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Research article

Stochastic modeling of the *Monkeypox* 2022 epidemic with cross-infection hypothesis in a highly disturbed environment

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Abstract: *Monkeypox* 2022, a new re-emerging disease, is caused by the *Monkeypox* virus. Structurally, this virus is related to the smallpox virus and infects the host in a similar way; however, the symptoms of *Monkeypox* are more severe. In this research work, a mathematical model for understanding the dynamics of *Monkeypox* 2022 is suggested that takes into account two modes of transmission: horizontal human dissemination and cross-infection between animals and humans. Due to lack of substantial knowledge about the virus diffusion and the effect of external perturbations, the model is extended to the probabilistic formulation with Lévy jumps. The proposed model is a two block compartmental system that requires the form of Itô-Lévy stochastic differential equations. Based on some assumptions and nonstandard analytical techniques, two principal asymptotic properties are proved: the eradication and continuation in the mean of *Monkeypox* 2022. The outcomes of the study reveals that the dynamical behavior of the proposed *Monkeypox* 2022 system is chiefly governed by some parameters that are precisely correlated with the noise intensities. To support the obtained theoretical finding, examples based on numerical simulations and real data are presented at the end of the study. The numerical simulations also exhibit the impact of the innovative adopted mathematical techniques on the findings of this work.

Keywords: *Monkeypox* 2022; stochastic analysis; extinction; persistence; numerical simulations

1. Introduction

Monkeypox is a contagious disease caused by an Orthopoxvirus. This zoonotic infection was initially transmitted to humans in the forest areas of central and western Africa by wild rodents; however, human-to-human spread (horizontal transmission) is also possible in this infection. The human-to-human spread is particularly observed within the family or in the context of care [1]. The *Monkeypox* virus can be diffused by immediate contact with lesions on the skin or mucous membranes of a sick person, as well as by droplets (sneezing, saliva, sputters, etc.) [2]. Generally, an individual can get the infection whenever he/she comes into the surrounding environment of a Monkeypox infected patient. It is therefore important to isolate such a patient throughout the duration of his/her illness. In many cases, humans are getting the infection due to their active contact with animals, rodents or monkeys [3].

Infection with the *Monkeypox* virus begins with a fever, often high, accompanied by headaches, body aches and weakness [4]. After about two days, a blistering rash appears, consisting of fluid-filled blisters that progress to dryness and crusting, then scarring and itching. The bubbles are most concentrated on the face, the forehands and the soles of the feet. The mucous membranes of the mouth and genital area are also affected [5], with swollen and painful lymph nodes under the jaw and neck. The disease is more severe in children and immunocompromised people. It can be complicated by secondary infection of skin lesions or damage to the respiratory, digestive, ophthalmic or nervous systems [2].

Historically, the earliest cases of the disease were discovered in the 1970*s* in DR Congo. The next outbreak of *Monkeypox* outside of Africa was detected in the United States, 2003 [6]. On *14 May 2022*, the United Kingdom Health Security Agency (UKHSA) reported two cases of *Monkeypox*. Since then, new cases have been identified in Belgium, France, Italy, Portugal, Spain, Sweden, Austria, Canada and the United States [1]. In total, 80 mild cases were recorded in ten countries, as of May 20, according to the World Health Organization. While transmission from animals to humans is acceptable in Africa, these emerging cases are associated with human-to-human contamination, most often observed in homosexual or bisexual men with skin lesions of the genitals and face [6].

The European Center for Disease Prevention and Control (ECDPC) recently announced that there is a potentially significant risk of human-to-animal spread of *Monkeypox* in Europe [7]. This has already been confirmed during recent years in West Africa, in particular, when a relationship exists between infected humans and susceptible domestic animals [8]. Actually, the hypothesis of cross-infection between human and animals cannot be neglected.

In this research, we propound a mathematical formulation of *Monkeypox* that simulates its strong spread at two levels: human-human transmission and animal-human infection. Our model is composed of two blocks with possible cross-infection between animals and humans. The first block describes the evolution of *Monkeypox* in the human population, which is divided into four classes: sensitive class $\mathbf{C}_{1,\mathfrak{h}}$, infected class $\mathbf{C}_{2,\mathfrak{h}}$, the isolated class $\mathbf{C}_{3,\mathfrak{h}}$ and the recovered class $\mathbf{C}_{4,\mathfrak{h}}$. The last block describes the evolution of the virus between animals, which are divided into two sub-populations: susceptible animals $\mathbf{C}_{1,\mathfrak{m}}$ and infected animals $\mathbf{C}_{2,\mathfrak{m}}$. Movements between these classes are characterized by the

following deterministic system:

$$\begin{cases} \mathrm{d}\mathbf{C}_{1,\mathfrak{h}}(t) = \Big\{\mathcal{A} - \zeta\mathbf{C}_{1,\mathfrak{h}}(t)\mathbf{C}_{2,\mathfrak{m}}(t) - \zeta_{\circ}\mathbf{C}_{1,\mathfrak{h}}(t)\mathbf{C}_{2,\mathfrak{h}}(t) - \mathfrak{u}\mathbf{C}_{1,\mathfrak{h}}(t) + \tau_1\mathbf{C}_{3,\mathfrak{h}}(t)\Big\}\mathrm{d}t, \\ \mathrm{d}\mathbf{C}_{2,\mathfrak{h}}(t) = \Big\{\zeta\mathbf{C}_{1,\mathfrak{h}}(t)\mathbf{C}_{2,\mathfrak{m}}(t) + \zeta_{\circ}\mathbf{C}_{1,\mathfrak{h}}(t)\mathbf{C}_{2,\mathfrak{h}}(t) - (\mathfrak{u} + \mathfrak{u}_0 + \varphi + r_1 + r_2)\,\mathbf{C}_{2,\mathfrak{h}}(t)\Big\}\mathrm{d}t, \\ \mathrm{d}\mathbf{C}_{3,\mathfrak{h}}(t) = \Big\{\varphi\mathbf{C}_{2,\mathfrak{h}}(t) - (\tau_1 + \tau_2 + \alpha_1 + \alpha_2)\,\mathbf{C}_{3,\mathfrak{h}}(t)\Big\}\mathrm{d}t, \\ \mathrm{d}\mathbf{C}_{4,\mathfrak{h}}(t) = \Big\{(r_1 + r_2)\,\mathbf{C}_{2,\mathfrak{h}}(t) + \tau_2\mathbf{C}_{3,\mathfrak{h}}(t) - \mathfrak{u}\mathbf{C}_{4,\mathfrak{h}}(t)\Big\}\mathrm{d}t, \\ - \\ \mathrm{d}\mathbf{C}_{1,\mathfrak{m}}(t) = \Big\{\mathcal{A}_{\star} - \zeta_{\star}\mathbf{C}_{1,\mathfrak{m}}(t)\mathbf{C}_{2,\mathfrak{h}}(t) - \mathfrak{u}_{\star}\mathbf{C}_{1,\mathfrak{m}}(t)\Big\}\mathrm{d}t, \\ \mathrm{d}\mathbf{C}_{2,\mathfrak{m}}(t) = \Big\{\zeta_{\star}\mathbf{C}_{1,\mathfrak{m}}(t)\mathbf{C}_{2,\mathfrak{h}}(t) - \mathfrak{u}_{\star}\mathbf{C}_{2,\mathfrak{m}}(t)\Big\}\mathrm{d}t, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ designates the recruitment rate of $\mathbf{C}_{1,\mathfrak{h}}$. ζ indicates the cross infection transmission rate between $\mathbf{C}_{1,\mathfrak{h}}$ and $\mathbf{C}_{2,\mathfrak{m}}$. $\zeta_{\circ}$ is the horizontal human-human dissemination rate. τ_1 is the transfer rate from $\mathbf{C}_{3,\mathfrak{h}}$ to $\mathbf{C}_{1,\mathfrak{h}}$. $\mathfrak{u}$ and $\mathfrak{u}_0$ are, respectively, the natural and the *Monkeypox*-induced death rates of the human individuals. φ is the quarantine rate of infected human individuals. r_1 is the human recuperation rate. r_2 is the the medical treatment rate. τ_2 is cure rate of isolated individuals. α_1 and α_2 are, respectively, the natural and the infection-induced death rates of the isolated human individuals. For the animal population, $\mathcal{A}_{\star}$ designates the recruitment rate of $\mathbf{C}_{1,\mathfrak{m}}$. $\zeta_{\star}$ is the the cross contamination rate between $\mathbf{C}_{1,\mathfrak{m}}$ and $\mathbf{C}_{2,\mathfrak{h}}$. $\mathfrak{u}_{\star}$ is the animal natural death rates. For ease of reading the remainder of this manuscript, we illustrate the transmission mechanisms of the above-mentioned model by the flow diagram shown in Figure 1.

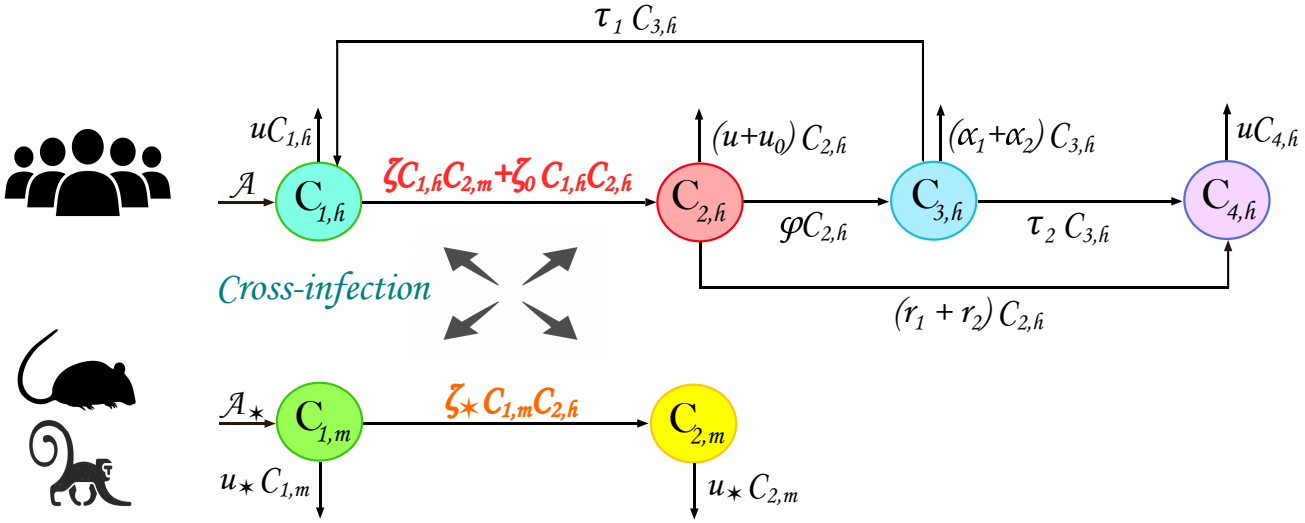

Figure 1. Flowchart of the studied *Monkeypox* 2022 model.

Epidemiologically, the basic reproduction ratio $\mathcal{S}_0$ is the number of secondary cases produced by one infected individual in an entirely susceptible population during its period as an infective. In the literature, several techniques have been proposed for the calculation of $\mathcal{S}_0$ ([9], chapter 5), but the most known is that of the next generation approach introduced by van den Driessche and Watmough in [10]. According to this method, which will be used in our case, $\mathcal{S}_0$ is the spectral radius ρ of the next generation matrix defined by $M = FV^{-1}$ where F and V are, respectively, the matrices expressing the infection transition and the emergence of new infected cases in the different contaminated compartments of the model. On account of the mathematical results presented in [6], the spread behavior of the aforesaid *Monkeypox* model is entirely determined by the deterministic sill $\mathcal{S}_0$, which is expressed in this case by

$$\mathcal{S}_0 = \frac{\zeta \mathcal{A} \zeta_\star \mathcal{A}_\star + \zeta_\circ \mathcal{A} \mathfrak{u}_\star^2}{\mathfrak{u}\left(\mathfrak{u} + \mathfrak{u}_0 + \varphi + r_1 + r_2\right) \mathfrak{u}_\star^2}.$$

In fact, external environmental disturbances affect the spread of the epidemic and make its behavior difficult to predict [11–16]. In such cases, deterministic systems, while capable of making very useful predictions, are not sufficiently adequate [17, 18]. Thus, there is an urgent need for a sophisticated and general mathematical model that takes into account the effect of randomness, especially in the context of studying the prevalence of a highly prevalent infectious disease such as Monkeypox [19,20]. In this context, a large number of authors have proposed and developed several stochastic models that describe the dynamics of epidemics from different angles and projections [21–23]. In all these previously mentioned works, the transition from the deterministic to the stochastic formula is carried out by assuming that the solution of this first naturally fluctuates around its value, which is often expressed by obfuscating some parameters of the system with white noise [24]. The addition of the latter is one of the most logical and remarkable ways of modeling any phenomenon in which a given quantity undergoes slight and continuous fluctuations [25]. For example, in the context of the spread of infectious diseases, environmental uncertainty is often represented in this way. Unfortunately, this approach is not adequate to model the impact of massive and sudden disturbances such as climate changes, economic crises, human interventions, etc. [26]. For this reason, we turn to Lévy processes, which are famous for their ability to properly formulate this type of issue. There are several reasons to enhance the classical stochastic paradigm. Models based on Lévy processes make it possible to deal with situations leading to heavy-tailed distributions. Moreover, they make it possible to exploit the full force of Markovian modeling because the most general Markov processes are solutions of stochastic differential equations driven by Lévy processes. As the Lévy processes admit jumps, they are well suited to describe complex problems in a discontinuous manner. In addition, several fractal patterns associated with white noise and Lévy jumps are observed in the living world. They seem, for example, to be present in domains as varied as biology and the economy. If the Lévy process acted at the individual level, these adaptation mechanisms could then also have effects at higher levels of organization and dynamics, even at the scale of ecosystems [27, 28]. So, we can expand the

model (1.1) to the probabilistic formulation bellow:

$$\begin{cases} \mathrm{d}\mathbf{C}_{1,\mathfrak{b}}(t) &= \Big\{\mathcal{A} - \zeta\mathbf{C}_{1,\mathfrak{b}}(t)\mathbf{C}_{2,\mathfrak{m}}(t) - \zeta_\circ\mathbf{C}_{1,\mathfrak{b}}(t)\mathbf{C}_{2,\mathfrak{b}}(t) - \mathfrak{u}\mathbf{C}_{1,\mathfrak{b}}(t) + \tau_1\mathbf{C}_{3,\mathfrak{b}}(t)\Big\}\mathrm{d}t + \mathrm{d}\Sigma_1(t),\\ \mathrm{d}\mathbf{C}_{2,\mathfrak{b}}(t) &= \Big\{\zeta\mathbf{C}_{1,\mathfrak{b}}(t)\mathbf{C}_{2,\mathfrak{m}}(t) + \zeta_\circ\mathbf{C}_{1,\mathfrak{b}}(t)\mathbf{C}_{2,\mathfrak{b}}(t) - (\mathfrak{u} + \mathfrak{u}_0 + \varphi + r_1 + r_2)\,\mathbf{C}_{2,\mathfrak{b}}(t)\Big\}\mathrm{d}t + \mathrm{d}\Sigma_2(t),\\ \mathrm{d}\mathbf{C}_{3,\mathfrak{b}}(t) &= \Big\{\varphi\mathbf{C}_{2,\mathfrak{b}}(t) - (\tau_1 + \tau_2 + \alpha_1 + \alpha_2)\,\mathbf{C}_{3,\mathfrak{b}}(t)\Big\}\mathrm{d}t + \mathrm{d}\Sigma_3(t),\\ \mathrm{d}\mathbf{C}_{4,\mathfrak{b}}(t) &= \Big\{(r_1 + r_2)\,\mathbf{C}_{2,\mathfrak{b}}(t) + \tau_2\mathbf{C}_{3,\mathfrak{b}}(t) - \mathfrak{u}\mathbf{C}_{4,\mathfrak{b}}(t)\Big\}\mathrm{d}t + \mathrm{d}\Sigma_4(t),\\ --\!-- &\quad ----------\\ \mathrm{d}\mathbf{C}_{1,\mathfrak{m}}(t) &= \Big\{\mathcal{A}_\star - \zeta_\star\mathbf{C}_{1,\mathfrak{m}}(t)\mathbf{C}_{2,\mathfrak{b}}(t) - \mathfrak{u}_\star\mathbf{C}_{1,\mathfrak{m}}(t)\Big\}\mathrm{d}t + \mathrm{d}\Sigma_5(t),\\ \mathrm{d}\mathbf{C}_{2,\mathfrak{m}}(t) &= \Big\{\zeta_\star\mathbf{C}_{1,\mathfrak{m}}(t)\mathbf{C}_{2,\mathfrak{b}}(t) - \mathfrak{u}_\star\mathbf{C}_{2,\mathfrak{m}}(t)\Big\}\mathrm{d}t + \mathrm{d}\Sigma_6(t), \end{cases} \tag{1.2}$$

where

$$\begin{cases} \mathrm{d}\Sigma_1(t) = \Upsilon_1\mathbf{C}_{1,\mathfrak{b}}(t)\mathrm{d}\mathcal{P}_1(t) + \displaystyle\int_{\mathcal{J}} \Xi_1(u)\mathbf{C}_{1,\mathfrak{b}}(t^-)\widetilde{C}[\mathrm{d}t, \mathrm{d}u],\\ \mathrm{d}\Sigma_2(t) = \Upsilon_2\mathbf{C}_{2,\mathfrak{b}}(t)\mathrm{d}\mathcal{P}_2(t) + \displaystyle\int_{\mathcal{J}} \Xi_2(u)\mathbf{C}_{2,\mathfrak{b}}(t^-)\widetilde{C}[\mathrm{d}t, \mathrm{d}u],\\ \mathrm{d}\Sigma_3(t) = \Upsilon_3\mathbf{C}_{3,\mathfrak{b}}(t)\mathrm{d}\mathcal{P}_3(t) + \displaystyle\int_{\mathcal{J}} \Xi_3(u)\mathbf{C}_{3,\mathfrak{b}}(t^-)\widetilde{C}[\mathrm{d}t, \mathrm{d}u],\\ \mathrm{d}\Sigma_4(t) = \Upsilon_4\mathbf{C}_{4,\mathfrak{b}}(t)\mathrm{d}\mathcal{P}_4(t) + \displaystyle\int_{\mathcal{J}} \Xi_4(u)\mathbf{C}_{4,\mathfrak{b}}(t^-)\widetilde{C}[\mathrm{d}t, \mathrm{d}u],\\ \mathrm{d}\Sigma_5(t) = \Upsilon_5\mathbf{C}_{1,\mathfrak{m}}(t)\mathrm{d}\mathcal{P}_5(t) + \displaystyle\int_{\mathcal{J}} \Xi_5(u)\mathbf{C}_{1,\mathfrak{m}}(t^-)\widetilde{C}[\mathrm{d}t, \mathrm{d}u],\\ \mathrm{d}\Sigma_6(t) = \Upsilon_6\mathbf{C}_{2,\mathfrak{m}}(t)\mathrm{d}\mathcal{P}_6(t) + \displaystyle\int_{\mathcal{J}} \Xi_6(u)\mathbf{C}_{2,\mathfrak{m}}(t^-)\widetilde{C}[\mathrm{d}t, \mathrm{d}u]. \end{cases}$$

Here, $\mathcal{P}_\ell$ ($\ell = 1, \cdots, 6$), are mutually independent Wiener processes of intensities $\Upsilon_\ell > 0$ ($\ell = 1, \cdots, 6$), respectively. These processes are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ endowed with a filtration that satisfies the habitual conditions. $\mathbf{C}_{1,\mathfrak{b}}(t^-), \mathbf{C}_{2,\mathfrak{b}}(t^-), \mathbf{C}_{3,\mathfrak{b}}(t^-), \mathbf{C}_{4,\mathfrak{b}}(t^-), \mathbf{C}_{1,\mathfrak{m}}(t^-)$ and $\mathbf{C}_{2,\mathfrak{m}}(t^-)$ denote the left limits of $\mathbf{C}_{1,\mathfrak{b}}(t), \mathbf{C}_{2,\mathfrak{b}}(t), \mathbf{C}_{3,\mathfrak{b}}(t), \mathbf{C}_{4,\mathfrak{b}}(t), \mathbf{C}_{1,\mathfrak{m}}(t)$ and $\mathbf{C}_{2,\mathfrak{m}}(t)$. C is a Poisson counting measure independent of $\mathcal{P}_\ell$ ($\ell = 1, \cdots, 6$) with a finite characteristic measure ϖ defined on a measurable set $\mathcal{J} \subset \mathbb{R}_+$. $\widetilde{C}$ is a compensating martingale, and it is presumed that ϖ is a specific Lévy measure that verifies $\varpi(\mathrm{d}u)\mathrm{d}t = C[\mathrm{d}t, \mathrm{d}u] - \widetilde{C}[\mathrm{d}t, \mathrm{d}u]$. Lastly, $\Xi_\ell : \mathcal{J} \to \mathbb{R}$ are the jumps, magnitude functions, which are supposed to be continuous on $\mathcal{J}$.

Regarding the present paper, our principal intention is to find sufficient criteria for eradication and continuation in the mean of the *Monkeypox* system (1.2). These two asymptotic properties are regarded to be sufficient for having an excellent view of the *Monkeypox* future pandemic situation. Our new approach presented in this research improves some standard methods presented, for example, in [29, 30]. Also, we adopt some new analytic techniques to get a more precise threshold for *Monkeypox* eradication. The analysis presented in this work seems to be very promising for studying other epidemic models, especially those which are perturbed with Lévy jumps.

The remainder of this research is ordered as follows: In Section 2, we start by giving some necessary lemmas and techniques. In Section 3, we treat the dynamical bifurcation by giving the

acute criteria of eradication and continuation of the *Monkeypox*. Section 4 numerically validates the mathematical outcomes and explores the effect of leaps on the long-run attitude of the *Monkeypox* infection model (1.2).

2. Hypothetical framework and required lemmas

We start this section by providing the hypothetical framework of our analysis. During this research, we presume the following.

⋆ $\mathbf{c_1}$: $\forall \ell \in \{1, \cdots, 6\}$, $\Xi_\ell(u) > -1$, and the quantities

$$\begin{cases} \max\limits_{1\leqslant\ell\leqslant 6}\left\{\displaystyle\int_{\mathcal{J}} \Xi_\ell^2(u)\varpi(\mathrm{d}u)\right\}, \\ \max\limits_{1\leqslant\ell\leqslant 6}\left\{\displaystyle\int_{\mathcal{J}} \Big(\Xi_\ell(u) - \ln\left(1+\Xi_\ell(u)\right)\Big)\varpi(\mathrm{d}u)\right\}, \\ \max\limits_{1\leqslant\ell\leqslant 6}\left\{\displaystyle\int_{\mathcal{J}} \Big(\big(1+\Xi_\ell(u)\big)^2 - 1\Big)^2\varpi(\mathrm{d}u)\right\}, \\ \max\limits_{1\leqslant\ell\leqslant 6}\left\{\displaystyle\int_{\mathcal{J}} \Big(\ln\big(1+\Xi_\ell(u)\big)\Big)^2\varpi(\mathrm{d}u)\right\}, \end{cases}$$

are finite.

• $\mathbf{c_2}$: $\exists v > 2$ such that $V_v = \min\left\{\mathfrak{u}, \mathfrak{u}_\star\right\} - 0.5(v-1)U - \dfrac{1}{v}W_v > 0$, where $U = \max\left\{\Upsilon_\ell^2 \mid \ell \in \{1, \cdots, 6\}\right\}$, $\widetilde{\Xi}(u) = \max\left\{\Xi_\ell(u) \mid \ell \in \{1, \cdots, 6\}\right\}$, $\Xi(u) = \min\left\{\Xi_\ell(u) \mid \ell \in \{1, \cdots, 6\}\right\}$, $\widetilde{\mathbf{\Theta}}_v(u) = \Big(1+\widetilde{\Xi}(u)\Big)^v - v \times \widetilde{\Xi}(u) - 1$, $\underset{\sim}{\mathbf{\Theta}}_v(u) = \Big(1+\underset{\sim}{\Xi}(u)\Big)^v - v \times \underset{\sim}{\Xi}(u) - 1$, $\mathbf{\Theta}_v(u) = \max\left\{\widetilde{\mathbf{\Theta}}_v(u), \underset{\sim}{\mathbf{\Theta}}_v(u)\right\}$, $W_v = \displaystyle\int_{\mathcal{J}} \mathbf{\Theta}_v(u)\varpi(\mathrm{d}u)$.

Now, the first question in exploring the dynamics of an epidemic model is whether it admits a unique and positive global solution over time. The following lemma shows these properties and ensures the well-posedness of the proposed probabilistic system (1.2).

Lemma 2.1. *Let* $\mathbf{c_1}$ *holds. Then, the stochastic system (1.2) is biologically and mathematically well-posed in the sense that it has a single solution which is positive and global in time.*

We define the total human population as follows: $T(t) = \mathbf{C}_{1,\mathfrak{h}}(t) + \mathbf{C}_{2,\mathfrak{h}}(t) + \mathbf{C}_{3,\mathfrak{h}}(t) + \mathbf{C}_{4,\mathfrak{h}}(t)$. Then, we explore some long-run characteristics of the boundary equations associated with model (1.2) in the case of $\mathbf{C}_{2,\mathfrak{h}}(t) = 0$ and $\mathbf{C}_{2,\mathfrak{m}}(t) = 0$ (absence of *Monkeypox*). Because of that reason, we use the following two-block auxiliary system with quadratic Lévy noise:

$$\begin{cases} \begin{cases} \mathrm{d}\mathbf{D}(t) = \Big(\mathcal{A} - \mathfrak{u}\mathbf{D}(t)\Big)\mathrm{d}t + \displaystyle\sum_{\ell=1}^{4}\mathrm{d}\Sigma_\ell(t), \\ \mathbf{D}(t) = T(0), \end{cases} \\ \\ \begin{cases} \mathrm{d}\mathbf{D}_\star(t) = \Big(\mathcal{A}_\star - \mathfrak{u}_\star\mathbf{D}_\star(t)\Big)\mathrm{d}t + \Upsilon_5\mathbf{D}_\star(t)\mathrm{d}\mathcal{P}_5(t) + \displaystyle\int_{\mathcal{J}}\Xi_5(u)\mathbf{D}_\star(t^-)\widetilde{C}[\mathrm{d}t, \mathrm{d}u], \\ \mathbf{D}(t) = \mathbf{C}_{1,\mathfrak{m}}(0). \end{cases} \end{cases} \tag{2.1}$$

Lemma 2.2. *Assume that* $\mathbf{c_1}$ *and* $\mathbf{c_2}$ *hold and let* $\big(\mathbf{D}(t), \mathbf{D}_\star(t)\big)$ *be two Markov processes that verify the two-block auxiliary system (2.1). Then, we have the following properties:*

1) $\Theta := 2\mathfrak{u} - U - \displaystyle\int_{\mathcal{J}} \max\left\{\widetilde{\Xi}^2(u), \widetilde{\Xi}^2(u)\right\}\varpi(\mathrm{d}u) > 0.$

2) $\displaystyle\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{D}(s)\,\mathrm{d}s = \frac{\mathcal{A}}{\mathfrak{u}}$ a.s

3) $\displaystyle\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{D}^2(s)\,\mathrm{d}s \leq \frac{2\mathcal{A}^2}{\mathfrak{u}\Theta}$ a.s

4) $\Theta_\star := 2\mathfrak{u}_\star - \Upsilon_5^2 - \displaystyle\int_{\mathcal{J}} \Xi_5^2(u)\varpi(\mathrm{d}u) > 0.$

5) $\displaystyle\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{D}_\star(s)\,\mathrm{d}s = \frac{\mathcal{A}_\star}{\mathfrak{u}_\star}$ a.s

6) $\displaystyle\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{D}_\star^2(s)\,\mathrm{d}s \leq \frac{2\mathcal{A}_\star^2}{\mathfrak{u}_\star\Theta_\star}$ a.s

The proof of this result is almost analogous to that of Lemma 2.1. in [12].

3. Potential scenarios of the *Monkeypox* infection

3.1. First scenario: eradication of the Monkeypox *infection*

In this part, we probe the eradication condition of the *Monkeypox* infection. For simplicity and comfort in reading the next results, we define the following quantities:

$\star$ $z^{\circ,\dagger} := \max\{0, z\} = 0.5\big(z + |z|\big)$, $\forall z \in \mathbb{R}$.

$\star$ $\underline{\mathrm{g}}(u) = \Big\{ \min\big\{\Xi_2(u), \Xi_5(u)\big\} - \ln\Big(1 + \min\big\{\Xi_2(u), \Xi_5(u)\big\}\Big)\Big\} \times \mathbb{1}_{\left\{\min\left\{\Xi_2(u),\Xi_5(u)\right\}>0\right\}}.$

$\star$ $\underline{\mathrm{g}}(u) = \Big\{ \max\big\{\Xi_2(u), \Xi_5(u)\big\} - \ln\Big(1 + \max\big\{\Xi_2(u), \Xi_5(u)\big\}\Big)\Big\} \times \mathbb{1}_{\left\{\max\left\{\Xi_2(u),\Xi_5(u)\right\}\leq 0\right\}}.$

$\star$ $\mathbf{H}_1 := \displaystyle\int_{\mathcal{J}} \Big(\overline{\mathrm{g}}(u) + \underline{\mathrm{g}}(u)\Big)\varpi(\mathrm{d}u).$

$\star$ $\mathbf{H}_2 = \dfrac{0.5(\Upsilon_2\Upsilon_5)^2}{\Upsilon_2^2 + \Upsilon_5^2}.$

$\star$ $\Delta^\star = \max\big\{\mathfrak{u} + \mathfrak{u}_0 + \varphi + r_1 + r_2, \mathfrak{u}_\star\big\} \times \Big\{\mathcal{S}_0^{0.5} - 1\Big\}^{\circ,\dagger}.$

$\star$ $\Delta_\star = \min\big\{\mathfrak{u} + \mathfrak{u}_0 + \varphi + r_1 + r_2, \mathfrak{u}_\star\big\} \times \Big\{1 - \mathcal{S}_0^{0.5}\Big\}^{\circ,\dagger}.$

$\star$ $\mathbf{H}_3 = \Delta^\star - \Delta_\star.$

Definition 3.1. *The population* $\mathcal{X}$ *is said to be exponentially extinct if* $\limsup_{t\to\infty} t^{-1}\ln\mathcal{X}(t) < 0$ *a.s.*

Theorem 3.1. *The solution of the probabilistic system* (1.2) *verifies the following estimate:*

$$\limsup_{t\to\infty}\ t^{-1}\ln\left(\frac{\zeta_\star\mathcal{A}_\star}{\mathfrak{u}_\star^2\left(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2\right)}\mathbf{C}_{2,\mathfrak{h}}(t)+\frac{\mathcal{S}_0^{0.5}}{\mathfrak{u}_\star}\mathbf{C}_{2,\mathfrak{m}}(t)\right)\leqslant\boldsymbol{\varsigma}\ \text{ a.s,}$$

where

$$\boldsymbol{\varsigma}=\mathbf{H}_1-\mathbf{H}_2-\mathbf{H}_3+\frac{\zeta_\circ\mathcal{A}}{\mathfrak{u}}+0.5\mathfrak{u}_\star\mathcal{S}_0^{0.5}\left(\frac{2\mathfrak{u}}{\Theta}-1\right)^{0.5}+0.5\left(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2\right)\mathcal{S}_0^{0.5}\left(\frac{2\mathfrak{u}_\star}{\Theta_\star}-1\right)^{0.5}.$$

Specifically, the eradication of Monkeypox *infection with full probability occurs when* $\boldsymbol{\varsigma}<0$.

Proof. First, we define the following function:

$$\mathcal{W}(\mathbf{C}_{2,\mathfrak{h}},\mathbf{C}_{2,\mathfrak{m}})=\ln\left(\underbrace{\frac{\zeta_\star\mathcal{A}_\star}{\mathfrak{u}_\star^2\left(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2\right)}}_{=\mathcal{Z}_1}\mathbf{C}_{2,\mathfrak{h}}+\underbrace{\frac{\mathcal{S}_0^{0.5}}{\mathfrak{u}_\star}}_{=\mathcal{Z}_2}\mathbf{C}_{2,\mathfrak{m}}\right).$$

By the use of Itô's rule, we obtain

$$\begin{aligned}\mathrm{d}\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{h}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)&=\mathcal{L}_{\text{Itô}}\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{h}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)\mathrm{d}t+\frac{\Big\{\mathcal{Z}_1\Upsilon_2\mathbf{C}_{2,\mathfrak{h}}(t)\mathrm{d}\mathcal{P}_2(t)+\mathcal{Z}_2\Upsilon_5\mathbf{C}_{2,\mathfrak{m}}(t)\mathrm{d}\mathcal{P}_5(t)\Big\}}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\\&\quad+\int_{\mathcal{J}}\ln\left(1+\frac{\mathcal{Z}_1\Xi_2(u)\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\Xi_5(u)\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\right)\widetilde{C}[\mathrm{d}t,\mathrm{d}u],\end{aligned}$$

where

$$\begin{aligned}&\mathcal{L}_{\text{Itô}}\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{h}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)\\&=\frac{\mathcal{Z}_1\Big(\zeta\mathbf{C}_{1,\mathfrak{h}}(t)\mathbf{C}_{2,\mathfrak{m}}(t)+\zeta_\circ\mathbf{C}_{1,\mathfrak{h}}(t)\mathbf{C}_{2,\mathfrak{h}}(t)-\left(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2\right)\mathbf{C}_{2,\mathfrak{h}}(t)\Big)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\\&\quad+\frac{\mathcal{Z}_2\Big(\zeta_\star\mathbf{C}_{1,\mathfrak{m}}(t)\mathbf{C}_{2,\mathfrak{h}}(t)-\mathfrak{u}_\star\mathbf{C}_{2,\mathfrak{m}}(t)\Big)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}-\frac{0.5\mathcal{Z}_1^2\Upsilon_2^2\mathbf{C}_{2,\mathfrak{h}}^2(t)+0.5\mathcal{Z}_2^2\Upsilon_5^2\mathbf{C}_{2,\mathfrak{m}}^2(t)}{\Big(\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)\Big)^2}\\&\quad+\int_{\mathcal{J}}\left\{\ln\left(1+\frac{\mathcal{Z}_1\Xi_2(u)\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\Xi_5(u)\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{h}}(t)}\right)-\frac{\mathcal{Z}_1\Xi_2(u)\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\Xi_5(u)\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\right\}\varpi(\mathrm{d}u).\end{aligned}\tag{3.1}$$

Obviously, we can see that

$$\begin{aligned}\Big(\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{h}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)\Big)^2&=\left(\frac{1}{\Upsilon_2}\mathcal{Z}_1\Upsilon_2\mathbf{C}_{2,\mathfrak{h}}(t)+\frac{1}{\Upsilon_5}\mathcal{Z}_2\Upsilon_5\mathbf{C}_{2,\mathfrak{m}}(t)\right)^2\\&\leqslant\left(\frac{1}{\Upsilon_2^2}+\frac{1}{\Upsilon_5^2}\right)\Big(\mathcal{Z}_1^2\Upsilon_2^2\mathbf{C}_{2,\mathfrak{h}}^2(t)+\mathcal{Z}_2^2\Upsilon_5^2\mathbf{C}_{2,\mathfrak{h}}^2(t)\Big).\end{aligned}$$

Thus,

$$-\frac{1}{\left(\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{b}}(t)\right)^2}\left(\mathcal{Z}_1^2\Upsilon_2^2\mathbf{C}_{2,\mathfrak{b}}^2(t)+\mathcal{Z}_2^2\Upsilon_5^2\mathbf{C}_{2,\mathfrak{m}}^2(t)\right)\leqslant-\frac{(\Upsilon_2\Upsilon_5)^2}{\Upsilon_2^2+\Upsilon_5^2}. \tag{3.2}$$

Furthermore, we can show that

$$\int_{\mathcal{J}}\left\{\ln\left(\frac{\mathcal{Z}_1\Xi_2(u)\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\Xi_5(u)\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}+1\right)-\frac{\mathcal{Z}_1\Xi_2(u)\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\Xi_5(u)\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\right\}\varpi(\mathrm{d}u)\leqslant-\mathbf{H}_1. \tag{3.3}$$

We amalgamate (3.2) and (3.3) with (3.1), and then

$$\begin{aligned}
&\mathcal{L}_{\mathrm{Itô}}\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)\\
&\leqslant\frac{\mathcal{Z}_1\Big(\zeta\mathbf{D}(t)\mathbf{C}_{2,\mathfrak{m}}(t)-(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2)\,\mathbf{C}_{2,\mathfrak{b}}(t)\Big)+\mathcal{Z}_2\Big(\zeta_\star\mathbf{D}_\star(t)\mathbf{C}_{2,\mathfrak{b}}(t)-\mathfrak{u}_\star\mathbf{C}_{2,\mathfrak{m}}(t)\Big)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}+\zeta_\circ\mathbf{D}(t)-\mathbf{H}_1-\mathbf{H}_2\\
&\leqslant\frac{\mathcal{Z}_1\left(\zeta\dfrac{\mathcal{A}}{\mathfrak{u}}\mathbf{C}_{2,\mathfrak{m}}(t)-(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2)\,\mathbf{C}_{2,\mathfrak{b}}(t)\right)+\mathcal{Z}_2\left(\zeta_\star\dfrac{\mathcal{A}_\star}{\mathfrak{u}_\star}\mathbf{C}_{2,\mathfrak{b}}(t)-\mathfrak{u}_\star\mathbf{C}_{2,\mathfrak{m}}(t)\right)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}+\zeta_\circ\mathbf{D}(t)-\mathbf{H}_1-\mathbf{H}_2\\
&\quad+\frac{\mathcal{Z}_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}(t)}+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\left(\mathbf{D}(t)-\frac{\mathcal{A}}{\mathfrak{u}}\right)+\frac{\mathcal{Z}_2\zeta_\star\mathbf{C}_{2,\mathfrak{b}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\left(\mathbf{D}_\star(t)-\frac{\mathcal{A}_\star}{\mathfrak{u}_\star}\right)\\
&\leqslant\frac{\left(\mathcal{Z}_1\zeta\dfrac{\mathcal{A}}{\mathfrak{u}}-\mathcal{Z}_2\mathfrak{u}_\star\right)\mathbf{C}_{2,\mathfrak{m}}(t)+\left(\mathcal{Z}_2\zeta_\star\dfrac{\mathcal{A}_\star}{\mathfrak{u}_\star}-\mathcal{Z}_1\,(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2)\right)\mathbf{C}_{2,\mathfrak{b}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}+\zeta_\circ\mathbf{D}(t)-\mathbf{H}_1-\mathbf{H}_2\\
&\quad+\frac{\mathcal{Z}_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\left\{\mathbf{D}(t)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}+\frac{\mathcal{Z}_2\zeta_\star\mathbf{C}_{2,\mathfrak{b}}}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\left\{\mathbf{D}_\star(t)-\frac{\mathcal{A}_\star}{\mathfrak{u}_\star}\right\}^{\circ,\dagger}.
\end{aligned}\tag{3.4}$$

Then, we have

$$\begin{aligned}
&\mathcal{L}_{\mathrm{Itô}}\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)\\
&\leqslant\frac{\left(\mathcal{S}_0-\mathcal{S}_0^{0.5}\right)\mathbf{C}_{2,\mathfrak{m}}(t)+\Big(\mathcal{S}_0^{0.5}\mathcal{Z}_1\,(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2)-\mathcal{Z}_1\,(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2)\Big)\mathbf{C}_{2,\mathfrak{b}}(t)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\\
&\quad+\zeta_\circ\mathbf{D}(t)-\mathbf{H}_1-\mathbf{H}_2+\frac{\mathcal{Z}_1\zeta}{\mathcal{Z}_2}\left\{\mathbf{D}(t)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}+\frac{\mathcal{Z}_2\zeta_\star}{\mathcal{Z}_1}\left\{\mathbf{D}_\star(t)-\frac{\mathcal{A}_\star}{\mathfrak{u}_\star}\right\}^{\circ,\dagger}\\
&\leqslant\frac{\left(\mathcal{S}_0^{0.5}-1\right)\Big(\mathcal{Z}_1\,(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2)\,\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathfrak{u}_\star\mathbf{C}_{2,\mathfrak{m}}(t)\Big)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(t)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}+\zeta_\circ\mathbf{D}(t)-\mathbf{H}_1-\mathbf{H}_2+\frac{\mathcal{Z}_1\zeta}{\mathcal{Z}_2}\left\{\mathbf{D}(t)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}\\
&\quad+\frac{\mathcal{Z}_2\zeta_\star}{\mathcal{Z}_1}\left\{\mathbf{D}_\star(t)-\frac{\mathcal{A}_\star}{\mathfrak{u}_\star}\right\}^{\circ,\dagger}\\
&\leqslant\mathbf{H}_3-\mathbf{H}_1-\mathbf{H}_2+\zeta_\circ\mathbf{D}(t)+\frac{\mathcal{Z}_1\zeta}{\mathcal{Z}_2}\left\{\mathbf{D}(t)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}+\frac{\mathcal{Z}_2\zeta_\star}{\mathcal{Z}_1}\left\{\mathbf{D}_\star(t)-\frac{\mathcal{A}_\star}{\mathfrak{u}_\star}\right\}^{\circ,\dagger}.
\end{aligned}$$

Consequently, we infer that

$$\begin{aligned}\mathrm{d}\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big) \leqslant & \left(\mathbf{H}_3-\mathbf{H}_1-\mathbf{H}_2+\zeta_{\circ}\mathbf{D}(t)+\frac{\mathcal{Z}_1\zeta}{\mathcal{Z}_2}\left\{\mathbf{D}(t)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}+\frac{\mathcal{Z}_2\zeta_{\star}}{\mathcal{Z}_1}\left\{\mathbf{D}_{\star}(t)-\frac{\mathcal{A}_{\star}}{\mathfrak{u}_{\star}}\right\}^{\circ,\dagger}\right)\mathrm{d}t\\ &+\frac{\Big(\mathcal{Z}_1\Upsilon_2\mathbf{C}_{2,\mathfrak{b}}\mathrm{d}\mathcal{P}_2(t)+\mathcal{Z}_2\Upsilon_5\mathbf{C}_{2,\mathfrak{m}}(t)\mathrm{d}\mathcal{P}_5(t)\Big)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(t)}\\ &+\int_{\mathcal{J}}\ln\Big\{1+\max\{\Xi_2(u),\Xi_5(u)\}\Big\}\widetilde{C}[\mathrm{d}t,\mathrm{d}u].\end{aligned}$$

A simple integration leads to

$$\begin{aligned}&\frac{\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)}{t}-\frac{\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{b}}(0),\mathbf{C}_{2,\mathfrak{m}}(0)\Big)}{t}\\ &\leqslant \mathbf{H}_3-\mathbf{H}_1-\mathbf{H}_2+\frac{\zeta_{\circ}}{t}\int_0^t\mathbf{D}(s)\mathrm{d}s\\ &\quad+\frac{\mathcal{Z}_1\zeta}{\mathcal{Z}_2t}\int_0^t\left\{\mathbf{D}(s)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}\mathrm{d}s+\frac{\mathcal{Z}_2\zeta_{\star}}{\mathcal{Z}_1t}\int_0^t\left\{\mathbf{D}_{\star}(s)-\frac{\mathcal{A}_{\star}}{\mathfrak{u}_{\star}}\right\}^{\circ,\dagger}\mathrm{d}s\\ &\quad+\underbrace{\frac{1}{t}\left(\int_0^t\frac{\mathcal{Z}_1\Upsilon_2\mathbf{C}_{2,\mathfrak{b}}(s)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(s)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(s)}\mathrm{d}\mathcal{P}_2(s)+\int_0^t\frac{\mathcal{Z}_2\Upsilon_5\mathbf{C}_{2,\mathfrak{m}}(s)}{\mathcal{Z}_1\mathbf{C}_{2,\mathfrak{b}}(s)+\mathcal{Z}_2\mathbf{C}_{2,\mathfrak{m}}(s)}\mathrm{d}\mathcal{P}_5(s)\right)}_{:=\mathbf{O}_1(t)}\\ &\quad+\underbrace{\frac{1}{t}\int_0^t\int_{\mathcal{J}}\ln\Big\{1+\max\{\Xi_2(u),\Xi_5(u)\}\Big\}\widetilde{C}[\mathrm{d}s,\mathrm{d}u]}_{:=\mathbf{O}_2(t)}.\end{aligned}\tag{3.5}$$

In line with Hölder's inequality, we acquire that

$$\begin{aligned}\frac{1}{t}\int_0^t\left\{\mathbf{D}(s)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}\mathrm{d}s&=\frac{0.5}{t}\int_0^t\left(\mathbf{D}(s)-\frac{\mathcal{A}}{\mathfrak{u}}\right)\mathrm{d}s+\frac{0.5}{t}\int_0^t\left|\mathbf{D}(s)-\frac{\mathcal{A}}{\mathfrak{u}}\right|\mathrm{d}s\\ &\leqslant\frac{0.5}{t}\int_0^t\left(\mathbf{D}(s)-\frac{\mathcal{A}}{\mathfrak{u}}\right)\mathrm{d}s+\frac{0.5}{t^{0.5}}\left(\int_0^t\left(\mathbf{D}(s)-\frac{\mathcal{A}}{\mathfrak{u}}\right)^2\mathrm{d}s\right)^{0.5}\\ &\leqslant 0.5\left(\frac{1}{t}\int_0^t\mathbf{D}(s)\,\mathrm{d}s-\frac{\mathcal{A}}{\mathfrak{u}}\right)+0.5\left(\frac{1}{t}\int_0^t\left(\mathbf{D}^2(s)-\frac{2\mathcal{A}}{\mathfrak{u}}\mathbf{D}(s)+\frac{\mathcal{A}^2}{\mathfrak{u}^2}\right)\mathrm{d}s\right)^{0.5}.\end{aligned}$$

As a result, we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left\{\mathbf{D}(s)-\frac{\mathcal{A}}{\mathfrak{u}}\right\}^{\circ,\dagger}\mathrm{d}s\leqslant 0.5\left(\frac{2\mathcal{A}^2}{\mathfrak{u}\Theta}-2\frac{\mathcal{A}^2}{\mathfrak{u}^2}+\frac{\mathcal{A}^2}{\mathfrak{u}^2}\right)^{0.5}=\frac{\mathcal{A}}{2\mathfrak{u}}\left(-1+\frac{2\mathfrak{u}}{\Theta}\right)^{0.5}\ \text{a.s.}\tag{3.6}$$

Using the same analytical treatment, we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left\{\mathbf{D}_{\star}(s)-\frac{\mathcal{A}_{\star}}{\mathfrak{u}_{\star}}\right\}^{\circ,\dagger}\mathrm{d}s\leqslant 0.5\left(\frac{2\mathcal{A}_{\star}^2}{\mathfrak{u}_{\star}\Theta_{\star}}-2\frac{\mathcal{A}_{\star}^2}{\mathfrak{u}_{\star}^2}+\frac{\mathcal{A}_{\star}^2}{\mathfrak{u}_{\star}^2}\right)^{0.5}=\frac{\mathcal{A}_{\star}}{2\mathfrak{u}_{\star}}\left(-1+\frac{2\mathfrak{u}_{\star}}{\Theta_{\star}}\right)^{0.5}\ \text{a.s.}\tag{3.7}$$

Now, by employing the well-known strong law of large numbers (for local martingales) and condition $\mathbf{c_1}$, we have

$$\lim_{t\to\infty} t^{-1}\mathbf{O}_1(t) = 0 \text{ and } \lim_{t\to\infty} t^{-1}\mathbf{O}_2(t) = 0 \quad \text{a.s.} \tag{3.8}$$

Eventually, we conclude that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\mathcal{W}\Big(\mathbf{C}_{2,\mathfrak{h}}(t), \mathbf{C}_{2,\mathfrak{m}}(t)\Big)}{t} \\
&\leqslant \mathbf{H}_3 - \mathbf{H}_1 - \mathbf{H}_2 + \frac{\zeta_\circ \mathcal{A}}{\mathfrak{u}} + \frac{\mathcal{Z}_1\zeta}{\mathcal{Z}_2}\frac{\mathcal{A}}{2\mathfrak{u}}\left(\frac{2\mathfrak{u}}{\Theta} - 1\right)^{0.5} + \frac{\mathcal{Z}_2\zeta_\star}{\mathcal{Z}_1}\frac{\mathcal{A}_\star}{2\mathfrak{u}_\star}\left(\frac{2\mathfrak{u}_\star}{\Theta_\star} - 1\right)^{0.5} \\
&= \mathbf{H}_3 - \mathbf{H}_1 - \mathbf{H}_2 + \frac{\zeta_\circ \mathcal{A}}{\mathfrak{u}} + 0.5\mathfrak{u}_\star \mathcal{S}_0^{0.5}\left(\frac{2\mathfrak{u}}{\Theta} - 1\right)^{0.5} + 0.5\left(\mathfrak{u} + \mathfrak{u}_0 + \varphi + r_1 + r_2\right)\mathcal{S}_0^{0.5}\left(\frac{2\mathfrak{u}_\star}{\Theta_\star} - 1\right)^{0.5} \\
&:= \boldsymbol{\varsigma}.
\end{aligned}$$

In accordance with the definition of the probabilistic eradication of the infection [12], we can say that when $\boldsymbol{\varsigma}$ is strictly negative, the *Monkeypox* epidemic will disappear in the population. □

Remark 3.1. *Needless to say, the preceding theorem implies the stochastic extinction of infected individuals, which implies in turn (by the positivity of the solution) that* $\lim_{t\to\infty} \mathbf{C}_{2,\mathfrak{h}}(t) = 0$ *and* $\lim_{t\to\infty} \mathbf{C}_{2,\mathfrak{m}}(t) = 0$ *a.s. Compared to the deterministic framework, we notice that the extinction criterion is mainly determined by the parameters closely related to the intensities of the disturbances. From the structure of the constant* $\boldsymbol{\varsigma}$*, we show that the stochastic extinction can occur although the deterministic solution is extinguishing (i.e.,* $\mathcal{S}_0 < 1$*). Therefore, we say that the Lévy noise has negative effects on the prevalence of* Monkeypox*. This means that jumps can change the asymptotic behavior of the* Monkeypox *model significantly.*

3.2. Second scenario: Continuation of the Monkeypox *infection*

In this part, we seek to explore the sufficient criteria for the continuation of the *Monkeypox* epidemic. For this target, we index the following notations:

$\star$ $I_1 = \mathfrak{u} + \dfrac{\Upsilon_1^2}{2} + \displaystyle\int_{\mathcal{J}} \Big(\Xi_1(u) - \ln\big(1 + \Xi_1(u)\big)\Big)\varpi(\mathrm{d}u)$.

$\star$ $I_2 = (\mathfrak{u} + \mathfrak{u}_0 + \varphi + r_1 + r_2) + \dfrac{\Upsilon_2^2}{2} + \displaystyle\int_{\mathcal{J}} \Big(\Xi_2(u) - \ln\big(1 + \Xi_2(u)\big)\Big)\varpi(\mathrm{d}u)$.

$\star$ $I_3 = \mathfrak{u}_\star + \dfrac{\Upsilon_5^2}{2} + \displaystyle\int_{\mathcal{J}} \Big(\Xi_5(u) - \ln\big(1 + \Xi_5(u)\big)\Big)\varpi(\mathrm{d}u)$.

$\star$ $I_4 = \mathfrak{u}_\star + \dfrac{\Upsilon_6^2}{2} + \displaystyle\int_{\mathcal{J}} \Big(\Xi_6(u) - \ln\big(1 + \Xi_6(u)\big)\Big)\varpi(\mathrm{d}u)$.

$\star$ $\widetilde{\mathcal{S}}_0 := \dfrac{\zeta\zeta_\star\mathcal{A}\mathcal{A}_\star}{I_1 I_2 I_3 I_4}$.

Definition 3.2. *The population $\mathcal{X}$ is said to be strongly persistent in the mean if* $\liminf_{t\to\infty} t^{-1}\int_0^t \mathcal{X}(s)\mathrm{d}s > 0$ *a.s.*

Theorem 3.2. *The sufficient condition of the* Monkeypox *persistence is* $\widetilde{\mathcal{S}}_0 > 1$.

Proof. We define the following function:

$$\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}},\mathbf{C}_{2,\mathfrak{b}},\mathbf{C}_{1,\mathfrak{m}},\mathbf{C}_{2,\mathfrak{m}}\Big) = -\mathfrak{Z}_1\ln\Big[\mathbf{C}_{1,\mathfrak{b}}\Big] - \mathfrak{Z}_2\ln\Big[\mathbf{C}_{2,\mathfrak{b}}\Big] - \mathfrak{Z}_3\ln\Big[\mathbf{C}_{1,\mathfrak{m}}\Big] - \mathfrak{Z}_4\ln\Big[\mathbf{C}_{2,\mathfrak{m}}\Big],$$

where $\mathfrak{Z}_1,\mathfrak{Z}_3,\mathfrak{Z}_4 > 0$ are three quantities to be determined later, and $\mathfrak{Z}_2 = 1$. By employing Itô's rule, we get

$$\begin{aligned}\mathrm{d}\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big) &= \mathcal{L}_{\mathrm{Itô}}\mathcal{W}_{\star\star}\big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\big)\,\mathrm{d}t\\ &\quad - \mathfrak{Z}_1\Upsilon_1\mathrm{d}\mathcal{P}_1(t) - \Upsilon_2\mathrm{d}\mathcal{P}_2(t) - \mathfrak{Z}_3\Upsilon_5\mathrm{d}\mathcal{P}_5(t) - \mathfrak{Z}_4\Upsilon_6\mathrm{d}\mathcal{P}_6(t)\\ &\quad - \mathfrak{Z}_1\int_{\mathcal{J}}\ln\big(1+\Xi_1(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u] - \int_{\mathcal{J}}\ln\big(1+\Xi_2(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u]\\ &\quad - \mathfrak{Z}_3\int_{\mathcal{J}}\ln\big(1+\Xi_5(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u] - \mathfrak{Z}_4\int_{\mathcal{J}}\ln\big(1+\Xi_6(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u],\end{aligned}$$

where $\mathcal{L}_{\mathrm{Itô}}\mathcal{W}_{\star\star}\big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\big)$ is expressed as follows:

$$\begin{aligned}&\mathcal{L}_{\mathrm{Itô}}\mathcal{W}_{\star\star}\big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\big)\\ &= -\mathfrak{Z}_1\frac{\mathcal{A}}{\mathbf{C}_{1,\mathfrak{b}}(t)} + \mathfrak{Z}_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t) + \mathfrak{Z}_1\zeta_\circ\mathbf{C}_{2,\mathfrak{b}}(t) + \mathfrak{Z}_1\mathfrak{u} - \frac{\tau_1\mathbf{C}_{3,\mathfrak{b}}(t)}{\mathbf{C}_{1,\mathfrak{b}}(t)} + \mathfrak{Z}_1\frac{\Upsilon_1^2}{2} + \mathfrak{Z}_1\int_{\mathcal{J}}\Big(\Xi_1(u) - \ln\big(1+\Xi_1(u)\big)\Big)\varpi(\mathrm{d}u)\\ &\quad - \frac{\zeta\mathbf{C}_{1,\mathfrak{b}}(t)\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathbf{C}_{2,\mathfrak{b}}} - \zeta_\circ\mathbf{C}_{1,\mathfrak{b}}(t) + (\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2) + \frac{\Upsilon_2^2}{2} + \int_{\mathcal{J}}\Big(\Xi_2(u) - \ln\big(1+\Xi_2(u)\big)\Big)\varpi(\mathrm{d}u)\\ &\quad - \mathfrak{Z}_3\frac{\mathcal{A}_\star}{\mathbf{C}_{1,\mathfrak{m}}(t)} + \mathfrak{Z}_3\zeta_\star\mathbf{C}_{2,\mathfrak{b}} + \mathfrak{Z}_3\mathfrak{u}_\star + \mathfrak{Z}_3\frac{\Upsilon_5^2}{2} + \mathfrak{Z}_3\int_{\mathcal{J}}\Big(\Xi_5(u) - \ln\big(1+\Xi_5(u)\big)\Big)\varpi(\mathrm{d}u)\\ &\quad - \frac{\mathfrak{Z}_4\zeta_\star\mathbf{C}_{1,\mathfrak{m}}(t)\mathbf{C}_{2,\mathfrak{b}}(t)}{\mathbf{C}_{2,\mathfrak{m}}(t)} + \mathfrak{Z}_4\mathfrak{u}_\star + \mathfrak{Z}_4\frac{\Upsilon_6^2}{2} + \mathfrak{Z}_4\int_{\mathcal{J}}\Big(\Xi_6(u) - \ln\big(1+\Xi_6(u)\big)\Big)\varpi(\mathrm{d}u).\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\mathcal{L}_{\mathrm{Itô}}\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)\\ &\leqslant -\mathfrak{Z}_1\frac{\mathcal{A}}{\mathbf{C}_{1,\mathfrak{b}}(t)} - \frac{\zeta\mathbf{C}_{1,\mathfrak{b}}(t)\mathbf{C}_{2,\mathfrak{m}}(t)}{\mathbf{C}_{2,\mathfrak{b}}} - \mathfrak{Z}_3\frac{\mathcal{A}_\star}{\mathbf{C}_{1,\mathfrak{m}}(t)} - \frac{\mathfrak{Z}_4\zeta_\star\mathbf{C}_{1,\mathfrak{m}}(t)\mathbf{C}_{2,\mathfrak{b}}(t)}{\mathbf{C}_{2,\mathfrak{m}}(t)} + \mathfrak{Z}_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t)\\ &\quad + \big(\mathfrak{Z}_1\zeta_\circ + \mathfrak{Z}_3\zeta_\star\big)\mathbf{C}_{2,\mathfrak{b}}(t) + \mathfrak{Z}_1\left\{\mathfrak{u} + \frac{\Upsilon_1^2}{2} + \int_{\mathcal{J}}\Big(\Xi_1(u) - \ln\big(1+\Xi_1(u)\big)\Big)\varpi(\mathrm{d}u)\right\}\\ &\quad + \left\{(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2) + \frac{\Upsilon_2^2}{2} + \int_{\mathcal{J}}\Big(\Xi_2(u) - \ln\big(1+\Xi_2(u)\big)\Big)\varpi(\mathrm{d}u)\right\}\\ &\quad + \mathfrak{Z}_3\left(\mathfrak{u}_\star + \frac{\Upsilon_5^2}{2} + \int_{\mathcal{J}}\Big(\Xi_5(u) - \ln\big(1+\Xi_5(u)\big)\Big)\varpi(\mathrm{d}u)\right)\end{aligned}$$

$$+\beth_4\left\{\mathfrak{u}_\star+\frac{\Upsilon_6^2}{2}+\int_{\mathcal{J}}\Big(\Xi_6(u)-\ln\big(1+\Xi_6(u)\big)\Big)\varpi(\mathrm{d}u)\right\}.$$

Hence, we derive that

$$\begin{aligned}
&\mathcal{L}_{\text{Itô}}\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)\\
&\leqslant -4\sqrt[4]{\beth_1\beth_3\beth_4\zeta\zeta_\star\mathcal{A}\mathcal{A}_\star}+\beth_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t)+\big(\beth_1\zeta_\circ+\beth_3\zeta_\star\big)\mathbf{C}_{2,\mathfrak{b}}(t)\\
&\quad+\beth_1\underbrace{\left\{\mathfrak{u}+\frac{\Upsilon_1^2}{2}+\int_{\mathcal{J}}\Big(\Xi_1(u)-\ln\big(1+\Xi_1(u)\big)\Big)\varpi(\mathrm{d}u)\right\}}_{=I_1}\\
&\quad+\underbrace{\left\{(\mathfrak{u}+\mathfrak{u}_0+\varphi+r_1+r_2)+\frac{\Upsilon_2^2}{2}+\int_{\mathcal{J}}\Big(\Xi_2(u)-\ln\big(1+\Xi_2(u)\big)\Big)\varpi(\mathrm{d}u)\right\}}_{=I_2}\\
&\quad+\beth_3\underbrace{\left\{\mathfrak{u}_\star+\frac{\Upsilon_5^2}{2}+\int_{\mathcal{J}}\Big(\Xi_5(u)-\ln\big(1+\Xi_5(u)\big)\Big)\varpi(\mathrm{d}u)\right\}}_{=I_3}\\
&\quad+\beth_4\underbrace{\left\{\mathfrak{u}_\star+\frac{\Upsilon_6^2}{2}+\int_{\mathcal{J}}\Big(\Xi_6(u)-\ln\big(1+\Xi_6(u)\big)\Big)\varpi(\mathrm{d}u)\right\}}_{=I_4}.
\end{aligned}$$

Now, we select $\beth_1=\dfrac{\zeta\zeta_\star\mathcal{A}\mathcal{A}_\star}{I_1^2I_3I_4}$, $\beth_3=\dfrac{\zeta\zeta_\star\mathcal{A}\mathcal{A}_\star}{I_1I_3^2I_4}$ and $\beth_4=\dfrac{\zeta\zeta_\star\mathcal{A}\mathcal{A}_\star}{I_1I_3I_4^2}$, and then

$$\begin{aligned}
\mathcal{L}_{\text{Itô}}\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)&\leqslant\beth_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t)+\beth_3\zeta_\star\mathbf{C}_{2,\mathfrak{b}}(t)+I_2-\frac{\zeta\zeta_\star\mathcal{A}\mathcal{A}_\star}{I_1I_3I_4}\\
&=\Big(\beth_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t)+\big(\beth_1\zeta_\circ+\beth_3\zeta_\star\big)\mathbf{C}_{2,\mathfrak{b}}(t)\Big)-I_2\Big\{\underbrace{\big(\frac{\zeta\zeta_\star\mathcal{A}\mathcal{A}_\star}{I_1I_2I_3I_4}}_{\widetilde{\mathcal{S}}_0}-1\Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{d}\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)&\leqslant\Big\{\Big(\beth_1\zeta\mathbf{C}_{2,\mathfrak{m}}(t)+\big(\beth_1\zeta_\circ+\beth_3\zeta_\star\big)\mathbf{C}_{2,\mathfrak{b}}(t)\Big)-I_2\Big(\widetilde{\mathcal{S}}_0-1\Big)\Big\}\mathrm{d}t\\
&\quad-\beth_1\Upsilon_1\mathrm{d}\mathcal{P}_1(t)-\Upsilon_2\mathrm{d}\mathcal{P}_2(t)-\beth_3\Upsilon_5\mathrm{d}\mathcal{P}_5(t)-\beth_4\Upsilon_6\mathrm{d}\mathcal{P}_6(t)\\
&\quad-\beth_1\int_{\mathcal{J}}\ln\big(1+\Xi_1(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u]-\int_{\mathcal{J}}\ln\big(1+\Xi_2(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u]\\
&\quad-\beth_3\int_{\mathcal{J}}\ln\big(1+\Xi_5(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u]-\beth_4\int_{\mathcal{J}}\ln\big(1+\Xi_6(u)\big)\widetilde{C}[\mathrm{d}t,\mathrm{d}u].
\end{aligned}\tag{3.9}$$

An integration from 0 to t gives

$$\frac{\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\Big)}{t}-\frac{\mathcal{W}_{\star\star}\Big(\mathbf{C}_{1,\mathfrak{b}}(0),\mathbf{C}_{2,\mathfrak{b}}(0),\mathbf{C}_{1,\mathfrak{m}}(0),\mathbf{C}_{2,\mathfrak{m}}(0)\Big)}{t}$$

$$
\begin{aligned}
&\leqslant \frac{\mathfrak{Z}_1\zeta}{t}\int_0^t \mathbf{C}_{2,\mathfrak{m}}(s)\,\mathrm{d}s + \frac{\left(\mathfrak{Z}_1\zeta_\circ + \mathfrak{Z}_3\zeta_\star\right)}{t}\int_0^t \mathbf{C}_{2,\mathfrak{b}}(s)\,\mathrm{d}s - I_2\left(\widetilde{\mathcal{S}}_0 - 1\right)\\
&\quad -\mathfrak{Z}_1\Upsilon_1\frac{\mathcal{P}_1(t)}{t} - \Upsilon_2\frac{\mathcal{P}_2(t)}{t} - \mathfrak{Z}_3\Upsilon_5\frac{\mathcal{P}_5(t)}{t} - \mathfrak{Z}_4\Upsilon_6\frac{\mathcal{P}_6(t)}{t}\\
&\quad - \frac{\mathfrak{Z}_1}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_1(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u] - \int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_2(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u]\\
&\quad - \frac{\mathfrak{Z}_3}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_5(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u] - \frac{\mathfrak{Z}_4}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_6(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u].
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{\mathcal{W}_{\star\star}\left(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\right)}{t} - \frac{\mathcal{W}_{\star\star}\left(\mathbf{C}_{1,\mathfrak{b}}(0),\mathbf{C}_{2,\mathfrak{b}}(0),\mathbf{C}_{1,\mathfrak{m}}(0),\mathbf{C}_{2,\mathfrak{m}}(0)\right)}{t}\\
&\leqslant \frac{\max\left\{\mathfrak{Z}_1\zeta, \mathfrak{Z}_1\zeta_\circ + \mathfrak{Z}_3\zeta_\star\right\}}{t}\int_0^t \left(\mathbf{C}_{2,\mathfrak{m}}(s) + \mathbf{C}_{2,\mathfrak{b}}(s)\right)\mathrm{d}s - I_2\left(\widetilde{\mathcal{S}}_0 - 1\right)\\
&\leqslant -\mathfrak{Z}_1\Upsilon_1\frac{\mathcal{P}_1(t)}{t} - \Upsilon_2\frac{\mathcal{P}_2(t)}{t} - \mathfrak{Z}_3\Upsilon_5\frac{\mathcal{P}_5(t)}{t} - \mathfrak{Z}_4\Upsilon_6\frac{\mathcal{P}_6(t)}{t}\\
&\quad - \frac{\mathfrak{Z}_1}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_1(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u] - \int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_2(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u]\\
&\quad - \frac{\mathfrak{Z}_3}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_5(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u] - \frac{\mathfrak{Z}_4}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_6(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{\max\left\{\mathfrak{Z}_1\zeta, \mathfrak{Z}_1\zeta_\circ + \mathfrak{Z}_3\zeta_\star\right\}}{t}\int_0^t \left(\mathbf{C}_{2,\mathfrak{m}}(s) + \mathbf{C}_{2,\mathfrak{b}}(s)\right)\mathrm{d}s\\
&\geqslant I_2\left(\widetilde{\mathcal{S}}_0 - 1\right) + \mathfrak{Z}_1\Upsilon_1\frac{\mathcal{P}_1(t)}{t} + \Upsilon_2\frac{\mathcal{P}_2(t)}{t} + \mathfrak{Z}_3\Upsilon_5\frac{\mathcal{P}_5(t)}{t} + \mathfrak{Z}_4\Upsilon_6\frac{\mathcal{P}_6(t)}{t}\\
&\quad + \frac{\mathfrak{Z}_1}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_1(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u] + \int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_2(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u])\\
&\quad + \frac{\mathfrak{Z}_3}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_5(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u] + \frac{\mathfrak{Z}_4}{t}\int_0^t\int_{\mathcal{J}} \ln\left(1+\Xi_6(u)\right)\widetilde{C}[\mathrm{d}s,\mathrm{d}u]\\
&\quad - \frac{\mathcal{W}_{\star\star}\left(\mathbf{C}_{1,\mathfrak{b}}(t),\mathbf{C}_{2,\mathfrak{b}}(t),\mathbf{C}_{1,\mathfrak{m}}(t),\mathbf{C}_{2,\mathfrak{m}}(t)\right)}{t} + \frac{\mathcal{W}_{\star\star}\left(\mathbf{C}_{1,\mathfrak{b}}(0),\mathbf{C}_{2,\mathfrak{b}}(0),\mathbf{C}_{1,\mathfrak{m}}(0),\mathbf{C}_{2,\mathfrak{m}}(0)\right)}{t}.
\end{aligned}
\tag{3.10}
$$

By taking the inferior limit, we infer that

$$
\liminf_{t\to\infty}\frac{1}{t}\int_0^t \left(\mathbf{C}_{2,\mathfrak{m}}(s) + \mathbf{C}_{2,\mathfrak{b}}(s)\right)\mathrm{d}s \geqslant \frac{I_2\left(\widetilde{\mathcal{S}}_0 - 1\right)}{\max\left\{\mathfrak{Z}_1\zeta, \mathfrak{Z}_1\zeta_\circ + \mathfrak{Z}_3\zeta_\star\right\}} > 0 \ \text{ a.s.},
$$

that is, the continuation of the *Monkeypox* infection occurs when $\widetilde{\mathcal{S}}_0 > 1$. □

Remark 3.2. *Persistence in the mean is an important concept in mathematical epidemiology. It captures the long-term survival of* Monkeypox *even when the population size is quite low at* $t = 0$*. Moreover, the persistence of the model refers to a situation where* Monkeypox *is endemic in a population.*

Table 1. Parameter values of the *Monkeypox* model (1.2).

Parameter	Test 1	Test 2	Test 3
$\mathcal{A}$	10	10	10
ζ	0.0001	0.0009	0.001
$\zeta_\circ$	0.0002	0.0002	0.0002
$\mathfrak{u}$	0.05	0.05	0.05
τ_1	0.32	0.32	0.32
τ_2	0.2	0.2	0.2
$\mathfrak{u}_0$	0.0003	0.0003	0.0003
φ	0.5	0.5	0.5
r_1	0.041	0.041	0.041
r_2	0.043	0.043	0.043
α_1	0.004	0.004	0.004
α_2	0.002	0.002	0.002
$\mathcal{A}_\star$	10	10	10
$\zeta_\star$	0.00027	0.00027	0.00027
$\mathfrak{u}_\star$	0.02	0.02	0.02
Υ_1	0.2	0.2	0.2
Υ_2	0.4	0.4	0.4
Υ_3	0.1	0.1	0.1
Υ_4	0.2	0.2	0.2
Υ_5	0.1	0.1	0.1
Υ_6	0.2	0.2	0.2
$\Xi_1(u)$	0.3	0.3	0.3
$\Xi_2(u)$	0.3	0.3	0.3
$\Xi_3(u)$	0.3	0.3	0.3
$\Xi_4(u)$	0.3	0.3	0.3
$\Xi_5(u)$	0.3	0.3	0.3
$\Xi_6(u)$	0.3	0.3	0.3

4. Numerical verification

In order to check and support the diverse analytical outcomes presented in this research, we implement some numerical examples by taking the deterministic and probabilistic parameter values from the simulated data presented in Table 1. All trajectories of the analyzed *Monkeypox* model are numerically simulated in our case with the following initial information: $\mathbf{C}_{1,\mathfrak{b}}(0) = 120, \mathbf{C}_{2,\mathfrak{b}}(0) = 70, \mathbf{C}_{3,\mathfrak{b}}(0) = 20, \mathbf{C}_{4,\mathfrak{b}}(0) = 10, \mathbf{C}_{1,\mathfrak{m}}(0) = 130$ and $\mathbf{C}_{2,\mathfrak{m}}(0) = 50$. In the following, the unit of time is 1 day, and the number of human and animal individuals is 1 million.

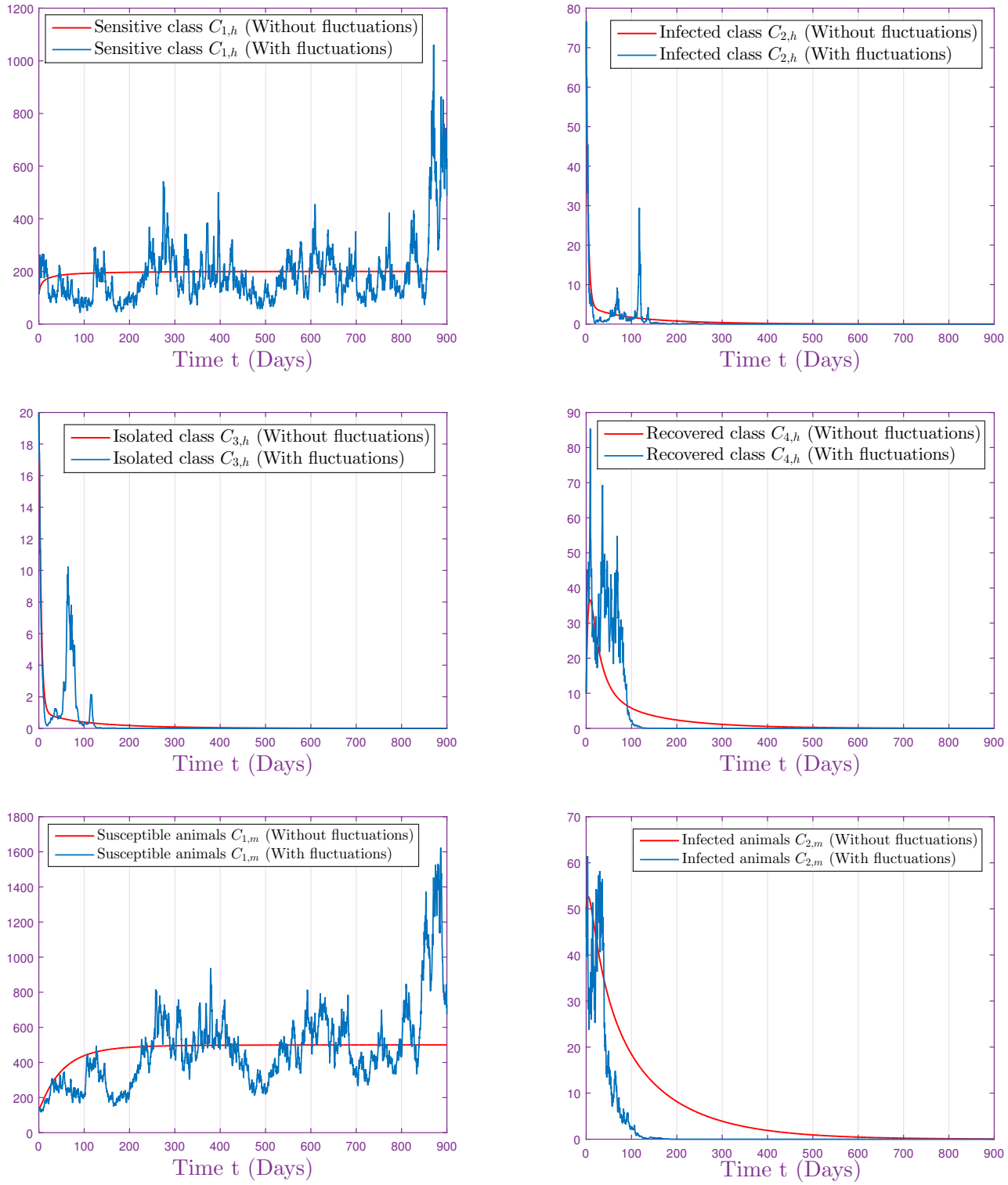

Figure 2. Solution paths of the *Monkeypox* model (1.2) when the numerical values are taken as shown in the second column of Table 1 ($\mathcal{S}_0 = 0.2759 < 1$ and $\varsigma = -0.0478 < 0$).

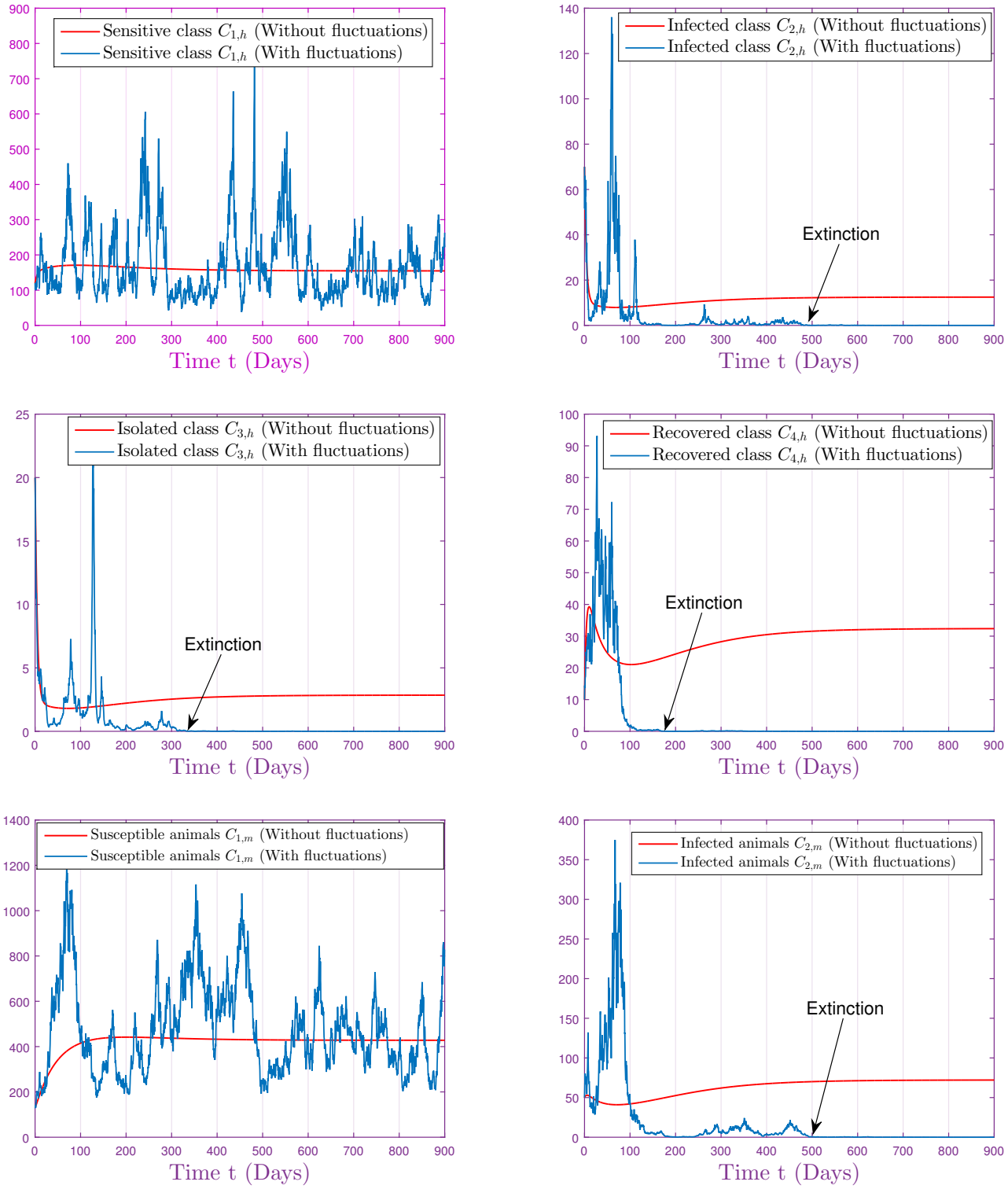

Figure 3. Solution paths of the *Monkeypox* model (1.2) when the numerical values are taken as shown in the third column of Table 1 ($\mathcal{S}_0 = 1.9786 > 1$ and $\varsigma = -0.0013 < 0$).

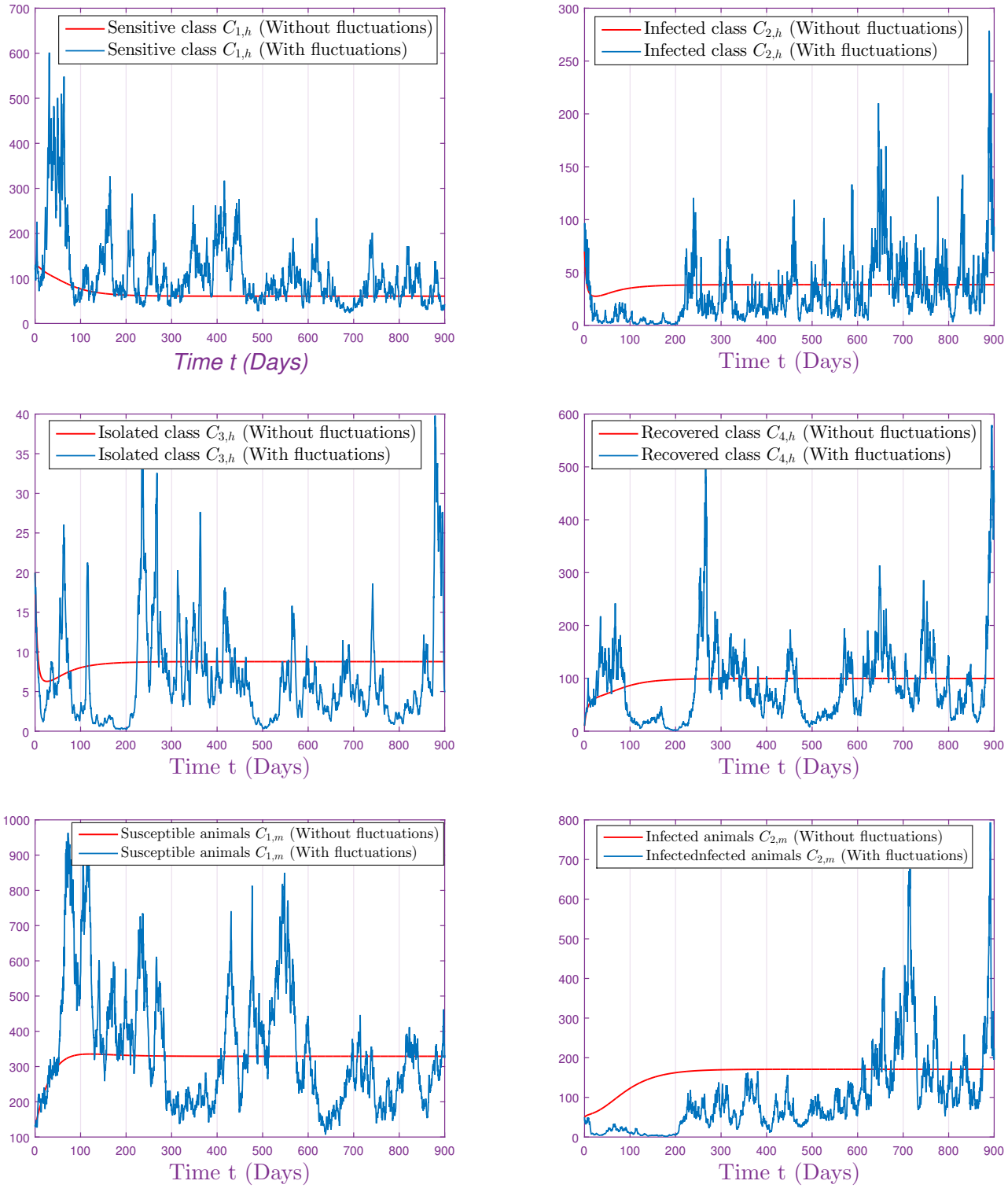

Figure 4. Solution paths of the *Monkeypox* model (1.2) when the numerical values are taken as shown in the fourth column of Table 1 ($\widetilde{\mathcal{S}}_0 = 1.2370 > 1$).

4.1. *First scenario: Eradication of the* Monkeypox *infection*

In this example, by choosing the parameter values presented in the second column, Test 1, in Table 1, and also taking $\mathcal{J} = \mathbb{R}_+$ with $\varpi(\mathcal{J}) = 1$, we will examine the theoretical results of Theorem 3.1. By performing some calculations, we get $\mathcal{S}_0 = 0.2759 < 1$ and $\varsigma = -0.0478 < 0$. So, by the virtue of Theorem 3.1, the *Monkeypox* pandemic will be eradicated almost surely, which is exactly shown in Figure 2. Now, we select the parameters from the third column, Test 2, in Table 1. Then, we obtain $\mathcal{S}_0 = 1.9786 > 1$ and $\varsigma = -0.0013 < 0$. From Theorem 3.1, we confirm the eradication of *Monkeypox*, which is depicted in Figure 3. We note from this figure that Poisson jumps lead to a lack of infection spreading, while the deterministic system predicts steady continuation. So, heavy jumps have passive influence on the diffusion of *Monkeypox*. This indicates that random fluctuations can notably alter the future of the epidemiological situation.

4.2. *Second scenario: Continuation of the* Monkeypox *infection*

In this part, we will illustrate the continuation of the *Monkeypox* infection. By selecting the parameters from Test 3 in Table 1, we can directly get $\widetilde{\mathcal{S}}_0 = 1.2370 > 1$. Consequently, by Theorem 3.2, the *Monkeypox* epidemic is persistent in the mean, which agrees well with the results shown in Figure 4.

5. Conclusions and discussion

In this paper, we analyzed a generalized stochastic *Monkeypox* epidemic model with cross-infection between animals and humans. Due to the complexity of disease, we captured the dynamics of the *Monkeypox* virus as a compartmental system that assumed the form of an Itô-Lévy stochastic differential equations system. Initially, we developed a deterministic system based on some assumptions and characteristics of the disease. We applied the nonstandard analytical techniques for proving two principal asymptotic properties: the eradication and continuation in the mean of *Monkeypox* 2022. The system is then converted to a stochastic model, and a number of sophisticated mathematical analyse are carried out that offer many insights related to *Monkeypox* propagation, including, notably, the results related to the long-run behavior. The main mathematical and biological results of this paper are as follows:

1) We have provided sufficient and threshold results for the *Monkeypox* eradication ($\varsigma < 0$).

2) We have proved results which ensure the continuation of the *Monkeypox* illness ($\widetilde{\mathcal{S}}_0 > 1$).

Roughly speaking, our obtained outcomes show that the eradication and permanence conditions rely fundamentally on white noise and leaps' intensities. Analytically, it was noticed that the obtained results enhance our comprehension related to the dynamics of the *Monkeypox* virus, which makes this paper more appropriate in providing a rich ground for further studies, especially in understanding the reappearance of the *Monkeypox* disease in many countries around the globe.

One limitation of this paper is that we obtained two separate critical conditions for extinction and persistence, which is less than the ideal when it comes to epidemiological models. Therefore, there is a significant gap between the defined criteria and the corresponding threshold value, which is still considered an open question. The authors are curious to deal with this problem in future work.

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Conflict of interest

The authors declare no conflicts of interest.

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