Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 19(12): 13421–13457.
DOI: 10.3934/mbe.2022628
Received: 29 June 2022
Revised: 21 August 2022
Accepted: 29 August 2022
Published: 14 September 2022

Theory article

On a two-species competitive predator-prey system with density-dependent diffusion

Pan Zheng[1,2,3,*]

[1] College of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, China

[2] Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong 999077, China

[3] School of Mathematics and Statistics, Yunnan University, Kunming 650091, China

* **Correspondence:** Email: zhengpan52@sina.com; Tel: +8615730493053.

Abstract: This paper deals with a two-species competitive predator-prey system with density-dependent diffusion, i.e.,

$$\begin{cases} u_t = \Delta(d_1(w)u) + \gamma_1 u F_1(w) - u h_1(u) - \beta_1 uv, & (x,t) \in \Omega \times (0,\infty), \\ v_t = \Delta(d_2(w)v) + \gamma_2 v F_2(w) - v h_2(v) - \beta_2 uv, & (x,t) \in \Omega \times (0,\infty), \\ w_t = D\Delta w - uF_1(w) - vF_2(w) + f(w), & (x,t) \in \Omega \times (0,\infty), \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^2$, with the nonnegative initial data $(u_0, v_0, w_0) \in (W^{1,p}(\Omega))^3$ with $p > 2$, where the parameters $D, \gamma_1, \gamma_2, \beta_1, \beta_2 > 0$, $d_1(w)$ and $d_2(w)$ are density-dependent diffusion functions, $F_1(w)$ and $F_2(w)$ are commonly called the functional response functions accounting for the intake rate of predators as the functions of prey density, $h_1(u)$ and $h_2(v)$ represent the mortality rates of predators, and $f(w)$ stands for the growth function of the prey. First, we rigorously prove the global boundedness of classical solutions for the above general model provided that the parameters satisfy some suitable conditions by means of L^p-estimate techniques. Moreover, in some particular cases, we establish the asymptotic stabilization and precise convergence rates of globally bounded solutions under different conditions on the parameters by constructing some appropriate Lyapunov functionals. Our results not only extend the previous ones, but also involve some new conclusions.

Keywords: boundedness; stabilization; predator-prey model; density-dependent diffusion; competition

1. Introduction

In 1987, Karevia and Odell [1] first proposed the following one-predator and one-prey model with prey-taxis in order to explain that an area-restricted search creates the following predator aggregation phenomenon

$$\begin{cases} u_t = \nabla \cdot (d(w)\nabla u) - \nabla \cdot (u\chi(w)\nabla w) + G_1(u,w), \\ w_t = D\Delta w + G_2(u,w), \end{cases} \tag{1.1}$$

where $D > 0$ is the diffusivity coefficient of preys, $d(w)$ denotes the motility function of predators, $\chi(w)$ represents the prey-taxis sensitivity coefficient, and the term $-\nabla \cdot (u\chi(w)\nabla w)$ stands for the tendency of the predator moving towards the increasing direction of the prey gradient, and it is viewed as the prey-taxis term. The functions $G_1(u,w)$ and $G_2(u,w)$ describe the predator-prey interactions, which include both intra-specific and inter-specific interactions. Generally the predator-prey interaction functions $G_1(u,w)$ and $G_2(u,w)$ possess the following prototypical forms.

$$G_1(u,w) = \gamma uF(w) - uh(u), \quad G_2(u,w) = -uF(w) + f(w), \tag{1.2}$$

where $\gamma > 0$ denotes the intrinsic predation rate, $uF(w)$ represents the inter-specific interaction and $uh(u)$ and $f(w)$ stand for the intra-specific interaction. Specifically, $F(w)$ is the functional response function accounting for the intake rate of predators as a function of prey density; it is often used in the following form in the literature [2–4]

$$\begin{aligned} &F(w) = w \text{ (Holling type I)}, \quad F(w) = \frac{w}{\lambda + w} \text{ (Holling type II)}, \\ &F(w) = \frac{w^m}{\lambda^m + w^m} \text{ (Holling type III)}, \quad F(w) = 1 - e^{-\lambda w} \text{ (Ivlev type)} \end{aligned} \tag{1.3}$$

with constants $\lambda > 0$ and $m > 1$; other types of functional response functions (e.g., Beddington-DeAngelis type in [5], Crowley-Martin type in [6]) and more predator-prey interactions can be found in [7–10]. The predator mortality rate function $h(u)$ is typically of the form

$$h(u) = \theta + \alpha u, \tag{1.4}$$

where $\theta > 0$ accounts for the natural death rate and $\alpha \geq 0$ denotes the rate of death resulting from the intra-specific competition, which is also called the density-dependent death [11]. The prey growth function $f(w)$ is usually assumed to be negative for large w due to the limitation of resources (or crowding effect), and its typical forms are

$$\begin{aligned} &f(w) = \mu w\left(1 - \frac{w}{K}\right) \text{ (Logistic type)}, \quad \text{or} \\ &f(w) = \mu w\left(1 - \frac{w}{K}\right)\left(\frac{w}{k} - 1\right) \text{ (Bistable or Allee effect type)}, \end{aligned} \tag{1.5}$$

where $\mu > 0$ is the intrinsic growth rate of prey, $K > 0$ is called the carrying capacity and $0 < k < K$. Now there exist many interesting results about global existence, uniform boundedness, asymptotic behavior, traveling waves and pattern formation of solutions to System (1.1) or its variants in [11–15].

When $d(w) = d > 0$ and $\chi(w) = \chi > 0$, $G_1(u, v)$ and $G_2(u, v)$ have the forms of (1.2), Wu et al. [15] obtained the global existence and uniform persistence of solutions to (1.1) in any dimension provided that χ is suitably small. Then, Jin and Wang [13] derived the global boundedness and asymptotic stability of solutions for System (1.1) without the smallness assumption on χ in a two-dimensional bounded domain. Moreover, Wang et al. [16] studied the nonconstant positive steady states and pattern formation of (1.1) in a one-dimensional bounded domain. Under the conditions that $d(w)$ and $\chi(w)$ are not constants and $h(u)$ is given by (1.4), Jin and Wang [14] established the global boundedness, asymptotic behavior and spatio-temporal patterns of solutions for (1.1) under some conditions on the parameters in a two-dimensional smooth bounded domain. For more related results in predator-prey models, we refer the readers to [17–29] and the references therein.

However, all the aforementioned works are devoted to studying prey-taxis models with one-predator and one prey. Now let us mention some predator-prey models with two-predator and one-prey. Recently, the following general two-predator and one-prey model with prey-taxis has attracted a lot of attention.

$$\begin{cases} u_t = \nabla \cdot (d_1(w)\nabla u) - \nabla \cdot (u\chi_1(w)\nabla w) + \gamma_1 u F_1(w) - u h_1(u) - \beta_1 uv, \\ v_t = \nabla \cdot (d_2(w)\nabla v) - \nabla \cdot (v\chi_2(w)\nabla w) + \gamma_2 v F_2(w) - v h_2(v) - \beta_2 uv, \\ w_t = D\Delta w - uF_1(w) - vF_2(w) + f(w), \end{cases} \tag{1.6}$$

as applied in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Given $d_1(w) = d_2(w) = 1$ and $\beta_1 = \beta_2 = 0$, Wang et al. [30] derived the global boundedness, nonconstant positive steady states and time-periodic patterns of solutions for System (1.6). Wang and Wang [31] studied the uniform boundedness and asymptotic stability of nonnegative spatially homogeneous equilibria for (1.6) in any dimension. Given $d_1(w) = d_2(w) = 1$, $\chi_i(w) = \chi_i > 0$ $(i = 1, 2)$ and $\beta_1 = \beta_2 = \beta > 0$, Mi et al. [32] obtained the global boundedness and stability of classical solutions in any dimension under suitable conditions of parameters. Under the conditions that $d_1(w)$ and $d_2(w)$ are non-constants and $\beta_1 = \beta_2 = 0$, Qiu et al. [33] rigorously proved the global existence, uniform boundedness and stabilization of classical solutions in any dimension with suitable conditions on motility functions and the coefficients of the logistic source. However, when $\beta_1, \beta_2 \neq 0$, the global existence and stabilization of solutions for (1.6) are still open. Given $\chi_i(w) = -d_i'(w) \geq 0$ if $d_i'(w) \leq 0$ $(i = 1, 2)$, the diffusion-advection terms in (1.6) can respectively become the forms $\Delta(d_1(w)u)$ and $\Delta(d_2(w)v)$, which could be interpreted as "density-suppressed motility" in [34, 35]. This means that the predator will reduce its motility when encountering the prey, which is a rather reasonable assumption that has very sound applications in the predator-prey systems. Since the possible degeneracy caused by the density-suppressed motility brings considerable challenges for mathematical analysis, many works have showed various interesting results, which can be found in [36–44]. Given $\chi_i(w) = -d_i'(w) \geq 0$ $(i = 1, 2)$, $F_1(w) = F_2(w) = w$, $h_1(u) = u, h_2(v) = v$ and $f(w) = \mu w(m(x) - w)$, System (1.6) can be simplified as

$$\begin{cases} u_t = \Delta(d_1(w)u) + u(\gamma_1 w - u - \beta_1 v), \\ v_t = \Delta(d_2(w)v) + v(\gamma_2 w - v - \beta_2 u), \\ w_t = D\Delta w - (u + v)w + \mu w(m(x) - w), \end{cases} \tag{1.7}$$

where the parameters $D, \mu, \gamma_i, \beta_i$ $(i = 1, 2)$ are positive, the dispersal rate functions $d_i(w)$ $(i = 1, 2)$ satisfy following the hypothesis: $d_i(w) \in C^2([0, \infty)), d_i'(w) \leq 0$ on $[0, \infty)$ and $d_i(w) > 0$. Wang and

Xu [45] have found some interesting results for System (1.7) in a two-dimensional smooth bounded domain. More specifically, when $D = 1$ and $m(x) = 1$, System (1.7) has a unique globally bounded classical solution. By constructing appropriate Lyapunov functionals and using LaSalle's invariant principle, the authors proved that the global bounded solution of (1.7) converges to the co-existence steady state exponentially or competitive exclusion steady state algebraically as time tends to infinity in different parameter regimes. For a prey's resource that is spatially heterogeneous (i.e., $m(x)$ is non-constant), the authors used numerical simulations to demonstrate that the striking phenomenon "slower diffuser always prevails" given in [46, 47] fails to appear if the non-random dispersal strategy is employed by competing species (i.e., either $d_1(w)$ or $d_2(w)$ is non-constant) while it still holds if both $d_1(w)$ and $d_2(w)$ are constants. However, there are few results about global boundedness and large time behavior of solutions for (1.7) in the general form.

Inspired by the above works, this paper is concerned with the following two-species competitive predator-prey system with the following density-dependent diffusion

$$\begin{cases} u_t = \Delta(d_1(w)u) + \gamma_1 u F_1(w) - u h_1(u) - \beta_1 uv, & (x,t) \in \Omega \times (0,\infty), \\ v_t = \Delta(d_2(w)v) + \gamma_2 v F_2(w) - v h_2(v) - \beta_2 uv, & (x,t) \in \Omega \times (0,\infty), \\ w_t = D\Delta w - uF_1(w) - vF_2(w) + f(w), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (u,v,w)(x,0) = (u_0,v_0,w_0)(x), & x \in \Omega, \end{cases} \tag{1.8}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outward normal vector ν of $\partial\Omega$, and the parameters D and γ_i, β_i $(i = 1, 2)$ are positive. The unknown functions $u = u(x,t)$ and $v = v(x,t)$ denote the densities of two-competing species (e.g., predators), and $w = w(x,t)$ represents the density of predators' resources (e.g., the prey) at a position x and time $t > 0$. When $d_1(w) = d_1 > 0$ and $d_2(w) = d_2 > 0$, System (1.8) becomes the well-known diffusive predator-prey system, which has been widely studied in [48–51]. However, to the best of our knowledge, the results of the two-predator and one-prey system given by (1.8) with density-suppressed motility (i.e., $d_1(w)$ and $d_2(w)$ are non-constants) indicate that the competition and general predator mortality rate $h_i(u)$ are almost vacant. The main aim of this paper is to explore the influence of the predation interaction, competition and general predator mortality on the dynamical behavior of System (1.8). Throughout this paper, we assume that the functions $d_i(s), F_i(s), h_i(s), (i = 1, 2), f(s)$ and initial data (u_0, v_0, w_0) mentioned in (1.8) satisfy the following hypotheses:

(H_1) $d_i(s) \in C^2([0,\infty))$ with $d_i(s) > 0$ and $d_i'(s) \le 0$, $i = 1, 2$ on $[0,\infty)$. The typical example is $d_i(s) = \frac{1}{(1+s)^{\kappa_i}}$ or $d_i(s) = \exp(-\kappa_i s)$ with $\kappa_i > 0$, $i = 1, 2$.

(H_2) $F_i(s) \in C^1([0,\infty))$, $F_i(0) = 0$, $F_i(s) > 0$ and $F_i'(s) > 0$, $i = 1, 2$ in $(0,\infty)$.

(H_3) $h_i(s) \in C^2([0,\infty))$ and there exist constants $\theta_i > 0$ and $\alpha_i \ge 0$ such that $h_i(s) \ge \theta_i$ and $h_i'(s) \ge \alpha_i$, $i = 1, 2$ for all $s > 0$.

(H_4) $f(s) \in C^1([0,\infty))$ with $f(0) = 0$, and there exist positive constants μ and K such that $f(s) \le \mu s$ for all $s \ge 0$, $f(K) = 0$ and $f(s) < 0$ for $s > K$.

(H_5) $(u_0, v_0, w_0) \in (W^{1,p}(\Omega))^3$ with $p > 2$ and $u_0, v_0, w_0 \ge 0$.

Here, we note that there are many candidates for the above functions $F_i(s)$, $h_i(s)$ and $f(s)$ as in (1.3)–(1.5). Due to the presence of the prey's density dependent diffusion coefficient, Model (1.8) is a cross-diffusion system and the parabolic comparison principle is no longer applicable. Moreover, when $\alpha_1 = \alpha_2 = 0$, the key L^2-spatiotemporal estimates of u and v cannot be directly derived; thus, the uniform boundedness of solutions is not an obvious result and needs to be justified. Based on the above hypotheses, the first main result of this paper asserts the global existence and boundedness of solutions for System (1.8) as follows.

Theorem 1.1. *Let $D, \gamma_i, \beta_i > 0$ $(i = 1, 2)$, $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and the hypotheses (H_1)–(H_5) hold. Then System (1.8) has a unique global nonnegative classical solution $(u, v, w) \in (C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)))^3$, which is uniformly bounded in time, i.e., there exists a constant $C > 0$ independent of t such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t > 0. \tag{1.9}$$

In particular, one has $0 \leq w(x, t) \leq K_0$ for all $(x, t) \in \Omega \times (0, \infty)$, where

$$K_0 := \max\{\|w_0\|_{L^\infty(\Omega)}, K\}. \tag{1.10}$$

Remark 1.1. For the special case $F_1(w) = F_2(w) = w$, $h_1(u) = u$, $h_2(v) = v$ and $f(w) = \mu w(1 - w)$, the results of Theorem 1.1 have been obtained in [45]. However, since α_1 and α_2 may be equal to zero in the hypotheses of this paper, the L^2-spatiotemporal estimates of u and v cannot be directly obtained by using the method in [45]. By means of the mechanism "density-suppressed motility", we invoke some ideas used in [14] and apply the self-adjoint realization of $\Delta + \delta$ with some $\delta > 0$ in $L^2(\Omega)$ to establish the key L^2-spatiotemporal estimates of u and v.

The second main aim of this paper is to study the role of non-random dispersal and competition between two predators in the asymptotic properties of the nonnegative spatial homogeneous equilibria of System (1.8). For simplicity, we let $F_1(w) = F_2(w) = w$, $h_1(u) = \theta_1 + \alpha_1 u$, $h_2(v) = \theta_2 + \alpha_2 v$ and $f(w) = \mu w(1-w)$, where $\theta_1 = \theta_2 = \theta > 0$ and $\alpha_1, \alpha_2, \mu > 0$; then, System system (1.8) can be simplified as

$$\begin{cases} u_t = \Delta(d_1(w)u) + \gamma_1 uw - u(\theta + \alpha_1 u) - \beta_1 uv, & (x,t) \in \Omega \times (0, \infty), \\ v_t = \Delta(d_2(w)v) + \gamma_2 vw - v(\theta + \alpha_2 v) - \beta_2 uv, & (x,t) \in \Omega \times (0, \infty), \\ w_t = D\Delta w - (u + v)w + \mu w(1 - w), & (x,t) \in \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, \infty), \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega. \end{cases} \tag{1.11}$$

Theorem 1.1 ensures that System (1.11) possesses a unique global bounded nonnegative classical solution (u, v, w) such that $0 \leq w(x, t) \leq K_0 := \max\{\|w_0\|_{L^\infty(\Omega)}, 1\}$ for all $(x, t) \in \Omega \times (0, \infty)$. Now we find some sufficient conditions of parameters so that (1.11) admits a positive constant solution (u^*, v^*, w^*) satisfying

$$\begin{cases} \gamma_1 w^* - \theta - \alpha_1 u^* - \beta_1 v^* &= 0, \\ \gamma_2 w^* - \theta - \alpha_2 v^* - \beta_2 u^* &= 0, \\ - u^* - v^* + \mu - \mu w^* &= 0, \end{cases} \tag{1.12}$$

i.e.,

$$AX = B, \tag{1.13}$$

where

$$A = \begin{bmatrix} -\alpha_1 & -\beta_1 & \gamma_1 \\ -\beta_2 & -\alpha_2 & \gamma_2 \\ 1 & 1 & \mu \end{bmatrix}, \ X = \begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix}, \ B = \begin{bmatrix} \theta \\ \theta \\ \mu \end{bmatrix}.$$

If the determinant Φ of the coefficient matrix A in (1.13) does not equal zero, it follows from Cramer's rule that

$$u^* = \frac{\Phi_u}{\Phi}, \quad v^* = \frac{\Phi_v}{\Phi}, \quad w^* = \frac{\Phi_w}{\Phi}, \tag{1.14}$$

where

$$\begin{cases} \Phi = (\gamma_1 + \beta_1\mu)(\alpha_2 - \beta_2) + (\gamma_2 + \alpha_2\mu)(\alpha_1 - \beta_1), \\ \Phi_u = \alpha_2\mu(\gamma_1 - \theta) - \beta_1\mu(\gamma_2 - \theta) + \theta(\gamma_1 - \gamma_2), \\ \Phi_v = \alpha_1\mu(\gamma_2 - \theta) - \beta_2\mu(\gamma_1 - \theta) + \theta(\gamma_2 - \gamma_1), \\ \Phi_w = (\theta + \beta_1\mu)(\alpha_2 - \beta_2) + (\theta + \alpha_2\mu)(\alpha_1 - \beta_1). \end{cases} \tag{1.15}$$

When $\beta_1 < \alpha_1$ and $\beta_2 < \alpha_2$, it follows that $\Phi > 0$ and $\Phi_w > 0$, and thus we know $w^* > 0$. Next, we shall discuss the sign of Φ_u and Φ_v. For convenience, we let

$$\beta_1^* := \frac{l_1}{\mu(\gamma_2 - \theta)} \tag{1.16}$$

and

$$\beta_2^* := \frac{l_2}{\mu(\gamma_1 - \theta)}, \tag{1.17}$$

where

$$l_1 := \alpha_2\mu(\gamma_1 - \theta) + \theta(\gamma_1 - \gamma_2) \tag{1.18}$$

and

$$l_2 := \alpha_1\mu(\gamma_2 - \theta) + \theta(\gamma_2 - \gamma_1). \tag{1.19}$$

It is not difficult to see that $u^* = \frac{\Phi_u}{\Phi}$, $v^* = \frac{\Phi_v}{\Phi}$ and $w^* = \frac{\Phi_w}{\Phi}$ are positive, if $\gamma_i > \theta$, $i = 1, 2$ and one of the following conditions holds:
(H_6) $\gamma_1 < \gamma_2$, $l_1 > 0$, $\beta_1 < \min\{\alpha_1, \beta_1^*\}$ and $\beta_2 < \min\{\alpha_2, \beta_2^*\}$;
(H_7) $\gamma_1 > \gamma_2$, $l_2 > 0$, $\beta_1 < \min\{\alpha_1, \beta_1^*\}$ and $\beta_2 < \min\{\alpha_2, \beta_2^*\}$;
(H_8) $\gamma_1 = \gamma_2$ and $\max\{\beta_1, \beta_2\} < \min\{\alpha_1, \alpha_2\}$.

Now, we give our main results on the asymptotic stability properties of the nonnegative spatial homogeneous equilibria of System (1.11) as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and the parameters $\gamma_i, \alpha_i, \beta_i$, $(i = 1, 2)$, θ, μ and D be positive. Assume that $d_1(w)$ and $d_2(w)$ satisfy (H_1), and that (u, v, w) is a global bounded classical solution of System (1.11). Suppose that $\gamma_i > \theta, i = 1, 2$,*

$$(\beta_1\gamma_2 + \beta_2\gamma_1)^2 < 4\gamma_1\gamma_2\alpha_1\alpha_2 \tag{1.20}$$

and

$$D > \max_{w\in[0,K_0]} \frac{w^2}{4w^*}\left[\frac{u^*|d_1'(w)|^2}{\gamma_1 d_1(w)} + \frac{v^*|d_2'(w)|^2}{\gamma_2 d_2(w)}\right] \tag{1.21}$$

as well as one of the conditions (H_6)–(H_8) holds. Then for all initial data (u_0, v_0, w_0) satisfying (H_5), there exist positive constants C and λ such that

$$\|u(\cdot,t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w^*\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \tag{1.22}$$

for all $t > 0$, where (u^*, v^*, w^*) is given by (1.14).

Remark 1.2. From a biological point of view, it is well known that the change of the predators comes from predation, competition and mortality in System (1.11). The parameters γ_i, β_i, α_i $(i = 1, 2)$ and θ respectively stand for the predation rate, competition strength, density-dependent death and natural death rate of the predators, which play a collective role in studying the dynamical behavior of (1.11). More specifically, when $\gamma_i > \theta$, it is called strong predation; otherwise, it is weak predation. Hence the results of Theorem 1.2 can tell us that if the predations of two predators are strong and the prey diffusion coefficient D is suitably large, all species can reach a coexistence state.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and the parameters $\gamma_i, \alpha_i, \beta_i$, $(i = 1, 2)$, θ, μ and D be positive. Assume that $d_1(w)$ and $d_2(w)$ satisfy (H_1), and the (u, v, w) is a global bounded classical solution of System (1.11). Suppose that we have (1.20) and*

$$D > \max_{w\in[0,K_0]} \frac{\overline{u}w^2|d_1'(w)|^2}{4\gamma_1\overline{w}d_1(w)} \tag{1.23}$$

as well as one of the following conditions holds:
(i) $\gamma_1 > \gamma_2 > \theta$, $l_2 \le 0$, $\beta_1 < \min\{\alpha_1, \beta_1^*\}$ and $\beta_2 < \alpha_2$;
(ii) $\gamma_1 > \gamma_2 > \theta$, $l_2 > 0$, $\beta_1 < \min\{\alpha_1, \beta_1^*\}$ and $\beta_2 \in [\beta_2^*, \alpha_2)$;
(iii) $\gamma_1 > \theta \ge \gamma_2$,
where

$$\overline{u} = \frac{\mu(\gamma_1 - \theta)}{\alpha_1\mu + \gamma_1} \quad and \quad \overline{w} = \frac{\alpha_1\mu + \theta}{\alpha_1\mu + \gamma_1}. \tag{1.24}$$

Then for all initial data (u_0, v_0, w_0) satisfying (H_5), one has

$$\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t) - \overline{w}\|_{L^\infty(\Omega)} \to 0 \ \ as \ \ t \to \infty, \tag{1.25}$$

exponentially if $\theta > \gamma_2\overline{w} - \beta_2\overline{u}$ or algebraically if $\theta = \gamma_2\overline{w} - \beta_2\overline{u}$.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and the parameters $\gamma_i, \alpha_i, \beta_i$, $(i = 1, 2)$, θ, μ and D be positive. Assume that $d_1(w)$ and $d_2(w)$ satisfy (H_1), and the (u, v, w) is a global bounded solution of System (1.11). Suppose that we have (1.20) and*

$$D > \max_{w\in[0,K_0]} \frac{\widetilde{v}w^2|d_2'(w)|^2}{4\gamma_2\widetilde{w}d_2(w)} \tag{1.26}$$

as well as one of the following conditions holds:
(i) $\gamma_2 > \gamma_1 > \theta$, $l_1 \le 0$, $\beta_2 < \min\{\alpha_2, \beta_2^*\}$ and $\beta_1 < \alpha_1$;

(ii) $\gamma_2 > \gamma_1 > \theta$, $l_1 > 0$, $\beta_2 < \min\{\alpha_2, \beta_2^*\}$ *and* $\beta_1 \in [\beta_1^*, \alpha_1)$;
(iii) $\gamma_2 > \theta \geq \gamma_1$,
where

$$\widetilde{v} = \frac{\mu(\gamma_2 - \theta)}{\alpha_2\mu + \gamma_2} \quad \text{and} \quad \widetilde{w} = \frac{\alpha_2\mu + \theta}{\alpha_2\mu + \gamma_2}. \tag{1.27}$$

Then for all initial data (u_0, v_0, w_0) *satisfying* (H_5)*, one has*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \widetilde{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \widetilde{w}\|_{L^\infty(\Omega)} \to 0 \ \text{ as } \ t \to \infty, \tag{1.28}$$

exponentially if $\theta > \gamma_1\widetilde{w} - \beta_1\widetilde{v}$ *or algebraically if* $\theta = \gamma_1\widetilde{w} - \beta_1\widetilde{v}$.

Remark 1.3. It follows from Theorem 1.3 that if the predator u is superior over v in the competition and the prey diffusion coefficient D is suitably large, the semi-trivial equilibrium $(\overline{u}, 0, \overline{w})$ is globally asymptotically stable. Similarly, we can obtain Theorem 1.4. Hence, we only give the conclusion of Theorem 1.4, without showing the details of the proof for brevity.

Theorem 1.5. *Let* $\Omega \subset \mathbb{R}^2$ *be a smooth bounded domain and the parameters* $\gamma_i, \alpha_i, \beta_i$, $(i = 1, 2)$, θ, μ *and* D *be positive. Assume that* $d_1(w)$ *and* $d_2(w)$ *satisfy* (H_1)*, and the* (u, v, w) *is a global bounded solution of System (1.11). Suppose that*

$$\gamma_i \leq \theta, \quad i = 1, 2.$$

Then for all initial data (u_0, v_0, w_0) *satisfying* (H_5)*, one has*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0 \ \text{ as } \ t \to \infty, \tag{1.29}$$

exponentially if $\gamma_i < \theta$, $i = 1, 2$ *or algebraically if* $\gamma_i = \theta$, $i = 1, 2$.

Remark 1.4. It follows from Theorem 1.5 that if the capture rates of the two predators are low (i.e. $\gamma_i \leq \theta$, $i = 1, 2$), the prey-only steady state $(0, 0, 1)$ is globally asymptotically stable regardless of the size of the prey diffusion coefficient D.

Remark 1.5. Compared with the previous results of [33] without competitive terms, the results of Theorems 1.2–1.5 indicate that the competition terms play a crucial role in the global stability of the constant steady states in (1.11). Moreover, under the condition of density-suppressed motility, our global stability results of Theorems 1.2–1.5 can also generalize the ranges of parameters α_i and β_i $(i = 1, 2)$ for two dimensional cases in [32]. However, since the heat-semigroup estimates of u and v are no longer applicable due to the appearance of density-suppressed motility, the global stability in L^∞-norm is still open in the higher-dimensional problem.

The rest of this paper is organized as follows. In Section 2, we first state the local existence of the classical solution to (1.8) and collect preliminary lemmas. In Section 3, we derive the global existence and boundedness of classical solutions for (1.8) and prove Theorem 1.1. Finally, we shall study the asymptotic stability of global bounded solutions for (1.11) and prove Theorems 1.2–1.5.

2. Local existence and preliminaries

In this section, we shall give the local existence and some preliminary lemmas. Firstly, we state the local existence of the classical solution to (1.8), as obtained by means of the abstract theory of quasilinear parabolic systems in [52].

Lemma 2.1. *Let $D, \gamma_i, \beta_i > 0$ $(i = 1, 2)$, $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and the hypotheses (H_1)–(H_5) hold. Then, there exists a $T_{\max} \in (0, \infty]$ such that System (1.8) possesses a unique classical solution $(u, v, w) \in (C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^3$ satisfying*

$$u, v \geq 0 \quad \textit{and} \quad 0 \leq w \leq K_0 := \max\{\|w_0\|_{L^\infty(\Omega)}, K\}. \tag{2.1}$$

In addition, the following extensibility criterion holds, i.e. if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \Big(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \Big) = \infty. \tag{2.2}$$

Proof. Let $\mathbf{z} = (u, v, w)^T$; then, System (1.8) can be rewritten as

$$\begin{cases} \mathbf{z}_t = \nabla \cdot (P(\mathbf{z}) \nabla \mathbf{z}) + Q(\mathbf{z}), & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial \mathbf{z}}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ \mathbf{z}(\cdot, 0) = \mathbf{z}_0 = (u_0, v_0, w_0), & x \in \Omega, \end{cases} \tag{2.3}$$

where

$$P(\mathbf{z}) = \begin{pmatrix} d_1(w) & 0 & u d_1'(w) \\ 0 & d_2(w) & v d_2'(w) \\ 0 & 0 & D \end{pmatrix} \text{ and } Q(\mathbf{z}) = \begin{pmatrix} \gamma_1 u F_1(w) - u h_1(u) - \beta_1 uv \\ \gamma_2 v F_2(w) - v h_2(v) - \beta_2 uv \\ -u F_1(w) - v F_2(w) + f(w) \end{pmatrix}. \tag{2.4}$$

According to the conditions that $D > 0$ and $d_i(w) > 0$ $(i = 1, 2)$, the matrix $P(\mathbf{z})$ is positively definite for the given initial data $\mathbf{z}_0$, which asserts that System (1.8) is normally parabolic. Thus it follows from Theorem 7.3 of [53] that there exists a $T_{\max} \in (0, \infty]$ such that System (1.8) admits a unique classical solution $(u, v, w) \in (C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^3$. The nonnegativity of (u, v, w) directly comes from the maximum principle [14, 45]. It similarly follows from Lemma 2.2 of [13] that $w \leq K_0 := \max\{\|w_0\|_{L^\infty(\Omega)}, K\}$. Since $P(\mathbf{z})$ is an upper triangular matrix, we can deduce from Theorem 5.2 of [54] that the extensibility criterion given by (2.2) holds. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smooth bounded domain, $D > 0$ and $T \in (0, \infty]$. Assume that $\varphi(x, t) \in C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$ satisfies*

$$\begin{cases} \varphi_t = D \Delta \varphi - \varphi + \psi, & (x, t) \in \Omega \times (0, T), \\ \dfrac{\partial \varphi}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, T), \end{cases} \tag{2.5}$$

where $\psi \in L^\infty((0, T); L^p(\Omega))$ with $p \geq 1$. Then there exists a positive constant C such that

$$\|\varphi(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \textit{for all} \quad t \in (0, T), \tag{2.6}$$

where

$$q \in \begin{cases} [1, \frac{np}{n-p}), & \text{if } p < n, \\ [1, \infty), & \text{if } p = n, \\ [1, \infty], & \text{if } p > n. \end{cases} \tag{2.7}$$

Proof. This lemma directly comes from Lemma 1 of [55]. □

Now, we give the following lemma ([56], Lemma 3.4) to derive some a priori estimates for w.

Lemma 2.3. *Let $T > 0, \tau \in (0, T)$ and $a, d > 0$, and assume that $y : [0, T) \to [0, \infty)$ is absolutely continuous. If there exists a nonnegative function $h \in L^1_{loc}([0, T))$ satisfying*

$$\int_t^{t+\tau} h(s)ds \le d \quad \text{for all } t \in [0, T - \tau) \tag{2.8}$$

and

$$y'(t) + ay(t) \le h(t), \tag{2.9}$$

one has

$$y(t) \le \max\left\{y(0) + d, \frac{d}{a\tau} + 2d\right\} \quad \text{for all } t \in [0, T). \tag{2.10}$$

Next, we give a basic property of the solution components u and v for System (1.8).

Lemma 2.4. *Let the assumptions of Lemma 2.1 hold. Then there exists a constant $C > 0$ such that*

$$\int_\Omega u + vdx \le C \quad \text{for all} \quad t \in (0, T_{\max}) \tag{2.11}$$

and

$$\int_t^{t+\tau} \int_\Omega u^2 + v^2 dxds \le C \quad \text{for all} \quad t \in (0, T_{\max} - \tau), \tag{2.12}$$

where $\tau = \min\{1, \frac{1}{2}T_{\max}\}$.

Proof. It follows from a direct computation for System (1.8) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \left(\frac{1}{\gamma_1}u + \frac{1}{\gamma_2}v + w\right) dx &= \int_\Omega f(w)dx - \frac{1}{\gamma_1}\int_\Omega uh_1(u)dx \\ &\quad - \frac{\beta_1}{\gamma_1}\int_\Omega uvdx - \frac{1}{\gamma_2}\int_\Omega vh_2(v)dx - \frac{\beta_2}{\gamma_2}\int_\Omega uvdx \\ &\le \mu \int_\Omega wdx - \frac{1}{\gamma_1}\int_\Omega u(\theta_1 + \alpha_1 u)dx \\ &\quad - \frac{1}{\gamma_2}\int_\Omega v(\theta_2 + \alpha_2 v)dx \\ &= \mu \int_\Omega wdx - \theta_1 \int_\Omega \frac{1}{\gamma_1} udx - \theta_2 \int_\Omega \frac{1}{\gamma_2} vdx \\ &\quad - \frac{\alpha_1}{\gamma_1}\int_\Omega u^2 dx - \frac{\alpha_2}{\gamma_2}\int_\Omega v^2 dx, \end{aligned} \tag{2.13}$$

for all $t \in (0, T_{\max})$, where we have applied (H_3), (H_4), $\beta_1, \beta_2 > 0$ and (2.1).

Let $\theta := \min\{\theta_1, \theta_2\}$, it follows from (2.1) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)dx+\frac{\alpha_1}{\gamma_1}\int_\Omega u^2dx+\frac{\alpha_2}{\gamma_2}\int_\Omega v^2dx\\&\leq-\theta\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)dx+(\mu K_0+\theta)|\Omega|,\end{aligned}\tag{2.14}$$

which leads to (2.11) by Gronwall's inequality. If $\alpha_i > 0$, $i = 1, 2$, then integrating (2.14) over $(t, t+\tau)$, we have (2.12) directly. If $\alpha_i = 0$, $i = 1, 2$, we can also prove (2.12) by means of the idea used in [14]. For the readers' convenience, we give the sketch of the proof.

Let $\mathcal{A}$ denote the self-adjoint realization of $-\Delta + \delta$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$, where $\delta \in \left(0, \min\{\frac{\theta_1}{d_1(0)}, \frac{\theta_2}{d_2(0)}\}\right)$. It follows from $\delta > 0$ that $\mathcal{A}$ has an order-preserving bounded inverse $\mathcal{A}^{-1}$ on $L^2(\Omega)$. Thus this allows us to obtain a positive constant c_1 such that

$$\|\mathcal{A}^{-1}\Psi\|_{L^2(\Omega)} \leq c_1\|\Psi\|_{L^2(\Omega)} \quad \text{for all } \Psi \in L^2(\Omega)\tag{2.15}$$

and

$$\|\mathcal{A}^{-\frac{1}{2}}\Psi\|^2_{L^2(\Omega)} = \int_\Omega \Psi\cdot\mathcal{A}^{-1}\Psi dx \leq c_1\|\Psi\|^2_{L^2(\Omega)} \quad \text{for all } \Psi \in L^2(\Omega).\tag{2.16}$$

By a simple calculation in (1.8), we have

$$\begin{aligned}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)_t=\Delta\left(\frac{1}{\gamma_1}d_1(w)u+\frac{1}{\gamma_2}d_2(w)v+Dw\right)-\frac{1}{\gamma_1}uh_1(u)-\frac{\beta_1}{\gamma_1}uv\\-\frac{1}{\gamma_2}vh_2(v)-\frac{\beta_2}{\gamma_2}uv+f(w),\end{aligned}\tag{2.17}$$

which can be written as

$$\begin{aligned}&\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)_t+\mathcal{A}\left(\frac{1}{\gamma_1}d_1(w)u+\frac{1}{\gamma_2}d_2(w)v+Dw\right)\\&=\delta\left(\frac{1}{\gamma_1}d_1(w)u+\frac{1}{\gamma_2}d_2(w)v+Dw\right)-\frac{1}{\gamma_1}uh_1(u)-\frac{\beta_1}{\gamma_1}uv\\&\quad-\frac{1}{\gamma_2}vh_2(v)-\frac{\beta_2}{\gamma_2}uv+f(w)\\&=\frac{1}{\gamma_1}u\left(\delta d_1(w)-h_1(u)\right)+\frac{1}{\gamma_2}v\left(\delta d_2(w)-h_2(v)\right)+\delta Dw+f(w)-\frac{\beta_1}{\gamma_1}uv-\frac{\beta_2}{\gamma_2}uv\\&\leq\frac{1}{\gamma_1}u\left(\delta d_1(0)-\theta_1\right)+\frac{1}{\gamma_2}v\left(\delta d_2(0)-\theta_2\right)+(\delta D+\mu)K_0\\&\leq(\delta D+\mu)K_0\\&:=c_2,\end{aligned}\tag{2.18}$$

where we have applied (H_1), (H_3), (H_4), (2.1) and $\delta \in \left(0, \min\{\frac{\theta_1}{d_1(0)}, \frac{\theta_2}{d_2(0)}\}\right)$. Hence, by multiplying

(2.18) by $\mathcal{A}^{-1}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\geq 0$ and integrating it over Ω, we derive

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega\left|\mathcal{A}^{-\frac{1}{2}}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\right|^2dx\\&\quad+\int_\Omega\left(\frac{1}{\gamma_1}d_1(w)u+\frac{1}{\gamma_2}d_2(w)v+Dw\right)\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)dx\\&\leq c_2\int_\Omega\mathcal{A}^{-1}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)dx.\end{aligned}\tag{2.19}$$

According to the fact that $0<d_i(K_0)\leq d_i(w)$, $i=1,2$, due to (H_1) and (2.1), and by letting $c_3:=\min\{d_1(K_0),d_2(K_0),D\}>0$, we deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega\left|\mathcal{A}^{-\frac{1}{2}}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\right|^2dx+2c_3\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)^2dx\\&\leq 2c_2\int_\Omega\mathcal{A}^{-1}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)dx.\end{aligned}\tag{2.20}$$

It follows from Hölder's and Young's inequality as well as (2.15) that

$$\begin{aligned}2c_2\int_\Omega\mathcal{A}^{-1}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)dx&\leq 2c_2|\Omega|^{\frac{1}{2}}\left(\int_\Omega\left|\mathcal{A}^{-1}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\right|^2dx\right)^{\frac{1}{2}}\\&\leq 2c_1c_2|\Omega|^{\frac{1}{2}}\left(\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)^2dx\right)^{\frac{1}{2}}\\&\leq\frac{c_3}{2}\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)^2dx+\frac{2c_1^2c_2^2|\Omega|}{c_3}.\end{aligned}\tag{2.21}$$

According to (2.16), we have

$$\frac{c_3}{2c_1}\int_\Omega\left|\mathcal{A}^{-\frac{1}{2}}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\right|^2\leq\frac{c_3}{2}\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)^2dx.\tag{2.22}$$

By combining (2.20)–(2.22), we derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega\left|\mathcal{A}^{-\frac{1}{2}}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\right|^2dx+\frac{c_3}{2c_1}\int_\Omega\left|\mathcal{A}^{-\frac{1}{2}}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\right|^2\\&\quad+c_3\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)^2dx\leq\frac{2c_1^2c_2^2|\Omega|}{c_3}.\end{aligned}\tag{2.23}$$

By the ordinary differential equations (ODE) argument, there exists a $c_4>0$ such that

$$\int_\Omega\left|\mathcal{A}^{-\frac{1}{2}}\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)\right|^2dx\leq c_4,\tag{2.24}$$

which implies that

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega \frac{1}{\gamma_1^2}u^2+\frac{1}{\gamma_2^2}v^2dxds &\le \int_t^{t+\tau}\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v\right)^2dxds\\ &\le \int_t^{t+\tau}\int_\Omega\left(\frac{1}{\gamma_1}u+\frac{1}{\gamma_2}v+w\right)^2dxds\\ &\le \frac{c_4}{c_3}+\frac{2c_1^2c_2^2|\Omega|}{c_3^2}.\end{aligned} \tag{2.25}$$

The proof of Lemma 2.4 is complete. □

Finally, we shall give the following key estimate of w, which plays a crucial role in the proof of Theorem 1.1.

Lemma 2.5. *Let the assumptions of Lemma 2.1 hold. Then there exists a constant $C > 0$ such that*

$$\int_\Omega |\nabla w|^2dx \le C \quad \textit{for all} \quad t\in(0,T_{\max}) \tag{2.26}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta w|^2dxds \le C \quad \textit{for all} \quad t\in(0,T_{\max}-\tau), \tag{2.27}$$

where $\tau=\min\{1,\frac{1}{2}T_{\max}\}$.

Proof. Multiplying the third equation of System (1.8) with $-2\Delta w$ and integrating it by parts, we deduce from Young's inequality and (2.1) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega|\nabla w|^2dx\\ &= -2D\int_\Omega|\Delta w|^2dx+2\int_\Omega(uF_1(w)+vF_2(w))\Delta wdx-2\int_\Omega f(w)\Delta wdx\\ &\le -D\int_\Omega|\Delta w|^2dx+\frac{2}{D}\int_\Omega(uF_1(w)+vF_2(w))^2dx+\frac{2}{D}\int_\Omega f^2(w)dx\\ &\le -D\int_\Omega|\Delta w|^2dx+\frac{4F_1^2(K_0)}{D}\int_\Omega u^2dx+\frac{4F_2^2(K_0)}{D}\int_\Omega v^2dx+\frac{2(\mu K_0)^2}{D}|\Omega|,\end{aligned} \tag{2.28}$$

where we have used the hypotheses (H_2) and (H_4).

It follows from $\frac{\partial w}{\partial \nu}=0$, Young's inequality and (2.1) that

$$\begin{aligned}\int_\Omega|\nabla w|^2dx = -\int_\Omega w\Delta wdx &\le \frac{D}{2}\int_\Omega|\Delta w|^2dx+\frac{1}{2D}\int_\Omega w^2dx\\ &\le \frac{D}{2}\int_\Omega|\Delta w|^2dx+\frac{K_0^2}{2D}|\Omega|.\end{aligned} \tag{2.29}$$

Let $c_5 := \max\left\{\frac{4F_1^2(K_0)}{D}, \frac{4F_2^2(K_0)}{D}\right\}$; we infer from (2.28) and (2.29) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\nabla w|^2 dx + \int_\Omega |\nabla w|^2 dx + \frac{D}{2}\int_\Omega |\Delta w|^2 dx\\ &\le c_5\int_\Omega u^2 + v^2 dx + c_6,\end{aligned} \tag{2.30}$$

where $c_6 := \frac{2(\mu K_0)^2}{D}|\Omega| + \frac{K_0^2}{2D}|\Omega|$. It follows from Lemma 2.3 and Lemma 2.4 that (2.26) holds. Then integrating (2.30) over $(t, t+\tau)$, we can deduce from (2.12) and (2.26) that (2.27) holds. □

3. Global boundedness of solutions

In this section, we shall study the global existence and uniform boundedness of solutions for system (1.8) when $n = 2$. To do this, we need the following lemmas.

Lemma 3.1. *Let the conditions of Theorem 1.1 hold. Then the solution (u, v, w) of system (1.8) satisfies*

$$\int_\Omega u^2 dx + \int_\Omega v^2 dx \le C \tag{3.1}$$

and

$$\|w(\cdot, t)\|_{W^{1,q}(\Omega)} \le C \tag{3.2}$$

for all $q \in [1, \infty)$ and $t \in (0, T_{\max})$, where $C > 0$ is a constant independent of t.

Proof. Multiplying the first equation of System (1.8) by $2u$ and integrating by parts, we deduce from Young's and Hölder's inequalities that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^2 dx &= -2\int_\Omega d_1(w)|\nabla u|^2 dx - 2\int_\Omega d_1'(w)u\nabla u\cdot\nabla w dx\\ &\quad + 2\gamma_1\int_\Omega u^2F_1(w)dx - 2\int_\Omega u^2h_1(u)dx - 2\beta_1\int_\Omega u^2 v dx\\ &\le -\int_\Omega d_1(w)|\nabla u|^2 dx + \int_\Omega \frac{|d_1'(w)|^2}{d_1(w)}u^2|\nabla w|^2 dx + 2\gamma_1 F_1(K_0)\int_\Omega u^2 dx\\ &\le -d_1(K_0)\int_\Omega |\nabla u|^2 dx + \mathcal{K}_1\int_\Omega u^2|\nabla w|^2 dx + 2\gamma_1F_1(K_0)\int_\Omega u^2 dx\\ &\le -d_1(K_0)\int_\Omega|\nabla u|^2dx + \mathcal{K}_1\left(\int_\Omega u^4 dx\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla w|^4 dx\right)^{\frac{1}{2}}\\ &\quad + 2\gamma_1F_1(K_0)\int_\Omega u^2 dx,\end{aligned} \tag{3.3}$$

where $\mathcal{K}_1 := \max_{w\in[0,K_0]}\frac{|d_1'(w)|^2}{d_1(w)}$ and we have applied (H_1)–(H_3) and (2.1).

By using the Gagliardo-Nirenberg inequality in two dimensions, there exists a $C_1 > 0$ such that

$$\left(\int_\Omega u^4 dx\right)^{\frac{1}{2}} = \|u\|^2_{L^4(\Omega)} \le C_1(\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}). \tag{3.4}$$

According to Lemma 2.5 of [19] when $n = 2$, it follows from Lemma 2.5 that

$$\begin{aligned}\left(\int_\Omega |\nabla w|^4 dx\right)^{\frac{1}{2}} &\le C_2(\|\Delta w\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)})\\ &\le C_3(\|\Delta w\|_{L^2(\Omega)} + 1),\end{aligned} \tag{3.5}$$

for all $t \in (0, T_{\max})$, where $C_2, C_3 > 0$. Thus, by combining (3.4) with (3.5), we infer from Young's inequality that

$$\begin{aligned}\mathcal{K}_1\left(\int_\Omega u^4 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4 dx\right)^{\frac{1}{2}} \le d_1(K_0)\int_\Omega |\nabla u|^2 dx + C_4\|u\|^2_{L^2(\Omega)}\|\Delta w\|^2_{L^2(\Omega)}\\ + C_5\|u\|^2_{L^2(\Omega)},\end{aligned} \tag{3.6}$$

where C_4, C_5 are positive constants. Thus it follows from (3.3) and (3.6) that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^2 dx \le C_6\int_\Omega u^2 dx\left(\int_\Omega |\Delta w|^2 dx + 1\right), \tag{3.7}$$

where $C_6 := \max\{C_4, C_5 + 2\gamma_1 F_1(K_0)\}$.

By Lemma 2.4, we can find $t_0 = t_0(t) \in ((t-\tau)_+, t)$ for any $t \in (0, T_{\max})$ such that there exists a $C_7 > 0$ satisfying

$$\int_\Omega u^2(x, t_0)dx \le C_7, \tag{3.8}$$

where τ is defined in Lemma 2.4. It follows from Lemma 2.5 that there exists a $C_8 > 0$ such that

$$\int_{t_0}^{t_0+\tau}\int_\Omega |\Delta w(x,t)|^2 dxdt \le C_8. \tag{3.9}$$

Therefore, integrating (3.7) over (t_0, t), we deduce from $t_0 < t < t_0 + \tau \le t_0 + 1$, (3.8) and (3.9) that

$$\int_\Omega u^2 dx \le \int_\Omega u^2(x, t_0)dx e^{C_6\int_{t_0}^t\left(\int_\Omega |\Delta w|^2 dx + 1\right)ds} \le C_7 e^{C_6(C_8+1)} \tag{3.10}$$

for all $t \in (0, T_{\max})$.

Similarly, we obtain

$$\int_\Omega v^2 dx \le C_9 \quad \text{for all } \ t \in (0, T_{\max}). \tag{3.11}$$

It follows from the third equation of System (1.8), we know that w solves the following problem

$$\begin{cases} w_t = D\Delta w - w + g(u, v, w), & (x,t) \in \Omega \times (0, T_{\max}),\\ \dfrac{\partial w}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}),\end{cases} \tag{3.12}$$

where $g(u, v, w) = w - uF_1(w) - vF_2(w) + f(w)$. According to (H_2), (H_3) and (2.1), we infer from (3.10) and (3.11) that

$$\|g(u, v, w)\|_{L^2(\Omega)} \leq C_{10}(\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} + 1) \leq C_{11} \tag{3.13}$$

for all $t \in (0, T_{\max})$. Hence, it follows from Lemma 2.2 in two dimensions that (3.2) holds. The proof of Lemma 3.1 is complete. □

Next, we shall prove the boundedness of w in $W^{1\infty}(\Omega)$.

Lemma 3.2. *Let the conditions of Theorem 1.1 hold. Then the solution component w of system (1.8) satisfies*

$$\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \tag{3.14}$$

for all $t \in (0, T_{\max})$, where $C > 0$ is a constant independent of t.

Proof. Multiplying the first equation of System (1.8) by u^2 and integrating by parts, we deduce from Young's and Hölder's inequalities that

$$\begin{aligned}
\frac{1}{3}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^3 dx &= -2\int_\Omega d_1(w)u|\nabla u|^2 dx - 2\int_\Omega d_1'(w)u^2\nabla u\cdot\nabla w dx \\
&\quad + \gamma_1\int_\Omega u^3 F_1(w)dx - \int_\Omega u^3 h_1(u)dx - \beta_1\int_\Omega u^3 v dx \\
&\leq -\int_\Omega d_1(w)u|\nabla u|^2 dx + \int_\Omega \frac{|d_1'(w)|^2}{d_1(w)}u^3|\nabla w|^2 dx \\
&\quad + \gamma_1 F_1(K_0)\int_\Omega u^3 dx - \theta_1\int_\Omega u^3 dx \\
&\leq -\frac{4d_1(K_0)}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 dx + \mathcal{K}_1\int_\Omega u^3|\nabla w|^2 dx \\
&\quad + \gamma_1 F_1(K_0)\int_\Omega u^3 dx - \theta_1\int_\Omega u^3 dx \\
&\leq -\frac{4d_1(K_0)}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 dx + \mathcal{K}_1\left(\int_\Omega u^6 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4 dx\right)^{\frac{1}{2}} \\
&\quad + \gamma_1 F_1(K_0)\int_\Omega u^3 dx - \theta_1\int_\Omega u^3 dx,
\end{aligned} \tag{3.15}$$

for all $t \in (0, T_{\max})$, where $\mathcal{K}_1$ is defined in the proof of Lemma 3.1 and we have applied (H_1)–(H_3) and (2.1).

It follows from Lemma 3.1 that there exist positive constants C_1 and C_2 such that $\|\nabla w\|_{L^4(\Omega)} \leq C_1$ and $\|u\|_{L^2(\Omega)} \leq C_2$ for all $t \in (0, T_{\max})$. Then by using the Gagliardo-Nirenberg inequality and Young's inequality, we can find positive constants C_i, $i = 3, \cdots, 6$ such that

$$\begin{aligned}\mathcal{K}_1\left(\int_\Omega u^6 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4 dx\right)^{\frac{1}{2}} &\le \mathcal{K}_1 C_1^2 \|u^{\frac{3}{2}}\|^2_{L^4(\Omega)} \\ &\le C_3\left(\|\nabla u^{\frac{3}{2}}\|^{\frac{4}{3}}_{L^2(\Omega)} \cdot \|u^{\frac{3}{2}}\|^{\frac{2}{3}}_{L^{\frac{4}{3}}(\Omega)} + \|u^{\frac{3}{2}}\|^2_{L^{\frac{4}{3}}(\Omega)}\right) \\ &\le \frac{2d_1(K_0)}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 dx + C_4\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\gamma_1 F_1(K_0)\int_\Omega u^3 dx &\le \gamma_1 F_1(K_0)\|u^{\frac{3}{2}}\|^2_{L^2(\Omega)} \\ &\le C_5\left(\|\nabla u^{\frac{3}{2}}\|^{\frac{2}{3}}_{L^2(\Omega)} \cdot \|u^{\frac{3}{2}}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)} + \|u^{\frac{3}{2}}\|^2_{L^{\frac{4}{3}}(\Omega)}\right) \\ &\le \frac{2d_1(K_0)}{9}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 dx + C_6\end{aligned} \tag{3.17}$$

for all $t \in (0, T_{\max})$.

Combining (3.15)–(3.17), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^3 dx + 3\theta_1\int_\Omega u^3 dx \le C_7 := 3(C_4 + C_6) \tag{3.18}$$

for all $t \in (0, T_{\max})$. By the ODE argument, we can derive

$$\int_\Omega u^3 dx \le \max\left\{\int_\Omega u_0^3 dx, \frac{C_7}{3\theta_1}\right\} \quad \text{for all} \quad t \in (0, T_{\max}). \tag{3.19}$$

Similarly, we also derive the boundedness of $\|v\|_{L^3(\Omega)}$. Then it follows from Lemma 2.2 in two dimensions that (3.14) holds. □

By means of the boundedness of $\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)}$, it follows from the Moser iteration of [45] that we can obtain the boundedness of $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ and $\|v(\cdot, t)\|_{L^\infty(\Omega)}$ for all $t \in (0, T_{\max})$.

Lemma 3.3. *Let the conditions of Theorem 1.1 hold. Then the solution component (u, v) of system (1.8) satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \le C \tag{3.20}$$

for all $t \in (0, T_{\max})$, where $C > 0$ is a constant independent of t.

Proof. Multiplying the first equation of System (1.8) by u^{p-1} with $p \ge 2$ and integrating by parts, we deduce from Young's inequality that

$$
\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^p dx &= -(p-1)\int_\Omega d_1(w)u^{p-2}|\nabla u|^2 dx-(p-1)\int_\Omega d_1'(w)u^{p-1}\nabla u\cdot\nabla w dx\\
&\quad+\gamma_1\int_\Omega u^p F_1(w)dx-\int_\Omega u^p h_1(u)dx-\beta_1\int_\Omega u^p v dx\\
&\le -\frac{p-1}{2}\int_\Omega d_1(w)u^{p-2}|\nabla u|^2 dx+\frac{p-1}{2}\int_\Omega \frac{|d_1'(w)|^2}{d_1(w)}u^p|\nabla w|^2 dx\\
&\quad+\gamma_1 F_1(K_0)\int_\Omega u^p dx\\
&\le -\frac{p-1}{2}d_1(K_0)\int_\Omega u^{p-2}|\nabla u|^2 dx+\frac{p-1}{2}\mathcal{K}_1\int_\Omega u^p|\nabla w|^2 dx\\
&\quad+\gamma_1 F_1(K_0)\int_\Omega u^p dx
\end{aligned}
\tag{3.21}
$$

for all $t\in(0,T_{\max})$, where $\mathcal{K}_1$ is defined in the proof of Lemma 3.1 and we have applied (H_1)–(H_3) and (2.1).

It follows from Lemma 3.2 that there exists a $C_1>0$ such that $\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)}\le C_1$ for all $t\in(0,T_{\max})$. Hence, we deduce from (3.21) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u^p dx+\frac{p(p-1)d_1(K_0)}{2}\int_\Omega u^{p-2}|\nabla u|^2 dx+p(p-1)\int_\Omega u^p dx\\
&\quad\le C_2 p(p-1)\int_\Omega u^p dx,
\end{aligned}
\tag{3.22}
$$

for all $t\in(0,T_{\max})$, where $C_2:=\frac{\mathcal{K}_1 C_1^2}{2}+\gamma_1 F_1(K_0)+1$ is independent of p. The rest can be handled exactly as the Moser iteration in Lemma 2.7 of [45] to derive the boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ for all $t\in(0,T_{\max})$. Similarly, we can obtain the boundedness of $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ for all $t\in(0,T_{\max})$. The proof of Lemma 3.3 is complete. □

Proof of Theorem 1.1. Theorem 1.1 is a direct consequence of Lemma 2.1, Lemma 3.2 and Lemma 3.3. □

4. Large time behavior

In this section, we shall study the asymptotic stability of global bounded solutions for System (1.11) by constructing energy functionals used in [13, 57]. To do this, we first give some regularity results of the solution (u,v,w) for System (1.11).

Lemma 4.1. *Let (u,v,w) be a global bounded classical solution for (1.11) ensured in Theorem 1.1. Then there exist $\sigma\in(0,1)$ and $C>0$ such that*

$$
\|u\|_{C^{\sigma,\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])}+\|v\|_{C^{\sigma,\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])}+\|w\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])}\le C\quad \textit{for all}\;\; t>1. \tag{4.1}
$$

Proof. This lemma can be verified by a similar argument in Lemma 4.1 of [14], so we omit the details here for brevity. □

Lemma 4.2. *Let (u, v, w) be a global bounded classical solution for (1.11) ensured in Theorem 1.1. Then there exists a $C > 0$ such that*

$$\|u(\cdot, t)\|_{W^{1,4}(\Omega)} + \|v(\cdot, t)\|_{W^{1,4}(\Omega)} \le C \quad \textit{for all} \;\; t > 0. \tag{4.2}$$

Proof. This lemma can be verified by a similar argument in Lemma 3.6 of [45], so we omit the details here for brevity. □

In order to prove the asymptotic stabilization of global bounded solutions for system (1.11), we provide the following lemma, which is proved in [57].

Lemma 4.3. *Let $\phi : (1, \infty) \to [0, \infty)$ be uniformly continuous such that $\int_1^\infty \phi(t)dt < \infty$. Then*

$$\phi(t) \to 0 \;\; as \;\; t \to \infty. \tag{4.3}$$

4.1. Proof of Theorem 1.2

In this subsection, we are devoted to studying the stabilization of the coexistence steady state (u^*, v^*, w^*) for some parameters cases. Let us introduce the following functionals

$$\begin{aligned}\mathcal{E}_1(t) =& \frac{1}{\gamma_1}\int_\Omega \Big(u - u^* - u^* \ln \frac{u}{u^*}\Big) dx + \frac{1}{\gamma_2}\int_\Omega \Big(v - v^* - v^* \ln \frac{v}{v^*}\Big) dx \\ &+ \int_\Omega \Big(w - w^* - w^* \ln \frac{w}{w^*}\Big) dx,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_1(t) =& \int_\Omega (u - u^*)^2 \, dx + \int_\Omega (v - v^*)^2 \, dx + \int_\Omega (w - w^*)^2 \, dx \\ &+ \int_\Omega \Big|\frac{\nabla u}{u}\Big|^2 dx + \int_\Omega \Big|\frac{\nabla v}{v}\Big|^2 dx + \int_\Omega |\nabla w|^2 dx,\end{aligned}$$

where (u^*, v^*, w^*) is given by (1.14).

Lemma 4.4. *Let the conditions of Theorem 1.2 hold. Then there exists a positive constant ε_1 independent of t such that*

$$\mathcal{E}_1(t) \ge 0 \tag{4.4}$$

and

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\varepsilon_1 \mathcal{F}_1(t) \quad \textit{for all}\; t > 0. \tag{4.5}$$

Proof. Let

$$\begin{aligned} I_1(t) &:= \tfrac{1}{\gamma_1}\int_\Omega \Big(u - u^* - u^* \ln \tfrac{u}{u^*}\Big) dx, \\ I_2(t) &:= \tfrac{1}{\gamma_2}\int_\Omega \Big(v - v^* - v^* \ln \tfrac{v}{v^*}\Big) dx, \\ I_3(t) &:= \int_\Omega \Big(w - w^* - w^* \ln \tfrac{w}{w^*}\Big) dx, \end{aligned}$$

then $\mathcal{E}_1(t)$ can be rewritten as

$$\mathcal{E}_1(t) = I_1(t) + I_2(t) + I_3(t) \quad \text{for all } \ t > 0.$$

Step 1: We shall prove the nonnegativity of $\mathcal{E}_1(t)$ for all $t > 0$. Let $H(\xi) := \xi - u^* \ln \xi$ for $\xi > 0$; it follows from Taylor's formula for all $x \in \Omega$ and each $t > 0$ that there exists a $\tau = \tau(x, t) \in (0, 1)$ such that

$$\begin{aligned} u - u^* - u^* \ln \frac{u}{u^*} &= H(u) - H(u^*) \\ &= H'(u^*) \cdot (u - u^*) + \frac{1}{2} H''(\tau u + (1-\tau)u^*) \cdot (u - u^*)^2 \\ &= \frac{u^*}{2(\tau u + (1-\tau)u^*)^2} (u - u^*)^2 \geq 0. \end{aligned}$$

Hence, we immediately derive that $I_1(t) = \int_\Omega (H(u) - H(u^*))\, dx \geq 0$. Similarly, we know that $I_2(t) \geq 0$ and $I_3(t) \geq 0$ for all $t > 0$. Thus, we know that (4.4) holds.

Step 2: Now, we further prove (4.5). By a series of simple calculations, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} I_1(t) &= \frac{1}{\gamma_1} \int_\Omega \frac{u - u^*}{u} u_t dx \\ &= \frac{1}{\gamma_1} \int_\Omega \frac{u - u^*}{u} (\Delta(d_1(w)u) + \gamma_1 uw - u(\theta + \alpha_1 u) - \beta_1 uv) dx \\ &= -\frac{u^*}{\gamma_1} \int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2} dx - \frac{u^*}{\gamma_1} \int_\Omega \frac{d_1'(w)\nabla u \cdot \nabla w}{u} dx \\ &\quad + \frac{1}{\gamma_1} \int_\Omega (u - u^*)(\gamma_1 w - \theta - \alpha_1 u - \beta_1 v) dx \\ &= -\frac{u^*}{\gamma_1} \int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2} dx - \frac{u^*}{\gamma_1} \int_\Omega \frac{d_1'(w)\nabla u \cdot \nabla w}{u} dx \\ &\quad + \int_\Omega (u - u^*)(w - w^*) dx - \frac{\alpha_1}{\gamma_1} \int_\Omega (u - u^*)^2 dx \\ &\quad - \frac{\beta_1}{\gamma_1} \int_\Omega (u - u^*)(v - v^*) dx, \end{aligned} \tag{4.6}$$

where we have used the fact that $\theta = \gamma_1 w^* - \alpha_1 u^* - \beta_1 v^*$.

Similarly, it follows from the identities $\theta = \gamma_2 w^* - \beta_2 u^* - \alpha_2 v^*$ and $\mu = u^* + v^* + \mu w^*$ that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} I_2(t) &= -\frac{v^*}{\gamma_2} \int_\Omega \frac{d_2(w)|\nabla v|^2}{v^2} dx - \frac{v^*}{\gamma_2} \int_\Omega \frac{d_2'(w)\nabla v \cdot \nabla w}{v} dx \\ &\quad + \int_\Omega (v - v^*)(w - w^*) dx - \frac{\alpha_2}{\gamma_2} \int_\Omega (v - v^*)^2 dx \\ &\quad - \frac{\beta_2}{\gamma_2} \int_\Omega (v - v^*)(u - u^*) dx, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} I_3(t) &= -Dw^* \int_\Omega \frac{|\nabla w|^2}{w^2} dx - \int_\Omega (w - w^*)(u - u^*) dx - \int_\Omega (w - w^*)(v - v^*) dx \\ &\quad - \mu \int_\Omega (w - w^*)^2 dx. \end{aligned} \tag{4.8}$$

Hence, by combining (4.6)–(4.8), we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_1(t) =& -\frac{\alpha_1}{\gamma_1}\int_\Omega (u-u^*)^2dx - \frac{\alpha_2}{\gamma_2}\int_\Omega (v-v^*)^2dx - \mu\int_\Omega (w-w^*)^2dx \\ &- \left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)\int_\Omega (u-u^*)(v-v^*)dx - \frac{u^*}{\gamma_1}\int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2}dx \\ &- \frac{v^*}{\gamma_2}\int_\Omega \frac{d_2(w)|\nabla v|^2}{v^2}dx - Dw^*\int_\Omega \frac{|\nabla w|^2}{w^2}dx - \frac{u^*}{\gamma_1}\int_\Omega \frac{d_1'(w)\nabla u\cdot\nabla w}{u}dx \\ &- \frac{v^*}{\gamma_2}\int_\Omega \frac{d_2'(w)\nabla v\cdot\nabla w}{v}dx \\ :=& -\int_\Omega X_1A_1X_1^Tdx - \int_\Omega Y_1B_1Y_1^Tdx,\end{aligned} \tag{4.9}$$

where $X_1 = (u-u^*, v-v^*, w-w^*)$ and $Y_1 = \left(\frac{\nabla u}{u}, \frac{\nabla v}{v}, \nabla w\right)$, as well as

$$A_1 = \begin{pmatrix} \frac{\alpha_1}{\gamma_1} & \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) & 0 \\ \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) & \frac{\alpha_2}{\gamma_2} & 0 \\ 0 & 0 & \mu \end{pmatrix}, B_1 = \begin{pmatrix} \frac{u^*d_1(w)}{\gamma_1} & 0 & \frac{u^*d_1'(w)}{2\gamma_1} \\ 0 & \frac{v^*d_2(w)}{\gamma_2} & \frac{v^*d_2'(w)}{2\gamma_2} \\ \frac{u^*d_1'(w)}{2\gamma_1} & \frac{v^*d_2'(w)}{2\gamma_2} & \frac{Dw^*}{w^2} \end{pmatrix}. \tag{4.10}$$

It follows from (1.20) that

$$\left|\frac{\alpha_1}{\gamma_1}\right| > 0 \quad \text{and} \quad \begin{vmatrix} \frac{\alpha_1}{\gamma_1} & \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) \\ \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) & \frac{\alpha_2}{\gamma_2} \end{vmatrix} = \frac{\alpha_1\alpha_2}{\gamma_1\gamma_2} - \frac{1}{4}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)^2 > 0 \tag{4.11}$$

as well as

$$|A_1| = \mu\left(\frac{\alpha_1\alpha_2}{\gamma_1\gamma_2} - \frac{1}{4}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)^2\right) > 0, \tag{4.12}$$

which implies that the matrix A_1 is positive definite as according to Sylvester's criterion. Similarly, we deduce from (1.21) that

$$\left|\frac{u^*d_1(w)}{\gamma_1}\right| > 0 \quad \text{and} \quad \begin{vmatrix} \frac{u^*d_1(w)}{\gamma_1} & 0 \\ 0 & \frac{v^*d_2(w)}{\gamma_2} \end{vmatrix} = \frac{u^*v^*d_1(w)d_2(w)}{\gamma_1\gamma_2} > 0 \tag{4.13}$$

as well as

$$|B_1| = \frac{u^*v^*w^*d_1(w)d_2(w)}{\gamma_1\gamma_2w^2}\left(D - \frac{u^*w^2|d_1'(w)|^2}{4\gamma_1w^*d_1(w)} - \frac{v^*w^2|d_2'(w)|^2}{4\gamma_2w^*d_2(w)}\right) > 0, \tag{4.14}$$

which implies that the matrix B_1 is positive definite. Thus there exist positive constants κ_1 and κ_2 such that

$$X_1A_1X_1^T \geq \kappa_1|X_1|^2 \quad \text{and} \quad Y_1B_1Y_1^T \geq \kappa_2|Y_1|^2 \tag{4.15}$$

for all $x \in \Omega$ and $t > 0$. Let $\varepsilon_1 := \min\{\kappa_1, \kappa_2\}$, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_1(t) \leq -\varepsilon_1\int_\Omega |X_1|^2 + |Y_1|^2dx \quad \text{for all} \;\; t > 0, \tag{4.16}$$

which implies that (4.5) holds. The proof of Lemma 4.4 is complete. □

With the aid of Lemma 4.4, we shall give the following large time behavior of global solutions for system (1.11).

Lemma 4.5. *Let the assumptions of Theorem 1.2 hold. Then the global bounded solution of (1.11) converges to the coexistence steady state* (u^*, v^*, w^*) *given by (1.14), i.e.,*

$$\|u(\cdot,t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w^*\|_{L^\infty(\Omega)} \to 0 \tag{4.17}$$

as $t \to \infty$.

Proof. It follows from Lemma 4.4 and integration over $(1,\infty)$ that

$$\int_1^\infty \mathcal{F}_1(t)dt \le \frac{\mathcal{E}_1(1)}{\varepsilon_1} < \infty.$$

According to Theorem 1.1 and Lemma 4.1, the bounded solution u, v and w are Hölder continuous in $\bar{\Omega} \times [t, t+1]$ with respect to $t > 1$. Thus we conclude that $\mathcal{F}_1(t)$ is uniformly continuous in $(1,\infty)$. Thus we infer from Lemma 4.3 that

$$\int_\Omega (u-u^*)^2\,dx + \int_\Omega (v-v^*)^2\,dx + \int_\Omega (w-w^*)^2\,dx \to 0 \tag{4.18}$$

as $t \to \infty$. By the Gagliardo-Nirenberg inequality in two dimensions, there exists a $C_1 > 0$ such that

$$\|u-u^*\|_{L^\infty(\Omega)} \le C_1 \|u-u^*\|_{W^{1,4}(\Omega)}^{\frac{2}{3}} \|u-u^*\|_{L^2(\Omega)}^{\frac{1}{3}}.$$

Moreover, it follows from Lemma 4.2 that $u(\cdot,t) - u^*$ is bounded in $W^{1,4}(\Omega)$; thus, we conclude from (4.18) that $u(\cdot,t) \to u^*$ in $L^\infty(\Omega)$ as $t \to \infty$. By the similar arguments for v and w, we derive (4.17). The proof of Lemma 4.5 is complete. □

Now, we give the convergence rate of the coexistence state (u^*, v^*, w^*) for System (1.11).

Lemma 4.6. *Let the assumptions of Theorem 1.2 hold; the global bounded solution* (u, v, w) *of (1.11) exponentially converges to the coexistence state* (u^*, v^*, w^*)*, i.e. there exist* $C > 0$ *and* $\lambda > 0$ *such that*

$$\|u(\cdot,t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w^*\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \tag{4.19}$$

for all $t > T_1$*, where* $T_1 > 0$ *is some fixed time.*

Proof. It follows from Lemma 4.5 that $\|u-u^*\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$. Therefore, we apply L'Hôpital's rule to obtain

$$\lim_{u\to u^*} \frac{u-u^*-u^*\ln\frac{u}{u^*}}{(u-u^*)^2} = \frac{1}{2u^*}, \tag{4.20}$$

which implies that there exists a $t_1 > 0$ such that

$$\frac{1}{4u^*}\int_\Omega (u-u^*)^2 dx \le \int_\Omega \left(u-u^*-u^*\ln\frac{u}{u^*}\right)dx \le \frac{3}{4u^*}\int_\Omega (u-u^*)^2 dx \tag{4.21}$$

for all $t > t_1$. Similarly, we can find $t_2 > 0$ satisfying

$$\frac{1}{4v^*}\int_\Omega (v-v^*)^2dx \le \int_\Omega \left(v - v^* - v^* \ln \frac{v}{v^*}\right)dx \le \frac{3}{4v^*}\int_\Omega (v-v^*)^2dx \tag{4.22}$$

and

$$\frac{1}{4w^*}\int_\Omega (w-w^*)^2dx \le \int_\Omega \left(w - w^* - w^* \ln \frac{w}{w^*}\right)dx \le \frac{3}{4w^*}\int_\Omega (w-w^*)^2dx \tag{4.23}$$

for all $t > t_2$. Let $T_1 := \max\{t_1, t_2\}$; by means of the definitions of $\mathcal{E}_1(t)$ and $\mathcal{F}_1(t)$, it follows from the second inequalities in (4.21)–(4.23) that there exists a $C_1 > 0$ such that

$$C_1\mathcal{E}_1(t) \le \mathcal{F}_1(t) \quad \text{for all} \quad t > T_1. \tag{4.24}$$

By Lemma 4.4, we derive

$$\mathcal{E}_1'(t) \le -\varepsilon_1\mathcal{F}_1(t) \le -\varepsilon_1 C_1\mathcal{E}_1(t) \quad \text{for all} \quad t > T_1, \tag{4.25}$$

which implies that there exist $C_2 > 0$ and $C_3 > 0$ such that

$$\mathcal{E}_1(t) \le C_2 e^{-C_3(t-T_1)} \quad \text{for all} \quad t > T_1. \tag{4.26}$$

Thus we deduce from the first inequalities in (4.21)–(4.23) that there exists a $C_4 > 0$ such that

$$\begin{aligned}&\int_\Omega (u(x,t)-u^*)^2dx + \int_\Omega (v(x,t)-v^*)^2dx + \int_\Omega (w(x,t)-w^*)^2dx \\ &\quad \le C_4\mathcal{E}_1(t) \le C_2C_4e^{-C_3(t-T_1)} \quad \text{for all} \quad t > T_1.\end{aligned} \tag{4.27}$$

It follows from the Gagliardo-Nirenberg inequality in two dimensions, Lemma 4.2 and Lemma 3.2 that there exist positive constants C_5 and C_6 such that

$$\begin{aligned}&\|u-u^*\|_{L^\infty(\Omega)} + \|v-v^*\|_{L^\infty(\Omega)} + \|w-w^*\|_{L^\infty(\Omega)} \\ &\le C_5\Big(\|u-u^*\|^{\frac{2}{3}}_{W^{1,4}(\Omega)}\|u-u^*\|^{\frac{1}{3}}_{L^2(\Omega)} + \|v-v^*\|^{\frac{2}{3}}_{W^{1,4}(\Omega)}\|v-v^*\|^{\frac{1}{3}}_{L^2(\Omega)} \\ &\quad + \|w-w^*\|^{\frac{2}{3}}_{W^{1,4}(\Omega)}\|w-w^*\|^{\frac{1}{3}}_{L^2(\Omega)}\Big) \\ &\le C_6\left(\int_\Omega (u-u^*)^2dx + \int_\Omega (v-v^*)^2dx + \int_\Omega (w-w^*)^2dx\right)^{\frac{1}{6}} \\ &\le C_6(C_2C_4)^{\frac{1}{6}}e^{\frac{-C_3(t-T_1)}{6}}\end{aligned} \tag{4.28}$$

for all $t > T_1$. The proof of Lemma 4.6 is complete. □

Proof of Theorem 1.2. The statement of Theorem 1.2 is a straightforward consequence of Lemma 4.6. □

4.2. Proof of Theorems 1.3 and 1.4

In this subsection, we shall study the stabilization of the semi-trivial steady state $(\overline{u}, 0, \overline{w})$ or $(0, \widetilde{v}, \widetilde{w})$ for some parameters cases. Since the methods of the proofs of Theorem 1.3 and Theorem 1.4 are very similar, we only give the proof of Theorem 1.3 for brevity. To do this, let us introduce the following functionals

$$\begin{aligned}\mathcal{E}_2(t) =& \frac{1}{\gamma_1}\int_\Omega \left(u - \overline{u} - \overline{u}\ln\frac{u}{\overline{u}}\right)dx + \frac{1}{\gamma_2}\int_\Omega vdx \\ &+ \int_\Omega \left(w - \overline{w} - \overline{w}\ln\frac{w}{\overline{w}}\right)dx,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_2(t) =& \int_\Omega (u-\overline{u})^2\,dx + \int_\Omega v^2dx + \int_\Omega (w-\overline{w})^2\,dx \\ &+ \int_\Omega \left|\frac{\nabla u}{u}\right|^2 dx + \int_\Omega |\nabla w|^2 dx,\end{aligned}$$

where $\overline{u} = \frac{\mu(\gamma_1-\theta)}{\alpha_1\mu+\gamma_1}$ and $\overline{w} = \frac{\alpha_1\mu+\theta}{\alpha_1\mu+\gamma_1}$.

Lemma 4.7. *Let the conditions of Theorem 1.3 hold. Then there exists a positive constant ε_2 independent of t such that*

$$\mathcal{E}_2(t) \geq 0 \tag{4.29}$$

and

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -\varepsilon_2\mathcal{F}_2(t) - \frac{1}{\gamma_2}(\theta - \gamma_2\overline{w} + \beta_2\overline{u})\int_\Omega vdx \quad \textit{for all } t > 0. \tag{4.30}$$

Proof. Let

$$\begin{aligned}J_1(t) &:= \tfrac{1}{\gamma_1}\textstyle\int_\Omega \left(u - \overline{u} - \overline{u}\ln\frac{u}{\overline{u}}\right)dx,\\ J_2(t) &:= \tfrac{1}{\gamma_2}\textstyle\int_\Omega vdx,\\ J_3(t) &:= \textstyle\int_\Omega \left(w - \overline{w} - \overline{w}\ln\frac{w}{\overline{w}}\right)dx,\end{aligned}$$

then $\mathcal{E}_2(t)$ can be represented as

$$\mathcal{E}_2(t) = J_1(t) + J_2(t) + J_3(t) \quad \text{for all } \ t > 0.$$

Firstly, we can prove the nonnegativity of $\mathcal{E}_2(t)$ for all $t > 0$ by the similar arguments used in Step 1 in Lemma 4.4. For brevity, we omit the details here. Now, we shall prove (4.30). By a series of simple calculations, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}J_1(t) =& -\frac{\overline{u}}{\gamma_1}\int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2}dx - \frac{\overline{u}}{\gamma_1}\int_\Omega \frac{d_1'(w)\nabla u\cdot\nabla w}{u}dx \\ &+ \int_\Omega (u-\overline{u})(w-\overline{w})dx - \frac{\alpha_1}{\gamma_1}\int_\Omega (u-\overline{u})^2dx \\ &- \frac{\beta_1}{\gamma_1}\int_\Omega (u-\overline{u})vdx,\end{aligned} \tag{4.31}$$

where we have used the fact that $\theta = \gamma_1\overline{w} - \alpha_1\overline{u}$.

Similarly, we can derive

$$\frac{\mathrm{d}}{\mathrm{dt}}J_2(t) = -\frac{\alpha_2}{\gamma_2}\int_\Omega v^2dx - \frac{\beta_2}{\gamma_2}\int_\Omega v(u-\overline{u})dx + \int_\Omega v(w-\overline{w})dx - \frac{1}{\gamma_2}(\theta - \gamma_2\overline{w} + \beta_2\overline{u})\int_\Omega vdx, \tag{4.32}$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}J_3(t) = -D\overline{w}\int_\Omega \frac{|\nabla w|^2}{w^2}dx - \int_\Omega (w-\overline{w})(u-\overline{u})dx - \int_\Omega (w-\overline{w})vdx - \mu\int_\Omega (w-\overline{w})^2dx, \tag{4.33}$$

where we have used the fact that $\mu = \overline{u} + \mu\overline{w}$. Thus it follows from (4.31)–(4.33) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_2(t) = &-\frac{\alpha_1}{\gamma_1}\int_\Omega (u-\overline{u})^2dx - \frac{\alpha_2}{\gamma_2}\int_\Omega v^2dx - \mu\int_\Omega (w-\overline{w})^2dx \\ &-\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)\int_\Omega (u-\overline{u})vdx - \frac{\overline{u}}{\gamma_1}\int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2}dx \\ &- D\overline{w}\int_\Omega \frac{|\nabla w|^2}{w^2}dx - \frac{\overline{u}}{\gamma_1}\int_\Omega \frac{d_1'(w)\nabla u\cdot\nabla w}{u}dx \\ &- \frac{1}{\gamma_2}(\theta-\gamma_2\overline{w}+\beta_2\overline{u})\int_\Omega vdx \\ := &-\int_\Omega X_2A_2X_2^Tdx - \int_\Omega Y_2B_2Y_2^Tdx - \frac{1}{\gamma_2}(\theta-\gamma_2\overline{w}+\beta_2\overline{u})\int_\Omega vdx,\end{aligned} \tag{4.34}$$

where $X_2 = (u-\overline{u}, v, w-\overline{w})$ and $Y_2 = \left(\frac{\nabla u}{u}, \nabla w\right)$, as well as

$$A_2 = \begin{pmatrix} \frac{\alpha_1}{\gamma_1} & \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) & 0 \\ \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) & \frac{\alpha_2}{\gamma_2} & 0 \\ 0 & 0 & \mu \end{pmatrix}, B_2 = \begin{pmatrix} \frac{\overline{u}d_1(w)}{\gamma_1} & \frac{\overline{u}d_1'(w)}{2\gamma_1} \\ \frac{\overline{u}d_1'(w)}{2\gamma_1} & \frac{D\overline{w}}{w^2} \end{pmatrix}. \tag{4.35}$$

It follows from (1.20) that

$$\left|\frac{\alpha_1}{\gamma_1}\right| > 0 \quad \text{and} \quad \begin{vmatrix} \frac{\alpha_1}{\gamma_1} & \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) \\ \frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) & \frac{\alpha_2}{\gamma_2} \end{vmatrix} = \frac{\alpha_1\alpha_2}{\gamma_1\gamma_2} - \frac{1}{4}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)^2 > 0 \tag{4.36}$$

as well as

$$|A_2| = \mu\left(\frac{\alpha_1\alpha_2}{\gamma_1\gamma_2} - \frac{1}{4}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)^2\right) > 0, \tag{4.37}$$

which implies that the matrix A_2 is positive definite as according to Sylvester's criterion. Similarly, we deduce from (1.23) that

$$\left|\frac{\overline{u}d_1(w)}{\gamma_1}\right| > 0 \quad \text{and} \quad |B_2| = \frac{\overline{uw}d_1(w)}{\gamma_1 w^2}\left(D - \frac{\overline{u}w^2|d_1'(w)|^2}{4\gamma_1\overline{w}d_1(w)}\right) > 0, \tag{4.38}$$

which implies that the matrix B_2 is positive definite. Thus there exist positive constants ι_1 and ι_2 such that

$$X_2 A_2 X_2^T \geq \iota_1 |X_2|^2 \quad \text{and} \quad Y_2 B_2 Y_2^T \geq \iota_2 |Y_2|^2 \tag{4.39}$$

for all $x \in \Omega$ and $t > 0$. Let $\varepsilon_2 \in (0, \min\{\iota_1, \iota_2\})$; we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_2(t) \leq -\varepsilon_2 \int_\Omega |X_2|^2 + |Y_2|^2 dx - \frac{1}{\gamma_2}(\theta - \gamma_2\overline{w} + \beta_2\overline{u}) \int_\Omega v dx \quad \text{for all} \quad t > 0, \tag{4.40}$$

which implies that (4.30) holds. The proof of Lemma 4.7 is complete. □

With the help of Lemma 4.7, we shall give the following stabilization of the semi-trivial steady state $(\overline{u}, 0, \overline{w})$ for System (1.11).

Lemma 4.8. *Let the assumptions of Theorem 1.3 hold. Then the global bounded solution* (u, v, w) *of (1.11) converges to the semi-trivial steady state* $(\overline{u}, 0, \overline{w})$ *given by (1.24), i.e.,*

$$\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \overline{w}\|_{L^\infty(\Omega)} \to 0 \tag{4.41}$$

as $t \to \infty$.

Proof. The proof of this lemma is similar to that of Lemma 4.5; here we omit the details. □

Now, we give the convergence rate of the semi-trivial steady state $(\overline{u}, 0, \overline{w})$ for System (1.11).

Lemma 4.9. *Let the assumptions of Theorem 1.3 hold; then, there exist positive constants* C *and* λ *such that:*

(a) when $\theta = \gamma_2\overline{w} - \beta_2\overline{u}$, *then*

$$\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \overline{w}\|_{L^\infty(\Omega)} \leq C(1 + t)^{-\lambda} \quad \text{for all} \quad t > T_2; \tag{4.42}$$

(b) when $\theta > \gamma_2\overline{w} - \beta_2\overline{u}$, *then*

$$\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \overline{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad \text{for all} \quad t > T_2, \tag{4.43}$$

where $T_2 > 0$ *is some fixed time.*

Proof. Let

$$\mathcal{F}_2^*(t) := \int_\Omega (u - \overline{u})^2\, dx + \int_\Omega v^2 dx + \int_\Omega (w - \overline{w})^2\, dx, \tag{4.44}$$

then it follows from Lemma 4.7 that there exists a $\varepsilon_2 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_2(t) \leq -\varepsilon_2 \mathcal{F}_2^*(t) - \frac{1}{\gamma_2}(\theta - \gamma_2\overline{w} + \beta_2\overline{u}) \int_\Omega v dx \quad \text{for all } t > 0. \tag{4.45}$$

We deduce from Lemma 4.8 that $\|u(\cdot, t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \overline{w}\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$. Hence, we apply L'Hôpital's rule to obtain

$$\lim_{u \to \overline{u}} \frac{u - \overline{u} - \overline{u}\ln\frac{u}{\overline{u}}}{(u - \overline{u})^2} = \frac{1}{2\overline{u}}, \tag{4.46}$$

which implies that there exists a $t_1' > 0$ such that

$$\frac{1}{4\overline{u}}\int_\Omega (u-\overline{u})^2dx \le \int_\Omega \left(u-\overline{u}-\overline{u}\ln\frac{u}{\overline{u}}\right)dx \le \frac{3}{4\overline{u}}\int_\Omega (u-\overline{u})^2dx \tag{4.47}$$

for all $t > t_1'$. Similarly, we can find $t_2' > 0$ satisfying

$$\frac{1}{4\overline{w}}\int_\Omega (w-\overline{w})^2dx \le \int_\Omega \left(w-\overline{w}-\overline{w}\ln\frac{w}{\overline{w}}\right)dx \le \frac{3}{4\overline{w}}\int_\Omega (w-\overline{w})^2dx \tag{4.48}$$

for all $t > t_2'$.

By using the fact that $\lim_{s\to 0}\frac{s}{s^2+s} = 1$, it follows from $\|v(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$ that there exists a $t_3' > 0$ such that

$$\frac{1}{2}\int_\Omega v^2 + vdx \le \int_\Omega vdx \le \frac{3}{2}\int_\Omega v^2 + vdx \tag{4.49}$$

for all $t > t_3'$.

(a) When $\theta = \gamma_2\overline{w} - \beta_2\overline{u}$, (4.45) can be turned into

$$\frac{d}{dt}\mathcal{E}_2(t) \le -\varepsilon_2\mathcal{F}_2^*(t) \quad \text{for all } t > 0. \tag{4.50}$$

Let $T_2 := \max\{t_1', t_2', t_3'\}$; by means of the definitions of $\mathcal{E}_2(t)$ and $\mathcal{F}_2^*(t)$, it follows from the second inequalities in (4.47) and (4.48) that there exist positive constants C_1 and C_2 such that

$$\begin{aligned}\mathcal{E}_2(t) &\le \frac{3}{4\gamma_1\overline{u}}\int_\Omega (u-\overline{u})^2dx + \frac{1}{\gamma_2}\int_\Omega vdx + \frac{3}{4\overline{w}}\int_\Omega (w-\overline{w})^2dx \\ &\le C_1\left(\int_\Omega (u-\overline{u})^2dx\right)^{\frac{1}{2}} + C_1\left(\int_\Omega v^2dx\right)^{\frac{1}{2}} + C_1\left(\int_\Omega (w-\overline{w})^2dx\right)^{\frac{1}{2}} \\ &\le C_2(\mathcal{F}_2^*(t))^{\frac{1}{2}},\end{aligned} \tag{4.51}$$

for all $t > T_2$, where we have used Hölder's inequality and the boundedness of (u, v, w) asserted by Theorem 1.1. Thus, we deduce from (4.50) that

$$\mathcal{E}_2'(t) \le -\frac{\varepsilon_2}{C_2^2}\mathcal{E}_2^2(t) \quad \text{for all } \ t > T_2, \tag{4.52}$$

which implies

$$\mathcal{E}_2(t) \le \frac{C_3}{t-T_2} \quad \text{for all } \ t > T_2, \tag{4.53}$$

with some positive constant C_3. Hence we infer from the first inequalities in (4.47)–(4.49) that there exists a $C_4 > 0$ such that

$$\begin{aligned}&\int_\Omega (u-\overline{u})^2dx + \int_\Omega v^2dx + \int_\Omega (w-\overline{w})^2dx \\ &\le C_4\mathcal{E}_2(t) \le \frac{C_3C_4}{t-T_2} \quad \text{for all } \ t > T_2.\end{aligned} \tag{4.54}$$

It follows from the Gagliardo-Nirenberg inequality in two dimensions, Lemma 4.2 and Lemma 3.2 that there exist positive constants C_5 and C_6 such that

$$\begin{aligned}
&\|u-\overline{u}\|_{L^\infty(\Omega)}+\|v\|_{L^\infty(\Omega)}+\|w-\overline{w}\|_{L^\infty(\Omega)}\\
&\le C_5\Big(\|u-\overline{u}\|^{\frac{2}{3}}_{W^{1,4}(\Omega)}\|u-\overline{u}\|^{\frac{1}{3}}_{L^2(\Omega)}+\|v\|^{\frac{2}{3}}_{W^{1,4}(\Omega)}\|v\|^{\frac{1}{3}}_{L^2(\Omega)}\\
&\quad+\|w-\overline{w}\|^{\frac{2}{3}}_{W^{1,4}(\Omega)}\|w-\overline{w}\|^{\frac{1}{3}}_{L^2(\Omega)}\Big)\\
&\le C_6\left(\int_\Omega(u-\overline{u})^2dx+\int_\Omega v^2dx+\int_\Omega(w-\overline{w})^2dx\right)^{\frac{1}{6}}\\
&\le C_6(C_3C_4)^{\frac{1}{6}}(t-T_2)^{-\frac{1}{6}}
\end{aligned}\tag{4.55}$$

for all $t>T_2$.

(*b*) When $\theta>\gamma_2\overline{w}-\beta_2\overline{u}$, let $T_2:=\max\{t_1',t_2',t_3'\}$; by means of the definitions of $\mathcal{E}_2(t)$ and $\mathcal{F}_2^*(t)$, it follows from the second inequalities in (4.47) and (4.48) that there exists a positive constant C_7 such that

$$\mathcal{E}_2(t)\le C_7\left(\mathcal{F}_2^*(t)+\int_\Omega vdx\right),\tag{4.56}$$

for all $t>T_2$.

By combining (4.45) with (4.56), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_2(t)\le-\frac{\varepsilon_2}{C_7}\mathcal{E}_2(t)-\frac{1}{\gamma_2}(\theta-\gamma_2\overline{w}+\beta_2\overline{u}-\gamma_2\varepsilon_2)\int_\Omega vdx\quad\text{for all } t>T_2.\tag{4.57}$$

Since $\theta>\gamma_2\overline{w}-\beta_2\overline{u}$, then we can select $\varepsilon_2\le\frac{1}{\gamma_2}(\theta-\gamma_2\overline{w}+\beta_2\overline{u})$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_2(t)\le-\frac{\varepsilon_2}{C_7}\mathcal{E}_2(t)\quad\text{for all } t>T_2,\tag{4.58}$$

which means that there exist $C_8>0$ and $C_9>0$ satisfying

$$\mathcal{E}_2(t)\le C_8e^{-C_9(t-T_2)}\quad\text{for all }\ t>T_2.\tag{4.59}$$

Thus we deduce from the first inequalities in (4.47)–(4.49) that there exists a $C_{10}>0$ such that

$$\begin{aligned}
&\int_\Omega(u-\overline{u})^2dx+\int_\Omega v^2dx+\int_\Omega(w-\overline{w})^2dx\\
&\quad\le C_{10}\mathcal{E}_2(t)\le C_8C_{10}e^{-C_9(t-T_2)}\quad\text{for all }\ t>T_2.
\end{aligned}\tag{4.60}$$

It follows from the Gagliardo-Nirenberg inequality in two dimensions, Lemma 4.2 and Lemma 3.2 that there exist positive constants C_{11} and C_{12} such that

$$\begin{aligned}
&\|u-\overline{u}\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w-\overline{w}\|_{L^\infty(\Omega)} \\
&\le C_{11}\Big(\|u-\overline{u}\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|u-\overline{u}\|_{L^2(\Omega)}^{\frac{1}{3}} + \|v\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|v\|_{L^2(\Omega)}^{\frac{1}{3}} \\
&\quad + \|w-\overline{w}\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|w-\overline{w}\|_{L^2(\Omega)}^{\frac{1}{3}}\Big) \\
&\le C_{12}\left(\int_\Omega (u-\overline{u})^2 dx + \int_\Omega v^2 dx + \int_\Omega (w-\overline{w})^2 dx\right)^{\frac{1}{6}} \\
&\le C_{12}(C_8C_{10})^{\frac{1}{6}} e^{\frac{-C_9(t-T_2)}{6}}
\end{aligned} \tag{4.61}$$

for all $t > T_2$. The proof of Lemma 4.9 is complete. □

Proof of Theorem 1.3. The statement of Theorem 1.3 is a direct consequence of Lemma 4.9. □

4.3. Proof of Theorem 1.5

In this subsection, we are devoted to discussing the asymptotic stability of the prey-only steady state $(0, 0, 1)$ under some suitable parameters conditions. To do this, let us denote the following functionals

$$\mathcal{E}_3(t) = \frac{1}{\gamma_1}\int_\Omega u dx + \frac{1}{\gamma_2}\int_\Omega v dx + \int_\Omega (w - 1 - \ln w)\, dx$$

and

$$\mathcal{F}_3(t) = \int_\Omega u^2 dx + \int_\Omega v^2 dx + \int_\Omega (w-1)^2\, dx + \int_\Omega |\nabla w|^2 dx,$$

we can derive the following estimates of $\mathcal{E}_3(t)$ and $\mathcal{F}_3(t)$.

Lemma 4.10. *Let the conditions of Theorem 1.5 hold. Then there exists a $\varepsilon_3 > 0$ independent of t such that*

$$\mathcal{E}_3(t) \ge 0 \tag{4.62}$$

and

$$\frac{d}{dt}\mathcal{E}_3(t) \le -\varepsilon_3\mathcal{F}_3(t) - \frac{1}{\gamma_1}(\theta-\gamma_1)\int_\Omega u dx - \frac{1}{\gamma_2}(\theta-\gamma_2)\int_\Omega v dx \quad \textit{for all } t > 0. \tag{4.63}$$

Proof. By the similar arguments as in the proofs of Lemma 4.4 and Lemma 4.7, we can derive (4.62) and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_3(t) =& -\frac{\alpha_1}{\gamma_1}\int_\Omega u^2dx - \frac{\alpha_2}{\gamma_2}\int_\Omega v^2dx - \mu\int_\Omega (w-1)^2dx \\ & -\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)\int_\Omega uvdx - D\int_\Omega \frac{|\nabla w|^2}{w^2}dx \\ & -\frac{1}{\gamma_1}(\theta-\gamma_1)\int_\Omega udx - \frac{1}{\gamma_2}(\theta-\gamma_2)\int_\Omega vdx \\ \leq & -\frac{\alpha_1}{\gamma_1}\int_\Omega u^2dx - \frac{\alpha_2}{\gamma_2}\int_\Omega v^2dx - \mu\int_\Omega (w-1)^2dx - \frac{D}{K_0^2}\int_\Omega |\nabla w|^2dx \\ & -\frac{1}{\gamma_1}(\theta-\gamma_1)\int_\Omega udx - \frac{1}{\gamma_2}(\theta-\gamma_2)\int_\Omega vdx,\end{aligned} \tag{4.64}$$

where we have used the fact that $w \leq K_0 = \max\{\|w_0\|_{L^\infty(\Omega)}, 1\}$. Let $\varepsilon_3 \in \left(0, \min\{\frac{\alpha_1}{\gamma_1}, \frac{\alpha_2}{\gamma_2}, \mu, \frac{D}{K_0^2}\}\right)$; we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_3(t) \leq -\varepsilon_3\mathcal{F}_3(t) - \frac{1}{\gamma_1}(\theta-\gamma_1)\int_\Omega udx - \frac{1}{\gamma_2}(\theta-\gamma_2)\int_\Omega vdx, \tag{4.65}$$

for all $t > 0$. The proof of Lemma 4.10 is complete. □

With the help of Lemma 4.10, we shall give the following stabilization of the prey-only steady state for System (1.11).

Lemma 4.11. *Let the assumptions of Theorem 1.5 hold. Then the global bounded solution of (1.11) converges to the prey-only steady state* $(0,0,1)$*, i.e.,*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)-1\|_{L^\infty(\Omega)} \to 0 \tag{4.66}$$

as $t \to \infty$.

Proof. The proof of this lemma is similar to that of Lemma 4.5; here we omit the details. □

Now, we give the convergence rate of the prey-only steady state $(0,0,1)$ for System (1.11).

Lemma 4.12. *Let the assumptions of Theorem 1.5 hold; then there exist positive constants C and* λ *such that:*

(a) when $\gamma_i = \theta$, $i = 1,2$, *then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)-1\|_{L^\infty(\Omega)} \leq C(1+t)^{-\lambda} \text{ for all } t > T_3; \tag{4.67}$$

(b) when $\gamma_i < \theta$, $i = 1,2$, *then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)-1\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \text{ for all } t > T_3, \tag{4.68}$$

where $T_3 > 0$ *is some fixed time.*

Proof. Let

$$\mathcal{F}_3^*(t) := \int_\Omega u^2dx + \int_\Omega v^2dx + \int_\Omega (w-1)^2\,dx, \tag{4.69}$$

then it follows from Lemma 4.10 that there exists a $\varepsilon_3 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_3(t) \le -\varepsilon_3\mathcal{F}_3^*(t) - \frac{1}{\gamma_1}(\theta-\gamma_1)\int_\Omega udx - \frac{1}{\gamma_2}(\theta-\gamma_2)\int_\Omega vdx \quad \text{for all } t>0. \tag{4.70}$$

By using the facts that $\lim_{s\to 0}\frac{s}{s^2+s} = 1$ and $\lim_{s\to 1}\frac{s-1-\ln s}{(s-1)^2} = \frac{1}{2}$, it follows from $\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)-1\|_{L^\infty(\Omega)} \to 0$ as $t\to\infty$, as asserted in Lemma 4.11 that there exists a $T_3 > 0$ such that

$$\frac{1}{2}\int_\Omega u^2+udx \le \int_\Omega udx \le \frac{3}{2}\int_\Omega u^2+udx \tag{4.71}$$

and

$$\frac{1}{2}\int_\Omega v^2+vdx \le \int_\Omega vdx \le \frac{3}{2}\int_\Omega v^2+vdx \tag{4.72}$$

as well as

$$\frac{1}{4}\int_\Omega (w-1)^2dx \le \int_\Omega (w-1-\ln w)\,dx \le \frac{3}{4}\int_\Omega (w-1)^2dx \tag{4.73}$$

for all $t > T_3$.

(a) When $\gamma_i = \theta$, $i = 1, 2$, (4.70) can be simplified as

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_3(t) \le -\varepsilon_3\mathcal{F}_3^*(t) \quad \text{for all } t>0. \tag{4.74}$$

By means of the definitions of $\mathcal{E}_3(t)$ and $\mathcal{F}_3^*(t)$, it follows from the second inequality in (4.73) that there exist positive constants C_1 and C_2 such that

$$\begin{aligned}\mathcal{E}_3(t) &\le \frac{1}{\gamma_1}\int_\Omega udx + \frac{1}{\gamma_2}\int_\Omega vdx + \frac{3}{4}\int_\Omega (w-1)^2dx\\ &\le C_1\left(\int_\Omega u^2dx\right)^{\frac{1}{2}} + C_1\left(\int_\Omega v^2dx\right)^{\frac{1}{2}} + C_1\left(\int_\Omega (w-1)^2dx\right)^{\frac{1}{2}}\\ &\le C_2(\mathcal{F}_3^*(t))^{\frac{1}{2}},\end{aligned} \tag{4.75}$$

for all $t > T_3$, where we have used Hölder's inequality and the boundedness of (u, v, w) asserted by Theorem 1.1. Thus we deduce from (4.74) that

$$\mathcal{E}_3'(t) \le -\frac{\varepsilon_3}{C_2^2}\mathcal{E}_3^2(t) \quad \text{for all } \ t > T_3, \tag{4.76}$$

which implies

$$\mathcal{E}_3(t) \le \frac{C_3}{t-T_3} \quad \text{for all } \ t > T_3, \tag{4.77}$$

with some positive constant C_3. Hence we infer from the first inequalities in (4.71)–(4.73) that there exists a $C_4 > 0$ such that

$$\begin{aligned}&\int_\Omega u^2dx + \int_\Omega v^2dx + \int_\Omega (w-1)^2dx\\ &\le C_4\mathcal{E}_3(t) \le \frac{C_3C_4}{t-T_3} \quad \text{for all } \ t > T_3.\end{aligned} \tag{4.78}$$

It follows from the Gagliardo-Nirenberg inequality in two dimensions, Lemma 4.2 and Lemma 3.2 that there exist positive constants C_5 and C_6 such that

$$\begin{aligned}
&\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} \\
&\le C_5\Big(\|u\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|u\|_{L^2(\Omega)}^{\frac{1}{3}} + \|v\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|v\|_{L^2(\Omega)}^{\frac{1}{3}} \\
&\quad + \|w-1\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|w-1\|_{L^2(\Omega)}^{\frac{1}{3}}\Big) \\
&\le C_6\left(\int_\Omega u^2dx + \int_\Omega v^2dx + \int_\Omega (w-1)^2dx\right)^{\frac{1}{6}} \\
&\le C_6(C_3C_4)^{\frac{1}{6}}(t-T_3)^{-\frac{1}{6}}
\end{aligned} \tag{4.79}$$

for all $t > T_3$.

(b) When $\gamma_i < \theta$, $i = 1, 2$, by means of the definitions of $\mathcal{E}_3(t)$ and $\mathcal{F}_3^*(t)$, it follows from the second inequalities in (4.71)–(4.73) that there exists a positive constant C_7 such that

$$\mathcal{E}_3(t) \le C_7\left(\mathcal{F}_3^*(t) + \int_\Omega udx + \int_\Omega vdx\right), \tag{4.80}$$

for all $t > T_3$.

By combining (4.70) with (4.80), we derive

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_3(t) \le -\frac{\varepsilon_3}{C_7}\mathcal{E}_3(t) - \frac{1}{\gamma_1}(\theta-\gamma_1-\varepsilon_3\gamma_1)\int_\Omega udx - \frac{1}{\gamma_2}(\theta-\gamma_2-\varepsilon_3\gamma_2)\int_\Omega vdx \tag{4.81}$$

for all $t > 0$. Since $\gamma_i < \theta$, $i = 1, 2$, we can select $\varepsilon_3 \le \min\left\{\frac{1}{\gamma_1}(\theta-\gamma_1), \frac{1}{\gamma_2}(\theta-\gamma_2)\right\}$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}_3(t) \le -\frac{\varepsilon_3}{C_7}\mathcal{E}_3(t) \quad \text{for all } t > T_3, \tag{4.82}$$

which means that there exist $C_8 > 0$ and $C_9 > 0$ satisfying

$$\mathcal{E}_3(t) \le C_8e^{-C_9(t-T_3)} \quad \text{for all } \ t > T_3. \tag{4.83}$$

Thus we deduce from the first inequalities in (4.71)–(4.73) that there exists a $C_{10} > 0$ such that

$$\begin{aligned}
&\int_\Omega u^2dx + \int_\Omega v^2dx + \int_\Omega (w-1)^2dx \\
&\le C_{10}\mathcal{E}_2(t) \le C_8C_{10}e^{-C_9(t-T_3)} \quad \text{for all } \ t > T_3.
\end{aligned} \tag{4.84}$$

It follows from the Gagliardo-Nirenberg inequality in two dimensions, Lemma 4.2 and Lemma 3.2 that there exist positive constants C_{11} and C_{12} such that

$$\begin{aligned}
&\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} \\
&\le C_{11}\Big(\|u\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|u\|_{L^2(\Omega)}^{\frac{1}{3}} + \|v\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|v\|_{L^2(\Omega)}^{\frac{1}{3}} \\
&\quad + \|w-1\|_{W^{1,4}(\Omega)}^{\frac{2}{3}}\|w-1\|_{L^2(\Omega)}^{\frac{1}{3}}\Big) \\
&\le C_{12}\left(\int_\Omega u^2dx + \int_\Omega v^2dx + \int_\Omega (w-1)^2dx\right)^{\frac{1}{6}} \\
&\le C_{12}(C_8C_{10})^{\frac{1}{6}}e^{\frac{-C_9(t-T_3)}{6}}
\end{aligned} \tag{4.85}$$

for all $t > T_3$. The proof of Lemma 4.12 is complete. □

Proof of Theorem 1.5. The statement of Theorem 1.5 is a direct consequence of Lemma 4.12. □

Acknowledgments

The author is grateful to the editor and three anonymous referees for their valuable comments, which greatly improved the exposition of our paper. The author is also deeply grateful to Professor Renjun Duan for the support and friendly hospitality in CUHK. The work was partially supported by the National Natural Science Foundation of China (Grant Nos: 11601053, 11526042), Natural Science Foundation of Chongqing (Grant No. cstc2019jcyj-msxmX0082), Chongqing Municipal Education Commission Science and Technology Research Project (Grant No. KJZD-K202200602), China-South Africa Young Scientist Exchange Project in 2020, The Hong Kong Scholars Program (Grant Nos: XJ2021042, 2021-005) and Young Hundred Talents Program of CQUPT from 2022–2024.

Conflict of interest

The author declares that there is no conflict of interest.

References

1. P. Kareiva, G. Odell, Swarms of predators exhibit preytaxis if individual predators use area-restricted search, *Amer. Nat.*, **130** (1987), 233–270. https://doi.org/10.1086/284707
2. A.J. Lotka, *Elements of Physical Biology*, Baltimore: Williams and Wilkins Co.,1925.
3. V. Volterra, Fluctuations in the abundance of a species considered mathematically, *Nature*, **118** (1926), 558–560. https://doi.org/10.1038/118558a0
4. C. Holling, The functional response of predators to prey density and its role in mimicry and population regulation, *Mem. Entom. Soc. Can.*, **45** (1965), 1–60. https://doi.org/10.4039/entm9745fv
5. C. Cosner, D. L. DeAngelis, J. S. Ault, D. Olson, Effects of spatial grouping on the functional response of predators, *Theor. Popul. Biol.*, **56** (1999), 65–75. https://doi.org/10.1006/tpbi.1999.1414
6. P. H. Crowley, E. K. Martin, Functional responses and interference within and between year classes of a dragonfly population, *J. North Amer. Benthol. Soc.*, **8** (1989), 211–221. https://doi.org/10.2307/1467324
7. C. Cosner, Reaction-diffusion-advection models for the effects and evolution of dispersal, *Discrete Contin. Dyn. Syst.*, **34** (2014), 1701–1745. https://doi.org/10.3934/dcds.2014.34.1701
8. W. W. Murdoch, C. J. Briggs, R. M. Nisbert, *Consumer-Resource Dynamics, Monographs in Population Biology*, Princeton University Press, 2003.
9. P. Turchin, *Complex Population Dynamics: A Theoretical/Empirical Synthesis, Monographs in Population Biology*, Princeton University Press, 2003.

10. G. T. Skalski, J. F. Gilliam, Functional responses with predator interference: Viable alternatives to the Holling type II model, *Ecology*, **82** (2001), 3083–3092. https://doi.org/10.1890/0012-9658
11. J. M. Lee, T. Hillen, M. A. Lewis, Pattern formation in prey-taxis systems, *J. Biol. Dyn.*, **3** (2009), 551–573. https://doi.org/10.1080/17513750802716112
12. J. M. Lee, T. Hillen, M. A. Lewis, Continuous traveling waves for prey-taxis, *Bull. Math. Biol.*, **70** (2008), 654–676. https://doi.org/10.1007/s11538-007-9271-4
13. H. Jin, Z. Wang, Global stability of prey-taxis systems, *J. Differ. Equations*, **262** (2017), 1257–1290. https://doi.org/10.1016/j.jde.2016.10.010
14. H. Jin, Z. Wang, Global dynamics and spatio-temporal patterns of predator-prey systems with density-dependent motion, *European J. Appl. Math.*, **32** (2021), 652–682. https://doi.org/10.1017/S0956792520000248
15. S. Wu, J. Shi, B. Wu, Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis, *J. Differ. Equations*, **260** (2016), 5847–5874. https://doi.org/10.1016/j.jde.2015.12.024
16. Q. Wang, Y. Song, L. Shao, Nonconstant positive steady states and pattern formation of 1D prey-taxis systems, *J. Nonlinear Sci.*, **27** (2017), 71–97. https://doi.org/10.1007/s00332-016-9326-5
17. W. Choi, I. Ahn, Predator invasion in predator-prey model with prey-taxis in spatially heterogeneous environment, *Nonlinear Anal. Real World Appl.*, **65** (2022), 103495. https://doi.org/10.1016/j.nonrwa.2021.103495
18. Y. Cai, Q. Cao, Z. Wang, Asymptotic dynamics and spatial patterns of a ratio-dependent predator-prey system with prey-taxis, *Appl. Anal.*, **101** (2022), 81–99. https://doi.org/10.1080/00036811.2020.1728259
19. H. Jin, Y. King Z. Wang, Boundedness, stabilization, and pattern formation driven by density suppressed motility, *SIAM J. Appl. Math.*, **78** (2018), 1632–1657. https://doi.org/10.1137/17M1144647
20. Y. Tao, Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis, *Nonlinear Anal. Real World Appl.*, **11** (2010), 2056–2064. https://doi.org/10.1016/j.nonrwa.2009.05.005
21. J. I. Tello, D. Wrzosek, Predator-prey model with diffusion and indirect prey-taxis, *Math. Models Methods Appl. Sci.*, **26** (2016), 2129–2162. https://doi.org/10.1142/S0218202516400108
22. J. Wang, M. Wang, The diffusive Beddington-DeAngelis predator-prey model with nonlinear prey-taxis and free boundary, *Math. Method. Appl. Sci.*, **41** (2018), 6741–6762. https://doi.org/10.1002/mma.5189
23. J. Wang, M. Wang, Global solution of a diffusive predator-prey model with prey-taxis, *Comput. Math. Appl.*, **77** (2019), 2676–2694. https://doi.org/10.1016/j.camwa.2018.12.042
24. J. Wang, M. Wang, The dynamics of a predator-prey model with diffusion and indirect prey-taxis, *J. Dyn. Differ. Equ.*, **32** (2020), 1291–1310. https://doi.org/10.1007/s10884-019-09778-7
25. M. Winkler, Asymptotic homogenization in a three-dimensional nutrient taxis system involving food-supported proliferation, *J. Differ. Equations*, **263** (2017), 4826–4869. https://doi.org/10.1016/j.jde.2017.06.002

26. T. Xiang, Global dynamics for a diffusive predator-prey model with prey-taxis and classical Lotka-Volterra kinetics, *Nonlinear Anal. Real World Appl.*, **39** (2018), 278–299. https://doi.org/10.1016/j.nonrwa.2017.07.001

27. P. Mishra, D. Wrzosek, Repulsive chemotaxis and predator evasion in predator-prey models with diffusion and prey-taxis, *Math. Models Methods Appl. Sci.*, **32** (2022), 1–42. https://doi.org/10.1142/S0218202522500014

28. L. Rodriguez Q., L. Gordillo, Density-dependent diffusion and refuge in a spatial Rosenzweig-MacArthur model: Stability results, *J. Math. Anal. Appl.*, **512** (2022), 126174. https://doi.org/10.1016/j.jmaa.2022.126174

29. L. Rodriguez Q., J. Zhao, L. Gordillo, The effects of simple density-dependent prey diffusion and refuge in a predator-prey system, *J. Math. Anal. Appl.*, **498** (2021), 124983. https://doi.org/10.1016/j.jmaa.2021.124983

30. K. Wang, Q. Wang, F. Yu, Stationary and time periodic patterns of two-predator and one-prey systems with prey-taxis, *Discrete Contin. Dyn. Syst.*, **37** (2017), 505–543. https://doi.org/10.3934/dcds.2017021

31. J. Wang, M. Wang, Boundedness and global stability of the two-predator and one-prey models with nonlinear prey-taxis, *Z. Angew. Math. Phys.*, **69** (2018), 63. https://doi.org/10.1007/s00033-018-0960-7

32. Y. Mi, C. Song, Z. Wang, Boundedness and global stability of the predator-prey model with prey-taxis and competition, *Nonlinear Anal. Real World Appl.*, **66** (2022), 103521. https://doi.org/10.1016/j.nonrwa.2022.103521

33. S. Qiu, C. Mu, X. Tu, Dynamics for a three-species predator-prey model with density-dependent motilities, *J. Dyn. Differ. Equations*, (2021). http://dx.doi.org/10.1007/s10884-021-10020-6

34. X. Fu, L.H. Tang, C. Liu, J. D. Huang, T. Hwa, P. Lenz, Stripe formation in bacterial system with density-suppressed motility, *Phys. Rev. Lett.*, **108** (2012), 198102. http://dx.doi.org/10.1103/PhysRevLett.108.198102

35. C. Liu, X. Fu, L. Liu, X. Ren, C. K. L. Chau, S. Li, et al., Sequential establishment of stripe patterns in an expanding cell population, *Science*, **334** (2011), 238–241. https://doi.org/10.1126/science.1209042

36. K. Fujie, J. Jiang, Global existence for a kinetic model of pattern formation with density-suppressed motilities, *J. Differ. Equations*, **269** (2020), 5338–5378. https://doi.org/10.1016/j.jde.2020.04.001

37. J. Jiang, P. Laurençot, Y. Zhang, Global existence, uniform boundedness, and stabilization in a chemotaxis system with density-suppressed motility and nutrient consumption, *Comm. Partial Differ. Equations*, **47** (2022), 1024–1069. https://doi.org/10.1080/03605302.2021.2021422

38. H. Jin, Z. Wang, Critical mass on the Keller-Segel system with signal-dependent motility, *Proc. Amer. Math. Soc.*, **148** (2020), 4855–4873. https://doi.org/10.1090/proc/15124

39. H. Jin, S. Shi, Z. Wang, Boundedness and asymptotics of a reaction-diffusion system with density-dependent motility, *J. Differ. Equations*, **269** (2020), 6758–6793. https://doi.org/10.1016/j.jde.2020.05.018

40. J. Li, Z. Wang, Traveling wave solutions to the density-suppressed motility model, *J. Differ. Equations*, **301** (2021), 1–36. https://doi.org/10.1016/j.jde.2021.07.038

41. M. Ma, R. Peng, Z. Wang, Stationary and non-stationary patterns of the density-suppressed motility model, *Physica D*, **402** (2020), 132259. https://doi.org/10.1016/j.physd.2019.132259

42. Z. Wang, X. Xu, Steady states and pattern formation of the density-suppressed motility model, *IMA J. Appl. Math.*, **86** (2021), 577603. https://doi.org/10.1093/imamat/hxab006

43. Y. Tao, M. Winkler, Effects of signal-dependent motilities in a Keller-Segel-type reaction-diffusion system, *Math. Models Methods Appl. Sci.*, **27** (2017), 1645–1683. https://doi.org/10.1142/S0218202517500282

44. P. Zheng, R. Willie, Dynamics in an attraction-repulsion Navier-Stokes system with signal-dependent motility and sensitivity, *J. Math. Phys.*, **62** (2021), 041503. https://doi.org/10.1063/5.0029161

45. Z. Wang, J. Xu, On the Lotka-Volterra competition system with dynamical resources and density-dependent diffusion, *J. Math. Biol.*, **82** (2021), 37. https://doi.org/10.1007/s00285-021-01562-w

46. J. Dockery, V. Hutson, K. Mischaikow, M. Pernarowski, The evolution of slow dispersal rates: a reaction diffusion model, *J. Math. Biol.*, **37** (1998), 61–83. https://doi.org/10.1007/s002850050120

47. Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Differ. Equations*, **223** (2006), 400–426. https://doi.org/10.1016/j.jde.2005.05.010

48. H. Berestycki, A. Zilio, Predators-prey models with competition, part I: Existence, bifurcation and qualitative properties, *Commun. Contemp. Math.*, **20** (2018), 1850010. https://doi.org/10.1142/S0219199718500104

49. H. Berestycki, A. Zilio, Predators-prey models with competition: The emergence of territoriality, *Amer. Nat.*, **193** (2019), 436–446. https://doi.org/10.1086/701670

50. J. Lin, W. Wang, C. Zhao, T. Yang, Global dynamics and traveling wave solutions of two predators-one prey models, *Discrete Contin. Dyn. Syst. Ser. B*, **20** (2015), 1135–1154. https://doi.org/10.3934/dcdsb.2015.20.1135

51. P. Pang, M. Wang, Strategy and stationary pattern in a three-species predator-prey model, *J. Differ. Equations*, **200** (2004), 245–273. https://doi.org/10.1016/j.jde.2004.01.004

52. H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in *Function Spaces, Differential Operators and Nonlinear Analysis*, (1993), 9–126. https://doi.org/10.1007/978-3-663-11336-2_1

53. H. Amann, Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems, *Differ. Integral Equations*, **3** (1990), 13–75.

54. H. Amann, Dynamic theory of quasilinear parabolic systems. III. Global existence, *Math. Z.*, **202** (1989), 219–250.

55. R. Kowalczyk, Z. Szymaáska, On the global existence of solutions to an aggregation model, *J. Math. Anal. Appl.*, **343** (2008), 379–398. https://doi.org/10.1016/j.jmaa.2008.01.005

56. C. Stinner, C. Surulescu, M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, *SIAM J. Math. Anal.*, **46** (2014), 1969–2007. https://doi.org/10.1137/13094058X

57. X. Bai, M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, *Indiana Univ. Math. J.*, **65** (2016), 553–583. https://doi.org/10.1512/iumj.2016.65.5776

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)