,–0.3)	1.504	0.778	1.057	1.057
	(1,5,–0.3)	0.834	0.566	0.682	0.682
	(1.5,5,–0.3)	0.517	0.404	0.456	0.456
	(2,5,–0.3)	0.32	0.274	0.296	0.296

From Table 2, we observe the following:

- For a small value of τ, the H–CE measure takes the largest values for all values of $(\vartheta,\delta,\upsilon)$, compared to other entropy measures, which leads to less information.
- The RE measure provides more information due to its small values compared with other measures at $\tau=0.5$ for all values of $(\vartheta,\delta,\upsilon)$.
- As the value of τ increases, the values of all entropy measures decrease.
- The TS entropy gets the smallest values for all values of $(\vartheta,\delta,\upsilon)$, at a large value of τ, which leads to less variability.

4. Acceptance sampling plans

We assume that the lifetime of a product follows the TPITL distribution with parameters (Θ) defined by (4) and that the specified median lifetime (med) of the units claimed by a producer is med_0. Our interest is to make an inference about the acceptance or rejection of the proposed lot based on the criterion that the actual med of the units is larger than the prescribed lifetime med_0. A common practice in life testing is to terminate the life test at a predetermined time t_0 and note the number of failures. Now to observe med, the experiment is run for a $t_0 = a^*$ med_0 units of time, multiple of claimed

med with any positive constant a. The idea of accepting the proposed lot based on the evidence that med $\geq$ med$_0$, given probability of at least p^* (consumer's risk) using a single ASP is as follows [31]: Draw a random sample of n number of units from the proposed lot and conduct an experiment for t_0 units of time. If during the experiment, c or less number of units (acceptance number) fail, then accept the whole lot; otherwise, reject the lot. Observe the probability of accepting a lot, and consider sufficiently large-sized lots so that the binomial distribution can be applied under the proposed sampling plan given by:

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i} \ , \ i = 1, \dots, n,$$

where $p = H(t_0; \vartheta)$, defined by (4). The function $L(p)$ is the operating characteristic function of the sampling plan, i.e., the acceptance probability of the lot as a function of the failure probability. Further, using $t_0 = a^* m_0$, p_0 can be written as follows:

$$p_0 = \left[1 - \left\{\frac{\left(1+2(ma_0)^{\delta}\right)^{\vartheta}}{\left(1+(ma_0)^{\delta}\right)^{2\vartheta}}\right\}\right]\left[1 + \upsilon\left\{\frac{\left(1+2(ma_0)^{\delta}\right)^{\vartheta}}{\left(1+(ma_0)^{\delta}\right)^{2\vartheta}}\right\}\right]. \quad (10)$$

Now, the problem is to determine for the given values of $p^*(0 < p^* < 1)$, t_0, and c, the smallest positive integer n such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \leq 1 - p^*, \quad (11)$$

where p_0 is given by (10). The minimum values of n satisfying the inequality (11) and its corresponding operating characteristic probability are obtained and displayed in Tables 3–8 for the assumed parameters as follows: $p^* =$ 0.2, 0.4, 0.6, 0.99, $c =$ 0.2 (0.2) 1 when $a = 1$, thus, $t_0 =$ med$_0$ = 0.5 for all parameter values. The assumed parameter values of TPITL distribution are:
(i) $(\upsilon, \delta, \vartheta) = (0.5, 0.5, 0.5)$, (ii) $(\upsilon, \delta, \vartheta) = (0.5, 0.5, 1.5)$, (iii) $(\upsilon, \delta, \vartheta) = (0.5, 1.5, 0.5)$, (iv) $(\upsilon, \delta, \vartheta) = (1.5, 0.5, 0.5)$, (v) $(\upsilon, \delta, \vartheta) = (1.5, 0.5, 1.5)$, and (vi) $(\upsilon, \delta, \vartheta) = (1.5, 1.5, 1.5)$. We provide the R codes of numerical value in Appendix A. From the results obtained in Tables 3–8, one can notice that:

- Regarding the ASP parameters, when p^* and c are increasing, the required sample sizes n and $L(p_0)$ are also increasing.
- As the value of a increases, the required sample sizes n and $L(p_0)$ decrease.
- Regarding the TPITL distribution parameters, the desired sample size n increases while $L(p_0)$ decreases with an increased value of ϑ for fixed υ and δ
- The required sample size n increases while $L(p_0)$ decreases as the value of δ increases for fixed values of υ and ϑ.
- The required sample size n increases while $L(p_0)$ decreases with increasing values of δ and fixed values of υ and ϑ.
- As the value of υ increases for fixed values of ϑ, and δ, the required sample size n increases, while $L(p_0)$ decreases.
- For all results, we have obtained and verified that $L(p_0) \leq 1 - p^*$. Also, when $a = 1$, we have $p_0 = 0.5$ as $t_0 = m_0$; hence, all results $(n, L(p_0))$ for any vector of parameters $(\upsilon, \delta, \vartheta)$ are similar.

Table 3. Single sampling plan for TPITL distribution at $\upsilon = 0.5, \delta = 0.5, \vartheta = 0.5$.

p^*	c	$a = 0.2$		$a = 0.4$		$a = 0.6$		$a = 0.8$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.2	0	1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2	6	0.8346	5	0.8354	4	0.9162	4	0.8941	4	0.8750
	4	11	0.8468	9	0.8457	8	0.8627	8	0.8152	7	0.8906
	10	29	0.8030	23	0.8085	20	0.8438	19	0.8256	18	0.8338
0.4	0	2	0.6983	2	0.6130	1	1.0000	1	1.0000	1	1.0000
	2	8	0.6432	6	0.7047	6	0.6158	5	0.7268	5	0.6875
	4	14	0.6491	11	0.6652	10	0.6504	9	0.6939	9	0.6367
	10	33	0.6359	26	0.6373	23	0.6486	21	0.6797	20	0.6762
0.6	0	3	0.4876	2	0.6130	2	0.5624	2	0.5270	2	0.5000
	2	10	0.4582	8	0.4484	7	0.4667	6	0.5504	6	0.5000
	4	17	0.4439	13	0.4753	12	0.4300	11	0.4448	10	0.5000
	10	38	0.4151	29	0.4539	26	0.4330	24	0.4387	23	0.4159
0.8	0	5	0.2377	4	0.2303	3	0.3163	3	0.2777	3	0.2500
	2	13	0.2487	10	0.2570	9	0.2419	8	0.2731	8	0.2266
	4	21	0.2324	16	0.2487	14	0.2562	13	0.2501	12	0.2744
	10	44	0.2077	34	0.2099	30	0.2071	27	0.2408	26	0.2122
0.99	0	13	0.0134	10	0.0122	9	0.0100	8	0.0113	7	0.0156
	2	25	0.0114	19	0.0109	16	0.0136	15	0.0111	14	0.0112
	4	35	0.0110	26	0.0132	23	0.0113	21	0.0113	19	0.0154
	10	62	0.0107	47	0.0114	41	0.0113	37	0.0135	35	0.0122

Table 4. Single sampling plan for TPITL distribution at $\upsilon = 0.5, \delta = 0.5, \vartheta = 1.5$.

p^*	c	$a = 0.2$		$a = 0.4$		$a = 0.6$		$a = 0.8$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.2	0	1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2	8	0.8058	5	0.8890	5	0.8148	4	0.9037	4	0.8750
	4	15	0.8099	10	0.8553	9	0.8183	8	0.8360	7	0.8906
	10	38	0.8055	26	0.8223	22	0.8127	19	0.8563	18	0.8338
0.4	0	3	0.6001	2	0.6668	1	1.0000	1	1.0000	1	1.0000
	2	11	0.6005	7	0.6806	6	0.6737	5	0.7473	5	0.6875
	4	19	0.6174	13	0.6318	11	0.6202	10	0.6016	9	0.6367
	10	44	0.6278	30	0.6362	25	0.6308	22	0.6507	20	0.6762
0.6	0	4	0.4649	3	0.4446	2	0.5949	2	0.5417	2	0.5000
	2	14	0.4118	9	0.4685	8	0.4088	7	0.4248	6	0.5000
	4	23	0.4261	16	0.4044	13	0.4237	11	0.4828	10	0.5000
	10	51	0.4092	34	0.4354	28	0.4368	25	0.4214	23	0.4159
0.8	0	7	0.2161	4	0.2964	4	0.2105	3	0.2934	3	0.2500
	2	18	0.2274	12	0.2343	10	0.2223	9	0.2062	8	0.2266
	4	29	0.2108	19	0.2314	16	0.2056	14	0.2100	12	0.2744
	10	59	0.2126	39	0.2313	32	0.2275	28	0.2357	26	0.2122
0.99	0	19	0.0101	12	0.0116	9	0.0157	8	0.0137	7	0.0156
	2	34	0.0121	22	0.0129	18	0.0111	15	0.0146	14	0.0112
	4	48	0.0109	31	0.0123	25	0.0118	22	0.0104	19	0.0154
	10	85	0.0101	55	0.0124	45	0.0104	39	0.0109	35	0.0122

Table 5. Single sampling plan for TPITL distribution at $\upsilon = 0.5, \delta = 1.5, \vartheta = 0.5$.

p^*	c	$a = 0.2$		$a = 0.4$		$a = 0.6$		$a = 0.8$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.2	0	5	0.8277	2	0.8220	1	1.0000	1	1.0000	1	1.0000
	2	34	0.8059	9	0.8432	6	0.8221	4	0.9270	4	0.8750
	4	68	0.8032	18	0.8285	11	0.8303	8	0.8854	7	0.8906
	10	178	0.8031	47	0.8163	28	0.8108	21	0.8344	18	0.8339
0.4	0	11	0.6234	3	0.6757	2	0.6890	1	1.0000	1	1.0000
	2	50	0.6045	13	0.6370	8	0.6221	6	0.6512	5	0.6875
	4	90	0.6072	24	0.6093	14	0.6210	10	0.6950	9	0.6367
	10	214	0.6045	56	0.6126	32	0.6386	24	0.6498	20	0.6762
0.6	0	20	0.4074	5	0.4566	3	0.4748	2	0.5821	2	0.5000
	2	67	0.4073	17	0.4386	10	0.4339	7	0.5073	6	0.5000
	4	113	0.4064	29	0.4263	17	0.4121	12	0.4836	10	0.5000
	10	249	0.4037	64	0.4207	37	0.4098	27	0.4468	23	0.4159
0.8	0	35	0.2005	9	0.2085	5	0.2254	3	0.3389	3	0.2500
	2	92	0.2033	23	0.2225	13	0.2274	9	0.2792	8	0.2266
	4	145	0.2011	37	0.2072	21	0.2062	15	0.2352	12	0.2744
	10	294	0.2030	75	0.2118	42	0.2269	31	0.2271	26	0.2122
0.99	0	98	0.0102	24	0.0110	13	0.0115	9	0.0132	7	0.0156
	2	179	0.0103	44	0.0115	24	0.0120	17	0.0129	14	0.0112
	4	248	0.0102	62	0.0104	34	0.0107	24	0.0124	19	0.0154
	10	432	0.0101	108	0.0110	60	0.0107	43	0.0119	35	0.0122

Table 6. Single sampling plan for TPITL distribution at $\upsilon = 1.5, \delta = 0.5, \vartheta = 0.5$.

p^*	c	$a = 0.2$		$a = 0.4$		$a = 0.6$		$a = 0.8$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.2	0	1	1.0000	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2	7	0.8405	5	0.8776	5	0.8063	4	0.9018	4	0.8750
	4	14	0.8097	10	0.8354	9	0.8067	8	0.8317	7	0.8906
	10	35	0.8142	25	0.8284	21	0.8469	19	0.8502	18	0.8338
0.4	0	2	0.7564	2	0.6543	1	1.0000	1	1.0000	1	1.0000
	2	10	0.6185	7	0.6557	6	0.6610	5	0.7430	5	0.6875
	4	17	0.6530	12	0.6795	11	0.6019	9	0.7174	9	0.6367
	10	41	0.6205	29	0.6344	25	0.6030	22	0.6399	20	0.6762
0.6	0	4	0.4327	3	0.4281	2	0.5877	2	0.5386	2	0.5000
	2	13	0.4107	9	0.4381	7	0.5188	7	0.4186	6	0.5000
	4	21	0.4409	15	0.4361	13	0.4036	11	0.4747	10	0.5000
	10	47	0.4152	33	0.4246	28	0.4068	25	0.4093	23	0.4159
0.8	0	6	0.2476	4	0.2801	4	0.2030	3	0.2901	3	0.2500
	2	17	0.2135	12	0.2085	10	0.2093	9	0.2011	8	0.2266
	4	27	0.2046	18	0.2463	15	0.2484	14	0.2035	12	0.2744
	10	55	0.2022	38	0.2163	32	0.2037	28	0.2258	26	0.2122
0.99	0	17	0.0115	11	0.0144	9	0.0142	8	0.0131	7	0.0156
	2	32	0.0103	21	0.0134	17	0.0144	15	0.0138	14	0.0112
	4	44	0.0113	30	0.0114	24	0.0142	21	0.0147	19	0.0154
	10	78	0.0105	53	0.0119	44	0.0107	38	0.0138	35	0.0122

Table 7. Single sampling plan for TPITL distribution at $\upsilon = 1.5, \delta = 0.5, \vartheta = 1.5$.

p^*	c	$a = 0.2$		$a = 0.4$		$a = 0.6$		$a = 0.8$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.2	0	2	0.8116	1	1.0000	1	1.0000	1	1.0000	1	1.0000
	2	9	0.8220	6	0.8359	5	0.8398	4	0.9104	4	0.8750
	4	17	0.8319	11	0.8484	9	0.8515	8	0.8504	7	0.8906
	10	45	0.8064	29	0.8062	23	0.8192	20	0.8219	18	0.8338
0.4	0	3	0.6588	2	0.6992	2	0.6170	1	1.0000	1	1.0000
	2	12	0.6560	8	0.6453	6	0.7115	5	0.7619	5	0.6875
	4	22	0.6392	14	0.6520	11	0.6751	10	0.6275	9	0.6367
	10	53	0.6120	33	0.6404	26	0.6530	23	0.6131	20	0.6762
0.6	0	5	0.4340	3	0.4889	2	0.6170	2	0.5526	2	0.5000
	2	16	0.4429	10	0.4608	8	0.4574	7	0.4467	6	0.5000
	4	28	0.4058	17	0.4472	14	0.4014	12	0.4039	10	0.5000
	10	61	0.4084	38	0.4200	30	0.4142	25	0.4640	23	0.4159
0.8	0	8	0.2320	5	0.2390	4	0.2349	3	0.3053	3	0.2500
	2	22	0.2147	14	0.2008	10	0.2651	9	0.2245	8	0.2266
	4	35	0.2059	21	0.2352	17	0.2035	14	0.2337	12	0.2744
	10	71	0.2089	44	0.2116	34	0.2239	29	0.2215	26	0.2122
0.99	0	23	0.0101	13	0.0137	10	0.0130	8	0.0157	7	0.0156
	2	42	0.0105	25	0.0116	19	0.0119	16	0.0112	14	0.0112
	4	58	0.0110	35	0.0114	27	0.0105	22	0.0134	19	0.0154
	10	102	0.0108	62	0.0111	48	0.0101	40	0.0112	35	0.0122

Table 8. Single sampling plan for TPITL distribution at $\upsilon = 1.5, \delta = 1.5, \vartheta = 1.5$.

p^*	c	$a = 0.2$		$a = 0.4$		$a = 0.6$		$a = 0.8$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.2	0	19	0.8043	3	0.8513	2	0.8010	1	1.0000	1	1.0000
	2	128	0.8027	21	0.8019	8	0.8536	5	0.8740	4	0.8750
	4	258	0.8008	41	0.8058	17	0.8012	10	0.8291	7	0.8906
	10	680	0.8003	107	0.8034	42	0.8220	25	0.8180	18	0.8338
0.4	0	43	0.6016	7	0.6169	3	0.6416	2	0.6505	1	1.0000
	2	190	0.6028	30	0.6086	12	0.6206	7	0.6481	5	0.6875
	4	345	0.6020	54	0.6087	21	0.6339	12	0.6696	9	0.6367
	10	821	0.6006	128	0.6061	50	0.6182	29	0.6182	20	0.6762
0.6	0	76	0.4036	12	0.4125	5	0.4116	3	0.4231	2	0.5000
	2	258	0.4018	40	0.4103	16	0.4017	9	0.4290	6	0.5000
	4	435	0.4017	68	0.4010	26	0.4255	15	0.4243	10	0.5000
	10	957	0.4012	149	0.4010	58	0.4019	33	0.4070	23	0.4159
0.8	0	134	0.2001	20	0.2166	8	0.2115	4	0.2752	3	0.2500
	2	355	0.2011	55	0.2017	21	0.2091	12	0.2011	8	0.2266
	4	558	0.2007	86	0.2043	33	0.2082	18	0.2361	12	0.2744
	10	1134	0.2003	175	0.2037	67	0.2113	38	0.2025	26	0.2122
0.99	0	381	0.0101	58	0.0102	21	0.0118	11	0.0136	7	0.0156
	2	696	0.0101	106	0.0103	39	0.0117	21	0.0123	14	0.0112
	4	962	0.0100	147	0.0101	55	0.0105	30	0.0103	19	0.0154
	10	1671	0.0100	256	0.0101	96	0.0112	53	0.0103	35	0.0122

5. Parameter estimation

In this section, the parameter estimation of the TPITL distribution is discussed using classical methods, including ML, MPS, LS, WLS, and CvM.

5.1. Maximum likelihood estimation

Let $T_1, T_2, \ldots T_n$ be the TPITL distribution's observed random sample. Then, the TPITL distribution's log-likelihood function, denoted by $\ln l$, for parameters, based on a complete sample, is given by

$$\ln l = n\ln(2\vartheta\delta) + (2\delta - 1)\sum_{i=1}^{n}\ln t_i - (2\vartheta + 1)\sum_{i=1}^{n}\ln W_i + (\vartheta - 1)\sum_{i=1}^{n}\ln S_i + \sum_{i=1}^{n}\ln\left[1 - \upsilon + 2\upsilon\left(S_i^{\vartheta}W_i^{-2\vartheta}\right)\right].$$

where, $W_i = W_i(\delta) = \left(1 + t_i^{\delta}\right)$ and $S_i = S_i(\delta) = \left(1 + 2t_i^{\delta}\right)$. The partial derivatives of $\ln l$ with respect to υ, δ and ϑ are given by:

$$\frac{\partial \ln l}{\partial \upsilon} = \sum_{i=1}^{n}\left[2\left(1 - S_i^{\vartheta}W_i^{-2\vartheta}\right) - 1\right]\left[1 - \upsilon + 2\upsilon\left(S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]^{-1},$$

$$\frac{\partial \ln l}{\partial \delta} = \frac{n}{\delta} + 2\sum_{i=1}^{n}\ln t_i - (2\vartheta + 1)\sum_{i=1}^{n}t_i^{\delta}\ln t_i W_i^{-1} + 2(\vartheta - 1)\sum_{i=1}^{n}t_i^{\delta}\ln t_i S_i^{-1}$$

$$+4\vartheta\upsilon\sum_{i=1}^{n}\left\{t_i^{\delta}\ln t_i\left[W_i^{-2\vartheta}S_i^{\vartheta-1} - S_i^{\vartheta}W_i^{-2\vartheta-1}\right]\right\}\left[1 - \upsilon + 2\upsilon\left(S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]^{-1},$$

$$\frac{\partial \ln l}{\partial \vartheta} = \frac{n}{\vartheta} - 2\sum_{i=1}^{n}\ln W_i + \sum_{i=1}^{n}\ln S_i + 2\upsilon\sum_{i=1}^{n}S_i^{\vartheta}W_i^{-2\vartheta}(\ln S_i - 2\ln W_i)\left[1 - \upsilon + 2\upsilon\left(S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]^{-1}.$$

To obtain the ML estimators of υ, δ and ϑ, say $\hat{\upsilon}, \hat{\delta}$ and $\hat{\vartheta}$ we use the nonlinear equations, $\partial\ln l/\partial\upsilon = 0$, $\partial\ln l/\partial\delta = 0$ and $\partial\ln l/\partial\vartheta = 0$ to solve them using an iterative approach.

5.2. Maximum product spacing

Reference [25] proposed the MPS estimation of population parameters. The notion of discrepancies between the CDF values at consecutive data points may be used to create this approach. Assume that $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$, are the ordered observations of TPITL distribution. Then, the uniform spacings may be determined based on a random sample of size n from the TPITL distribution, as follows:

$$D_i(\Theta) = H\left(t_{(i)}|\Theta\right) - H\left(t_{(i-1)}|\Theta\right),\ i = 1, 2, \ldots, n + 1,\ H\left(t_{(0)}|\Theta\right) = 0,\ H\left(t_{(n+1)}|\Theta\right) = 1,\ \sum_{i=1}^{n+1}D_i(\Theta) = 1.$$

For simplicity of notation, we write $t_i = t_{(i)}$, hence, the MPS estimates obtained by $\hat{\upsilon}_{MPS}, \hat{\delta}_{MPS}$ and $\hat{\vartheta}_{MPS}$ can be obtained by maximizing the geometric mean of the spacings

$$GM(\Theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\{\left[\left(1 - S_i^{\vartheta}W_i^{-2\vartheta}\right)\left(1 + \upsilon - \upsilon\left(1 - S_i^{\vartheta}W_i^{-2\vartheta}\right)\right)\right] - \left[\left(1 - S_{i-1}^{\vartheta}W_{i-1}^{-2\vartheta}\right)\left(1 + \upsilon - \upsilon\left(1 - S_{i-1}^{\vartheta}W_{i-1}^{-2\vartheta}\right)\right)\right]\right\},$$

with respect to υ, δ and ϑ, or by solving the following equations:

$$\frac{\partial GM(\Theta)}{\partial \upsilon}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\omega_1(t_i|\Theta)-\omega_1(t_{i-1}|\Theta)}{D_i(\Theta)}\ ,\ \frac{\partial GM(\Theta)}{\partial \vartheta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\omega_2(t_i|\Theta)-\omega_2(t_{i-1}|\Theta)}{D_i(\Theta)},$$

$$\frac{\partial GM(\Theta)}{\partial \delta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\omega_3(t_i|\Theta)-\omega_3(t_{i-1}|\Theta)}{D_i(\Theta)}.$$

$$\omega_1(t_i|\Theta)=\left(1-S_i^{\vartheta}W_i^{-2\vartheta}\right)\left(S_i^{\vartheta}W_i^{-2\vartheta}\right), \tag{12}$$

$$\omega_2(t_i|\Theta)=S_i^{\vartheta}W_i^{-2\vartheta}\left\{\left(1-S_i^{\vartheta}W_i^{-2\vartheta}\right)\upsilon[\ln S_i-2\ln W_i]+\left(1+\upsilon S_i^{\vartheta}W_i^{-2\vartheta}\right)[2\ln W_i-\ln S_i]\right\}, \tag{13}$$

$$\omega_3(t_i|\Theta)=2\vartheta t_i^{\delta}\ln t_i\, S_i^{\vartheta}W_i^{-2\vartheta}[W_i^{-1}-S_i^{-1}][\upsilon(S_i^{-1}-W_i^{-1})]. \tag{14}$$

5.3. LS and WLS estimation

Let $T_{(1)},T_{(2)},\ldots.T_{(n)}$ be the observed ordered sample and $T_1,T_2,\ldots.T_n$ a TPITL-generated random sample of size *n*. Then, the LS estimates (LSEs) and WLS estimates (WLSEs) of υ,δ and ϑ can be obtained by minimizing the following function with respect to υ,δ and ϑ:

$$\Lambda(\Theta)=\sum_{i=1}^{n}u_i\left\{\left[\left(1-S_i^{\vartheta}W_i^{-2\vartheta}\right)\left(1+\upsilon S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]-\frac{i}{n+1}\right\}^2.$$

where $t_i=t_{(i)}$, we can get the LSEs designated by $\hat{\upsilon}_{LS},\hat{\delta}_{LS}$ and $\hat{\vartheta}_{LS}$ by setting $u_i=1$, whereas we can get the WLSEs denoted by $\hat{\upsilon}_{WLS},\hat{\delta}_{WLS}$ and $\hat{\vartheta}_{WLS}$ by setting $u_i=\frac{(n+1)^2(n+2)}{i(n-i+1)}$. These estimates can also be obtained by solving the equations stated below:

$$\frac{\partial\Lambda(\Theta)}{\partial\upsilon}=\sum_{i=1}^{n}2u_i\left\{\left[\left(1-S_i^{\vartheta}W_i^{-2\vartheta}\right)\left(1+\upsilon S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]-\frac{i}{n+1}\right\}\omega_1(t_i|\Theta),$$

$$\frac{\partial\Lambda(\Theta)}{\partial\vartheta}=\sum_{i=1}^{n}2u_i\left\{\left[\left(1-S_i^{\vartheta}W_i^{-2\vartheta}\right)\left(1+\upsilon S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]-\frac{i}{n+1}\right\}\omega_2(t_i|\Theta),$$

and

$$\frac{\partial\Lambda(\Theta)}{\partial\delta}=\sum_{i=1}^{n}2u_i\left\{\left[\left(1-S_i^{\vartheta}W_i^{-2\vartheta}\right)\left(1+\upsilon S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]-\frac{i}{n+1}\right\}\omega_3(t_i|\Theta).$$

$\omega_1(t_i|\Theta)$, $\omega_2(t_i|\Theta)$ and $\omega_3(t_i|\Theta)$ are defined in (12)–(14).

5.4. The Cramer-von Mises estimation

The CvM estimates (CvMEs) denoted by $\hat{\upsilon}_{CME},\hat{\delta}_{CME}$ and $\hat{\vartheta}_{CME}$ of υ,δ and ϑ can be obtained by minimizing the following function with respect to υ,δ and ϑ:

$$\mathrm{CvM}(\Theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left\{\left[\left(1-S_i^{\vartheta}W_i^{-2\vartheta}\right)\left(1+\upsilon S_i^{\vartheta}W_i^{-2\vartheta}\right)\right]-\frac{2i-1}{2n}\right\}^2,$$

where $t_i = t_{(i)}$. These estimates can also be obtained by solving the following equations:

$$\frac{\partial \text{CvM}(\Theta)}{\partial \upsilon} = \sum_{i=1}^{n} 2\left\{\left[\left(1 - S_i^{\vartheta} W_i^{-2\vartheta}\right)\left(1 + \upsilon S_i^{\vartheta} W_i^{-2\vartheta}\right)\right] - \frac{2i-1}{2n}\right\} \omega_1(t_i|\Theta),$$

$$\frac{\partial \text{CvM}(\Theta)}{\partial \vartheta} = \sum_{i=1}^{n} 2\left\{\left[\left(1 - S_i^{\vartheta} W_i^{-2\vartheta}\right)\left(1 + \upsilon S_i^{\vartheta} W_i^{-2\vartheta}\right)\right] - \frac{2i-1}{2n}\right\} \omega_2(t_i|\Theta),$$

and

$$\frac{\partial \text{CvM}(\Theta)}{\partial \delta} = \sum_{i=1}^{n} 2\left\{\left[\left(1 - S_i^{\vartheta} W_i^{-2\vartheta}\right)\left(1 + \upsilon S_i^{\vartheta} W_i^{-2\vartheta}\right)\right] - \frac{2i-1}{2n}\right\} \omega_3(t_i|\Theta).$$

where $\omega_1(t_i|\Theta)$, $\omega_2(t_i|\Theta)$ and $\omega_3(t_i|\Theta)$ are defined in (12)–(14).

6. Simulation study

As mentioned in the previous section, expressions for the derived estimators are hard to obtain. Therefore, we designed a simulation study to clarify the theoretical results. The behavior of estimates was examined in terms of their mean squared error (MSE) and standard error (SE). We performed the following steps:

Step 1: Random samples (10,000) of sizes 20, 30, 50, 75 and 150 were generated from the TPITL distribution. The chosen parameter values are as follows: $(\delta = 0.75, \vartheta = 0.25, \upsilon = -0.25)$, $(\delta = 1.5, \vartheta = 0.5, \upsilon = -0.25)$, $(\delta = 0.75, \vartheta = 0.25, \upsilon = -0.75)$, and $(\delta = 1.5, \vartheta = 0.5, \upsilon = -0.75)$.

Step 2: ML estimate (MLE), MPSE, LSE, WLSE, and CvMEs of the parameters were obtained.

Step 3: We computed MSEs and SEs of all estimates, and the results are listed in Tables 9–12. We noticed the following about the performance of estimates:

- For all estimating methods, it was evident that MSEs and SEs reduced with n.
- The MSE and SE for $\hat{\upsilon}$ performed better than the other estimates as the sample size increased.
- The LSE, WLSE, WLSE, MPSE, and CvMEs for υ and ϑ had analogous results, while MLE had a slightly different result in terms of MSE and SE (Figure 3).

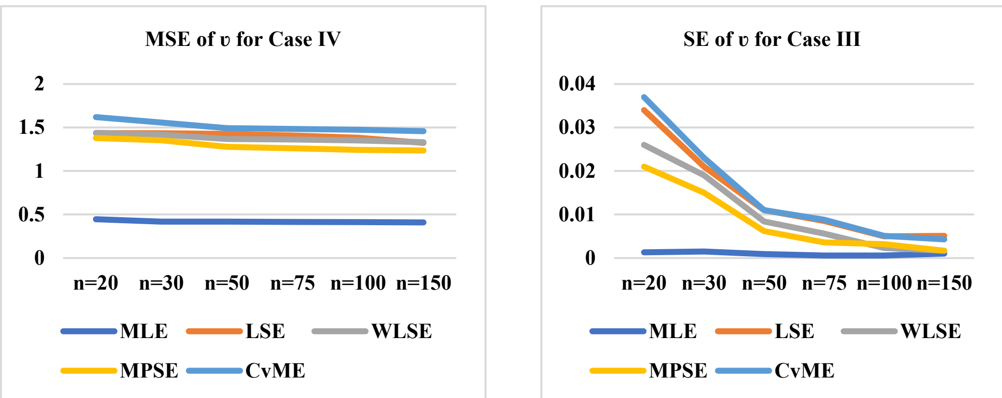

Figure 3. The MSE and SE for υ based on different methods.

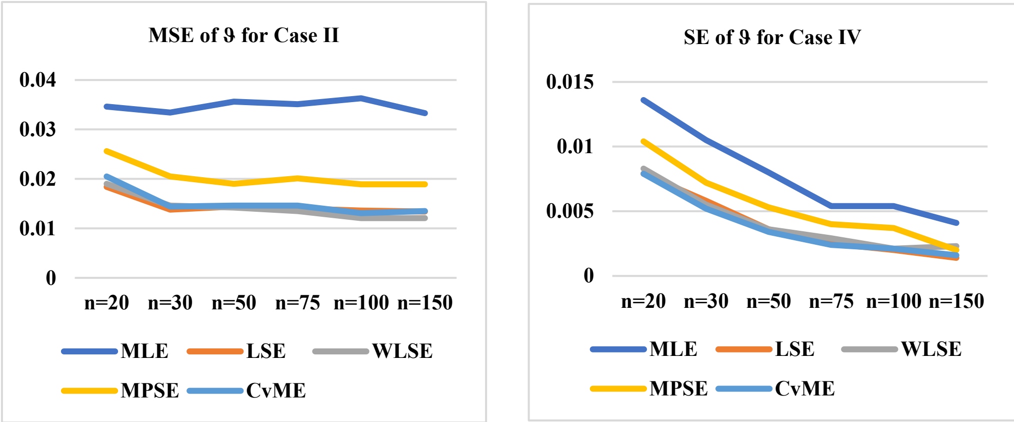

Figure 4. MSE and SE for ϑ based on different methods.

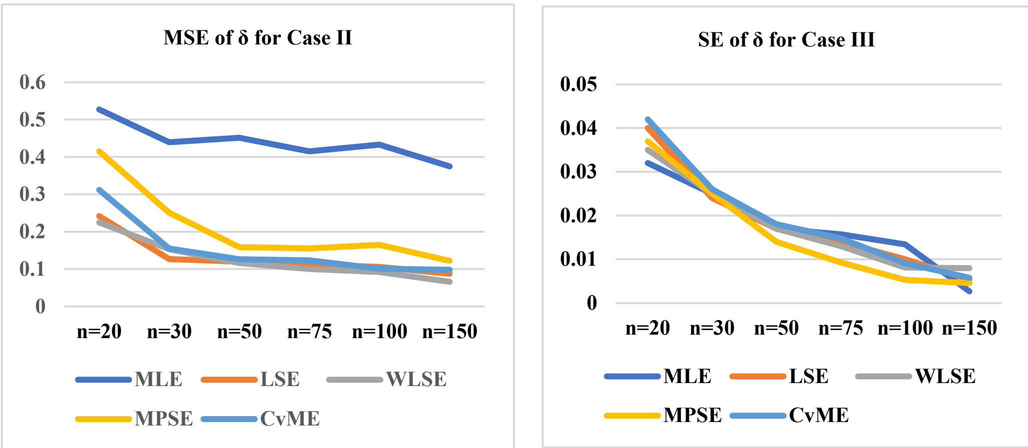

Figure 5. The MSE and SE for δ based on different methods.

- As shown in the left panel of Figures 4–5, the LSE, WLSE, MPSE, and CvME outperformed MLE in terms of MSE, especially for ϑ and δ.
- Generally, under all approaches, we noticed that MSE of $\hat{\upsilon}$ had the least value, while MSE of $\hat{\vartheta}_{WLS}$ took the smallest value.
- For a fixed value of the parameters (δ, ϑ) and as υ increased, the MSEs and SEs of υ estimate increased while the MSEs and SEs of (δ, ϑ) estimate decreased for different methods.
- The general conclusion from the aforementioned figures is that as the sample size increases, all MSE and SE graphs for all parameters will reach zero. This confirms the accuracy of these estimation methods.
- It should be emphasized that, in most situations, the ML technique outperformed other methods in terms of MSE.
- In some situations, MPSE comes in as the second-best performing estimator, followed by LSE.

Table 9. The MSE and SE for different estimates of the TPITL distribution.

n	Parameters	MLE		LSE		WLSE		MPSE		CvME	
		MSE	SE	MSE	SE	MSE	SE	MSE	SE	MSE	SE
20	$\delta = 0.75$	0.0634	0.0512	0.0462	0.0471	0.0321	0.0392	0.0318	0.0385	0.0457	0.0476
	$\vartheta = 0.25$	4.2070*	0.0104	5.9938*	0.0168	3.7685*	0.0135	1.8682*	9.6702*	5.7922*	0.0166
	$\upsilon = -0.25$	0.0563	3.5521*	0.1807	0.0249	0.1933	0.0222	0.2111	0.0204	0.2275	0.0281
30	$\delta = 0.75$	0.0316	0.0295	0.0327	0.0309	0.0242	0.0270	0.0193	0.0248	0.0328	0.0321
	$\vartheta = 0.25$	3.5439*	7.0577*	3.0803*	9.8808*	2.7990*	9.6063*	1.0694*	5.5847*	4.3042*	0.0118
	$\upsilon = -0.25$	0.0558	2.0435*	0.1807	0.0176	0.1915	0.0147	0.1973	0.0116	0.2110	0.0199
50	$\delta = 0.75$	0.0317	0.0213	0.0162	0.0174	0.0101	0.0140	0.0157	0.0165	0.0162	0.0178
	$\vartheta = 0.25$	3.8108*	4.7838*	1.1283*	4.7452*	0.5372*	3.0750*	1.1926*	3.7336*	1.1844*	4.8565*
	$\upsilon = -0.25$	0.0561	1.3275*	0.1649	8.5725*	0.1800	5.9981*	0.1874	5.1355*	0.1824	8.9142*
75	$\delta = 0.75$	0.0400	0.0202	0.0101	0.0104	6.1135*	8.6267*	0.0106	0.0112	8.3405*	9.6595*
	$\vartheta = 0.25$	3.6780*	3.5785*	0.4110*	2.3153*	0.3427*	1.8551*	1.0702*	2.6413*	0.3617*	2.1469*
	$\upsilon = -0.25$	0.0555	1.4264*	0.1659	5.9550*	0.1788	3.7775*	0.1838	3.2474*	0.1763	5.9837*
100	$\delta = 0.75$	0.0194	0.0113	7.9590*	7.7147*	5.3663*	6.9967*	8.3169*	8.5896*	7.3736*	7.4625*
	$\vartheta = 0.25$	3.1715*	2.4667*	0.4031*	1.9978*	0.3735*	1.7288*	0.9853*	2.0216*	0.3708*	1.9209*
	$\upsilon = -0.25$	0.0558	0.7247*	0.1641	4.0990*	0.1780	2.5736*	0.1811	2.2216*	0.1713	4.2937*
150	$\delta = 0.75$	0.0174	8.6607*	4.7108*	4.4809*	3.5370*	4.4433*	4.2869*	5.3664*	4.3355*	4.4987*
	$\vartheta = 0.25$	3.0945*	1.8702*	0.2082*	1.1216*	0.2407*	0.8882*	0.6736*	1.4636*	0.2130*	1.1105*
	$\upsilon = -0.25$	0.0557	0.4741*	0.1543	3.4320*	0.1758	1.6379*	0.1533	1.7328*	0.1703	3.1378*

Table 10. The MSE and SE for different estimates of the TPITL distribution.

n	Parameters	MLE		LSE		WLSE		MPSE		CvME	
		MSE	SE	MSE	SE	MSE	SE	MSE	SE	MSE	SE
20	$\delta = 1.5$	0.5272	0.1158	0.2419	0.0810	0.2248	0.0707	0.4151	0.1004	0.3123	0.0825
	$\vartheta = 0.5$	0.0346	0.0142	0.0184	6.5972*	0.0190	5.5531*	0.0256	0.0107	0.0205	6.0177*
	$\upsilon = -0.25$	0.0375	9.2001*	0.4592	0.0446	0.4543	0.0345	0.3840	0.0235	0.5443	0.0447
30	$\delta = 1.5$	0.4394	0.0833	0.1267	0.0477	0.1531	0.0502	0.2508	0.0594	0.1548	0.0482
	$\vartheta = 0.5$	0.0334	0.0100	0.0138	6.2299*	0.0146	5.9041*	0.0205	8.3477*	0.0144	5.2513*
	$\upsilon = -0.25$	0.0336	0.0101	0.4306	0.0254	0.4516	0.0219	0.4126	0.0164	0.4962	0.0278
50	$\delta = 1.5$	0.4512	0.0672	0.1199	0.0348	0.1158	0.0321	0.1586	0.0376	0.1261	0.0324
	$\vartheta = 0.5$	0.0356	8.5213*	0.0144	4.0590*	0.0142	3.7567*	0.0190	4.9860*	0.0146	3.3391*
	$\upsilon = -0.25$	0.0353	6.0854*	0.4172	0.0154	0.4337	0.0122	0.3878	9.5065*	0.4603	0.0164
75	$\delta = 1.5$	0.4150	0.0512	0.1116	0.0226	0.1003	0.0193	0.1555	0.0300	0.1234	0.0216
	$\vartheta = 0.5$	0.0351	6.0012*	0.0140	2.7917*	0.0135	2.6442*	0.0201	4.4084*	0.0146	2.4824*
	$\upsilon = -0.25$	0.0358	4.3681*	0.4250	0.0103	0.4419	8.0752*	0.3809	6.4523*	0.4510	0.0106
100	$\delta = 1.5$	0.4328	0.0426	0.1056	0.0171	0.0920	0.0172	0.1647	0.0282	0.1011	0.0153
	$\vartheta = 0.5$	0.0363	5.5815*	0.0136	2.3123*	0.0121	2.7129*	0.0189	4.3207*	0.0131	1.8163*
	$\upsilon = -0.25$	0.0339	3.6981*	0.4570	7.3339*	0.4564	5.6197*	0.3973	5.2128*	0.4761	7.1705*
150	$\delta = 1.5$	0.3750	0.0254	0.0874	6.4564*	0.0663	6.6803*	0.1223	0.0146	0.0980	7.1989*
	$\vartheta = 0.5$	0.0333	3.0071*	0.0134	1.1206*	0.0121	1.2057*	0.0189	3.0867*	0.0135	1.1276*
	$\upsilon = -0.25$	0.0336	1.9440*	0.4335	2.6577*	0.4322	1.7794*	0.3768	2.8150*	0.4433	2.4982*

Table 11. The MSE and SE for different estimates of the TPITL distribution.

n	Parameters	MLE		LSE		WLSE		MPSE		CvME	
		MSE	SE	MSE	SE	MSE	SE	MSE	SE	MSE	SE
20	$\delta = 0.75$	0.0276	0.0320	0.0650	0.0400	0.0521	0.0350	0.0312	0.0370	0.0656	0.0420
	$\vartheta = 0.25$	2.9819*	9.6410*	8.3113*	0.0200	6.9943*	0.0180	2.1731*	0.0100	0.0109	0.0220
	$\upsilon = -0.75$	0.5469	1.3110*	0.9473	0.0340	0.9252	0.0260	0.8917	0.0210	1.0580	0.0370
30	$\delta = 0.75$	0.0240	0.0250	0.0523	0.0240	0.0435	0.0250	0.0217	0.0250	0.0539	0.0260
	$\vartheta = 0.25$	3.4648*	7.9550*	5.8641*	0.0140	5.8410*	0.0140	2.2441*	6.4740*	6.4372*	0.0140
	$\upsilon = -0.75$	0.5458	1.4640*	0.9186	0.0210	0.9199	0.0190	0.8715	0.0150	0.9848	0.0230
50	$\delta = 0.75$	0.0173	0.0170	0.0441	0.0170	0.0377	0.0170	0.0129	0.0140	0.0452	0.0180
	$\vartheta = 0.25$	3.8093*	5.0600*	3.6101*	8.5390*	4.1827*	9.1340*	2.3664*	3.4760*	3.8059*	8.7550*
	$\upsilon = -0.75$	0.5438	0.9780*	0.8972	0.0110	0.8821	8.3640*	0.8377	6.2060*	0.9392	0.0110
75	$\delta = 0.75$	0.0238	0.0157	0.0410	0.0140	0.0307	0.0130	9.5963*	9.3015*	0.0413	0.0148
	$\vartheta = 0.25$	2.9710*	4.3074*	3.6810*	7.0170*	3.2999*	6.5592*	1.9662*	2.2798*	4.7127*	7.9185*
	$\upsilon = -0.75$	0.5459	0.6425*	0.9595	8.5172*	0.8979	5.5828*	0.8356	3.6097*	0.9863	8.7994*
100	$\delta = 0.75$	0.0227	0.0134	0.0298	0.0100	0.0137	7.9828*	4.3208*	5.2667*	0.0297	9.1020*
	$\vartheta = 0.25$	2.3568*	4.0206*	2.8294*	5.5969*	1.5028*	2.2740*	1.5569*	1.4654*	2.5345*	5.3057*
	$\upsilon = -0.75$	0.5416	0.6058*	0.9242	5.0194*	0.8670	2.3480*	0.8194	3.1607*	0.9360	5.1240*
150	$\delta = 0.75$	9.1118*	2.7224*	0.0249	5.3565*	8.5464*	7.9564*	4.1972*	4.5636*	0.0241	5.8262*
	$\vartheta = 0.25$	1.7642*	0.9286*	1.5173*	1.8264*	0.8638*	0.4991*	1.0472*	0.9626*	1.0430*	2.3932*
	$\upsilon = -0.75$	0.5329	1.0329*	0.9163	5.0756*	0.8485	1.7051*	0.7717	1.7450*	0.9272	4.3014*

Table 12. The MSE and SE for different estimates of the TPITL distribution.

n	Parameters	MLE		LSE		WLSE		MPSE		CvME	
		MSE	SE	MSE	SE	MSE	SE	MSE	SE	MSE	SE
20	$\delta = 1.5$	0.3166	0.1074	0.0752	0.0611	0.0688	0.0584	0.1069	0.0688	0.0855	0.0628
	$\vartheta = 0.5$	0.0371	0.0136	0.0161	7.9040*	0.0156	8.2771*	0.0170	0.0104	0.0172	7.9114*
	$\upsilon = -0.75$	0.4460	0.0347	1.4360	0.0421	1.4374	0.0389	1.3799	0.0355	1.6197	0.0471
30	$\delta = 1.5$	0.2908	0.0852	0.0450	0.0385	0.0401	0.0359	0.0525	0.0419	0.0446	0.0385
	$\vartheta = 0.5$	0.0361	0.0105	0.0136	5.8095*	0.0121	5.5192*	0.0125	7.1990*	0.0141	5.2089*
	$\upsilon = -0.75$	0.4180	0.0143	1.4336	0.0260	1.4175	0.0237	1.3529	0.0213	1.5567	0.0286
50	$\delta = 1.5$	0.2901	0.0662	0.0281	0.0235	0.0269	0.0219	0.0337	0.0256	0.0273	0.0234
	$\vartheta = 0.5$	0.0352	7.9137*	0.0134	3.6008*	0.0111	3.5586*	0.0121	5.2926*	0.0138	3.4227*
	$\upsilon = -0.75$	0.4176	0.0131	1.4264	0.0148	1.3670	0.0112	1.2780	0.0106	1.4927	0.0155
75	$\delta = 1.5$	0.1622	0.0418	0.0195	0.0158	0.0229	0.0153	0.0315	0.0192	0.0185	0.0157
	$\vartheta = 0.5$	0.0301	5.3795*	0.0131	2.5415*	0.0103	2.9067*	0.0112	3.9476*	0.0134	2.4295*
	$\upsilon = -0.75$	0.4141	9.7076*	1.4082	0.0101	1.3617	6.9812*	1.2608	6.7749*	1.4829	0.0101
100	$\delta = 1.5$	0.2589	0.0436	0.0151	0.0122	0.0181	0.0107	0.0204	0.0132	0.0159	0.0126
	$\vartheta = 0.5$	0.0357	5.4308*	0.0129	2.0247*	9.3446*	2.0512*	0.0106	3.7361*	0.0132	2.0816*
	$\upsilon = -0.75$	0.4123	6.1602*	1.3819	7.3598*	1.3518	5.2508*	1.2430	5.0394*	1.4739	7.9705*
150	$\delta = 1.5$	0.1303	0.0258	0.0113	8.2565*	0.0176	0.0105	0.0189	7.0110*	0.0134	9.3264*
	$\vartheta = 0.5$	0.0312	4.0948*	0.0124	1.4042*	7.8324*	2.2745*	8.8525*	2.0483*	0.0128	1.6189*
	$\upsilon = -0.75$	0.4096	4.3745*	1.3245	5.5974*	1.3308	3.0557*	1.2360	2.7320*	1.4591	5.2175*

*Note: * Indicate that the value multiply 10^{-3} in Tables 9–12.

7. Real data analysis

This section discussed two applications for the TPITL model using two different real datasets. The first dataset represents 30 observations of the March precipitation (in inches) in Minneapolis/St Paul. The proposed dataset was first recorded in [32] and recently used to fit the PITL distribution in [22].

We checked whether the TPITL distribution is suitable for analyzing this dataset. We reported the MLEs of the parameters and the values of the negative −2 log-likelihood $(-2\log L)$, Akaike's information criterion (AIC), Bayesian IC (BIC), corrected AIC (CAIC), and the Kolmogorov–Smirnov (K–S) test statistic, as well as associated P-value to judge the goodness of fit with comparison to PITL, generalized inverse exponential (GIE), inverse Weibull (in-We), inverse gamma (in-Ga) and Burr XII distributions. The lower the values of these criteria, the better the fit. Table 13 shows the parameter estimators, SE, and some goodness of fit statistics. Plots of estimated PDF and CDF are provided in Figure 6. The probability–probability (PP) plots of estimated densities are given in Figure 7.

The second dataset represents the survival times (in days) of 72 guinea pigs infected with virulent *tubercle bacilli*. The above dataset was first recorded in [33] and was recently used to fit the inverse Weibull distribution by reference [34] under Type-II right censoring. Reference [35] also used it to fit extended inverse power Lomax under Type-I right censoring.

We checked whether the TPITL distribution is suitable for analyzing this dataset. We reported the MLEs of the parameters and the values of the –2log L, AIC, BIC, and the K–S test statistics to judge the goodness of fit with comparison to PITL, log-normal (log-N), Weibull (We), generalized exponential (Ge-Ex), and extended inverse exponential (EIE) distributions. The lower the values of these criteria, the better the fit. In Table 14, MLEs, SE, and some measures are obtained. Plots of estimated PDF and CDF are provided in Figure 8. The PP plots of estimated densities are given in Figure 9. It can be seen that the TPITL provides the overall best fit based on Tables 13 and 14. The R codes of data analysis are provided in Appendix B.

Table 13. The MLEs, SEs of all model parameters and goodness of fit measures for first data.

Model	MLE	SE	-2LogL	AIC	BIC	CAIC	K-S	P-value
TPITL	$\hat{\upsilon}=-0.373$	0.610	77.89	82.66	85.09	83.10	0.1069	0.8831
	$\hat{\delta}=1.723$	0.587						
	$\hat{\vartheta}=1.758$	1.003						
PITL	$\hat{\alpha}=1.313$	0.322	78.66	83.32	85.46	83.77	0.1182	0.7961
	$\hat{\vartheta}=1.974$	0.365						
GIE	$\hat{\alpha}=3.300$	1.065	79.32	83.89	86.13	84.81	0.1266	0.7224
	$\hat{\lambda}=2.199$	0.433						
Burr XII	$\hat{\vartheta}=3.216$	0.645	80.55	84.55	87.36	85	0.1287	0.7033
	$\hat{\beta}=0.563$	0.137						
in-Ga	$\hat{\lambda}=2.596$	0.631	80.61	85.61	87.42	85.06	0.1375	0.6218
	$\hat{\beta}=2.966$	0.795						
in-We	$\hat{\alpha}=1.550$	0.198	83.83	87.83	90.64	88.28	0.1523	0.4896
	$\hat{\lambda}=1.025$	0.203						

Table 14. The MLEs, SEs of all model parameters and goodness of fit measures for second data.

Model	MLE	SE	-2LogL	AIC	BIC	CAIC	K-S	P-value
TPITL	$\hat{\upsilon} = -0.794$	0.193	194.002	198.002	202.610	198.171	0.089	0.601
	$\hat{\delta} = 1.640$	0.270						
	$\hat{\vartheta} = 2.010$	0.407						
Ge-Ex	$\hat{\alpha} = 3.565$	0.696	195.184	200.889	205.498	201.026	0.090	0.588
	$\hat{\lambda} = 0.891$	0.103						
We	$\hat{\eta} = 1.820$	0.156	196.889	201.184	208.096	201.558	0.100	0.448
	$\hat{\beta} = 1.989$	0.134						
log-N	$\hat{\mu} = 0.393$	0.073	200.565	204.565	209.174	204.734	0.101	0.437
	$\hat{\sigma} = 0.632$	0.052						
PITL	$\hat{\alpha} = 1.142$	0.189	200.604	204.604	209.213	204.774	0.118	0.254
	$\hat{\vartheta} = 2.154$	0.274						
EIE	$\hat{\alpha} = 2.858$	0.586	221.525	225.525	230.133	225.694	0.145	0.072
	$\hat{\lambda} = 2.082$	0.271						

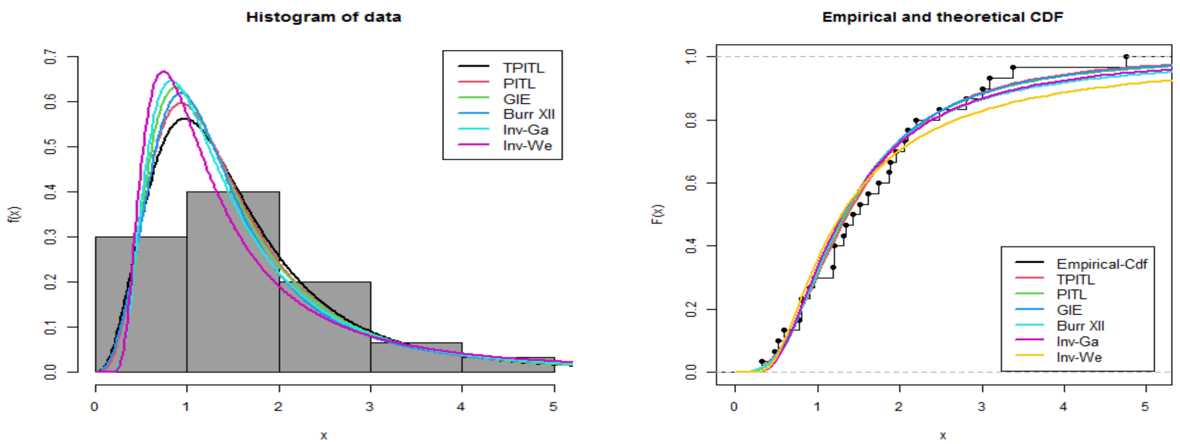

Figure 6. Estimated PDF and CDF of models for the first data.

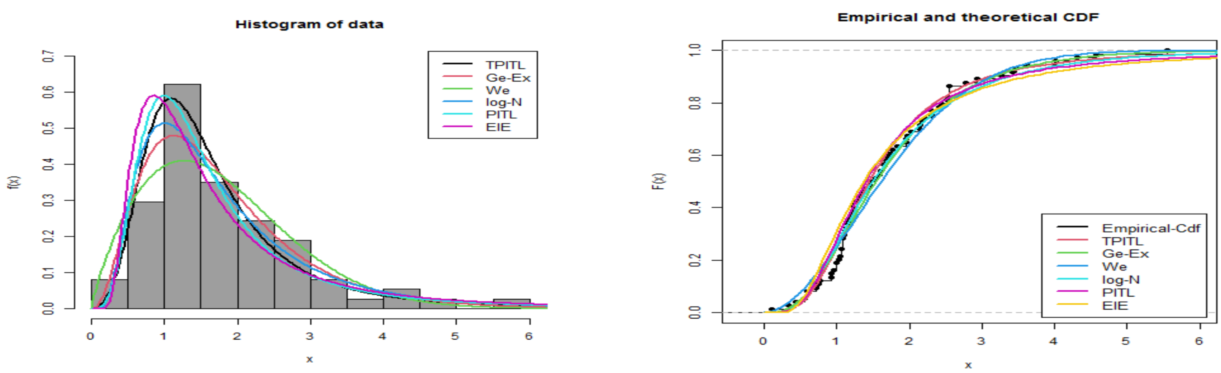

Figure 7. Estimated PDF and CDF of all models for second data.

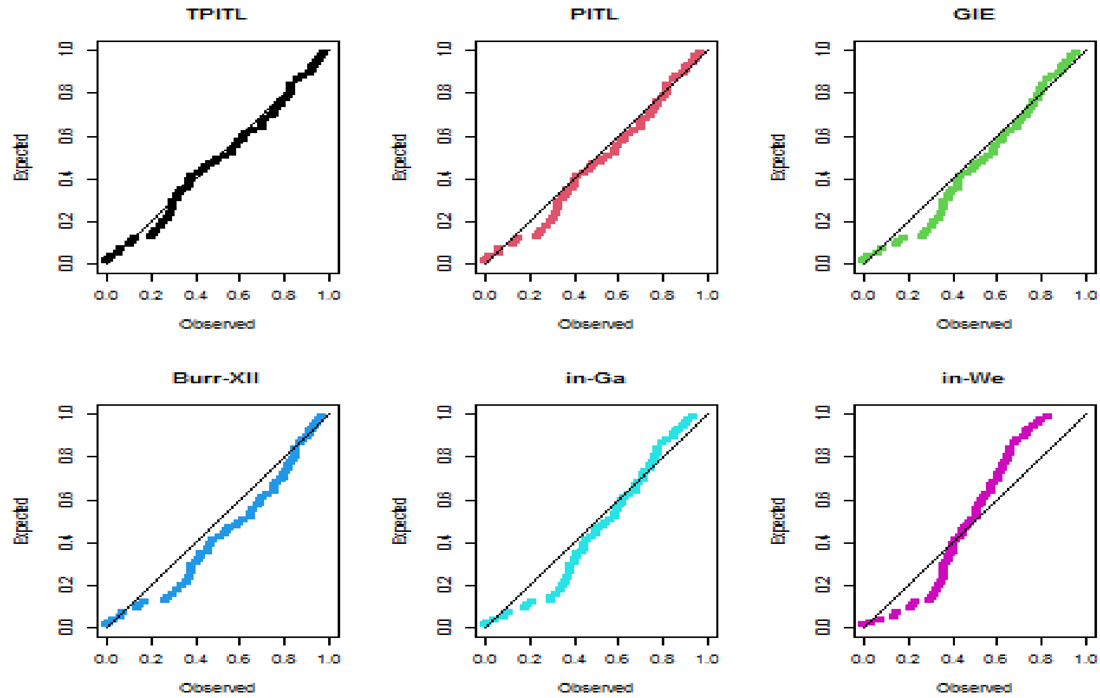

Figure 8. The PP plots of all models for the first data.

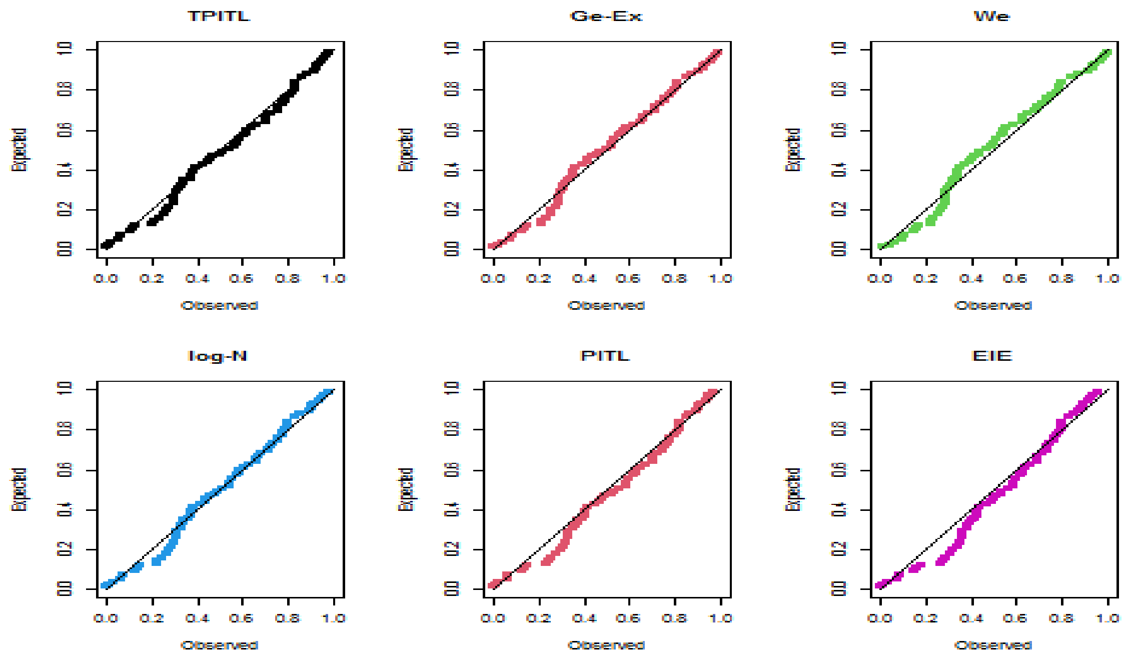

Figure 9. The PP plots of all models for the second data.

Figures 6 and 7 demonstrate that, among the two datasets, the TPITL model provided an excellent fit to the observed distribution (i.e., to the histogram). Also, we conclude that from these figures, the TPITL model offered the best agreement to the empirical CDF for the two datasets. Figures 8 and 9 also show that the TPITL model matches the two real-world datasets more closely than the other competing models. Consequently, the TPITL distribution can be chosen as a suitable model when compared with other distributions for explaining the studied data.

8. Conclusions

The QRTM is used to suggest a novel extension of the power inverted Topp–Leone distribution in this article. The developed distribution is known as the TPITL distribution. The PITL and ITL are included in the proposed distribution as specific models. The significant characteristics of the provided model, including quantile function, moments and incomplete moments, SO, and some uncertainty measures, are also discussed. The ASPs are created for the TPITL model, assuming the life test is terminated at a specific period. The median lifetime of the TPITL distribution with pre-specified variables is used to calculate the truncation time. The smallest sample size is necessary to get the claimed life test at a given consumer risk. Different estimating approaches are used to analyze the TPITL distribution characteristics. To assess the efficacy of estimations based on precision criteria, a comprehensive simulation research is conducted.

Regarding the simulation outcomes, it should be noted that the ML estimation approach often outperforms other methods. In some cases, the next best performing estimator is MPSE, followed by LSE. Finally, we observed that MSE decreases across all estimation techniques as the sample size grows, indicating that all estimation approaches are consistent. Two real data analyses are used to compare the proposed model's validity and adaptability to the alternative models, such as the PITL, log-normal, Weibull, GIE, inverse Weibull, inverse gamma, Ge-Ex, and EIE models. In the future, we will examine the statistical inference of this suggested model using Bayesian estimates under various censored schemes. Other researchers can employ this model in the future to investigate its statistical inference using Bayesian and E-Bayesian estimations under various complete and censored schemes.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: (22UQU4340290DSR02). Additionally, the authors thank the editor and the reviewers for their careful reading of the research article and for their constructive comments that greatly improved this paper. The authors also thank all those who help in creating or revising this work. Thanks, is also extended to Enago® Office for proofreading and editing the paper. The authors extend their appreciation to the Deanship of Scientific Research at Umm Al-Qura University for supporting and funding this work this work by Grant Code: (22UQU4340290DSR02).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix A

```
######## Compute sampling plan and probability of acceptance
accept <- function(pstar, v, delta, vartheta, a, c)
{
######## Compute median life time from quantile function (t_0 )
  m0 <- t_0(0.5, v, delta, vartheta)

######## Compute truncation time at a
  t0 <- a*m0
######## Compute probability (p) from cdf
  p0 <- Fx(t0, v, delta, vartheta)
######## Determine optimal sample size (n)
  for(n0 in 1:10^9)
  {
    s=0
    l_p=0
    for(i in 0:c)
    {
      s    = s+choose(n0,i)*p0^i*(1-p0)^(n0-i)
      l_p = l_p+choose((n0-1),i)*p0^i*(1-p0)^((n0-1)-i)
    }
######## print minimum sample size
    if (s<=1-pstar)
    {
      print(c(c=c, n=n0, OC=l_p))
      return(c(c, n0, l_p))
      break()
    }
  }
}
```

Appendix B

```
######## Define pdf and cdf of PITL distribution
Pdf_PITL <- function(parm,x){
  alpha <- parm[1]
```

```
  theta <- parm[2]
  a <- 2*alpha*theta*x^(2*theta-1)
  b <- (1+x^theta)^(-2*alpha-1)
  c <- (1+2*x^theta)^(alpha-1)
  Pdf_PITL <- a*b*c
}
Cdf_PITL <- function(parm,x){
  alpha <- parm[1]
  theta <- parm[2]
  a <- (1+x^theta)^(2*alpha)
  b <- (1+2*x^theta)^alpha
  Cdf_PITL <- 1-(b/a)
}
######## define pdf and cdf of inv. Weibull (invWe) Distribution
Pdf_inWE <- function(parm,x){
  lambda <- parm[1]
  beta     <- parm[2]
  Pdf_inWE <- lambda*beta*x^(-beta-1)*exp(-lambda/x^beta)
}
Cdf_inWE <- function(parm,x){
  lambda <- parm[1]
  beta     <- parm[2]
  Cdf_inWE <- exp(-lambda/x^beta)
}
######## define pdf and cdf for inverse gamma (invGa) Distribution
Pdf_inGa <- function(parm,x){
  alpha   <- parm[1]
  lambda <- parm[2]
  Pdf_inGa <- dinvgamma(x, alpha, rate=lambda)
}
Cdf_inGa <- function(parm,x){
  alpha   <- parm[1]
  lambda <- parm[2]
  Cdf_inGa <- pinvgamma(x,alpha, rate=lambda)
}
######## define pdf and cdf for Weibull Distribution
Pdf_We <- function(parm,x){
  alpha      <- parm[1]
  lambda     <- parm[2]
  Pdf_We <- (alpha*x^(alpha-1)*exp(-(x/lambda)^alpha))/lambda^alpha
}
Cdf_We <- function(parm,x){
  alpha     <- parm[1]
  lambda   <- parm[2]
```

```
  Cdf_We       <- 1-exp(-(x/lambda)^alpha)
}
######## define pdf and cdf for log-Normal Distribution
Pdf_log <- function(parm,x){
  alpha       <- parm[1]
  lambda      <- parm[2]
  Pdf_log <- dlnorm(x,alpha,lambda)
}
Cdf_log <- function(parm,x){
  alpha       <- parm[1]
  lambda      <- parm[2]
  Cdf_log <- plnorm(x,alpha,lambda)
}
######## define pdf and cdf for Gamma Distribution
Pdf_Ga <- function(parm,x){
  alpha   <- parm[1]
  lambda <- parm[2]
  Pdf_Ga   <- dgamma(x,shape = alpha,scale = lambda)
}
Cdf_Ga <- function(parm,x){
  alpha   <- parm[2]
  lambda <- parm[1]
  Cdf_Ga <- pgamma(x,shape = alpha,scale =lambda)
}
######## define pdf and cdf for Generalized Exponential Distribution
Pdf_Gex <- function(parm,x){
  alpha   <- parm[1]
  lambda <- parm[2]
  Pdf_Gex <- (alpha/lambda)*exp(-x/lambda)*(1-exp(-x/lambda))^(alpha-1)
}
Cdf_Gex <- function(parm,x){
  alpha   <- parm[1]
  lambda <- parm[2]
  Cdf_Gex <- (1-exp(-x/lambda))^alpha
}
######## define pdf and cdf for Generalized inverted Exponential (GIE) Distribution
Pdf_GIE <- function(parm,x){
  alpha   <- parm[1]
  lambda <- parm[2]
  Pdf_GIE <- (alpha*lambda/x^2)*exp(-lambda/x)*(1-exp(-lambda/x))^(alpha-1)
}
Cdf_GIE <- function(parm,x){
  alpha   <- parm[1]
  lambda <- parm[2]
```

```
  Cdf_GIE <- 1-(1-exp(-lambda/x))^alpha
}
######## define pdf and cdf for Extendend inverse exponential (EIE) Distribution
Pdf_EIE <- function(parm,x){
  alpha   <- parm[1]
  theta <- parm[2]
  a <- exp(-theta/x)
  Pdf_EIE <- ((alpha*theta)/x^2)*a*(1/(1-a)^2)*exp(-alpha*(a/(1-a)))
}
Cdf_EIE <- function(parm,x){
  alpha   <- parm[1]
  theta   <- parm[2]
  a <- exp(-theta/x)
  Cdf_EIE <- 1-exp(-alpha*(a/(1-a)))
}
######## Define TPITL distribution
Pdf_TPITL <- function(parm,x){
  v          <- parm[1]
  delta      <- parm[2]
  vartheta <- parm[3]
  a = (1+2*x^delta)^vartheta
  b = (1+x^delta)^(2*vartheta)
  d = (1-(a/b))
  aa = 2*vartheta*delta*x^(2*delta-1)*(1+x^delta)^(-2*vartheta-1)*(1+2*x^delta)^(vartheta-1)
  Pdf_TPITL <- aa*(1+v-2*v*d)
}
Cdf_TPITL <- function(parm,x){
  v          <- parm[1]
  delta      <- parm[2]
  vartheta <- parm[3]
    a = (1+2*x^delta)^vartheta
  b = (1+x^delta)^(2*vartheta)
  d = (1-(a/b))
  Cdf_TPITL <- d*(1+v-v*(d))
}
######## define pdf and cdf of Burr XII dist.
Pdf_Burr <- function(parm,x){
  theta <- parm[1]
  beta   <- parm[2]
  Pdf_Burr <- theta*beta*x^(theta-1)*(1+x^theta)^(-(beta+1))
}
Cdf_Burr <- function(parm,x){
  theta <- parm[1]
  beta   <- parm[2]
```

```
   Cdf_Burr <- 1-(1+x^theta)^(-beta)
}
######## log-likhood
LL_PITL    <- function(alpha,theta){-sum(log(Pdf_PITL(c(alpha,theta),x)))}
LL_inWE    <- function(lambda,beta){-sum(log(Pdf_inWE(c(lambda,beta),x)))}
LL_inGa    <- function(alpha,lambda){-sum(log(Pdf_inGa(c(alpha,lambda),x)))}
LL_GIE     <- function(alpha,lambda){-sum(log(Pdf_GIE(c(alpha,lambda),x)))}
LL_EIE     <- function(alpha,theta){-sum(log(Pdf_EIE(c(alpha,theta),x)))}
LL_TPITL <- function(v,delta,vartheta){-sum(log(Pdf_TPITL(c(v,delta,vartheta),x)))}
LL_Burr    <- function(theta,beta){-sum(log(Pdf_Burr(c(theta,beta),x)))}
LL_We          <- function(alpha,lambda){-sum(log(Pdf_We(c(alpha,lambda),x)))}
LL_Ga          <- function(alpha,lambda){-sum(log(Pdf_Ga(c(alpha,lambda),x)))}
LL_Gex          <- function(alpha,lambda){-sum(log(Pdf_Gex(c(alpha,lambda),x)))}
LL_log          <- function(alpha,lambda){-sum(log(Pdf_log(c(alpha,lambda),x)))}
################################################################################
######## Real data set I: ------------------------------------------------------------
# The data contain 30 observations of the March precipitation (in inches) in Minneapolis/St Paul
######## D. Hinkley, "On quick choice of power transformation," Applied Statistics, vol. 26, no. 1,
X1 = c(0.77, 1.74, 0.81, 1.20, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1,0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81,
1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05)
######## Real data set II: ------------------------------------------------------------
######## The survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli,
# observed and reported by Bjerkedal (1960).
X2 = c(0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09,
1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53,1.59, 1.6, 1.63, 1.63, 1.68,
1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3,2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78,
2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55,2.54, 0.77)
######## Use one data set:
x = X1
######## obtain MLE
fit_TPITL      <- mle2(minuslog=LL_TPITL,start=list(v=1,delta=0.6,vartheta=0.05),
                              data=list(x),method = "Nelder-Mead")
fit_invWe    <- mle2(minuslog=LL_inWE,start=list(lambda=1,beta=1),
                             data= list(x), method = "Nelder-Mead")
fit_invGa      <- mle2(minuslog=LL_inGa,start=list(alpha=1,lambda=3),
                              data=list(x),method = "Nelder-Mead")
fit_GIE          <- mle2(minuslog=LL_GIE,start=list(alpha=1,lambda=1),
                              data=list(x),method = "Nelder-Mead")
fit_EIE          <- mle2(minuslog=LL_EIE,start=list(alpha=1,theta=1),
                              data=list(x),method = "Nelder-Mead")
fit_Burr       <- mle2(minuslog=LL_Burr,start=list(theta=1,beta=1),
                             data=list(x),method = "Nelder-Mead")
fit_We <- mle2(minuslog=LL_We,start=list(alpha=1,lambda=1),
                         data=list(x),method = "Nelder-Mead")
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fit_Ga <- mle2(minuslog=LL_Ga,start=list(alpha=1,lambda=1),
               data=list(x),method = "Nelder-Mead")
fit_Gex <- mle2(minuslog=LL_Gex,start=list(alpha=1,lambda=1),
               data=list(x),method = "Nelder-Mead")
fit_log <- mle2(minuslog=LL_log,start=list(alpha=1,lambda=1),
                data=list(x),method = "Nelder-Mead")
##################################################################
######## goodness of fit
gof.test_TPITL <- goodness.fit(Pdf_TPITL,Cdf_TPITL,starts=c(coef(fit_TPITL)),data=x,
                               method = "N",domain = c(0,1000))
gof.test_invWE   <- goodness.fit(Pdf_inWE,Cdf_inWE,starts=c(coef(fit_invWe)),data=x,
                               method = "N", domain = c(0,10000))
gof.test_invGa   <- goodness.fit(Pdf_inGa,Cdf_inGa,starts=c(coef(fit_invGa)),data=x,
                               lim_inf = c(0.001,0.001),lim_sup = c(10,10))
gof.test_GIE      <- goodness.fit(Pdf_GIE,Cdf_GIE,starts=c(coef(fit_GIE)),data=x,
                               lim_inf = c(0.001,0.001),lim_sup = c(10,20))
gof.test_EIE      <- goodness.fit(Pdf_EIE,Cdf_EIE,starts=c(coef(fit_EIE)),data=x,
                               lim_inf = c(0.00001,0.00001),lim_sup = c(10,10))
gof.test_Burr     <- goodness.fit(Pdf_Burr,Cdf_Burr,starts=c(coef(fit_Burr)),data=x,
                               lim_inf = c(0.001,0.001),lim_sup = c(10,10))
gof.test_We    <- goodness.fit(Pdf_We,Cdf_We,starts=c(coef(fit_We)),data=x,
                            lim_inf = c(0.001,0.001),lim_sup = c(10,10))
gof.test_Ga    <- goodness.fit(Pdf_Ga,Cdf_Ga,starts=c(coef(fit_Ga)),data=x,
                               lim_inf = c(0.001,0.001),lim_sup = c(10,10))
gof.test_Gex    <- goodness.fit(Pdf_Gex,Cdf_Gex,starts=c(coef(fit_Gex)),data=x,
                            lim_inf = c(0.001,0.001),lim_sup = c(10,10))
gof.test_log    <- goodness.fit(Pdf_log,Cdf_log,starts=c(coef(fit_log)),data=x,
                             lim_inf = c(0.001,0.001),lim_sup = c(10,10))
```