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Research article

Boundedness and stabilization of a predator-prey model with attraction-repulsion taxis in all dimensions

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Abstract: This paper establishes the existence of globally bounded classical solutions to a predator-prey model with attraction-repulsion taxis in a smooth bounded domain of any dimensions with Neumann boundary conditions. Moreover, the global stabilization of solutions with convergence rates to constant steady states is obtained. Using the local time integrability of the L^2-norm of solutions, we build up the basic energy estimates and derive the global boundedness of solutions by the Moser iteration. The global stability of constant steady states is established based on the Lyapunov functional method.

Keywords: predator-prey; attraction-repulsion; global boundedness; global stability

1. Introduction and main results

A taxis is the movement of an organism in response to a stimulus such as chemical signal or the presence of food. Taxes can be classified based on the types of stimulus, such as chemotaxis, prey-taxis, galvanotaxis, phototaxis and so on. According to the direction of movements, the taxis is said to be attractive (resp. repulsive) if the organism moves towards (resp. away from) the stimulus. In the ecosystem, a widespread phenomenon is the prey-taxis, where predators move up the prey density gradient, which is often referred to as the *direct* prey-taxis. However some predators may approach the prey by tracking the chemical signals released by the prey, such as the smell of blood or specific odo, and such movement is called *indirect* prey-taxis (cf. [1]). Since the pioneering modeling work by Kareiva and Odell [2], prey-taxis models have been widely studied in recent years (cf. [3–12]), followed by numerous extensions, such as three-species prey-taxis models (cf. [13–15]) and predator-taxis models (cf. [16, 17]). The indirect prey-taxis models have also been well studied (cf. [18–20]).

Recently, a predator-prey model with attraction-repulsion taxis mechanisms was proposed by Bell and Haskell in [21] to describe the interaction between direct prey-taxis and indirect chemotaxis, where the direct prey-taxis describes the predator's directional movement towards the prey density gradient, while the indirect chemotaxis models a defense mechanism in which the prey repels the predator by releasing odour chemicals (like a fox breaking wind in order to escape from hunting dogs). The model reads as

$$\begin{cases} u_t = d\Delta u + u\left(a_1 - a_2 u - a_3 v\right), & x \in \Omega,\ t > 0, \\ v_t = \nabla \cdot (\nabla v + \chi v \nabla w - \xi v \nabla u) + \rho v(1 - v) + e a_3 u v, & x \in \Omega,\ t > 0, \\ w_t = \eta \Delta w + r u - \gamma w, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where the unknown functions $u(x,t)$, $v(x,t)$ and $w(x,t)$ denote the densities of the prey, predator and prey-derived chemical repellent, respectively, at position $x \in \Omega$ and time $t > 0$. Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain (habitat of species) with smooth boundary $\partial\Omega$, and ν is the unit outer normal vector of $\partial\Omega$. The parameters d, η, χ, ξ, a_1, a_2, a_3, e, ρ, r, γ are all positive, where $\chi > 0$ and $\xi > 0$ denote the (attractive) prey-taxis and (repulsive) chemotaxis coefficients, respectively. The predator v is assumed to be a generalist, so that it has a logistic growth term $\rho v(1 - v)$ with intrinsic growth rate $\rho > 0$. More modeling details with biological interpretations are referred to in [21]. We remark that the predator-prey model with attraction-repulsion taxes has some similar structures to the so-called attraction-repulsion chemotaxis model proposed originally in [22], where the species elicit both attractive and repulsive chemicals (see [23–26] and references therein for some mathematical studies).

The initial data satisfy the following conditions:

$$v_0 \in C^0(\overline{\Omega}),\ u_0,\ w_0 \in W^{1,\infty}(\Omega),\ \text{and}\ u_0,\ v_0,\ w_0 \gneqq 0 \text{ in } \overline{\Omega}. \tag{1.2}$$

In [21], the global existence of strong solutions to (1.1) was established in one dimension ($n = 1$), and the existence of nontrivial steady state solutions alongside pattern formation was studied by the bifurcation theory. The main purpose of this paper is to study the global dynamics of (1.1) in higher dimensional spaces, which are usually more physical in the real world. Specifically, we shall show the existence of global classical solutions in all dimensions and explore the global stability of constant steady states, by which we may see how parameter values play roles in determining these dynamical properties of solutions.

The first main result is concerned with the global existence and boundedness of solutions to (1.1). For the convenience of presentation, we let

$$K_1 = \max\left\{\frac{a_1}{a_2}, \|u_0\|_{L^\infty(\Omega)}\right\},\quad K_2 = \max\left\{a_1 K_1 + a_2 K_1^2, a_3 K_1\right\} \tag{1.3}$$

and

$$\begin{aligned} K_3(z) = &\frac{2^{\frac{3z-1}{2}} z}{d^z}\left(\frac{n + 2(z-1)K_2^2}{z+1}\right)^{\frac{z+1}{2}}\left((z-1)(4z^2+n)K_1^2\right)^{\frac{z-1}{2}} \\ &+ \frac{2^{\frac{3}{z}} z^2}{d^{\frac{1}{z}}}\left(\frac{(z-1)\xi^2}{z+1}\right)^{\frac{z+1}{z}}\left((4z^2+n)K_1^2\right)^{\frac{1}{z}}. \end{aligned} \tag{1.4}$$

Then, the result on the global boundedness of solutions to (1.1) is stated as follows.

Theorem 1.1 (Global existence). *Let $\Omega \subset \mathbb{R}^n (n \geqslant 1)$ be a bounded domain with smooth boundary and parameters d, η, χ, ξ, a_1, a_2, a_3, e, ρ, r, γ be positive. If*

$$\rho \begin{cases} > 0, & n \leqslant 2, \\ \geqslant \frac{2K_3([\frac{n}{2}]+1)}{[\frac{n}{2}]+1}, & n > 2, \end{cases}$$

where $K_3(p)$ is defined in (1.4)*, then for any initial data (u_0, v_0, w_0) satisfying* (1.2)*, the system* (1.1) *admits a unique classical solution (u, v, w) satisfying*

$$u,\ v,\ w \in C^0(\overline{\Omega} \times [0, +\infty)) \cap C^{2,1}(\overline{\Omega} \times (0, +\infty)),$$

and $u, v, w > 0$ in $\Omega \times (0, +\infty)$. Moreover, there exists a constant $C > 0$ independent of t such that

$$\|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leqslant C \quad \text{for all } t > 0.$$

Our next goal is to explore the large-time behavior of solutions to (1.1). Simple calculations show the system (1.1) has four possible homogeneous equilibria as classified below:

$$\begin{cases} (0,0,0),\ (0,1,0),\ (\frac{a_1}{a_2}, 0, \frac{ra_1}{\gamma a_2}), & \text{if } a_1 \leqslant a_3, \\ (0,0,0),\ (0,1,0),\ (\frac{a_1}{a_2}, 0, \frac{ra_1}{\gamma a_2}), (u_*, v_*, w_*), & \text{if } a_1 > a_3, \end{cases}$$

with

$$u_* = \frac{\rho(a_1 - a_3)}{\rho a_2 + e a_3^2}, \quad v_* = \frac{e a_1 a_3 + \rho a_2}{\rho a_2 + e a_3^2}, \quad w_* = \frac{r\rho(a_1 - a_3)}{\gamma(\rho a_2 + e a_3^2)} \tag{1.5}$$

where the trivial equilibrium $(0, 0, 0)$ is called the extinction steady state, $(0, 1, 0)$ is the predator-only steady state, and (u_*, v_*, w_*) is the coexistence steady state. We shall show that if $a_1 > a_3$, then the coexistence steady state is globally asymptotically stable with exponential convergence rate, provided that ξ and χ are suitably small, while if $a_1 \leqslant a_3$, the predator-only steady state is globally asymptotically stable with exponential or algebraic convergence rate when ξ and χ are suitably small. To state our results, we denote

$$\Gamma = \frac{4d\rho(a_1 - a_3)}{K_1^2(e a_1 a_3 + \rho a_2)}, \quad \Phi = \frac{2a_2\rho}{a_3^2} + e, \quad \Psi = \frac{\gamma\eta a_3^2 K_1^2(\rho a_2 + e a_3^2)}{d\rho^2 r^2(a_1 - a_3)} \tag{1.6}$$

and

$$A = \frac{\xi^2}{4d}, \quad B = \frac{e a_2}{a_1}, \quad D = \frac{16\eta\gamma a_1}{r^2}, \tag{1.7}$$

where K_1 is defined in (1.3). Then, the global stability result is stated in the following theorem.

Theorem 1.2 (Global stability). *Let the assumptions in Theorem 1.1 hold. Then, the following results hold.*

(1) Let $a_1 > a_3$. If ξ and χ satisfy

$$\xi^2 < \Gamma\Big(\Phi + \sqrt{\Phi^2 - e^2}\Big) \text{ and } \chi^2 < \Psi \max_{y \in [a,b]} \frac{(\Gamma y - \xi^2)(-y^2 + 2\Phi y - e^2)}{y},$$

where $a = \max\{\frac{\xi^2}{\Gamma}, \Phi - \sqrt{\Phi^2 - e^2}\}, b = \Phi + \sqrt{\Phi^2 - e^2}$, then there exist some constants T_*, C, $\alpha > 0$ such that the solution (u, v, w) obtained in Theorem 1.1 satisfies for all $t \geqslant T_*$

$$\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v_*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w_*\|_{L^\infty(\Omega)} \leqslant Ce^{-\alpha t}.$$

(2) Let $a_1 \leqslant a_3$, If ξ and χ satisfy

$$\xi^2 < \frac{4dea_2}{a_1} \quad \text{and} \quad \chi^2 < D\left(A + B - 2\sqrt{AB}\right),$$

then there exist some constants T^*, C, $\beta > 0$ such that the solution (u, v, w) obtained in Theorem 1.1 satisfies, for all $t \geqslant T^*$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \begin{cases} Ce^{-\beta t} & \text{if } a_1 < a_3, \\ C(t+1)^{-1} & \text{if } a_1 = a_3. \end{cases}$$

Remark 1.1. *In the biological view, the relative sizes of a_1 and a_2 determine the coexistence of the system. The results indicated that a large $\frac{a_1}{a_2}$ facilitates the coexistence of the species.*

The rest of this paper is organized as follows. In Section 2, we state the local existence of solutions to (1.1) with extensibility conditions. Then, we deduce some *a priori* estimates and prove Theorem 1.1 in Section 3. Finally, we show the global convergence to the constant steady states and prove Theorem 1.2 in Section 4.

2. Preliminary

For convenience, in what follows we shall use $C_i (i = 1, 2, \cdots)$ to denote a generic positive constant which may vary from line to line. For simplicity, we abbreviate $\int_0^t \int_\Omega f(\cdot, s)dxds$ and $\int_\Omega f(\cdot, t)dx$ as $\int_0^t \int_\Omega f$ and $\int_\Omega f$, respectively. The local existence and extensibility result of problem (1.1) can be directly established by the well-known Amman's theory for triangular parabolic systems (cf. [27,28]). Below, we shall present the local existence theorem without proof for brevity, and we refer to [21] for the proof in one dimension as a reference.

Lemma 2.1 (Local existence and extensibility)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. The parameters d, η, χ, ξ, a_1, a_2, a_3, e, ρ, r, γ are positive. Then, for the initial data (u_0, v_0, w_0) satisfying (1.2), there exists $T_{max} \in (0, \infty]$ such that the system (1.1) admits a unique classical solution (u, v, w) satisfying*

$$u,\ v,\ w \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$

and $u, v, w > 0$ in $\Omega \times (0, T_{max})$. Moreover, we have

$$\text{either } T_{max} = +\infty \text{ or } \limsup_{t \nearrow T_{max}} \left(\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}\right) = +\infty.$$

We recall some well-known results which will be used later frequently. The first one is an uniform Grönwall inequality [29].

Lemma 2.2. *Let $T_{max} > 0$, $\tau \in (0, T_{max})$. Suppose that c_1, c_2, y are three positive locally integrable functions on $(0, T_{max})$ such that y' is locally integrable on $(0, T_{max})$ and satisfies*

$$y'(t) \leqslant c_1(t)y(t) + c_2(t) \quad \text{for all } t \in (0, T_{max}).$$

If

$$\int_t^{t+\tau} c_1 \leqslant C_1, \quad \int_t^{t+\tau} c_2 \leqslant C_2, \quad \int_t^{t+\tau} y \leqslant C_3 \quad \text{for all } t \in [0, T_{max} - \tau),$$

where $C_i (i = 1, 2, 3)$ *are positive constants, then*

$$y(t) \leqslant \left(\frac{C_3}{\tau} + C_2\right) e^{C_1} \quad \text{for all } t \in [\tau, T_{max}).$$

Next, we recall a basic inequality [30].

Lemma 2.3. *Let* $p \in [1, \infty)$. *Then, the following inequality holds:*

$$\int_\Omega |\nabla u|^{2(p+1)} \leqslant 2\left(4p^2 + n\right) \|u\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla u|^{2(p-1)} |D^2 u|^2$$

for any $u \in C^2(\bar{\Omega})$ *satisfying* $\frac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$, *where* $D^2 u$ *denotes the Hessian of* u.

The last one is a Gagliardo-Nirenberg type inequality shown in [31, Lemma 2.5].

Lemma 2.4. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary. Then, for any* $\varphi \in W^{2,2}(\Omega)$ *satisfying* $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0$, *there exists a positive constant* C *depending only on* Ω *such that*

$$\|\nabla\varphi\|_{L^4(\Omega)} \leq C(\|\Delta\varphi\|^{\frac{1}{2}}_{L^2(\Omega)} \|\nabla\varphi\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)}). \tag{2.1}$$

3. Global existence

In this section, we establish the global boundedness of solutions to the system (1.1). To this end, we will proceed with several steps to derive *a priori* estimates for the solution of the system (1.1). The first one is the uniform-in-time $L^\infty(\Omega)$ boundedness of u.

Lemma 3.1. *Let* (u, v, w) *be the solution of* (1.1) *and* K_1 *be as defined in* (1.3). *Then, we have*

$$\|u\|_{L^\infty(\Omega)} \leqslant K_1 \quad \text{for all } t \in (0, T_{max}).$$

Furthermore, there is a constant $C > 0$ *such that for any* $0 < \tau < \min\{T_{max}, 1\}$, *it follows that*

$$\int_t^{t+\tau} |\nabla u|^2 \leq C \ \text{ for all } \ t \in (0, T_{max} - \tau).$$

Proof. The result is a direct consequence of the maximum principle applied to the first equation in (1.1). Indeed, if we let $\bar{u} = \max\left\{\frac{a_1}{a_2}, \|u_0\|_{L^\infty(\Omega)}\right\}$, then $\bar{u}$ satisfies

$$\begin{cases} \bar{u}_t \geqslant d\Delta\bar{u} + \bar{u}\left(a_1 - a_2\bar{u} - a_3 v\right), & x \in \Omega, t > 0, \\ \nabla\bar{u} \cdot \nu = 0, & x \in \partial\Omega, t > 0, \\ \bar{u}(x, 0) \geqslant u_0(x), & x \in \Omega. \end{cases}$$

Apparently, the comparison principle of parabolic equations gives $u \leqslant \bar{u}$ on $\Omega \times (0, T_{\max})$.

Next, we multiply the first equation of (1.1) by u and integrate the result to get

$$\frac{d}{dt}\int_\Omega u^2 + d\int_\Omega |\nabla u|^2 = a_1 \int_\Omega u^2 - \int_\Omega u(a_2 u + a_3 v) \leq a_1 K_1^2 |\Omega|.$$

Then, the integration of the above inequality with respect to t over $(t, t + \tau)$ completes the proof by noting that $\int_\Omega u_0^2$ is bounded.

Having at hand the uniform-in-time $L^\infty(\Omega)$ boundedness of u, the *a priori* estimate of w follows immediately.

Lemma 3.2. *Let* (u, v, w) *be the solution of* (1.1)*. We can find a constant* $C > 0$ *satisfying*

$$\|w\|_{W^{1,\infty}(\Omega)} \leqslant C \quad \text{for all } t \in (0, T_{max}).$$

Proof. Noting the boundedness of $\|u\|_{L^\infty(\Omega)}$ from Lemma 3.1, we get the desired result from the third equation of (1.1) and the regularity theorem [32, Lemma 1].

Now, the *a priori* estimate of v can be obtained as below.

Lemma 3.3. *Let* (u, v, w) *be the solution of* (1.1)*. Then, there exists a constant* $C > 0$ *such that*

$$\int_\Omega v \leqslant C \quad \text{for all } t \in (0, T_{max}), \tag{3.1}$$

and

$$\int_t^{t+\tau} \int_\Omega v^2 \leqslant C \quad \text{for all } t \in (0, T_{max} - \tau), \tag{3.2}$$

where τ *is a constant such that* $0 < \tau < \min\{T_{max}, 1\}$.

Proof. Integrating the second equation of (1.1) over Ω by parts, using Young's inequality and Lemma 3.1, we find some constant $C_1 > 0$ such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega v =&\rho\int_\Omega v - \rho\int_\Omega v^2 + ea_3\int_\Omega uv \\ \leqslant&\left(\rho + ea_3 \sup_{t\in(0,T_{max})}\|u\|_{L^\infty(\Omega)}\right)\int_\Omega v - \rho\int_\Omega v^2 \\ \leqslant& -\int_\Omega v - \frac{\rho}{2}\int_\Omega v^2 + C_1 \quad \text{for all } t \in (0, T_{max}).\end{aligned} \tag{3.3}$$

Hence, (3.1) is obtained by the Grönwall inequality. Integrating (3.3) over $(t, t + \tau)$, we get (3.2) immediately.

Due to the estimates of u and v obtained in Lemmas 3.1 and 3.3 respectively, we have the following improved uniform-in-time $L^2(\Omega)$ boundedness of ∇u and the space-time L^2 boundedness of Δu when $n = 2$.

Lemma 3.4. *Let* (u, v, w) *be the solution of* (1.1)*. If* $n = 2$*, then we can find a constant* $C > 0$ *such that*

$$\int_\Omega |\nabla u|^2 \leqslant C \quad \text{for all } t \in (0, T_{max}) \tag{3.4}$$

and

$$\int_t^{t+\tau} \int_\Omega |\Delta u|^2 \leqslant C \quad \text{for all } t \in (0, T_{max} - \tau), \tag{3.5}$$

where τ *is defined in Lemma 3.3.*

Proof. Integrating the first equation of (1.1) by parts and using Lemma 3.1, we find a constant $C_1 > 0$ such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla u|^2 &= 2\int_\Omega \nabla u\cdot\nabla u_t = -2\int_\Omega u_t\Delta u\\ &= -2\int_\Omega \Delta u\left(d\Delta u + a_1 u - a_2 u^2 - a_3 uv\right)\\ &\leqslant -2d\int_\Omega |\Delta u|^2 + C_1\int_\Omega (v+1)|\Delta u| \quad \text{for all } t\in(0,T_{max}).\end{aligned} \tag{3.6}$$

The Gagliardo-Nirenberg inequality in Lemma 2.4, Young's inequality and Lemma 3.1 yield some constants $C_2, C_3 > 0$ satisfying

$$\int_\Omega |\nabla u|^2 = \|\nabla u\|^2_{L^2(\Omega)} \leqslant C_2\left(\|\Delta u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|^2_{L^\infty(\Omega)}\right) \leqslant \frac{d}{2}\int_\Omega |\Delta u|^2 + C_3$$

and

$$C_1\int_\Omega (v+1)|\Delta u| \leqslant \frac{d}{2}\int_\Omega |\Delta u|^2 + C_3\int_\Omega v^2 + C_3 \quad \text{for all } t\in(0,T_{max}),$$

which along with (3.6) imply

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u|^2 + d\int_\Omega |\Delta u|^2 \leqslant C_3\int_\Omega v^2 + 2C_3 \quad \text{for all } t\in(0,T_{max}). \tag{3.7}$$

Then, applications of Lemma 2.2, 3.1 and 3.3 give (3.4). Finally, (3.5) can be obtained by integrating (3.7) over $(t, t+\tau)$.

Now, the uniform-in-time boundedness of v in $L^2(\Omega)$ can be established when $n = 2$.

Lemma 3.5. *Let (u, v, w) be the solution of (1.1). If $n = 2$, then there exists a constant $C > 0$ such that*

$$\int_\Omega v^2 \leqslant C \quad \textit{for all } t\in(0,T_{max}).$$

Proof. Multiplying the second equation of (1.1) by v, integrating the result by parts and using Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega v^2 + 2\int_\Omega |\nabla v|^2 =& -2\chi\int_\Omega v\nabla v\cdot\nabla w + 2\xi\int_\Omega v\nabla u\cdot\nabla v\\ &+ 2\rho\int_\Omega v^2 - 2\rho\int_\Omega v^3 + 2ea_3\int_\Omega uv^2\\ \leqslant& \int_\Omega |\nabla v|^2 + 2\chi^2\|\nabla w\|^2_{L^\infty(\Omega)}\int_\Omega v^2 + 2\xi^2\int_\Omega v^2|\nabla u|^2\\ &+ 2\rho\int_\Omega v^2 - 2\rho\int_\Omega v^3 + 2ea_3\|u\|_{L^\infty(\Omega)}\int_\Omega v^2,\end{aligned}$$

which along with Lemma 3.1 and Lemma 3.2 gives some constant $C_1 > 0$ such that

$$\frac{d}{dt}\int_\Omega v^2 + \int_\Omega |\nabla v|^2 \leqslant 2\xi^2\int_\Omega v^2|\nabla u|^2 + C_1\int_\Omega v^2 - 2\rho\int_\Omega v^3 \quad \text{for all } t\in(0,T_{max}). \tag{3.8}$$

Using Lemmas 3.1 and 3.3, Hölder's inequality, Lemma 2.4 and Young's inequality, we find some constants $C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}
&2\xi\int_\Omega v^2|\nabla u|^2 \leqslant 2\xi\|v\|^2_{L^4(\Omega)}\|\nabla u\|^2_{L^4(\Omega)}\\
\leqslant& C_2\left(\|\nabla v\|^{\frac{1}{2}}_{L^2(\Omega)}\|v\|^{\frac{1}{2}}_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right)^2\left(\|\Delta u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\right)^2\\
\leqslant& C_3\Big(\|\nabla v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\
&\quad + \|\Delta u\|_{L^2(\Omega)}\|v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\Big)\\
\leqslant& \|\nabla v\|^2_{L^2(\Omega)} + C_4\left(1 + \|\Delta u\|^2_{L^2(\Omega)}\right)\|v\|^2_{L^2(\Omega)} \quad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{3.9}$$

Furthermore, Young's inequality yields some constant $C_5 > 0$ such that

$$C_1\int_\Omega v^2 - 2\rho\int_\Omega v^3 \leqslant C_5 \quad \text{for all } t \in (0, T_{max}). \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we get

$$\frac{d}{dt}\int_\Omega v^2 \leqslant C_4\left(1 + \|\Delta u\|^2_{L^2(\Omega)}\right)\|v\|^2_{L^2(\Omega)} + C_5 \quad \text{for all } t \in (0, T_{max}),$$

which alongside Lemma 2.2, Lemma 3.3 and Lemma 3.4 completes the proof.

To get the global existence of solutions in any dimensions, we derive the following functional inequality which gives an *a priori* estimate on ∇u.

Lemma 3.6. *Let* (u, v, w) *be the solution of* (1.1) *and* $q \geqslant 2$. *If* $n \geqslant 1$, *then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^{2q} + dq\int_\Omega |\nabla u|^{2(q-1)}|D^2u|^2\\
\leqslant& \frac{q(n + 2(q-1))K_2^2}{d}\int_\Omega \left(v^2 + 1\right)|\nabla u|^{2(q-1)} + C \quad \textit{for all } t \in (0, T_{max}),
\end{aligned}$$

where K_2 *is defined in* (1.3).

Proof. From the first equation of (1.1) and the fact $2\nabla u \cdot \nabla\Delta u = \Delta|\nabla u|^2 - 2|D^2u|^2$, it follows that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla u|^{2q} =& 2q\int_\Omega |\nabla u|^{2(q-1)}\nabla u \cdot \nabla u_t\\
=& 2q\int_\Omega |\nabla u|^{2(q-1)}\nabla u \cdot \nabla\left(d\Delta u + a_1u - a_2u^2 - a_3uv\right)\\
=& dq\int_\Omega |\nabla u|^{2(q-1)}\Delta|\nabla u|^2 - 2dq\int_\Omega |\nabla u|^{2(q-1)}|D^2u|^2\\
&+ 2q\int_\Omega |\nabla u|^{2(q-1)}\nabla u \cdot \nabla\left(a_1u - a_2u^2 - a_3uv\right)
\end{aligned}$$

which implies

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^{2q} + 2dq\int_\Omega |\nabla u|^{2(q-1)}|D^2u|^2\\
=&dq\int_\Omega |\nabla u|^{2(q-1)}\Delta|\nabla u|^2 + 2q\int_\Omega |\nabla u|^{2(q-1)}\nabla u\cdot\nabla\left(a_1u - a_2u^2 - a_3uv\right)\\
=&:I_1 + I_2 \quad \text{for all } t\in(0,T_{max}).
\end{aligned}
\tag{3.11}
$$

Now, we estimate the right hand side of (3.11). Choosing $s \in (0, \frac{1}{2})$ and

$$
\theta = \frac{\frac{1}{2} - \frac{s+\frac{1}{2}}{n} - q}{\frac{1}{2} - \frac{1}{n} - q} \in (0,1),
$$

we get

$$
\frac{1}{2} - \frac{s+\frac{1}{2}}{n} = \theta\left(\frac{1}{2} - \frac{1}{n}\right) + (1-\theta)q,
$$

which, along with the Gagliardo-Nirenberg inequality, Young's inequality and the embedding

$$
W^{s+\frac{1}{2},2}(\Omega) \subset W^{s,2}(\partial\Omega) \subset L^2(\partial\Omega),
$$

gives some constants $C_1, C_2, C_3, C_4 > 0$ such that

$$
\begin{aligned}
\int_{\partial\Omega}|\nabla u|^{2(q-1)}\frac{\partial|\nabla u|^2}{\partial\nu} \leqslant& C_1\int_{\partial\Omega}|\nabla u|^{2q} = C_1\left\||\nabla u|^q\right\|^2_{L^2(\partial\Omega)}\\
\leqslant& C_2\left\||\nabla u|^q\right\|^2_{W^{s+\frac{1}{2},2}(\Omega)}\\
\leqslant& C_3\left\|\nabla|\nabla u|^q\right\|^{2\theta}_{L^2(\Omega)}\left\||\nabla u|^q\right\|^{2(1-\theta)}_{L^{\frac{1}{q}}(\Omega)} + C_3\left\||\nabla u|^q\right\|^2_{L^{\frac{1}{q}}(\Omega)}\\
\leqslant& \frac{2(q-1)}{q^2}\left\|\nabla|\nabla u|^q\right\|^2_{L^2(\Omega)} + C_4 \quad \text{for all } t\in(0,T_{max}).
\end{aligned}
$$

Therefore, it holds that

$$
\begin{aligned}
I_1 =& dq\int_{\partial\Omega}|\nabla u|^{2(q-1)}\frac{\partial|\nabla u|^2}{\partial\nu} - dq\int_\Omega \nabla|\nabla u|^{2(q-1)}\cdot\nabla|\nabla u|^2\\
\leqslant& \frac{2d(q-1)}{q}\int_\Omega|\nabla|\nabla u|^q|^2 + C_4dq - \frac{4d(q-1)}{q}\int_\Omega|\nabla|\nabla u|^q|^2\\
\leqslant& -\frac{2d(q-1)}{q}\int_\Omega|\nabla|\nabla u|^q|^2 + C_4dq \quad \text{for all } t\in(0,T_{max}).
\end{aligned}
$$

Owning to the fact $|\Delta u| \leqslant \sqrt{n}|D^2u|$, Young's inequality and Lemma 3.1, we have

$$
\begin{aligned}
I_2 =& -2q(q-1)\int_\Omega\left(a_1u - a_2u^2 - a_3uv\right)|\nabla u|^{2(q-2)}\nabla|\nabla u|^2\cdot\nabla u\\
&-2q\int_\Omega\left(a_1u - a_2u^2 - a_3uv\right)|\nabla u|^{2(q-1)}\Delta u
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 2q(q-1)K_2\int_\Omega (v+1)|\nabla u|^{2(q-2)}\left|\nabla|\nabla u|^2\right||\nabla u| \\
&\quad + 2q\sqrt{n}K_2\int_\Omega (v+1)|\nabla u|^{2(q-1)}|D^2u| \\
&\leqslant \frac{qd(q-1)}{2}\int_\Omega |\nabla u|^{2(q-2)}\left|\nabla|\nabla u|^2\right|^2 + \frac{2q(q-1)K_2^2}{d}\int_\Omega \left(v^2+1\right)|\nabla u|^{2(q-1)} \\
&\quad + dq\int_\Omega |\nabla u|^{2(q-1)}|D^2u|^2 + \frac{qnK_2^2}{d}\int_\Omega \left(v^2+1\right)|\nabla u|^{2(q-1)} \\
&= \frac{2d(q-1)}{q}\int_\Omega |\nabla|\nabla u|^q|^2 + dq\int_\Omega |\nabla u|^{2(q-1)}|D^2u|^2 \\
&\quad + \frac{q(n+2(q-1))K_2^2}{d}\int_\Omega \left(v^2+1\right)|\nabla u|^{2(q-1)} \quad \text{for all } t\in(0,T_{max}),
\end{aligned}
$$

where K_2 is defined in (1.3). Hence, substituting the estimates I_1 and I_2 into (3.11), we finish the proof of the lemma.

Now, we show the following functional inequality to derive the *a priori estimate* on v in any dimensions.

Lemma 3.7. *Let* (u, v, w) *be the solution of* (1.1) *and* $q \geqslant 2$. *If* $n \geqslant 1$, *we can find a constant* $C > 0$ *such that*

$$
\frac{d}{dt}\int_\Omega v^q + \frac{2(q-1)}{q}\int_\Omega \left|\nabla v^{\frac{q}{2}}\right|^2 + \rho q\int_\Omega v^{q+1} \leqslant q(q-1)\xi^2\int_\Omega v^q|\nabla u|^2 + C\int_\Omega v^q
$$

for all $t \in (0, T_{max})$.

Proof. Utilizing the second equation of (1.1) and integration by parts, we get

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega v^q &= q\int_\Omega v^{q-1}v_t = q\int_\Omega v^{q-1}\left(\nabla\cdot(\nabla v+\chi v\nabla w-\xi v\nabla u)+v\left(\rho(1-v)+ea_3u\right)\right) \\
&= -q(q-1)\int_\Omega v^{q-2}|\nabla v|^2 - \chi q(q-1)\int_\Omega v^{q-1}\nabla w\cdot\nabla v \qquad (3.12)\\
&\quad + \xi q\,(q-1)\int_\Omega v^{q-1}\nabla u\cdot\nabla v + \rho q\int_\Omega v^q - \rho q\int_\Omega v^{q+1} + ea_3q\int_\Omega uv^q.
\end{aligned}
$$

Now, we estimate the right hand side of (3.12). An application of Young's inequality and Lemma 3.2 yields some constant $C_1 > 0$ such that

$$
\begin{aligned}
-\chi q(q-1)\int_\Omega v^{q-1}\nabla w\cdot\nabla v &\leqslant \chi q(q-1)\sup_{t\in(0,T_{max})}\|\nabla w\|_{L^\infty(\Omega)}\int_\Omega v^{q-1}|\nabla v| \\
&\leqslant \frac{q(q-1)}{4}\int_\Omega v^{q-2}|\nabla v|^2 + C_1\int_\Omega v^q
\end{aligned}
$$

and

$$
\xi q(q-1)\int_\Omega v^{q-1}\nabla u\cdot\nabla v \leqslant \frac{q(q-1)}{4}\int_\Omega v^{q-2}|\nabla v|^2 + q(q-1)\xi^2\int_\Omega v^q|\nabla u|^2,
$$

which along with (3.12), Lemma 3.1 and the fact

$$v^{q-2}|\nabla v|^2 = \frac{4}{q^2}\left|\nabla v^{\frac{q}{2}}\right|^2$$

gives a constant $C_2 > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v^q + \frac{2(q-1)}{q}\int_\Omega \left|\nabla v^{\frac{q}{2}}\right|^2 \\
\leqslant& q(q-1)\xi^2\int_\Omega v^q|\nabla u|^2 + (\rho q + C_1)\int_\Omega v^q - \rho q\int_\Omega v^{q+1} + e a_3 q\int_\Omega u v^q \\
\leqslant& q(q-1)\xi^2\int_\Omega v^q|\nabla u|^2 - \rho q\int_\Omega v^{q+1} + C_2\int_\Omega v^q \quad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

Hence, we finish the proof of the lemma.

Combining Lemmas 3.6 and 3.7, we have the following inequality which can help us to achieve the global existence of solutions in any dimensions.

Lemma 3.8. *Let (u, v, w) be the solution of (1.1) and $p \geqslant 2$. If $n \geqslant 1$, we can find a constant $C > 0$ such that*

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega |\nabla u|^{2p} + \int_\Omega v^p\right) + \frac{2(p-1)}{p}\int_\Omega \left|\nabla v^{\frac{p}{2}}\right|^2 + \int_\Omega |\nabla u|^{2p} + \int_\Omega v^p \\
\leqslant& \left(K_3(p) - \frac{\rho p}{2}\right)\int_\Omega v^{p+1} + C \quad \textit{for all } t\in(0,T_{max}),
\end{aligned}$$

where $K_3(p)$ is defined in (1.4).

Proof. Combining Lemmas 3.6 and 3.7, we see for any $p = q \geqslant 2$ there exists a constant $C_1 > 0$ such that for all $t \in (0, T_{max})$

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega |\nabla u|^{2p} + \int_\Omega v^p\right) + \frac{2(p-1)}{p}\int_\Omega \left|\nabla v^{\frac{p}{2}}\right|^2 + dp\int_\Omega |\nabla u|^{2(p-1)}|D^2 u|^2 + \rho p\int_\Omega v^{p+1} \\
\leqslant& \frac{p(n+2(p-1))K_2^2}{d}\int_\Omega v^2|\nabla u|^{2(p-1)} + p(p-1)\xi^2\int_\Omega v^p|\nabla u|^2 \\
&+ C_1\int_\Omega |\nabla u|^{2(p-1)} + C_1\int_\Omega v^p + C_1.
\end{aligned} \tag{3.13}$$

Now, we estimate the right hand side of the above inequality. Indeed, owing to Lemma 2.3 and Young's inequality, for all $t \in (0, T_{max})$, we have

$$\begin{aligned}
&\frac{p(n+2(p-1))K_2^2}{d}\int_\Omega v^2|\nabla u|^{2(p-1)} \\
\leqslant& \frac{dp}{8(4p^2+n)\|u\|^2_{L^\infty(\Omega)}}\int_\Omega |\nabla u|^{2(p+1)} \\
&+ \frac{2}{p+1}\left(\frac{dp(p+1)}{8(p-1)(4p^2+n)\|u\|^2_{L^\infty(\Omega)}}\right)^{-\frac{p-1}{2}}\left(\frac{p(n+2(p-1))K_2^2}{d}\right)^{\frac{p+1}{2}}\int_\Omega v^{p+1} \\
\leqslant& \frac{dp}{4}\int_\Omega |\nabla u|^{2(p-1)}|D^2u|^2 + \frac{2^{\frac{3p-1}{2}}p}{d^p}\left(\frac{n+2(p-1)K_2^2}{p+1}\right)^{\frac{p+1}{2}}\left((p-1)(4p^2+n)K_1^2\right)^{\frac{p-1}{2}}\int_\Omega v^{p+1}
\end{aligned}$$

and

$$
\begin{aligned}
&p(p-1)\xi^2\int_\Omega v^p|\nabla u|^2\\
\leqslant&\frac{dp}{8(4p^2+n)\|u\|^2_{L^\infty(\Omega)}}\int_\Omega|\nabla u|^{2(p+1)}\\
&+\frac{p}{p+1}\left(\frac{dp(p+1)}{8(4p^2+n)\|u\|^2_{L^\infty(\Omega)}}\right)^{-\frac{1}{p}}\left(p(p-1)\xi^2\right)^{\frac{p+1}{p}}\int_\Omega v^{p+1}\\
\leqslant&\frac{dp}{4}\int_\Omega|\nabla u|^{2(p-1)}|D^2u|^2+\frac{2^{\frac{3}{p}}p^2}{d^{\frac{1}{p}}}\left(\frac{(p-1)\xi^2}{p+1}\right)^{\frac{p+1}{p}}\left((4p^2+n)K_1^2\right)^{\frac{1}{p}}\int_\Omega v^{p+1},
\end{aligned}
$$

where K_1 and K_2 are defined in (1.3). Similarly, we can find a constant $C_2>0$ such that

$$
\begin{aligned}
C_1\int_\Omega|\nabla u|^{2(p-1)}\leqslant&\frac{dp}{8(4p^2+n)\|u\|^2_{L^\infty(\Omega)}}\int_\Omega|\nabla u|^{2(p+1)}+C_2\\
\leqslant&\frac{dp}{4}\int_\Omega|\nabla u|^{2(p-1)}|D^2u|^2+C_2\quad\text{for all } t\in(0,T_{max})\,.
\end{aligned}
$$

Substituting the above estimates into (3.13), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega|\nabla u|^{2p}+\int_\Omega v^p\right)+\frac{2(p-1)}{p}\int_\Omega\left|\nabla v^{\frac{p}{2}}\right|^2\\
&+\frac{dp}{4}\int_\Omega|\nabla u|^{2(p-1)}|D^2u|^2+\rho p\int_\Omega v^{p+1}\\
\leqslant&K_3(p)\int_\Omega v^{p+1}+C_1\int_\Omega v^p+C_1+C_2\quad\text{for all } t\in(0,T_{max})\,,
\end{aligned}
\tag{3.14}
$$

where $K_3(p)$ is given in (1.4). Furthermore, we can use Young's inequality and Lemma 2.3 to get a constant $C_3>0$ such that

$$
(C_1+1)\int_\Omega v^p\leqslant\frac{\rho p}{2}\int_\Omega v^{p+1}+C_3,
$$

and

$$
\begin{aligned}
\int_\Omega|\nabla u|^{2p}\leqslant&\frac{dp}{8\,(4p^2+n)\,\|u\|^2_{L^\infty(\Omega)}}\int_\Omega|\nabla u|^{2(p+1)}+C_3\\
\leqslant&\frac{dp}{4}\int_\Omega|\nabla u|^{2(p-1)}|D^2u|^2+C_3\quad\text{for all } t\in(0,T_{max})\,,
\end{aligned}
$$

which together with (3.14) finishes the proof.

Next, we shall deduce a criterion of global boundedness of solutions for the system (1.1) inspired by an idea of [33].

Lemma 3.9. *Let $n\geqslant1$. If there exist $M>0$ and $p_0>\frac{n}{2}$ such that*

$$
\int_\Omega v^{p_0}\leqslant M\quad\textit{for all } t\in(0,T_{max})\,,
\tag{3.15}
$$

then $T_{max}=+\infty$. Moreover, there exists $C>0$ such that

$$
\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}\leqslant C\quad\textit{for all } t>0.
$$

Proof. We divide the proof into two steps.

Step 1: We claim that there exists a constant $C_1 > 0$ such that

$$\int_\Omega v^{2p_0} \leqslant C_1 \quad \text{for all } t \in (0, T_{max}).$$

Indeed, due to Lemma 3.8, for any $p = 2p_0$, there exists a constant $C_2 > 0$ such that

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega |\nabla u|^{4p_0} + \int_\Omega v^{2p_0}\right) + \frac{2p_0 - 1}{p_0}\int_\Omega |\nabla v^{p_0}|^2 + \int_\Omega |\nabla u|^{4p_0} + \int_\Omega v^{2p_0} \\ &\leqslant (K_3(2p_0) - \rho p_0)\int_\Omega v^{2p_0+1} + C_2 \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.16}$$

Let

$$\theta = \frac{n}{n+2}\frac{2p_0+2}{2p_0+1} \in (0,1).$$

Then, $\frac{2p_0+1}{2p_0}\theta < 1$ due to $p_0 > \frac{n}{2}$. By the Gagliardo-Nirenberg inequality, Young's inequality and (3.15), we can find some constants $C_3, C_4 > 0$ such that

$$\begin{aligned} (K_3(2p_0) - \rho p_0)\int_\Omega v^{2p_0+1} &= (K_3(2p_0) - \rho p_0)\,\|v^{p_0}\|^{\frac{2p_0+1}{p_0}}_{L^{\frac{2p_0+1}{p_0}}(\Omega)} \\ &\leqslant C_3\left(\|v^{p_0}\|^{\frac{2p_0+1}{p_0}(1-\theta)}_{L^1(\Omega)}\,\|\nabla v^{p_0}\|^{\frac{2p_0+1}{p_0}\theta}_{L^2(\Omega)} + \|v^{p_0}\|^{\frac{2p_0+1}{p_0}}_{L^1(\Omega)}\right) \\ &\leqslant C_3\left(M^{\frac{2p_0+1}{p_0}(1-\theta)}\,\|\nabla v^{p_0}\|^{\frac{2p_0+1}{p_0}\theta}_{L^2(\Omega)} + M^{\frac{2p_0+1}{p_0}}\right) \\ &\leqslant \frac{2p_0-1}{p_0}\int_\Omega |\nabla v^{p_0}|^2 + C_4 \quad \text{for all } t \in (0, T_{max}), \end{aligned}$$

which along with (3.16) implies

$$\frac{d}{dt}\left(\int_\Omega |\nabla u|^{4p_0} + \int_\Omega v^{2p_0}\right) + \int_\Omega |\nabla u|^{4p_0} + \int_\Omega v^{2p_0} \leqslant C_2 + C_4 \quad \text{for all } t \in (0, T_{max}).$$

Therefore, the claim follows from the Grönwall inequality applied to the above inequality.

Step 2: Thanks to the regularity theorem [32, Lemma 1], we can find a constant $C_5 > 0$ such that $\|\nabla u\|_{L^\infty(\Omega)} \leqslant C_5$ due to $2p_0 > n$. With (3.12) and Lemmas 3.1 and 3.2, we get a constant $C_6 > 0$ such that for any $p \geqslant 2$

$$\frac{d}{dt}\int_\Omega v^p + p(p-1)\int_\Omega v^{p-2}|\nabla v|^2 \leqslant p(p-1)(C_6\chi + C_5\xi)\int_\Omega v^{p-1}|\nabla v| + p(\rho + ea_3K_1)\int_\Omega v^p. \tag{3.17}$$

Thanks to Young's inequality, we find a constant $C_7 > 0$ such that

$$p(p-1)(C_6\chi + C_5\xi)\int_\Omega v^{p-1}|\nabla v| \leqslant \frac{p(p-1)}{2}\int_\Omega v^{p-2}|\nabla v|^2 + C_7 p(p-1)\int_\Omega v^p,$$

which together with (3.17) implies

$$\frac{d}{dt}\int_\Omega v^p + p(p-1)\int_\Omega v^p + \frac{2(p-1)}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 \leqslant p(p-1)C_8\int_\Omega v^p, \tag{3.18}$$

with $C_8 = C_7 + \rho + ea_3K_1 + 1$. Applying $1 + p^n \leqslant (1 + p)^n$ and the following inequality [34]

$$\|f\|_{L^2}^2 \leqslant \varepsilon\|\nabla f\|_{L^2}^2 + C_9(1 + \varepsilon^{-\frac{n}{2}})\|f\|_{L^1}^2,$$

with $f = v^{\frac{p}{2}}$ and $\varepsilon = \frac{2}{p^2C_8}$, we find a constant $C_{10} > 0$ such that

$$p(p-1)C_8\int_\Omega u^p \leqslant \frac{2(p-1)}{p}\int_\Omega\left|\nabla u^{\frac{p}{2}}\right|^2 + C_{10}p(p-1)(1+p^n)\left(\int_\Omega u^{\frac{p}{2}}\right)^2. \tag{3.19}$$

Substituting (3.19) into (3.18), we have

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \leqslant C_{10}p(p-1)(1+p)^n\left(\int_\Omega u^{\frac{p}{2}}\right)^2.$$

Then, employing the standard Moser iteration in [35] or a similar argument as in [34], we can prove that there exists a constant $C_{11} > 0$ such that

$$\|v\|_{L^\infty(\Omega)} \leqslant C_{11} \quad \text{for all } t \in (0, T_{max}).$$

Thus, with the help of Lemma 3.2, we finish the proof.

Now, utilizing the criterion in Lemma 3.9, we prove the global existence and boundedness of solutions for the system (1.1).

Proof of Theorem 1.1. If $n \leqslant 2$, then the conclusion of the theorem can be obtained by Lemmas 3.3, 3.5 and 3.9. If $n \geqslant 3$ and

$$\rho \geqslant \frac{2K_3\left(\left[\frac{n}{2}\right]+1\right)}{\left[\frac{n}{2}\right]+1},$$

then according to Lemma 3.8, by fixing $p = [\frac{n}{2}] + 1$ we can find a constant $C_1 > 0$ such that

$$\frac{d}{dt}\left(\int_\Omega |\nabla u|^{2[\frac{n}{2}]+2} + \int_\Omega v^{[\frac{n}{2}]+1}\right) + \int_\Omega |\nabla u|^{2[\frac{n}{2}]+2} + \int_\Omega v^{[\frac{n}{2}]+1} \leqslant C_1 \quad \text{for all } t \in (0, T_{max}),$$

which along with the Grönwall inequality gives a constant $C_2 > 0$,

$$\int_\Omega v^{[\frac{n}{2}]+1} \leqslant C_2 \quad \text{for all } t \in (0, T_{max}).$$

Together with Lemma 3.9, we finish the proof by Lemma 2.1.

4. Stabilization

In this section, we will employ suitable Lyapunov functionals to study the large-time behavior of u, v and w. We first improve the regularity of the solution.

Lemma 4.1. *There exist constants $\theta_1, \theta_2, \theta_3 \in (0, 1)$ and $C > 0$ such that*

$$\|u\|_{C^{2+\theta_1,1+\frac{\theta_1}{2}}(\bar{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\theta_2,1+\frac{\theta_2}{2}}(\bar{\Omega}\times[t,t+1])} + \|w\|_{C^{2+\theta_3,1+\frac{\theta_3}{2}}(\bar{\Omega}\times[t,t+1])} \leqslant C \quad \textit{for all } t > 1.$$

In particular, one can find $C > 0$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)} \leqslant C \quad \textit{for all } t > 1.$$

Proof. The conclusion is a consequence of the regularity of parabolic equations in [36].

We split our analysis into two cases: $a_1 > a_3$ and $a_1 \leqslant a_3$.

4.1. Coexistence: $a_1 > a_3$

We know that there are four homogeneous equilibria $(0,0,0)$, $(0,1,0)$, $\left(\frac{a_1}{a_2}, 0, \frac{ra_1}{\gamma a_2}\right)$ and (u_*, v_*, w_*) when $a_1 > a_3$, where u_*, v_* and w_* are defined in (1.5). In this case, we shall prove the coexistence steady state (u_*, v_*, w_*) is globally exponentially stable under certain conditions. Define an energy functional for (1.1) as follows:

$$\mathcal{F}(t) = \varepsilon_1 \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*}\right) + \frac{\varepsilon_2}{2} \int_\Omega (w - w_*)^2,$$

where ε_1 and ε_2 are to be determined below.

Proof of Theorem 1.2–(1). We complete the proof in four steps.

Step 1: The parameters ε_1 and ε_2 can be chosen in the following way. First, we recall from (1.5) and (1.6) that

$$\Gamma = \frac{4du_*}{K_1^2 v_*}, \Phi = \frac{2a_2\rho}{a_3^2} + e, \Psi = \frac{\gamma\eta a_3^2 K_1^2}{d\rho^2 r^2 u_*}. \tag{4.1}$$

Let

$$f(y) = \frac{\Psi(\Gamma y - \xi^2)(-y^2 + 2\Phi y - e^2)}{y}, \quad y > 0.$$

It is clear that $f \in C^0((0, +\infty))$. Then, if

$$\frac{\xi^2}{\Gamma} < \Phi + \sqrt{\Phi^2 - e^2},$$

the following holds:

$$\frac{\xi^2 K_1^2 v_*}{4du_*} < \frac{2a_2\rho}{a_3^2} + e + \frac{2}{a_3}\sqrt{a_2\rho\left(\frac{a_2\rho}{a_3^2} + e\right)}. \tag{4.2}$$

Under (4.2), we let $a = \max\left\{\frac{\xi^2}{\Gamma}, \Phi - \sqrt{\Phi^2 - e^2}\right\}$ and $b = \Phi + \sqrt{\Phi^2 - e^2}$ with $a < b$. Then, $f(y)$ is continuous on $[a, b]$ with $f(a) = f(b) = 0$, and consequently $f(y)$ must attain the maximum at some point, say ε_1, in (a, b), namely $f(\varepsilon_1) = \max_{y\in[a,b]} f(y)$. Then, $a < \varepsilon_1 < b$, or equivalently (see (4.1))

$$\max\left\{\frac{\xi^2 u^2 v_*}{4du_*}, \frac{2a_2\rho}{a_3^2} + e - \frac{2}{a_3}\sqrt{a_2\rho\left(\frac{a_2\rho}{a_3^2} + e\right)}\right\} < \varepsilon_1 < \frac{2a_2\rho}{a_3^2} + e + \frac{2}{a_3}\sqrt{a_2\rho\left(\frac{a_2\rho}{a_3^2} + e\right)}. \tag{4.3}$$

Next, we assume $\chi > 0$ is suitably small such that

$$\begin{aligned}\chi^2 < f(\varepsilon_1) =& \frac{\gamma\eta a_3^2 K_1^2}{d\rho r^2 u_* \varepsilon_1}\left(\frac{4du_*\varepsilon_1}{v_* K_1^2} - \xi^2\right)\left(-\varepsilon_1^2 + 2\left(\frac{2a_2\rho}{a_3^2} + e\right)\varepsilon_1 - e^2\right)\\ =& \frac{4\gamma\eta}{d\rho r^2 u_* v_* \varepsilon_1}\left(4du_*\varepsilon_1 - \xi^2 v_* K_1^2\right)\left(a_2\rho\varepsilon_1 - \frac{a_3^2(\varepsilon_1 - e)^2}{4}\right),\end{aligned}$$

which implies

$$\frac{d\chi^2 u_* v_*^2 \varepsilon_1}{\eta\left(4du_* v_*\varepsilon_1 - \xi^2 v_*^2 K_1^2\right)} < \frac{4\gamma}{\rho r^2}\left(a_2\rho\varepsilon_1 - \frac{a_3^2(\varepsilon_1 - e)^2}{4}\right).$$

Hence, there exists a constant $\varepsilon_2 > 0$ such that

$$\frac{d\chi^2 u_* v_*^2 \varepsilon_1}{\eta\left(4du_* v_* \varepsilon_1 - \xi^2 v_*^2 K_1^2\right)} < \varepsilon_2 < \frac{4\gamma}{\rho r^2}\left(a_2\rho\varepsilon_1 - \frac{a_3^2(\varepsilon_1 - e)^2}{4}\right)$$

which along with Lemma 3.1 yields

$$\frac{d\chi^2 u_* v_*^2 \varepsilon_1}{\eta\left(4du_* v_* \varepsilon_1 - \xi^2 v_*^2 u^2\right)} < \varepsilon_2 < \frac{4\gamma}{\rho r^2}\left(a_2\rho\varepsilon_1 - \frac{a_3^2(\varepsilon_1 - e)^2}{4}\right). \tag{4.4}$$

Step 2: We claim

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to +\infty.$$

Indeed, using the equations in system (1.1) along with integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(u - u_* - u_* \ln\frac{u}{u_*}\right) = \int_\Omega \frac{u - u_*}{u} u_t \\
&= - du_* \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega (u - u_*)(a_1 - a_2 u - a_3 v) \\
&= - du_* \int_\Omega \frac{|\nabla u|^2}{u^2} - a_2\int_\Omega (u - u_*)^2 - a_3\int_\Omega (u - u_*)(v - v_*).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(v - v_* - v_* \ln\frac{v}{v_*}\right) = \int_\Omega \frac{v - v_*}{v} v_t \\
&= - v_* \int_\Omega \frac{|\nabla v|^2}{v^2} - \chi v_* \int_\Omega \frac{\nabla v \cdot \nabla w}{v} + \xi v_* \int_\Omega \frac{\nabla u \cdot \nabla v}{v} + \int_\Omega (v - v_*)(\rho - \rho v + e a_3 u) \\
&= - v_* \int_\Omega \frac{|\nabla v|^2}{v^2} - \chi v_* \int_\Omega \frac{\nabla v \cdot \nabla w}{v} + \xi v_* \int_\Omega \frac{\nabla u \cdot \nabla v}{v} \\
&\quad - \rho\int_\Omega (v - v_*)^2 + e a_3 \int_\Omega (u - u_*)(v - v_*)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (w - w_*)^2 &= 2\int_\Omega (w - w_*)\, w_t = 2\int_\Omega (w - w_*)\,(\eta\Delta w + ru - \gamma w) \\
&= - 2\eta\int_\Omega |\nabla w|^2 + 2r\int_\Omega (u - u_*)(w - w_*) - 2\gamma\int_\Omega (w - w_*)^2 \quad \text{for all } t > 0.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(t) = &- du_*\varepsilon_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - v_* \int_\Omega \frac{|\nabla v|^2}{v^2} - \eta\varepsilon_2 \int_\Omega |\nabla w|^2 \\
&- \chi v_* \int_\Omega \frac{\nabla v \cdot \nabla w}{v} + \xi v_* \int_\Omega \frac{\nabla u \cdot \nabla v}{v}
\end{aligned}$$

$$
\begin{aligned}
&- a_2\varepsilon_1 \int_\Omega (u-u_*)^2 - \rho \int_\Omega (v-v_*)^2 - \gamma\varepsilon_2 \int_\Omega (w-w_*)^2 \\
&- a_3(\varepsilon_1 - e) \int_\Omega (u-u_*)(v-v_*) + r\varepsilon_2 \int_\Omega (u-u_*)(w-w_*) \\
=&: -X^T S X - Y^T T Y,
\end{aligned}
$$

where $X = (\nabla u, \nabla v, \nabla w)$, $Y = (u-u_*, v-v_*, w-w_*)$, and

$$
S = \begin{bmatrix} \frac{du_*\varepsilon_1}{u^2} & -\frac{\xi v_*}{2v} & 0 \\ -\frac{\xi v_*}{2v} & \frac{v_*}{v^2} & \frac{\chi v_*}{2v} \\ 0 & \frac{\chi v_*}{2v} & \eta\varepsilon_2 \end{bmatrix}, \quad T = \begin{bmatrix} a_2\varepsilon_1 & \frac{a_3(\varepsilon_1-e)}{2} & -\frac{r\varepsilon_2}{2} \\ \frac{a_3(\varepsilon_1-e)}{2} & \rho & 0 \\ -\frac{r\varepsilon_2}{2} & 0 & \gamma\varepsilon_2 \end{bmatrix}.
$$

Note that (4.3) yields

$$
\frac{du_* v_* \varepsilon_1}{u^2 v^2} - \frac{\xi^2 v_*^2}{4v^2} > \frac{v_*^2}{4v^2}\Big(\frac{4du_*\varepsilon}{K_1^2} - \xi^2\Big) > 0,
$$

and (4.4) gives

$$
\frac{\eta d u_* v_* \varepsilon_1 \varepsilon_2}{u^2 v^2} - \frac{d\chi^2 u_* v_*^2 \varepsilon_1}{4u^2 v^2} - \frac{\eta \xi^2 v_*^2 \varepsilon_2}{4v^2} > 0.
$$

The above results indicate that matrix S is positive definite. Using (4.3) and (4.4) again, we observe that

$$
a_2\rho\varepsilon_1 - \frac{a_3^2(\varepsilon_1 - e)^2}{4} > 0,
$$

and

$$
a_2\rho\gamma\varepsilon_1\varepsilon_2 - \frac{\rho r^2 \varepsilon_2^2}{4} - \frac{a_3^2\gamma(\varepsilon_1 - e)^2\varepsilon_2}{4} > 0,
$$

which imply that matrix T is positive definite. Therefore, one can choose a constant $C_1 > 0$ such that

$$
\frac{d}{dt}\mathcal{F}(t) \leqslant -C_1\left(\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2\right) \quad \text{for all } t > 0. \tag{4.5}
$$

Integrating the above inequality with respect to time, we get a constant $C_2 > 0$ satisfying

$$
\int_1^{+\infty}\int_\Omega (u-u_*)^2 + \int_1^{+\infty}\int_\Omega (v-v_*)^2 + \int_1^{+\infty}\int_\Omega (w-w_*)^2 \leqslant C_2,
$$

which together with the uniform continuity of u, v and w due to Lemma 4.1 yields

$$
\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2 \to 0, \quad \text{as } t \to +\infty. \tag{4.6}
$$

By the Gagliardo-Nirenberg inequality, we can find a constant $C_3 > 0$ such that

$$
\|u - u_*\|_{L^\infty(\Omega)} \leqslant C_3 \|u-u_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|u-u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}}, \tag{4.7}
$$

$$
\|v - v_*\|_{L^\infty(\Omega)} \leqslant C_3 \|v-v_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|v-v_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \tag{4.8}
$$

and

$$\|w - w_*\|_{L^\infty(\Omega)} \leqslant C_3 \|w - w_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|w - w_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \quad \text{for all } t > 1, \tag{4.9}$$

which along with (4.6) and Lemma 4.1 prove the claim.

Step 3: From the L'Hôpital rule, it holds that for any $s_0 > 0$

$$\lim_{s \to s_0} \frac{s - s_0 - s_0 \ln \frac{s}{s_0}}{(s - s_0)^2} = \lim_{s \to s_0} \frac{1 - \frac{s_0}{s}}{2(s - s_0)} = \lim_{s \to s_0} \frac{1}{2s} = \frac{1}{2s_0},$$

which gives a constant $\eta > 0$ such that for all $|s - s_0| \leqslant \eta$

$$\frac{1}{4s_0}(s - s_0)^2 \leqslant s - s_0 - s_0 \ln \frac{s}{s_0} \leqslant \frac{1}{s_0}(s - s_0)^2. \tag{4.10}$$

By (4.6), there exists $T_1 > 1$ such that

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} \leqslant \eta \quad \text{for all } t \geqslant T_1.$$

Therefore, by (4.10), we get

$$\frac{1}{4u_*} \int_\Omega (u - u_*)^2 \leqslant \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*} \right) \leqslant \frac{1}{u_*} \int_\Omega (u - u_*)^2 \quad \text{for all } t \geqslant T_1 \tag{4.11}$$

and

$$\frac{1}{4v_*} \int_\Omega (v - v_*)^2 \leqslant \int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*} \right) \leqslant \frac{1}{v_*} \int_\Omega (v - v_*)^2 \quad \text{for all } t \geqslant T_1. \tag{4.12}$$

Step 4: From (4.11) and (4.12), it follows that

$$\mathcal{F}(t) \leqslant \max \left\{ \frac{\varepsilon_1}{u_*}, \frac{1}{v_*}, \frac{\varepsilon_2}{2} \right\} \left(\int_\Omega (u - u_*)^2 + \int_\Omega (v - v_*)^2 + \int_\Omega (w - w_*)^2 \right),$$

which alongside (4.5) yields a constant $C_4 > 0$ such that

$$\frac{d}{dt} \mathcal{F}(t) \leqslant -C_4 \mathcal{F}(t) \quad \text{for all } t \geqslant T_1.$$

This immediately gives a constant $C_5 > 0$ such that

$$\mathcal{F}(t) \leqslant C_5 e^{-C_4 t} \quad \text{for all } t \geqslant T_1.$$

Hence, utilizing (4.11) and (4.12) again, one obtains a constant $C_6 > 0$ such that

$$\int_\Omega (u - u_*)^2 + \int_\Omega (v - v_*)^2 + \int_\Omega (w - w_*)^2 \leqslant C_6 e^{-C_4 t} \quad \text{for all } t \geqslant T_1.$$

Finally, by (4.7)–(4.9) and Lemma 4.1, we get the decay rates of $\|u - u_*\|_{L^\infty(\Omega)}$, $\|v - v_*\|_{L^\infty(\Omega)}$ and $\|w - w_*\|_{L^\infty(\Omega)}$, as claimed in Theorem 1.2–(1).

4.2. *Predator-only:* $a_1 \leqslant a_3$

In this case, there are three homogeneous equilibria $(0,0,0)$, $(0,1,0)$ and $\left(\frac{a_1}{a_2}, 0, \frac{ra_1}{\gamma a_2}\right)$, and we shall show that the steady state $(0,1,0)$ is global asymptotically stable, where the convergence rate is exponential if $a_1 < a_3$ and algebraic if $a_1 = a_3$. Define an energy functional for (1.1) as follows:

$$G(t) = e\int_\Omega u + \frac{\zeta_1}{2}\int_\Omega u^2 + \int_\Omega (v - 1 - \ln v) + \frac{\zeta_2}{2}\int_\Omega w^2,$$

where ζ_1 and ζ_2 will be determined below.

Proof of Theorem 1.2–(2). We divide the proof into five steps.

Step 1: We shall choose the appropriate parameters ζ_1 and ζ_2. By the definitions of A and B in (1.7), since $A < B$, we have

$$\left(\frac{\xi^2}{4d}\right)^2 < \frac{\xi^2 e a_2}{4 d a_1} < \left(\frac{e a_2}{a_1}\right)^2. \tag{4.13}$$

Let

$$g(y) = \frac{16\eta\gamma}{dr^2}\frac{(dy - \frac{\xi^2}{4})(ea_2 - a_1 y)}{y}, \quad \frac{\xi^2}{4d} < y < \frac{ea_2}{a_1}.$$

Then, $g \in C^1\left(\left(\frac{\xi^2}{4d}, \frac{ea_2}{a_1}\right)\right)$, and $g(y) > 0$ in $\left(\frac{\xi^2}{4d}, \frac{ea_2}{a_1}\right)$. We further observe that

$$g\left(\frac{\xi}{2}\sqrt{\frac{ea_2}{da_1}}\right) = D\left(A + B - 2\sqrt{AB}\right)$$

which along with $\chi^2 < D\left(A + B - 2\sqrt{AB}\right)$ implies

$$\chi^2 < g\left(\frac{\xi}{2}\sqrt{\frac{ea_2}{da_1}}\right).$$

By the definition of g, one has

$$g'(y_0) = \frac{16\eta\gamma}{dr^2}\left(-da_1 + \frac{\xi^2 e a_2}{4y_0^2}\right) = 0,$$

which alongside (4.13) gives $y_0 = \frac{\xi}{2}\sqrt{\frac{ea_2}{da_1}} \in \left(\frac{\xi^2}{4d}, \frac{ea_2}{a_1}\right)$. Thus, $g(y)$ is increasing in $\left(\frac{\xi^2}{4d}, \frac{\xi}{2}\sqrt{\frac{ea_2}{da_1}}\right)$ and decreasing in $\left(\frac{\xi}{2}\sqrt{\frac{ea_2}{da_1}}, \frac{ea_2}{a_1}\right)$. We can find a constant $\zeta_1 > 0$ such that

$$\frac{\xi}{2}\sqrt{\frac{ea_2}{da_1}} < \zeta_1 < \frac{ea_2}{a_1} \tag{4.14}$$

and

$$0 = g\left(\frac{ea_2}{a_1}\right) < \chi^2 < g(\zeta_1) < g\left(\frac{\xi}{2}\sqrt{\frac{ea_2}{da_1}}\right).$$

With the definition of g, we get

$$\frac{d\chi^2\zeta_1}{4\eta(d\zeta_1 - \frac{\xi^2}{4})} < \frac{4\gamma}{r^2}(ea_2 - a_1\zeta_1),$$

which implies that there exists $\zeta_2 > 0$ such that

$$\frac{d\chi^2\zeta_1}{4\eta(d\zeta_1 - \frac{\xi^2}{4})} < \zeta_2 < \frac{4\gamma}{r^2}(ea_2 - a_1\zeta_1). \tag{4.15}$$

One can verify that

$$\eta d\zeta_1\zeta_2 - \frac{d\chi^2}{4}\zeta_1 - \frac{\eta\xi^2}{4}\zeta_2 > 0, \tag{4.16}$$

and

$$(ea_2 - a_1\zeta_1)\rho\gamma\zeta_2 - \frac{\rho r^2}{4}\zeta_2^2 > 0. \tag{4.17}$$

Thanks to (4.13) and (4.14), one obtains

$$\frac{\xi^2}{4d} < \zeta_1 < \frac{ea_2}{a_1}. \tag{4.18}$$

Step 2: We claim

$$\|u\|_{L^\infty(\Omega)} + \|v - 1\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to +\infty. \tag{4.19}$$

Indeed, if (u, v, w) is the solution of system (1.1), then we get

$$\frac{d}{dt}\int_\Omega u = a_1\int_\Omega u - a_2\int_\Omega u^2 - a_3\int_\Omega uv, \tag{4.20}$$

$$\frac{d}{dt}\int_\Omega u^2 = 2\int_\Omega uu_t = -2d\int_\Omega |\nabla u|^2 + 2a_1\int_\Omega u^2 - 2a_2\int_\Omega u^3 - 2a_3\int_\Omega u^2 v, \tag{4.21}$$

$$\begin{aligned}\frac{d}{dt}\int_\Omega (v - 1 - \ln v) &= \int_\Omega \frac{v-1}{v}v_t\\ = &-\int_\Omega \frac{|\nabla v|^2}{v^2} - \chi\int_\Omega \frac{\nabla v\cdot\nabla w}{v} + \xi\int_\Omega \frac{\nabla u\cdot\nabla v}{v} + \int_\Omega (v-1)(\rho - \rho v + ea_3 u)\\ = &-\int_\Omega \frac{|\nabla v|^2}{v^2} - \chi\int_\Omega \frac{\nabla v\cdot\nabla w}{v} + \xi\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \rho\int_\Omega (v-1)^2 + ea_3\int_\Omega uv - ea_3\int_\Omega u\end{aligned} \tag{4.22}$$

and

$$\frac{d}{dt}\int_\Omega w^2 = 2\int_\Omega ww_t = -2\eta\int_\Omega |\nabla w|^2 + 2r\int_\Omega uw - 2\gamma\int_\Omega w^2 \quad \text{for all } t > 0. \tag{4.23}$$

Then, combining (4.20), (4.21), (4.22) and (4.23), we have from the definition of $G(t)$ that

$$\begin{aligned}\frac{d}{dt}G(t) \leqslant &- d\zeta_1\int_\Omega |\nabla u|^2 - \int_\Omega \frac{|\nabla v|^2}{v^2} - \eta\zeta_2\int_\Omega |\nabla w|^2\\ &-\chi\int_\Omega \frac{\nabla v\cdot\nabla w}{v} + \xi\int_\Omega \frac{\nabla u\cdot\nabla v}{v} + e(a_1 - a_3)\int_\Omega u\\ &-(ea_2 - a_1\zeta_1)\int_\Omega u^2 - \rho\int_\Omega (v-1)^2 - \gamma\zeta_2\int_\Omega w^2 + r\zeta_2\int_\Omega uw\\ =: &-X^T PX - Y^T QY + e(a_1 - a_3)\int_\Omega u,\end{aligned} \tag{4.24}$$

where $X = (\nabla u, \nabla v, \nabla w)$, $Y = (u, v-1, w)$,

$$P = \begin{bmatrix} d\zeta_1 & -\frac{\xi}{2v} & 0 \\ -\frac{\xi}{2v} & \frac{1}{v^2} & \frac{\chi}{2v} \\ 0 & \frac{\chi}{2v} & \eta\zeta_2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} ea_2 - a_1\zeta_1 & 0 & -\frac{r\zeta_2}{2} \\ 0 & \rho & 0 \\ -\frac{r\zeta_2}{2} & 0 & \gamma\zeta_2 \end{bmatrix}.$$

It can be checked that (4.16) and (4.18) ensure that the matrix P is positive definite while (4.17) and (4.18) guarantee that the matrix Q is positive definite. Thus, there is a constant $C_1 > 0$ such that if $a_1 < a_3$, then

$$\frac{d}{dt}G(t) \leqslant -C_1\left(\int_\Omega u + \int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega w^2\right) \quad \text{for all } t > 0, \tag{4.25}$$

and if $a_1 = a_3$, then

$$\frac{d}{dt}G(t) \leqslant -C_1\left(\int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega w^2\right) \quad \text{for all } t > 0. \tag{4.26}$$

Integrating the above inequalities with respect to time, we find a constant $C_2 > 0$ satisfying

$$\int_1^{+\infty}\int_\Omega u^2 + \int_1^{+\infty}\int_\Omega (v-1)^2 + \int_1^{+\infty}\int_\Omega w^2 \leqslant C_2,$$

which together with the uniform continuity of u, v and w due to Lemma 4.1 yields

$$\int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega w^2 \to 0, \quad \text{as } t \to +\infty. \tag{4.27}$$

Thus, (4.19) is obtained by the Gagliardo-Nirenberg inequality and Lemma 4.1.

Step 3: By the L'Hôpital rule, we get

$$\lim_{s\to 1}\frac{s-1-\ln s}{(s-1)^2} = \lim_{s\to 1}\frac{1-\frac{1}{s}}{2(s-1)} = \lim_{s\to 1}\frac{1}{2s} = \frac{1}{2},$$

which gives a constant $\varepsilon > 0$ such that

$$\frac{1}{4}(s-1)^2 \leqslant s-1-\ln s \leqslant (s-1)^2 \ \text{ for all } \ |s-1| \leqslant \varepsilon. \tag{4.28}$$

By (4.19), there exists $T_1 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \leqslant \varepsilon \quad \text{for all } t \geqslant T_1. \tag{4.29}$$

Therefore, it follows from (4.28) that

$$\frac{1}{4}\int_\Omega (v-1)^2 \leqslant \int_\Omega (v-1-\ln v) \leqslant \int_\Omega (v-1)^2 \quad \text{for all } t \geqslant T_1. \tag{4.30}$$

Step 4: If $a_1 < a_3$, from the definition of $G(t)$ and (4.30), one has

$$G(t) \leqslant \max\left\{e, \frac{\zeta_1}{2}, \frac{\zeta_2}{2}, 1\right\}\left(\int_\Omega u + \int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega w^2\right),$$

which along with (4.25) yields a constant $C_3 > 0$ such that

$$\frac{d}{dt}G(t) \leqslant -C_3 G(t) \quad \text{for all } t \geqslant T_1.$$

This gives a constant $C_4 > 0$ such that

$$G(t) \leqslant C_4 e^{-C_3 t} \quad \text{for all } t \geqslant T_1.$$

Hence, utilizing (4.30) again, we find a constant $C_5 > 0$ such that

$$\int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega w^2 \leqslant C_5 e^{-C_3 t} \quad \text{for all } t \geqslant T_1.$$

Then, by the Gagliardo-Nirenberg inequality and Lemma 4.1, we get the exponential convergence for $\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}$.

Step 5: If $a_1 = a_3$, we use (4.29), (4.30) and Young's inequality to find a constant $C_6 > 0$:

$$\begin{aligned} G^2(t) \leqslant & C_6 \left(\int_\Omega u + \int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega w^2 \right)^2 \\ \leqslant & C_6 (\varepsilon + 1)^2 \left(\int_\Omega u + \int_\Omega (v-1) + \int_\Omega w \right)^2 \\ \leqslant & 3C_6 (\varepsilon + 1)^2 |\Omega| \left(\int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega w^2 \right) \quad \text{for all } t \geqslant T_1, \end{aligned}$$

which alongside (4.26) implies some constant $C_7 > 0$

$$\frac{d}{dt}G(t) \leqslant -C_7 G^2(t) \quad \text{for all } t \geqslant T_1.$$

Solving the above inequality directly yields a constant $C_8 > 0$ such that

$$G(t) \leqslant C_8 (t+1)^{-1} \quad \text{for all } t \geqslant T_1.$$

Similar to the case $a_1 < a_3$, we can use (4.30), the Gagliardo-Nirenberg inequality and Lemma 4.1 to get the convergence rate of $\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}$.

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Conflict of interest

The author declares there is no conflict of interest.

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