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Research article

Fuzzy optimal harvesting of a prey-predator model in the presence of toxicity with prey refuge under imprecise parameters

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Abstract: The objective of this paper is to investigate the dynamic behaviors of a prey-predator model incorporating the effect of toxic substances with prey refuge under imprecise parameters. We handle these biological parameters in model by using interval numbers. The existence together with stability of biological equilibria are obtained. We also analyze the existence conditions of the bionomic equilibria. The optimal harvesting strategy is explored by taking into account instantaneous annual discount rate under fuzzy conditions. Three numeric examples are performed to illustrate our analytical findings.

Keywords: prey-predator model; imprecise parameter; refuge; toxicity; equilibria; stability; fuzzy optimal harvesting

1. Introduction

The study of theoretical ecology originated from Lotka [1] and Volterra [2]. In the long run, over-exploitation of living resources such as fisheries and forestry will threaten biodiversity and put humanity in a very alarming situation. Bioeconomic models involving scientific management of renewable resource exploitation have been drawing attention to the interest of many researchers [3–7] and the references therein. For example, Clark [3,4] set the foundation in this work domain. And Kar and Chaudhuri [5] investigated the existence of bioeconomic equilibrium as well as the optimal harvesting policy of a multispecies harvesting model with interference. He and Zhou [6] presented an optimal harvesting problem for a class of hierarchical age-structured model. Lately, Wang [7] established a predator-prey model with prey refuge in fuzzy environment, simultaneously investigated the problem on fuzzy optimal harvesting.

In recent years, the influence of toxicants on ecosystem have turned into a major environmental problem. Mathematical modelling for handling such problems began with the work of Hallam and Clark [8], Dubey and Hussain [9], Kar and Chaudhuri [10]. Since toxin released by one species not only affects the growth of species themselves but also may impact that of the other species, majority of these models manage single species or two species models in general. Maynard-Smith [11] introduced the effect of toxin in a Lotka-Volterra competition model due to each species producing toxin to the other only when the other exists. Chattopadhyay et al. [12] investigated a mathematical model in view of field observations in the Ridiga area of Tulsa in the bay of West Bengal, India and indicated that toxin producing can be used as a biological control of planktonic blooms. As is known, toxicity may reduce the amount of population while refuge often increases the survival rate of prey, many preys spend most of their lives nearby or in refuges, such as holes, crevices, thick vegetation or shells, to avoid predation. The concept of prey refuge has been concerned by scholars since Maynard-Smith [11] and Gause et al. [13] introduced a quantity x_R on prey involving refuges into the models. Extensive literature shows that prey refuges have vital impact on population dynamics, see González-Olivares and Ramos-Jiliberto [14], Kar [15], Li et al. [16], Han et al. [17], Qi and Meng [18], Lu and Xia [19] in details.

Note that the biological parameters in most literature are fixed constants. Nevertheless, any species will unavoidably be influenced by the complexity of ecosystem itself. As a matter of fact, in real ecosystem many biological parameters may fluctuate collaboratively with the periodically changing environment which plays an vital effect on population growth, maturity, predation, interspecific competition etc. These uncertainties mainly come from both natural factor and human factor, such as forest fire, earthquake, changing climate, measuring error, limitations of tools, and missing experimental data. The dynamic behaviors caused by these phenomenon can be investigated through the model incorporating with imprecise biological parameters. To deal with the issue, several approaches such as interval approach, fuzzy approach and stochastic approach have been adopted by researchers to depict the imprecise parameters. To the best of our knowledge, stochastic approach is widely used by Liu and Bai [20, 21], Liu et al [22], Qi et al. [23], Xie et al. [24], Zhang [25, 26]. The imprecise parameters in stochastic approach are taken the place of random variables possessing known probability distributions, while that in fuzzy approach are substituted by fuzzy sets or fuzzy numbers with known membership functions. Bassanezi et al. [27] laid the foundation by employing fuzzy differential equations to investigate the stability of dynamical system. Mizukoshi et al. [28], on account of initial conditions under fuzzy conditions, discussed the stability of fuzzy dynamical systems. Bede and Gal [29], based on generalized differentiability, considered the solutions of fuzzy differential equations. Guo et al. [30] applied fuzzy impulsive functional differential equations in Gompertz model and logistic model. Due to the difficulties for constructing a suitable probability distribution or membership function, Pal et al. [31] first introduce interval approach into an imprecise prey-predator harvesting model. Later, Sharma and Samanta [32] came up with a two species competition harvesting model with interval parameters. Pal et al. [33] considered parameter uncertainty in biomathematical model described by two-prey one-predator system with mutualism. In this paper, both interval approach and fuzzy approach are considered to characterize the parameters. We assume biological parameters involved in our model are imprecise in nature and depicted by interval number. Since the instantaneous annual rate of discount is the difference of the inflation and discount rates which are fuzzy in economic perspective, we consider it as fuzzy and expressed by trapezoidal fuzzy number due to intuitive, use friendly, and computationally simple in promoting representation.

Motivated by the model construction in [33], we analogously care to establish a two competing and continuously harvested preys and one predator depending on two preys. Different from model (1) in [33], we further consider the following three aspects: the refuges on preys are introduced to protect the preys from predation; the effects of toxic substances from environmental point of view on prey and predator as the hot topic of the moment are considered in our model; not only predator mortality but also intraspecific competition are taken into account which makes dynamic behaviors become more complicated. Based on the above, our model is established as follows

$$\begin{cases} \dfrac{dx_1}{dt} = r_1 x_1\Big(1 - \dfrac{x_1}{K_1}\Big) - \alpha_1 x_1 x_2 - c_1(1-m_1)x_1 y - \gamma_1 x_1^3 - q_1 E_1 x_1, \\ \dfrac{dx_2}{dt} = r_2 x_2\Big(1 - \dfrac{x_2}{K_2}\Big) - \alpha_2 x_1 x_2 - c_2(1-m_2)x_2 y - q_2 E_2 x_2, \\ \dfrac{dy}{dt} = -dy - sy^2 + e_1(1-m_1)x_1 y + e_2(1-m_2)x_2 y - \gamma_2 y^2, \end{cases} \tag{1.1}$$

with initial value $x_1(0) > 0$, $x_2(0) > 0$, $y(0) > 0$. Here, $x_1(t)$, $x_2(t)$ and $y(t)$ denote the biomass densities of two competing preys and one predator at time t. r_1 and r_2 represent the intrinsic growth rates of two preys, and K_1, K_2 are the carrying capacity of two preys, respectively. α_1 and α_2 stand for the interspecific competition between x_1 and x_2. c_1, c_2 are the coefficients of predation and e_1, e_1 are the coefficients of conversion. m_1 and m_2 severally denote the refuge of two preys. d acts as the mortality rate of predator and s shows the intra-specific competition rate of predator. γ_1 and γ_2 were regarded as the coefficients of toxicity to the prey and predator, respectively. Since prey is directly infected by some external toxic substances, while predator that feed on these infected preys is indirectly affected by the toxic substances, we call $0 < \gamma_2 < \gamma_1 < 1$. These terms for toxicity $\gamma_1 x_1^3$ and $\gamma_2 y^2$ are first proposed by Das et al. [34]. q_1, q_2 and E_1, E_2 severally denote the catchability coefficients and harvesting efforts of two preys, and also the catch rate functions $q_1 E_1 x_1$ and $q_2 E_2 x_2$ satisfy the catch-per-unit-effort hypothesis [3]. We consider, in this paper, parameters involved are replaced by interval numbers because of the imprecision of the parameters. Again when studying the optimal harvesting strategy of the model, the instantaneous annual discount rate is considered as fuzziness.

The rest of this paper is emerged as follows. Section 2 presents the formulation of model with interval-valued parameters. The positivity and boundedness of model together with the existence of biological equilibria are discussed in Sections 3 and 4, respectively. In Section 5, we analyze the stability of all biological equilibria. Also the existence conditions of bionomic equilibria in four cases are obtained in Section 6. In Section 7, considering the fuzzy inflation net discount rate as a trapezoidal fuzzy number, we investigate the optimal harvesting in fuzzy environment. Three numerical examples and a brief summary are displayed in Sections 8 and 9, respectively. In the end, Appendices A, B and C show some definitions and a method which will be used in previous sections.

2. Formulation of model

In classical deterministic differential model, the parameters, for instance the rates of species growth and death, are identified as fixed constants. Nevertheless, the values of the parameters do not always remain fixed on account of the lack of sufficient information or inaccurate understanding of ecological phenomenon. Here we consider model (1.1) has imprecise parameters, and replace the fixed positive constant r_i, α_i, c_i, e_i, γ_i, d, s $(i = 1, 2)$ by interval-valued parameters $\bar{r}_i$, $\bar{\alpha}_i$, $\bar{c}_i$, $\bar{d}$, $\bar{s}$, $\bar{e}_i$, $\bar{\gamma}_i$, respectively (see Appendix A). Thus model (1.1) can be represented as:

$$\begin{cases} \frac{dx_1}{dt} = \bar{r}_1 x_1 - \frac{\bar{r}_1}{K_1} x_1^2 - \bar{\alpha}_1 x_1 x_2 - \bar{c}_1 (1-m_1) x_1 y - \bar{\gamma}_1 x_1^3 - q_1 E_1 x_1, \\ \frac{dx_2}{dt} = \bar{r}_2 x_2 - \frac{\bar{r}_2}{K_2} x_2^2 - \bar{\alpha}_2 x_1 x_2 - \bar{c}_2 (1-m_2) x_2 y - q_2 E_2 x_2, \\ \frac{dy}{dt} = -\bar{d} y - \bar{s} y^2 + \bar{e}_1 (1-m_1) x_1 y + \bar{e}_2 (1-m_2) x_2 y - \bar{\gamma}_2 y^2, \end{cases} \tag{2.1}$$

where $\bar{r}_i \in [r_{il}, r_{iu}]$, $\bar{\alpha}_i \in [\alpha_{il}, \alpha_{iu}]$, $\bar{c}_i \in [c_{il}, c_{iu}]$, $\bar{e}_i \in [e_{il}, e_{iu}]$, $\bar{\gamma}_i \in [\gamma_{il}, \gamma_{iu}]$, $\bar{d} \in [d_l, d_u]$, and $\bar{s} \in [s_l, s_u]$, and all above interval-valued parameters are covered in the first quadrant.

Similar to the conversion method in [31], we can write model (2.1) as the following form for $p \in [0, 1]$:

$$\begin{cases} \frac{dx_1(t;p)}{dt} = r_{1l}^{1-p} r_{1u}^{p} x_1 - \frac{r_{1l}^{p} r_{1u}^{1-p}}{K_1} x_1^2 - \alpha_{1l}^{p} \alpha_{1u}^{1-p} x_1 x_2 - c_{1l}^{p} c_{1u}^{1-p} (1-m_1) x_1 y - \gamma_{1l}^{p} \gamma_{1u}^{1-p} x_1^3 - q_1 E_1 x_1, \\ \frac{dx_2(t;p)}{dt} = r_{2l}^{1-p} r_{2u}^{p} x_2 - \frac{r_{2l}^{p} r_{2u}^{1-p}}{K_2} x_2^2 - \alpha_{2l}^{p} \alpha_{2u}^{1-p} x_1 x_2 - c_{2l}^{p} c_{2u}^{1-p} (1-m_2) x_2 y - q_2 E_2 x_2, \\ \frac{dy(t;p)}{dt} = -d_l^{p} d_u^{1-p} y - s_l^{p} s_u^{1-p} y^2 + e_{1l}^{1-p} e_{1u}^{p} (1-m_1) x_1 y + e_{2l}^{1-p} e_{2u}^{p} (1-m_2) x_2 y - \gamma_{2l}^{p} \gamma_{2u}^{1-p} y^2. \end{cases} \tag{2.2}$$

Clearly, if we neglect the prey refuges and toxicity effect, and the mortality rate of the predator, model (2.2) can be simplified as model (3) in [33]. Similarly, we do not consider the toxicity effect and the prey species x_2, while consider the harvesting of predator, then model (2.2) turns into system (5) in [35].

3. Positivity and boundedness

This section guarantees the positive properties and boundedness of model (2.2) which are necessary preparation for the subsequent results. Therefore, the theorem is proposed as follows.

Theorem 3.1. *Any solution $(x_1(t), x_2(t), y(t))$ of model (2.2) is positive and bounded for all $t > 0$ if initial conditions $x_1(0) > 0$, $x_2(0) > 0$ and $y(0) > 0$ exist.*

Proof. The right side of model (2.2) satisfies continuity and Local Lipschitz condition on C, the unique solution $(x_1(t), x_2(t), y(t))$ of model (2.2) meeting initial conditions $x_1(0) > 0$, $x_2(0) > 0$, $y(0) > 0$ exist on $[0, \xi]$, where $0 < \xi < +\infty$. From model (2.2), we gain the following equations

$$\begin{aligned} x_1(t) &= x_1(0)\left[\exp \int_0^t \left\{ r_{1l}^{1-p} r_{1u}^{p} - \frac{r_{1l}^{p} r_{1u}^{1-p}}{K_1} x_1 - \alpha_{1l}^{p} \alpha_{1u}^{1-p} x_2 - c_{1l}^{p} c_{1u}^{1-p} (1-m_1) y - \gamma_{1l}^{p} \gamma_{1u}^{1-p} x_1^2 - q_1 E_1 \right\} ds \right] > 0, \\ x_2(t) &= x_2(0)\left[\exp \int_0^t \left\{ r_{2l}^{1-p} r_{2u}^{p} - \frac{r_{2l}^{p} r_{2u}^{1-p}}{K_2} x_2 - \alpha_{2l}^{p} \alpha_{2u}^{1-p} x_1 - c_{2l}^{p} c_{2u}^{1-p} (1-m_2) y - q_2 E_2 \right\} ds \right] > 0, \\ y(t) &= y(0)\left[\exp \int_0^t \left\{ -d_l^{p} d_u^{1-p} - (s_l^{p} s_u^{1-p} + \gamma_{2l}^{p} \gamma_{2u}^{1-p}) y + e_{1l}^{1-p} e_{1u}^{p} (1-m_1) x_1 + e_{2l}^{1-p} e_{2u}^{p} (1-m_2) x_2 \right\} ds \right] > 0, \end{aligned} \tag{3.1}$$

which together with initial conditions imply the positivity of model (2.2) for all $t > 0$. Under the positivity of x_1, x_2 and y, we have

$$\frac{dx_i}{dt} \le x_i\Big(r_{il}^{1-p} r_{iu}^{p} - \frac{r_{il}^{p} r_{iu}^{1-p}}{K_i} x_i\Big), \quad i = 1, 2. \tag{3.2}$$

It is easy to see that

$$\limsup_{t\to\infty} x_i(t) \le \frac{r_{il}^{1-p} r_{iu}^{p}}{r_{il}^{p} r_{iu}^{1-p}} K_i \equiv k_i.$$

Then we get the following inequation from the third equation of (2.2)

$$\begin{aligned} \frac{dy}{dt} &\le y[e_{1l}^{1-p} e_{1u}^{p}(1-m_1)x_1 + e_{2l}^{1-p} e_{2u}^{p}(1-m_2)x_2 - (s_l^p s_u^{1-p} + \gamma_{2l}^{p}\gamma_{2u}^{1-p})y] \\ &\le y[e_{1l}^{1-p} e_{1u}^{p}(1-m_1)k_1 + e_{2l}^{1-p} e_{2u}^{p}(1-m_2)k_2 - (s_l^p s_u^{1-p} + \gamma_{2l}^{p}\gamma_{2u}^{1-p})y]. \end{aligned} \tag{3.3}$$

We obtain that

$$\limsup_{t\to\infty} y(t) \le \frac{e_{1l}^{1-p} e_{1u}^{p}(1-m_1)k_1 + e_{2l}^{1-p} e_{2u}^{p}(1-m_2)k_2}{s_l^p s_u^{1-p} + \gamma_{2l}^{p}\gamma_{2u}^{1-p}}.$$

The solution of model (2.2) is bounded. Therefore, the theorem is proved. □

4. Existence of biological equilibria

In this part, the existence of all possible biological equilibria of model (2.2) are discussed in detail. For convenience, some notations are introduced in the following

$$\begin{aligned} &\Psi = \alpha_{1l}^{p}\alpha_{1u}^{1-p}\alpha_{2l}^{p}\alpha_{2u}^{1-p}, \ \ \Omega = s_l^p s_u^{1-p} + \gamma_{2l}^{p}\gamma_{2u}^{1-p}, \\ &\Xi_1 = d_l^p d_u^{1-p}/(e_{1l}^{1-p} e_{1u}^{p}(1-m_1)), \ \ \Phi_1 = c_{1l}^{p} c_{1u}^{1-p} e_{2l}^{1-p} e_{2u}^{p}(1-m_1)(1-m_2), \\ &\Xi_2 = d_l^p d_u^{1-p}/(e_{2l}^{1-p} e_{2u}^{p}(1-m_2)), \ \ \Phi_2 = c_{2l}^{p} c_{2u}^{1-p} e_{1l}^{1-p} e_{1u}^{p}(1-m_1)(1-m_2), \\ &\Gamma_1 = r_{1l}^{p} r_{1u}^{1-p}/K_1, \ \ \Upsilon_1 = r_{1l}^{1-p} r_{1u}^{p} - q_1E_1, \ \ \Lambda_1 = c_{1l}^{p} c_{1u}^{1-p} e_{1l}^{1-p} e_{1u}^{p}(1-m_1)^2/\Omega, \\ &\Gamma_2 = r_{2l}^{p} r_{2u}^{1-p}/K_2, \ \ \Upsilon_2 = r_{2l}^{1-p} r_{2u}^{p} - q_2E_2, \ \ \Lambda_2 = c_{2l}^{p} c_{2u}^{1-p} e_{2l}^{1-p} e_{2u}^{p}(1-m_2)^2/\Omega. \end{aligned} \tag{4.1}$$

Similar to the concept of BTP (biotechnical productivity) in [31], we also define BTP representing the ratio of biotic potential to catchability coefficient, i.e., $BTP = r_l^{1-p} r_u^{p}/q$.

By a tedious calculation model (2.2) exists the following biological equilibria:

(1) Trivial equilibrium $P_1(0, 0, 0)$ is obviously existing.

(2) Axial equilibrium $P_2(x_1^{\theta}, 0, 0)$, where $x_1^{\theta} = \Big(-\Gamma_1 + \sqrt{{\Gamma_1}^2 + 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Upsilon_1}\Big)\Big/\Big(2\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Big)$ exists if $\Upsilon_1 > 0$ (i.e., $E_1 < (BTP)_{x_1}$) holds.

(3) Axial equilibrium $P_3(0, x_2^{\psi}, 0)$, where $x_2^{\psi} = \Upsilon_2/\Gamma_2$ exists if $\Upsilon_2 > 0$ (i.e., $E_2 < (BTP)_{x_2}$) holds.

(4) Axial equilibrium $P_4(x_1^{\xi}, 0, y^{\xi})$, where

$$\begin{aligned} x_1^{\xi} &= \Big[-(\Gamma_1 + \Lambda_1) + \sqrt{(\Gamma_1+\Lambda_1)^2 + 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}(\Xi_1\Lambda_1 + \Upsilon_1)}\Big]\Big/\Big(2\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Big), \\ y^{\xi} &= e_{1l}^{1-p} e_{1u}^{p}(1-m_1)(x_1^{\xi} - \Xi_1)/\Omega, \end{aligned} \tag{4.2}$$

exists if $\Upsilon_1 > \gamma_{1l}^{p}\gamma_{1u}^{1-p}\Xi_1^2 + \Gamma_1\Xi_1$ is satisfied.

(5) Axial equilibrium $P_5(0, x_2^{\eta}, y^{\eta})$, where

$$x_2^{\eta} = (\Xi_2\Lambda_2 + \Upsilon_2)/(\Gamma_2 + \Lambda_2) \text{ and } y^{\eta} = e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)(x_2^{\eta} - \Xi_2)/\Omega, \tag{4.3}$$

exists if $\Upsilon_2 > \Gamma_2\Xi_2$ is satisfied.

(6) Axial equilibrium $P_6(x_1^{\nu}, x_2^{\nu}, 0)$, where

$$\begin{aligned} x_1^{\nu} &= \Big[(\Psi - \Gamma_1\Gamma_2) + \sqrt{(\Psi - \Gamma_1\Gamma_2)^2 + 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Gamma_2(\Upsilon_1\Gamma_2 - \alpha_{1l}^{p}\alpha_{1u}^{1-p}\Upsilon_2)}\Big]\Big/\Big(2\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Gamma_2\Big), \\ x_2^{\nu} &= (\Upsilon_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1^{\nu})/\Gamma_2, \end{aligned} \tag{4.4}$$

exists if either

$$\begin{aligned} &\Psi > \Gamma_1\Gamma_2, \;\; (\alpha_{2l}^{p}\alpha_{2u}^{1-p})^2\Upsilon_1 < \Upsilon_2(\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Upsilon_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Gamma_1), \\ &(\Psi - \Gamma_1\Gamma_2)^2 \geq 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Gamma_2(\alpha_{1l}^{p}\alpha_{1u}^{1-p}\Upsilon_2 - \Upsilon_1\Gamma_2) \end{aligned}$$

or

$$\Psi \leq \Gamma_1\Gamma_2, \;\; \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Upsilon_2\Psi < (\alpha_{2l}^{p}\alpha_{2u}^{1-p})^2\Gamma_2\Upsilon_1 < \Gamma_2\Upsilon_2(\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Upsilon_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Gamma_1)$$

hold.

(7) Axial equilibrium $P_7(x_1^{\iota}, x_2^{\iota}, 0)$, where

$$\begin{aligned} x_1^{\iota} &= \Big[(\Psi - \Gamma_1\Gamma_2) - \sqrt{(\Psi - \Gamma_1\Gamma_2)^2 + 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Gamma_2(\Upsilon_1\Gamma_2 - \alpha_{1l}^{p}\alpha_{1u}^{1-p}\Upsilon_2)}\Big]\Big/\Big(2\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Gamma_2\Big), \\ x_2^{\iota} &= (\Upsilon_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1^{\iota})/\Gamma_2, \end{aligned} \tag{4.5}$$

exists if the following conditions are satisfied

$$\begin{aligned} &\Psi > \Gamma_1\Gamma_2, \;\; (\alpha_{2l}^{p}\alpha_{2u}^{1-p})^2\Upsilon_1 > \Upsilon_2(\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Upsilon_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Gamma_1), \\ &(\Psi - \Gamma_1\Gamma_2)^2 \geq 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Gamma_2(\alpha_{1l}^{p}\alpha_{1u}^{1-p}\Upsilon_2 - \Upsilon_1\Gamma_2) > 0. \end{aligned} \tag{4.6}$$

(8) Interior equilibrium $P_8(x_1^{\vartheta}, x_2^{\vartheta}, y^{\vartheta})$, where

$$\begin{aligned} x_1^{\vartheta} &= \frac{-b + \sqrt{b^2 - 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}c}}{2\gamma_{1l}^{p}\gamma_{1u}^{1-p}}, \;\; x_2^{\vartheta} = \frac{-(\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_1^{\vartheta} + (\Xi_2\Lambda_2 + \Upsilon_2)\Omega}{(\Lambda_2 + \Gamma_2)\Omega}, \\ y^{\vartheta} &= \frac{[e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)\Gamma_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)]x_1^{\vartheta} + e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)(\Upsilon_2 - \Xi_2\Gamma_2)}{(\Lambda_2 + \Gamma_2)\Omega}. \end{aligned} \tag{4.7}$$

In addition, b and c are expressed as

$$b = \frac{(\Gamma_1\Gamma_2 + \Gamma_1\Lambda_2 + \Gamma_2\Lambda_1 - \Psi)\Omega - (\alpha_{1l}^{p}\alpha_{1u}^{1-p}\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Phi_1)}{(\Lambda_2 + \Gamma_2)\Omega} \tag{4.8}$$

and

$$c = \frac{\Phi_1\Upsilon_2 - \Phi_1\Xi_2\Gamma_2 + (\alpha_{1l}^{p}\alpha_{1u}^{1-p}\Upsilon_2 + \alpha_{1l}^{p}\alpha_{1u}^{1-p}\Xi_2\Lambda_2 - \Gamma_2\Upsilon_1 - \Lambda_2\Upsilon_1)\Omega}{(\Lambda_2 + \Gamma_2)\Omega}. \tag{4.9}$$

The equilibrium P_8 exists if (III) and (IV) hold, and simultaneously one of the conditions (I) and (II) is satisfied

$$\begin{aligned} &\text{(I) } b \geq 0, \;\; c < 0, \;\; \text{(II) } b < 0, \;\; b^2 \geq 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}c, \;\; \text{(III) } (\Xi_2\Lambda_2 + \Upsilon_2)\Omega > (\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_1^{\vartheta}, \\ &\text{(IV) } [e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)\Gamma_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)]x_1^{\vartheta} + e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)(\Upsilon_2 - \Xi_2\Gamma_2) > 0. \end{aligned} \tag{4.10}$$

(9) Interior equilibrium $P_9(x_1^{\varsigma}, x_2^{\varsigma}, y^{\varsigma})$,where

$$
\begin{aligned}
x_1^{\varsigma} &= \frac{-b - \sqrt{b^2 - 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}c}}{2\gamma_{1l}^{p}\gamma_{1u}^{1-p}}, \quad x_2^{\varsigma} = \frac{-(\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_1^{\varsigma} + (\Xi_2\Lambda_2 + \Upsilon_2)\Omega}{(\Lambda_2 + \Gamma_2)\Omega}, \\
y^{\varsigma} &= \frac{[e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)\Gamma_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)]x_1^{\varsigma} + e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)(\Upsilon_2 - \Xi_2\Gamma_2)}{(\Lambda_2 + \Gamma_2)\Omega},
\end{aligned} \tag{4.11}
$$

and b, c are defined in (4.8) and (4.9). The existence of equilibrium P_9 is obvious if the conditions (I′) – (III′) are satisfied

$$
\begin{aligned}
&(\mathrm{I}')\ b < 0,\ \ c > 0,\ \ b^2 \geq 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}c,\ \ (\mathrm{II}')\ (\Xi_2\Lambda_2 + \Upsilon_2)\Omega > (\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_1^{\varsigma}, \\
&(\mathrm{III}')\ [e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)\Gamma_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)]x_1^{\varsigma} + e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)(\Upsilon_2 - \Xi_2\Gamma_2) > 0.
\end{aligned} \tag{4.12}
$$

Remark 4.1. *Model (2.2) has only a unique interior equilibrium $P^*(x_1^*, x_2^*, y^*)$ if the following conditions* (I″), (III″), (IV″) *or* (II″), (III″), (IV″) *are satisfied*

$$
\begin{aligned}
&(\mathrm{I}'')\ b \geq 0,\ \ c < 0,\ \ (\mathrm{II}'')\ b = -2\sqrt{\gamma_{1l}^{p}\gamma_{1u}^{1-p}c},\ \ c > 0,\ \ (\mathrm{III}'')\ (\Xi_2\Lambda_2 + \Upsilon_2)\Omega > (\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_1^*, \\
&(\mathrm{IV}'')\ [e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)\Gamma_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)]x_1^* + e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)(\Upsilon_2 - \Xi_2\Gamma_2) > 0,
\end{aligned} \tag{4.13}
$$

where

$$
\begin{aligned}
x_1^* &= \frac{-b}{2\gamma_{1l}^{p}\gamma_{1u}^{1-p}}, \quad x_2^* = \frac{-(\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_1^* + (\Xi_2\Lambda_2 + \Upsilon_2)\Omega}{(\Lambda_2 + \Gamma_2)\Omega}, \\
y^* &= \frac{[e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)\Gamma_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)]x_1^* + e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)(\Upsilon_2 - \Xi_2\Gamma_2)}{(\Lambda_2 + \Gamma_2)\Omega}.
\end{aligned} \tag{4.14}
$$

5. Stability analysis

In this section, by applying Jacobian matrix we analyze the local stability of all biological equilibria, and then investigate interior equilibrium P_8 is globally stable through constructing Lyapunov functions. Here, the notations in (4.1) are also used in this part.

5.1. Local stability

The Jacobian matrix of model (2.2) is given by

$$
M = \begin{pmatrix} M_{11} & -\alpha_{1l}^{p}\alpha_{1u}^{1-p}x_1 & -c_{1l}^{p}c_{1u}^{1-p}(1 - m_1)x_1 \\ -\alpha_{2l}^{p}\alpha_{2u}^{1-p}x_2 & M_{22} & -c_{2l}^{p}c_{2u}^{1-p}(1 - m_2)x_2 \\ e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)y & e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)y & M_{33} \end{pmatrix}, \tag{5.1}
$$

where

$$
\begin{aligned}
M_{11} &= r_{1l}^{1-p}r_{1u}^{p} - 2r_{1l}^{p}r_{1u}^{1-p}\frac{x_1}{K_1} - \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_2 - c_{1l}^{p}c_{1u}^{1-p}(1 - m_1)y - 3\gamma_{1l}^{p}\gamma_{1u}^{1-p}x_1^2 - q_1E_1, \\
M_{22} &= r_{2l}^{1-p}r_{2u}^{p} - 2r_{2l}^{p}r_{2u}^{1-p}\frac{x_2}{K_2} - \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1 - c_{2l}^{p}c_{2u}^{1-p}(1 - m_2)y - q_2E_2, \\
M_{33} &= -d_l^{p}d_u^{1-p} - 2s_l^{p}s_u^{1-p}y + e_{1l}^{1-p}e_{1u}^{p}(1 - m_1)x_1 + e_{2l}^{1-p}e_{2u}^{p}(1 - m_2)x_2 - 2\gamma_{2l}^{p}\gamma_{2u}^{1-p}y.
\end{aligned} \tag{5.2}
$$

We analyze the local stability conditions of all biological equilibria displayed in Section 4.

Theorem 5.1. *Assume that all equilibria exist, the following conclusions are true:*
(1) *Trivial equilibrium* $P_1(0,0,0)$ *is locally asymptotically stable under the following circumstances*

$$(BTP)_{x_1} - E_1 < 0 \text{ and } (BTP)_{x_2} - E_2 < 0. \tag{5.3}$$

(2) *Axial equilibrium* $P_2(x_1^\theta, 0, 0)$ *is locally asymptotically stable when*

$$(BTP)_{x_2} - E_2 < \frac{\alpha_{2l}^p \alpha_{2u}^{1-p}}{q_2} x_1^\theta < \frac{\alpha_{2l}^p \alpha_{2u}^{1-p} \Xi_1}{q_2}. \tag{5.4}$$

(3) *Axial equilibrium* $P_3(0, x_2^\psi, 0)$ *is locally asymptotically stable in case of*

$$(BTP)_{x_1} - E_1 < \frac{\alpha_{1l}^p \alpha_{1u}^{1-p}}{q_1} x_2^\psi < \frac{\alpha_{1l}^p \alpha_{1u}^{1-p} \Xi_2}{q_1}. \tag{5.5}$$

(4) *Axial equilibrium* $P_4(x_1^\xi, 0, y^\xi)$ *is locally asymptotically stable in the situation that*

$$(BTP)_{x_2} - E_2 < \frac{\alpha_{2l}^p \alpha_{2u}^{1-p} x_1^\xi + c_{2l}^p c_{2u}^{1-p}(1-m_2) y^\xi}{q_2}. \tag{5.6}$$

(5) *Axial equilibrium* $P_5(0, x_2^\eta, y^\eta)$ *is locally asymptotically stable provided that*

$$(BTP)_{x_1} - E_1 < \frac{\alpha_{1l}^p \alpha_{1u}^{1-p} x_2^\eta + c_{1l}^p c_{1u}^{1-p}(1-m_1) y^\eta}{q_1}. \tag{5.7}$$

(6) *Axial equilibrium* $P_6(x_1^\nu, x_2^\nu, 0)$ *is locally asymptotically stable on condition that*

$$e_{1l}^{1-p} e_{1u}^p (1-m_1) x_1^\nu + e_{2l}^{1-p} e_{2u}^p (1-m_2) x_2^\nu < d_l^p d_u^{1-p} \tag{5.8}$$

and $P_7(x_1^\iota, x_2^\iota, 0)$ *is unstable.*
(7) *Interior equilibrium* $P_8(x_1^\vartheta, x_2^\vartheta, y^\vartheta)$ *is locally asymptotically stable supposing that*

$$\vartheta_1 \vartheta_2 > \vartheta_3, \tag{5.9}$$

where

$$\begin{aligned}
\vartheta_1 =& (\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) x_1^\vartheta + \Gamma_2 x_2^\vartheta + \Omega y^\vartheta, \\
\vartheta_2 =& [\Gamma_2(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) - \Psi] x_1^\vartheta x_2^\vartheta + [(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) + \Lambda_1] \Omega x_1^\vartheta y^\vartheta + (\Gamma_2 + \Lambda_2) \Omega x_2^\vartheta y^\vartheta, \\
\vartheta_3 =& [(\Gamma_2 + \Lambda_2)(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) \Omega + \Gamma_2 \Lambda_1 \Omega - (\alpha_{2l}^p \alpha_{2u}^{1-p} \Phi_1 + \alpha_{1l}^p \alpha_{1u}^{1-p} \Phi_2) - \Psi \Omega] x_1^\vartheta x_2^\vartheta y^\vartheta,
\end{aligned} \tag{5.10}$$

and $P_9(x_1^\varsigma, x_2^\varsigma, y^\varsigma)$ *is unstable.*

Proof. (1) Three eigenvalues of the variational matrix $M(0,0,0)$ represent as follows

$$\lambda_1^1 = -d_l^p d_u^{1-p} < 0, \quad \lambda_1^2 = r_{1l}^{1-p} r_{1u}^p - q_1 E_1 \text{ and } \lambda_1^3 = r_{2l}^{1-p} r_{2u}^p - q_2 E_2,$$

then P_1 is locally asymptotically stable under the circumstances

$$r_{1l}^{1-p} r_{1u}^p - q_1 E_1 < 0 \text{ and } r_{2l}^{1-p} r_{2u}^p - q_2 E_2 < 0,$$

i.e.,

$$(BTP)_{x_1} - E_1 < 0 \text{ and } (BTP)_{x_2} - E_2 < 0.$$

We omit the proofs of (2), (3), (4) and (5) which are similar to that of (1).

(6) One eigenvalue of variational matrix $M(x_1^\nu, x_2^\nu, 0)$ is expressed as

$$\lambda_6^1 = -d_l^p d_u^{1-p} + e_{1l}^{1-p} e_{1u}^p (1 - m_1) x_1^\nu + e_{2l}^{1-p} e_{2u}^p (1 - m_2) x_2^\nu.$$

Other two eigenvalues of variational matrix $M(x_1^\nu, x_2^\nu, 0)$ are two roots of quadratic equation

$$\lambda^2 + [(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\nu) x_1^\nu + \Gamma_2 x_2^\nu]\lambda + [\Gamma_2(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\nu) - \Psi] x_1^\nu x_2^\nu = 0, \tag{5.11}$$

where

$$\begin{aligned}
&(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\nu) x_1^\nu + \Gamma_2 x_2^\nu > 0, \\
&\Gamma_2(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\nu) - \Psi = \sqrt{(\Psi - \Gamma_1\Gamma_2)^2 + 4\gamma_{1l}^p \gamma_{1u}^{1-p} \Gamma_2 (\Upsilon_1 \Gamma_2 - \alpha_{1l}^p \alpha_{1u}^{1-p} \Upsilon_2)} > 0.
\end{aligned}$$

By Routh-Hurwitz condition P_6 is locally asymptotically stable provided that

$$e_{1l}^{1-p} e_{1u}^p (1 - m_1) x_1^\nu + e_{2l}^{1-p} e_{2u}^p (1 - m_2) x_2^\nu < d_l^p d_u^{1-p}.$$

Analogously, we proof $P_7(x_1^\iota, x_2^\iota, 0)$ is unstable.

(7) By simple calculation, we get the following determinant

$$|\lambda E - M| = \begin{vmatrix} \lambda - M_{11} & \alpha_{1l}^p \alpha_{1u}^{1-p} x_1 & c_{1l}^p c_{1u}^{1-p} (1 - m_1) x_1 \\ \alpha_{2l}^p \alpha_{2u}^{1-p} x_2 & \lambda - M_{22} & c_{2l}^p c_{2u}^{1-p} (1 - m_2) x_2 \\ -e_{1l}^{1-p} e_{1u}^p (1 - m_1) y & -e_{2l}^{1-p} e_{2u}^p (1 - m_2) y & \lambda - M_{33} \end{vmatrix}. \tag{5.12}$$

Then substituting interior equilibrium $P_8(x_1^\vartheta, x_2^\vartheta, y^\vartheta)$ of model (2.2) and simplifying M_{11}, M_{22} and M_{33}, it yields that

$$|\lambda E - M(x_1^\vartheta, x_2^\vartheta, y^\vartheta)| = \begin{vmatrix} \lambda + (\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) x_1^\vartheta & \alpha_{1l}^p \alpha_{1u}^{1-p} x_1^\vartheta & c_{1l}^p c_{1u}^{1-p} (1 - m_1) x_1^\vartheta \\ \alpha_{2l}^p \alpha_{2u}^{1-p} x_2^\vartheta & \lambda + \Gamma_2 x_2^\vartheta & c_{2l}^p c_{2u}^{1-p} (1 - m_2) x_2^\vartheta \\ -e_{1l}^{1-p} e_{1u}^p (1 - m_1) y^\vartheta & -e_{2l}^{1-p} e_{2u}^p (1 - m_2) y^\vartheta & \lambda + \Omega y^\vartheta \end{vmatrix}.$$

The form of characteristic equation is written as

$$\lambda^3 + \vartheta_1 \lambda^2 + \vartheta_2 \lambda + \vartheta_3 = 0,$$

where

$$\begin{aligned}
\vartheta_1 &= (\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) x_1^\vartheta + \Gamma_2 x_2^\vartheta + \Omega y^\vartheta > 0, \\
\vartheta_2 &= [\Gamma_2(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) - \Psi] x_1^\vartheta x_2^\vartheta + [(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta) + \Lambda_1] \Omega x_1^\vartheta y^\vartheta + (\Gamma_2 + \Lambda_2) \Omega x_2^\vartheta y^\vartheta, \\
\vartheta_3 &= [(\Gamma_2 + \Lambda_2)(\Gamma_1 + 2\gamma_{1l}^p \gamma_{1u}^{1-p} x_1^\vartheta)\Omega + \Gamma_2 \Lambda_1 \Omega - (\alpha_{2l}^p \alpha_{2u}^{1-p} \Phi_1 + \alpha_{1l}^p \alpha_{1u}^{1-p} \Phi_2) - \Psi\Omega] x_1^\vartheta x_2^\vartheta y^\vartheta \\
&= \sqrt{b^2 - 4\gamma_{1l}^p \gamma_{1u}^{1-p} c} (\Gamma_2 + \Lambda_2) \Omega x_1^\vartheta x_2^\vartheta y^\vartheta > 0.
\end{aligned}$$

Applying Routh-Hurwitz condition $P_8(x_1^\vartheta, x_2^\vartheta, y^\vartheta)$ is locally asymptotically stable if $\vartheta_1\vartheta_2 > \vartheta_3$. Also, we demonstrate the unstability of $P_9(x_1^\varsigma, x_2^\varsigma, y^\varsigma)$ with the same method. □

5.2. Global stability

Next, let us consider the global stability of nontrivial equilibrium P_8. The appropriate Lyapunov function is constructed as follows

$$V(x_1, x_2, y) = x_1 - x_1^{\vartheta} - x_1^{\vartheta} \ln\Big(\frac{x_1}{x_1^{\vartheta}}\Big) + l_1\Big[x_2 - x_2^{\vartheta} - x_2^{\vartheta} \ln\Big(\frac{x_2}{x_2^{\vartheta}}\Big)\Big] + l_2\Big[y - y^{\vartheta} - y^{\vartheta} \ln\Big(\frac{y}{y^{\vartheta}}\Big)\Big]. \tag{5.13}$$

Obviously $x_i - x_i^{\vartheta} - x_i^{\vartheta} \ln\Big(\frac{x_i}{x_i^{\vartheta}}\Big) \geq 0$ $(i = 1, 2)$ and $y - y^{\vartheta} - y^{\vartheta} \ln\Big(\frac{y}{y^{\vartheta}}\Big) \geq 0$, therefore we have $V \geq 0$. Differentiating two sides of (5.13) with respect to t, we derive that

$$\begin{aligned}\frac{dV}{dt} =& \frac{x_1 - x_1^{\vartheta}}{x_1}\frac{dx_1}{dt} + \frac{l_1(x_2 - x_2^{\vartheta})}{x_2}\frac{dx_2}{dt} + \frac{l_2(y - y^{\vartheta})}{y}\frac{dy}{dt}\\ =& -\{[\Gamma_1 + \gamma_{1l}^{p}\gamma_{1u}^{1-p}(x_1 + x_1^{\vartheta})](x_1 - x_1^{\vartheta})^2 + (\alpha_{1l}^{p}\alpha_{1u}^{1-p} + l_1\alpha_{2l}^{p}\alpha_{2u}^{1-p})(x_1 - x_1^{\vartheta})(x_2 - x_2^{\vartheta})\\ &+ (c_{1l}^{p}c_{1u}^{1-p} - l_2 e_{1l}^{1-p}e_{1u}^{p})(1 - m_1)(x_1 - x_1^{\vartheta})(y - y^{\vartheta}) + l_1\Gamma_2(x_2 - x_2^{\vartheta})^2\\ &+ (l_1 c_{2l}^{p}c_{2u}^{1-p} - l_2 e_{2l}^{1-p}e_{2u}^{p})(1 - m_2)(x_2 - x_2^{\vartheta})(y - y^{\vartheta}) + l_2\Omega(y - y^{\vartheta})^2\}.\end{aligned} \tag{5.14}$$

We choose $l_1 = c_{1l}^{p}c_{1u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}/(c_{2l}^{p}c_{2u}^{1-p}e_{1l}^{1-p}e_{1u}^{p})$ and $l_2 = c_{1l}^{p}c_{1u}^{1-p}/(e_{1l}^{1-p}e_{1u}^{p})$, then (5.14) becomes

$$\begin{aligned}\frac{dV}{dt} =& -\{[\Gamma_1 + \gamma_{1l}^{p}\gamma_{1u}^{1-p}(x_1 + x_1^{\vartheta})](x_1 - x_1^{\vartheta})^2 + l_1\Gamma_2(x_2 - x_2^{\vartheta})^2 + l_2\Omega(y - y^{\vartheta})^2\\ &+ (\alpha_{1l}^{p}\alpha_{1u}^{1-p} + l_1\alpha_{2l}^{p}\alpha_{2u}^{1-p})(x_1 - x_1^{\vartheta})(x_2 - x_2^{\vartheta})\}\\ =& -Y^T BY,\end{aligned}$$

where

$$Y^T = [(x_1 - x_1^{\vartheta}), (x_2 - x_2^{\vartheta}), (y - y^{\vartheta})]$$

and

$$B = \begin{pmatrix} \Gamma_1 + \gamma_{1l}^{p}\gamma_{1u}^{1-p}(x_1 + x_1^{\vartheta}) & (\alpha_{1l}^{p}\alpha_{1u}^{1-p} + l_1\alpha_{2l}^{p}\alpha_{2u}^{1-p})/2 & 0\\ (\alpha_{1l}^{p}\alpha_{1u}^{1-p} + l_1\alpha_{2l}^{p}\alpha_{2u}^{1-p})/2 & l_1\Gamma_2 & 0\\ 0 & 0 & l_2\Omega \end{pmatrix}.$$

Thus $\frac{dV}{dt} < 0$ if $4l_1\Gamma_2(\Gamma_1 + \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_1^{\vartheta}) > (\alpha_{1l}^{p}\alpha_{1u}^{1-p} + l_1\alpha_{2l}^{p}\alpha_{2u}^{1-p})^2$, then $P_8(x_1^{\vartheta}, x_2^{\vartheta}, y^{\vartheta})$ is globally asymptotically stable.

6. Bionomic equilibria

The biological equilibrium is provided by $\frac{dx_1}{dt} = 0, \frac{dx_2}{dt} = 0$ and $\frac{dy}{dt} = 0$ while the economic equilibrium means that the total revenue is equal to the total expenses. Combine biological equilibrium and economic equilibrium to form bionomic equilibrium, we will investigate all possible bionomic equilibria in different cases for model (2.2) in this section.

Suppose C_1 and C_2 are the unit fishing cost for preys x_1 and x_2, and p_1 and p_2 say the unit biomass price for preys x_1 and x_2, respectively. Therefore economic rent (π) is yield to $\pi(x_1, x_2, y, E_1, E_2) = \pi_1 + \pi_2$, where

$$\pi_1 = (p_1q_1x_1 - C_1)E_1, \quad \pi_2 = (p_2q_2x_2 - C_2)E_2. \tag{6.1}$$

The bionomic equilibria are constant solutions of the following equations

$$\begin{cases} r_{1l}^{1-p}r_{1u}^{p}x_1 - r_{1l}^{p}r_{1u}^{1-p}\dfrac{x_1^2}{K_1} - \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_1x_2 - c_{1l}^{p}c_{1u}^{1-p}(1-m_1)x_1y - \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_1^3 - q_1E_1x_1 = 0, \\ r_{2l}^{1-p}r_{2u}^{p}x_2 - r_{2l}^{p}r_{2u}^{1-p}\dfrac{x_2^2}{K_2} - \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1x_2 - c_{2l}^{p}c_{2u}^{1-p}(1-m_2)x_2y - q_2E_2x_2 = 0, \\ -d_l^{p}d_u^{1-p}y - s_l^{p}s_u^{1-p}y^2 + e_{1l}^{1-p}e_{1u}^{p}(1-m_1)x_1y + e_{2l}^{1-p}e_{2u}^{p}(1-m_2)x_2y - \gamma_{2l}^{p}\gamma_{2u}^{1-p}y^2 = 0, \\ \pi = (p_1q_1x_1 - C_1)E_1 + (p_2q_2x_2 - C_2)E_2 = 0. \end{cases} \tag{6.2}$$

The notations $\Psi, \Omega, \Xi_1, \Xi_2, \Phi_1, \Phi_2, \Gamma_1, \Gamma_2, \Lambda_1, \Lambda_2$ used here are given in (4.1). Similar to the methods in [33], we discuss possible bionomic equilibria in different cases.

Case 1. If $C_2 > p_2q_2x_2$, i.e., the fishing cost is higher than the earning for prey x_2. Thus we have to stop fishing prey x_2 ($E_{2\infty} = 0$) and maintain the fishing of prey x_1 ($C_1 < p_1q_1x_1$). Based on $x_{1\infty} = \frac{C_1}{p_1q_1}$ and $E_{2\infty} = 0$, let us divide into the following four situations to investigate the values of $x_{2\infty}, y_\infty$ and $E_{1\infty}$.

Situation 1. If $x_{2\infty} = 0$ and $y_\infty = 0$, solving the first equation of (6.2) yields

$$E_{1\infty} = (r_{1l}^{1-p}r_{1u}^{p} - \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^2 - \Gamma_1x_{1\infty})/q_1, \tag{6.3}$$

which is positive provided that $r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^2 + \Gamma_1x_{1\infty}$.

Situation 2. If $x_{2\infty} = 0$, we solve the first and third equation of (6.2) and obtain that

$$\begin{aligned} &y_\infty = e_{1l}^{1-p}e_{1u}^{p}(1-m_1)(x_{1\infty} - \Xi_1)/\Omega, \\ &E_{1\infty} = [r_{1l}^{1-p}r_{1u}^{p} - \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^2 - (\Gamma_1 + \Lambda_1)x_{1\infty} + \Lambda_1\Xi_1]/q_1, \end{aligned} \tag{6.4}$$

which are positive provided that $r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^2 + (\Gamma_1 + \Lambda_1)x_{1\infty} - \Lambda_1\Xi_1$ and $x_{1\infty} > \Xi_1$.

Situation 3. If $y_\infty = 0$, calculating the first and second equation gives

$$\begin{aligned} x_{2\infty} &= \frac{r_{2l}^{1-p}r_{2u}^{p} - \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_{1\infty}}{\Gamma_2}, \\ E_{1\infty} &= \frac{r_{1l}^{1-p}r_{1u}^{p}\Gamma_2 - \gamma_{1l}^{p}\gamma_{1u}^{1-p}\Gamma_2x_{1\infty}^2 + (\Psi - \Gamma_1\Gamma_2)x_{1\infty} - r_{2l}^{1-p}r_{2u}^{p}\alpha_{1l}^{p}\alpha_{1u}^{1-p}}{q_1\Gamma_2}, \end{aligned} \tag{6.5}$$

which are positive if $r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^2 - \left(\frac{\Psi}{\Gamma_2} - \Gamma_1\right)x_{1\infty} + \frac{r_{2l}^{1-p}r_{2u}^{p}\alpha_{1l}^{p}\alpha_{1u}^{1-p}}{\Gamma_2}$ and $r_{2l}^{1-p}r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_{1\infty}$.

Situation 4. By a tedious calculation, it follows from the first three equations that

$$\begin{aligned} x_{2\infty} &= \frac{r_{2l}^{1-p}r_{2u}^{p}\Omega + (\Phi_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_{1\infty} + c_{2l}^{p}c_{2u}^{1-p}d_l^{p}d_u^{1-p}(1-m_2)}{(\Gamma_2 + \Lambda_2)\Omega}, \\ y_\infty &= \frac{r_{2l}^{1-p}r_{2u}^{p}e_{2l}^{1-p}e_{2u}^{p}(1-m_2) + [\Gamma_2e_{1l}^{1-p}e_{1u}^{p}(1-m_1) - \alpha_{2l}^{p}\alpha_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)]x_{1\infty} - \Gamma_2d_l^{p}d_u^{1-p}}{(\Gamma_2 + \Lambda_2)\Omega}, \\ E_{1\infty} &= \frac{r_{1l}^{1-p}r_{1u}^{p} - \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^2 - \Gamma_1x_{1\infty} - \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_{2\infty} - c_{1l}^{p}c_{1u}^{1-p}(1-m_1)y_\infty}{q_1}, \end{aligned} \tag{6.6}$$

if $r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^2 + \Gamma_1x_{1\infty} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_{2\infty} + c_{1l}^{p}c_{1u}^{1-p}(1-m_1)y_\infty$ and $r_{2l}^{1-p}r_{2u}^{p} > \max\Big\{\Big(\alpha_{2l}^{p}\alpha_{2u}^{1-p} - \frac{\Phi_2}{\Omega}\Big)x_{1\infty} - \Lambda_2\Xi_2, \Big[\alpha_{2l}^{p}\alpha_{2u}^{1-p} - \frac{\Gamma_2e_{1l}^{1-p}e_{1u}^{p}(1-m_1)}{e_{2l}^{1-p}e_{2u}^{p}(1-m_2)}\Big]x_{1\infty} + \Gamma_2\Xi_2\Big\}$ are satisfied.

Case 2. If $C_1 > p_1q_1x_1$, i.e., the fishing cost is greater than the revenue for prey x_1. Then prey x_1 is not fit to be fished ($E_{1\infty} = 0$). Only prey x_2 keeps normal capture, i.e., $p_2q_2x_2 < C_2$. According to $x_{2\infty} = \frac{C_2}{p_2q_2}$ and $E_{1\infty} = 0$, we consider four situations as follows:
Situation 1. If $x_{1\infty} = 0$, $y_\infty = 0$, solving the second equation of (6.2) obtains

$$E_{2\infty} = (r_{2l}^{1-p} r_{2u}^{p} - \Gamma_2 x_{2\infty})/q_2, \tag{6.7}$$

which is positive provided that $r_{2l}^{1-p} r_{2u}^{p} > \Gamma_2 x_{2\infty}$.
Situation 2. If $x_{1\infty} = 0$ holds, y_∞ and $E_{2\infty}$ can be calculated through the second and third equation of (6.2)

$$\begin{aligned} y_\infty &= e_{2l}^{1-p} e_{2u}^{p} (1 - m_2)(x_{2\infty} - \Xi_2)/\Omega, \\ E_{2\infty} &= [r_{2l}^{1-p} r_{2u}^{p} - (\Gamma_2 + \Lambda_2)x_{2\infty} + \Lambda_2\Xi_2]/q_2, \end{aligned} \tag{6.8}$$

which are positive if $r_{2l}^{1-p} r_{2u}^{p} > (\Gamma_2 + \Lambda_2)x_{2\infty} - \Lambda_2\Xi_2$ and $x_{2\infty} > \Xi_2$.
Situation 3. If $y_\infty = 0$, it follows from the first and second equation that

$$\begin{aligned} x_{1\infty} &= \frac{-\Gamma_1 + \sqrt{\Gamma_1^2 + 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}(r_{1l}^{1-p} r_{1u}^{p} - \alpha_{1l}^{p}\alpha_{1u}^{1-p} x_{2\infty})}}{2r_{1l}^{1-p} r_{1u}^{p}}, \\ E_{2\infty} &= \frac{r_{2l}^{1-p} r_{2u}^{p} - \alpha_{2l}^{p}\alpha_{2u}^{1-p} x_{1\infty} - \Gamma_2 x_{2\infty}}{q_1}, \end{aligned} \tag{6.9}$$

which are positive if $r_{1l}^{1-p} r_{1u}^{p} > \alpha_{1l}^{p}\alpha_{1u}^{1-p} x_{2\infty}$ and $r_{2l}^{1-p} r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p} x_{1\infty} + \Gamma_2 x_{2\infty}$ are satisfied.
Situation 4. By a careful calculation, we obtain the results as follows

$$\begin{aligned} x_{1\infty} &= \frac{-(\Gamma_1 + \Lambda_1) + \sqrt{(\Gamma_1 + \Lambda_1)^2 + 4\gamma_{1l}^{p}\gamma_{1u}^{1-p}\Big[r_{1l}^{1-p} r_{1u}^{p} + \frac{d_l^p d_u^{1-p}}{\Omega} - \Big(\frac{\Phi_1}{\Omega} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}\Big)x_{2\infty}\Big]}}{2\gamma_{1l}^{p}\gamma_{1u}^{1-p}}, \\ y_\infty &= \frac{e_{1l}^{1-p} e_{1u}^{p}(1 - m_1)x_{1\infty} + e_{2l}^{1-p} e_{2u}^{p}(1 - m_2)x_{2\infty} - d_l^p d_u^{1-p}}{\Omega}, \\ E_{2\infty} &= \frac{r_{2l}^{1-p} r_{2u}^{p} - \alpha_{2l}^{p}\alpha_{2u}^{1-p} x_{1\infty} - \Gamma_2 x_{2\infty} - c_{2l}^{p} c_{2u}^{1-p}(1 - m_2)y_\infty}{q_2}, \end{aligned} \tag{6.10}$$

which are positive provided that $r_{1l}^{1-p} r_{1u}^{p} > \Big(\frac{\Phi_1}{\Omega} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}\Big)x_{2\infty} - \frac{d_l^p d_u^{1-p}}{\Omega}$, $r_{2l}^{1-p} r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p} x_{1\infty} + \Gamma_2 x_{2\infty} + c_{2l}^{p} c_{2u}^{1-p}(1 - m_2)y_\infty$ and $e_{1l}^{1-p} e_{1u}^{p}(1 - m_1)x_{1\infty} + e_{2l}^{1-p} e_{2u}^{p}(1 - m_2)x_{2\infty} > d_l^p d_u^{1-p}$.
Case 3. If $C_1 > p_1q_1x_1$ and $C_2 > p_2q_2x_2$, the fishing cost of the preys x_1 and x_2 are greater than the revenue. The harvesting of preys x_1 and x_2 are unworkable, so we cannot but stop the fishing of the preys, that is, $E_{1\infty} = E_{2\infty} = 0$. The existence of bionomic equilibrium is the same as that of biological equilibria in Section 4.
Case 4. If $C_1 < p_1q_1x_1$ and $C_2 < p_2q_2x_2$, i.e., the income of both preys is greater than the capture cost, so the system will continue to be in operation. Therefore $x_{1\infty} = \frac{C_1}{p_1q_1}$ and $x_{2\infty} = \frac{C_2}{p_2q_2}$. We substitute $x_{1\infty}$ and $x_{2\infty}$ into (6.2) and consider the following two situations.
Situation 1. If $y_\infty = 0$, $E_{1\infty}$ and $E_{2\infty}$ can be calculated by the first and second equation of (6.2)

$$\begin{aligned} E_{1\infty} &= (r_{1l}^{1-p} r_{1u}^{p} - \gamma_{1l}^{p}\gamma_{1u}^{1-p} x_{1\infty}^2 - \Gamma_1 x_{1\infty} - \alpha_{1l}^{p}\alpha_{1u}^{1-p} x_{2\infty})/q_1, \\ E_{2\infty} &= (r_{2l}^{1-p} r_{2u}^{p} - \alpha_{2l}^{p}\alpha_{2u}^{1-p} x_{1\infty} - \Gamma_2 x_{2\infty})/q_2, \end{aligned} \tag{6.11}$$

which are positive if $r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} + \Gamma_1 x_{1\infty} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_{2\infty}$ and $r_{2l}^{1-p}r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_{1\infty} + \Gamma_2 x_{2\infty}$ hold.
Situation 2. According to the first three equations of (6.2), it yields that

$$\begin{aligned} y_\infty &= \frac{e_{1l}^{1-p}e_{1u}^{p}(1-m_1)x_{1\infty} + e_{2l}^{1-p}e_{2u}^{p}(1-m_2)x_{2\infty} - d_l^p d_u^{1-p}}{\Omega}, \\ E_{1\infty} &= \frac{r_{1l}^{1-p}r_{1u}^{p}\Omega - \gamma_{1l}^{p}\gamma_{1u}^{1-p}\Omega x_{1\infty}^{2} - (\Gamma_1+\Lambda_1)\Omega x_{1\infty} - (\Phi_1 + \alpha_{1l}^{p}\alpha_{1u}^{1-p}\Omega)x_{2\infty} + d_l^p d_u^{1-p}}{q_1\Omega}, \\ E_{2\infty} &= \frac{r_{2l}^{1-p}r_{2u}^{p}\Omega - (\Phi_2 + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\Omega)x_{1\infty} - (\Gamma_2+\Lambda_2)\Omega x_{2\infty} + d_l^p d_u^{1-p}}{q_2\Omega}, \end{aligned} \tag{6.12}$$

exist if $r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} + (\Gamma_1+\Lambda_1)x_{1\infty} + \left(\frac{\Phi_1}{\Omega} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}\right)x_{2\infty} - \frac{d_l^p d_u^{1-p}}{\Omega}$, $r_{2l}^{1-p}r_{2u}^{p} > \left(\frac{\Phi_2}{\Omega} + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\right)x_{1\infty} + (\Gamma_2+\Lambda_2)x_{2\infty} - \frac{d_l^p d_u^{1-p}}{\Omega}$ and $e_{1l}^{1-p}e_{1u}^{p}(1-m_1)x_{1\infty} + e_{2l}^{1-p}e_{2u}^{p}(1-m_2)x_{2\infty} > d_l^p d_u^{1-p}$ are satisfied.

Theorem 6.1. *The existence conditions of bionomic equilibria are displayed in Table 1:*

Table 1. The existence conditions of bionomic equilibria in four cases.

Theorem	Conditions
$(x_{1\infty}, 0, 0, E_{1\infty}, 0)$	$r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} + \Gamma_1 x_{1\infty}$
$(x_{1\infty}, 0, y_\infty, E_{1\infty}, 0)$	$r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} + (\Gamma_1+\Lambda_1)x_{1\infty} - \Lambda_1\Xi_1$, $x_{1\infty} > \Xi_1$
$(x_{1\infty}, x_{2\infty}, 0, E_{1\infty}, 0)$	$r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} - \left(\frac{\Psi}{\Gamma_2} - \Gamma_1\right)x_{1\infty} + \frac{r_{2l}^{1-p}r_{2u}^{p}\alpha_{1l}^{p}\alpha_{1u}^{1-p}}{\Gamma_2}$, $r_{2l}^{1-p}r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_{1\infty}$
$(x_{1\infty}, x_{2\infty}, y_\infty, E_{1\infty}, 0)$	$r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} + \Gamma_1 x_{1\infty} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_{2\infty} + c_{1l}^{p}c_{1u}^{1-p}(1-m_1)y_\infty$, $r_{2l}^{1-p}r_{2u}^{p} > \max\left\{\left(\alpha_{2l}^{p}\alpha_{2u}^{1-p} - \frac{\Phi_2}{\Omega}\right)x_{1\infty} - \Lambda_2\Xi_2, \left[\alpha_{2l}^{p}\alpha_{2u}^{1-p} - \frac{\Gamma_2 e_{1l}^{1-p}e_{1u}^{p}(1-m_1)}{e_{2l}^{1-p}e_{2u}^{p}(1-m_2)}\right]x_{1\infty} + \Gamma_2\Xi_2\right\}$
$(0, x_{2\infty}, 0, 0, E_{2\infty})$	$r_{2l}^{1-p}r_{2u}^{p} > \Gamma_2 x_{2\infty}$
$(0, x_{2\infty}, y_\infty, 0, E_{2\infty})$	$r_{2l}^{1-p}r_{2u}^{p} > (\Gamma_2+\Lambda_2)x_{2\infty} - \Lambda_2\Xi_2$, $x_{2\infty} > \Xi_2$
$(x_{1\infty}, x_{2\infty}, 0, 0, E_{2\infty})$	$r_{1l}^{1-p}r_{1u}^{p} > \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_{2\infty}$, $r_{2l}^{1-p}r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_{1\infty} + \Gamma_2 x_{2\infty}$
$(x_{1\infty}, x_{2\infty}, y_\infty, 0, E_{2\infty})$	$r_{1l}^{1-p}r_{1u}^{p} > \left(\frac{\Phi_1}{\Omega} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}\right)x_{2\infty} - \frac{d_l^p d_u^{1-p}}{\Omega}$, $r_{2l}^{1-p}r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_{1\infty} + \Gamma_2 x_{2\infty} + c_{2l}^{p}c_{2u}^{1-p}(1-m_2)y_\infty$, $e_{1l}^{1-p}e_{1u}^{p}(1-m_1)x_{1\infty} + e_{2l}^{1-p}e_{2u}^{p}(1-m_2)x_{2\infty} > d_l^p d_u^{1-p}$
$(x_{1\infty}, x_{2\infty}, 0, E_{1\infty}, E_{2\infty})$	$r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} + \Gamma_1 x_{1\infty} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_{2\infty}$, $r_{2l}^{1-p}r_{2u}^{p} > \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_{1\infty} + \Gamma_2 x_{2\infty}$
$(x_{1\infty}, x_{2\infty}, y_\infty, E_{1\infty}, E_{2\infty})$	$r_{1l}^{1-p}r_{1u}^{p} > \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_{1\infty}^{2} + (\Gamma_1+\Lambda_1)x_{1\infty} + \left(\frac{\Phi_1}{\Omega} + \alpha_{1l}^{p}\alpha_{1u}^{1-p}\right)x_{2\infty} - \frac{d_l^p d_u^{1-p}}{\Omega}$, $r_{2l}^{1-p}r_{2u}^{p} > \left(\frac{\Phi_2}{\Omega} + \alpha_{2l}^{p}\alpha_{2u}^{1-p}\right)x_{1\infty} + (\Gamma_2+\Lambda_2)x_{2\infty} - \frac{d_l^p d_u^{1-p}}{\Omega}$, $e_{1l}^{1-p}e_{1u}^{p}(1-m_1)x_{2\infty} + e_{2l}^{1-p}e_{2u}^{p}(1-m_2)x_{2\infty} > d_l^p d_u^{1-p}$

7. Fuzzy optimal harvesting

Use r and k to represent the inflation and discount rates, respectively. They are often considered as fuzzy parameters since the imprecision of the environment. Therefore, let the notations $\widetilde{r}$ and $\widetilde{k}$ represent trapezoidal fuzzy number. $\widetilde{\delta}$ is the difference value of $\widetilde{r}$ and $\widetilde{k}$ standing for the fuzzy inflation net discount rate and can be also regarded as trapezoidal fuzzy number, i.e., $\widetilde{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4)$ (see Appendix B). And a continuous time stream of revenues $\widetilde{J}$ is yield to

$$\widetilde{J} = \int_0^\infty e^{-\widetilde{\delta}t}[(p_1q_1x_1 - C_1)E_1(t) + (p_2q_2x_2 - C_2)E_2(t)]dt. \tag{7.1}$$

The aim of this part is to maximize $\widetilde{J}$ yield to the state Eq (2.2). The control variables $E_i(t)$ (i=1,2) are subjected to the constrains $0 \le E_i(t) \le E_i^{\max}$ (i=1,2). On account of the method of Maity and Maiti [36], Sadhukhan et al. [37] and Pal and Mahapatra [33], we apply the value α to cut trapezoidal fuzzy number obtaining an interval number $[\delta_L, \delta_R]$, where $\delta_L = \delta_1 + \alpha(\delta_2 - \delta_1)$ and $\delta_R = \delta_4 - \alpha(\delta_4 - \delta_3)(0 \le \alpha \le 1)$. Maximization of the $\widetilde{J}$ translate into maximization of $[J_L, J_R]$ as follows

$$\text{Max}[J_L, J_R] = \int_0^\infty e^{-[\delta_L,\delta_R]}[(p_1q_1x_1 - C_1)E_1(t) + (p_2q_2x_2 - C_2)E_2(t)]dt, \tag{7.2}$$

where

$$J_L = \int_0^\infty e^{-\delta_R t}[(p_1q_1x_1 - C_1)E_1(t) + (p_2q_2x_2 - C_2)E_2(t)]dt,$$
$$J_R = \int_0^\infty e^{-\delta_L t}[(p_1q_1x_1 - C_1)E_1(t) + (p_2q_2x_2 - C_2)E_2(t)]dt,$$

subject to the constrains (2.2). Consider nonnegative numbers ω_1 and ω_2 meeting $\omega_1 + \omega_2 = 1$ as two weights as well as the method of weighted sum (see Appendix C), thus $\text{Max}[J_L, J_R]$ can be written as

$$\text{Max}\widetilde{J} = \text{Max}[J_L, J_R] = \text{Max}(\omega_1 J_L + \omega_2 J_R). \tag{7.3}$$

We first construct the Hamiltonian provided by

$$\begin{aligned} H =&(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})[(p_1q_1x_1 - C_1)E_1 + (p_2q_2x_2 - C_2)E_2] \\ &+ \lambda_1\Big[r_{1l}^{1-p}r_{1u}^{p}x_1 - r_{1l}^{p}r_{1u}^{1-p}\frac{x_1^2}{k_1} - \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_1x_2 - c_{1l}^{p}c_{1u}^{1-p}(1-m_1)x_1y - \gamma_{1l}^{p}\gamma_{1u}^{1-p}x_1^3 - q_1E_1x_1\Big] \\ &+ \lambda_2\Big[r_{2l}^{1-p}r_{2u}^{p}x_2 - r_{2l}^{p}r_{2u}^{1-p}\frac{x_2^2}{k_2} - \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1x_2 - c_{2l}^{p}c_{2u}^{1-p}(1-m_2)x_2y - q_2E_2x_2\Big] \\ &+ \lambda_3[-d_l^{p}d_u^{1-p}y - s_l^{p}s_u^{1-p}y^2 + e_{1l}^{1-p}e_{1u}^{p}(1-m_1)x_1y + e_{2l}^{1-p}e_{2u}^{p}(1-m_2)x_2y - \gamma_{2l}^{p}\gamma_{2u}^{1-p}y^2], \end{aligned} \tag{7.4}$$

where λ_1, λ_2 and λ_3 denote the adjoint variables. Based on Pontryagin's maximum principle [38], the adjoint equations are expressed as follows

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial x_3}. \tag{7.5}$$

Together with (7.4) and (7.5), it yields to

$$\begin{aligned} \frac{d\lambda_1}{dt} =&\lambda_2\alpha_{2l}^{p}\alpha_{2u}^{1-p}x_2 - \lambda_3 e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y - p_1q_1(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})E_1 \\ &- \lambda_1\Big[r_{1l}^{1-p}r_{1u}^{p} - 2\frac{r_{1l}^{p}r_{1u}^{1-p}}{k_1}x_1 - \alpha_{1l}^{p}\alpha_{1u}^{1-p}x_2 - c_{1l}^{p}c_{1u}^{1-p}(1-m_1)y - 3\gamma_{1l}^{p}\gamma_{1u}^{1-p}x_1^2 - q_1E_1\Big], \\ \frac{d\lambda_2}{dt} =&\lambda_1\alpha_{1l}^{p}\alpha_{1u}^{1-p}x_1 - \lambda_3 e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y - p_2q_2(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})E_2 \\ &- \lambda_2\Big[r_{2l}^{1-p}r_{2u}^{p} - 2\frac{r_{2l}^{p}r_{2u}^{1-p}}{k_2}x_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1 - c_{2l}^{p}c_{2u}^{1-p}(1-m_2)y - q_2E_2\Big], \\ \frac{d\lambda_3}{dt} =&\lambda_1 c_{1l}^{p}c_{1u}^{1-p}(1-m_1)x_1 + \lambda_2 c_{2l}^{p}c_{2u}^{1-p}(1-m_2)x_2y \\ &- \lambda_3[-d_l^{p}d_u^{1-p} - 2s_l^{p}s_u^{1-p}y + e_{1l}^{1-p}e_{1u}^{p}(1-m_1)x_1 + e_{2l}^{1-p}e_{2u}^{p}(1-m_2)x_2 - 2\gamma_{2l}^{p}\gamma_{2u}^{1-p}y]. \end{aligned} \tag{7.6}$$

Introducing the interior equilibrium into (7.6), we have

$$
\begin{aligned}
\frac{d\lambda_1}{dt} =&\lambda_1\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)x_1 + \lambda_2\alpha_{2l}^p\alpha_{2u}^{1-p}x_2 - \lambda_3 e_{1l}^{1-p}e_{1u}^p(1-m_1)y - p_1q_1(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})E_1,\\
\frac{d\lambda_2}{dt} =& \lambda_1\alpha_{1l}^p\alpha_{1u}^{1-p}x_1 + \lambda_2\frac{r_{2l}^p r_{2u}^{1-p}}{k_2}x_2 - \lambda_3 e_{2l}^{1-p}e_{2u}^p(1-m_2)y - p_2q_2(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})E_2,\\
\frac{d\lambda_3}{dt} =& \lambda_1 c_{1l}^p c_{1u}^{1-p}(1-m_1)x_1 + \lambda_2 c_{2l}^p c_{2u}^{1-p}(1-m_2)x_2 + \lambda_3(s_l^p s_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})y.
\end{aligned} \tag{7.7}
$$

We get a third order differential equation with respect to λ_3 by deleting λ_1 and λ_2 in Eq (7.7)

$$(a_0D^3 + a_1D^2 + a_2D + a_3)\lambda_3 = M_{3L}e^{-\delta_R t} + M_{3R}e^{-\delta_L t}, \tag{7.8}$$

where

$$
\begin{aligned}
D =&\frac{d}{dt},\quad a_0 = 1,\quad a_1 = -\Big[\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)x_1 + \frac{r_{2l}^p r_{2u}^{1-p}}{k_2}x_2 + (s_l^p s_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})y\Big],\\
a_2 =&\Big[\frac{r_{2l}^p r_{2u}^{1-p}}{k_2}\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big) - \alpha_{1l}^p\alpha_{1u}^{1-p}\alpha_{2l}^p\alpha_{2u}^{1-p}\Big]x_1x_2\\
&+ \Big[c_{2l}^p c_{2u}^{1-p}e_{2l}^{1-p}e_{2u}^p(1-m_2)^2 + \frac{r_{2l}^p r_{2u}^{1-p}}{k_2}(s_l^p s_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})\Big]x_2y\\
&+ \Big[(s_l^p s_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big) + c_{1l}^p c_{1u}^{1-p}e_{1l}^{1-p}e_{1u}^p(1-m_1)^2\Big]x_1y,\\
a_3 =&\Big\{(s_l^p s_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})\Big[\alpha_{1l}^p\alpha_{1u}^{1-p}\alpha_{2l}^p\alpha_{2u}^{1-p} - \frac{r_{2l}^p r_{2u}^{1-p}}{k_2}\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)\Big]\\
&+ e_{1l}^{1-p}e_{1u}^p(1-m_1)\Big[\alpha_{1l}^p\alpha_{1u}^{1-p}c_{2l}^p c_{2u}^{1-p}(1-m_2) - \frac{r_{2l}^p r_{2u}^{1-p}}{k_2}c_{1l}^p c_{1u}^{1-p}(1-m_1)\Big]\\
&+ e_{2l}^{1-p}e_{2u}^p(1-m_2)\Big[\alpha_{2l}^p\alpha_{2u}^{1-p}c_{1l}^p c_{1u}^{1-p}(1-m_1) - c_{2l}^p c_{2u}^{1-p}(1-m_2)\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)\Big]\Big\}x_1x_2y,\\
M_{3L} =&\omega_1\Big[p_1q_1\Big\{\delta_R c_{1l}^p c_{1u}^{1-p}(1-m_1)x_1 - \Big[\alpha_{1l}^p\alpha_{1u}^{1-p}c_{2l}^p c_{2u}^{1-p}(1-m_2) - \frac{r_{2l}^p r_{2u}^{1-p}}{k_2}c_{1l}^p c_{1u}^{1-p}(1-m_1)\Big]x_1x_2\Big\}E_1\\
&+ p_2q_2\Big\{-\Big[\alpha_{2l}^p\alpha_{2u}^{1-p}c_{1l}^p c_{1u}^{1-p}(1-m_1) - c_{2l}^p c_{2u}^{1-p}(1-m_2)\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)\Big]x_1x_2\\
&+ \delta_R c_{2l}^p c_{2u}^{1-p}(1-m_2)x_2\Big\}E_2\Big],\\
M_{3R} =&\omega_2\Big[p_1q_1\Big\{\delta_L c_{1l}^p c_{1u}^{1-p}(1-m_1)x_1 - \Big[\alpha_{1l}^p\alpha_{1u}^{1-p}c_{2l}^p c_{2u}^{1-p}(1-m_2) - \frac{r_{2l}^p r_{2u}^{1-p}}{k_2}c_{1l}^p c_{1u}^{1-p}(1-m_1)\Big]x_1x_2\Big\}E_1\\
&+ p_2q_2\Big\{-\Big[\alpha_{2l}^p\alpha_{2u}^{1-p}c_{1l}^p c_{1u}^{1-p}(1-m_1) - c_{2l}^p c_{2u}^{1-p}(1-m_2)\Big(\frac{r_{1l}^p r_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)\Big]x_1x_2\\
&+ \delta_L c_{2l}^p c_{2u}^{1-p}(1-m_2)x_2\Big\}E_2\Big].
\end{aligned}
$$

The solution of (7.8) is written as follows

$$\lambda_3 = F_1 e^{\mu_1 t} + F_2 e^{\mu_2 t} + F_3 e^{\mu_3 t} + \frac{M_{3L}}{N_L} e^{-\delta_R t} + \frac{M_{3R}}{N_R} e^{-\delta_L t}, \tag{7.9}$$

where $F_i (i = 1, 2, 3)$ are arbitrary constants and $\mu_i (i = 1, 2, 3)$ are the roots of the cubic equation

$$a_0 \mu^3 + a_1 \mu^2 + a_2 \mu + a_3 = 0 \tag{7.10}$$

and

$$N_L = -(a_0 \delta_R^3 - a_1 \delta_R^2 + a_2 \delta_R - a_3) \neq 0, \quad N_R = -(a_0 \delta_L^3 - a_1 \delta_L^2 + a_2 \delta_L - a_3) \neq 0.$$

It follows from (7.9) that λ_3 is bounded if and only if

$$\mu_i < 0 \ \text{ or } \ F_i = 0 \ \ (i = 1, 2, 3).$$

The Hurwitz matrix is displayed as follows

$$\begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{pmatrix},$$

and assign

$$\Delta_1 = a_1, \quad \Delta_2 = a_1 a_2 - a_3, \quad \Delta_3 = a_3(a_1 a_2 - a_3).$$

Thus the roots of (7.10) are negative real number or complex conjugate whose real part is negative if and only if $\Delta_i > 0$ $(i = 1, 2, 3)$. However $\Delta_1 < 0$, so it is hard to make sure whether $\mu_i < 0$, we have to take $F_i = 0$ into account. Then (7.6) can be simplified as

$$\lambda_3 = \frac{M_{3L}}{N_L} e^{-\delta_R t} + \frac{M_{3R}}{N_R} e^{-\delta_L t}. \tag{7.11}$$

Analogously, we obtain that

$$\lambda_1 = \frac{M_{1L}}{N_L} e^{-\delta_R t} + \frac{M_{1R}}{N_R} e^{-\delta_L t} \tag{7.12}$$

and

$$\lambda_2 = \frac{M_{2L}}{N_L} e^{-\delta_R t} + \frac{M_{2R}}{N_R} e^{-\delta_L t}, \tag{7.13}$$

where

$$
\begin{aligned}
M_{1L} =&\omega_1\Big[p_2q_2\{\delta_R\alpha_{2l}^p\alpha_{2u}^{1-p}x_2 + [c_{2l}^pc_{2u}^{1-p}e_{1l}^pe_{1u}^{1-p}(1-m_1)(1-m_2) + \alpha_{2l}^p\alpha_{2u}^{1-p}(s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})]x_2y\}E_2 \\
&- p_1q_1\Big\{\Big[\frac{r_{2l}^pr_{2u}^{1-p}}{k_2}(s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p}) + c_{2l}^pc_{2u}^{1-p}e_{2l}^pe_{2u}^{1-p}(1-m_2)^2\Big]x_2y \\
&+ \delta_R\Big[\frac{r_{2l}^pr_{2u}^{1-p}}{k_2}x_2 + (s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})y + \delta_R\Big]\Big\}E_1\Big], \\
M_{1R} =&\omega_2\Big[p_2q_2\{\delta_L\alpha_{2l}^p\alpha_{2u}^{1-p}x_2 + [c_{2l}^pc_{2u}^{1-p}e_{1l}^pe_{1u}^{1-p}(1-m_1)(1-m_2) + \alpha_{2l}^p\alpha_{2u}^{1-p}(s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})]x_2y\}E_2 \\
&- p_1q_1\Big\{\Big[\frac{r_{2l}^pr_{2u}^{1-p}}{k_2}(s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p}) + c_{2l}^pc_{2u}^{1-p}e_{2l}^pe_{2u}^{1-p}(1-m_2)^2\Big]x_2y \\
&+ \delta_L\Big[\frac{r_{2l}^pr_{2u}^{1-p}}{k_2}x_2 + (s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})y + \delta_L\Big]\Big\}E_1\Big], \\
M_{2L} =&\omega_1\Big[p_1q_1\{\delta_R\alpha_{1l}^p\alpha_{1u}^{1-p}x_1 + [c_{1l}^pc_{1u}^{1-p}e_{1l}^pe_{1u}^{1-p}(1-m_1)(1-m_2) + \alpha_{1l}^p\alpha_{1u}^{1-p}(s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})]x_1y\}E_1 \\
&- p_2q_2\Big\{\Big[c_{1l}^pc_{1u}^{1-p}e_{1l}^pe_{1u}^{1-p}(1-m_1)^2 + (s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})\Big(\frac{r_{1l}^pr_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)\Big]x_1y \\
&+ \delta_R\Big[\Big(\frac{r_{1l}^pr_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)x_1 + (s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})y + \delta_R\Big]\Big\}E_2\Big], \\
M_{2R} =&\omega_2\Big[p_1q_1\{\delta_L\alpha_{1l}^p\alpha_{1u}^{1-p}x_1 + [c_{1l}^pc_{1u}^{1-p}e_{1l}^pe_{1u}^{1-p}(1-m_1)(1-m_2) + \alpha_{1l}^p\alpha_{1u}^{1-p}(s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})]x_1y\}E_1 \\
&- p_2q_2\Big\{\Big[c_{1l}^pc_{1u}^{1-p}e_{1l}^pe_{1u}^{1-p}(1-m_1)^2 + (s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})\Big(\frac{r_{1l}^pr_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)\Big]x_1y \\
&+ \delta_L\Big[\Big(\frac{r_{1l}^pr_{1u}^{1-p}}{k_1} + 2\gamma_{1l}^p\gamma_{1u}^{1-p}x_1\Big)x_1 + (s_l^ps_u^{1-p} + \gamma_{2l}^p\gamma_{2u}^{1-p})y + \delta_L\Big]\Big\}E_2\Big].
\end{aligned}
$$

The shadow prices $\lambda_i e^{\delta_L t}$ $(i = 1, 2, 3)$ of the three species remain bounded as $t \to \infty$, that is it satisfies the transversality consider at ∞. The Hamiltonian should be maximized for $E_i \in [0, E_i^{\max}]$. Suppose that the optimal equilibrium occurs at neither $E_i = 0$ nor $E_i = E_i^{\max}$, we therefore consider the singular control

$$
\begin{aligned}
\frac{\partial H}{\partial E_1} &= (\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})(p_1q_1x_1 - C_1) - \lambda_1 q_1 x_1 = 0, \\
\frac{\partial H}{\partial E_2} &= (\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})(p_2q_2x_2 - C_2) - \lambda_2 q_2 x_2 = 0,
\end{aligned}
\tag{7.14}
$$

i.e.,

$$
\begin{aligned}
\lambda_1 &= (\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})(p_1 - C_1/(q_1x_1)), \\
\lambda_2 &= (\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})(p_2 - C_2/(q_2x_2)).
\end{aligned}
\tag{7.15}
$$

Substituting the values of λ_1 and λ_2 in (7.12) and (7.13) into (7.15), we get

$$
\begin{aligned}
M_L^1 e^{-\delta_R t} + M_R^1 e^{-\delta_L t} &= C_1(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t}), \\
M_L^2 e^{-\delta_R t} + M_R^2 e^{-\delta_L t} &= C_2(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t}),
\end{aligned}
\tag{7.16}
$$

where

$$
\begin{aligned}
M_L^1 &= \Big(\omega_1 p_1 - \tfrac{M_{1L}}{N_L}\Big)q_1x_1, \quad M_R^1 = \Big(\omega_2 p_1 - \tfrac{M_{1R}}{N_R}\Big)q_1x_1, \\
M_L^2 &= \Big(\omega_1 p_2 - \tfrac{M_{2L}}{N_L}\Big)q_2x_2, \quad M_R^2 = \Big(\omega_2 p_2 - \tfrac{M_{2R}}{N_R}\Big)q_2x_2.
\end{aligned}
\tag{7.17}
$$

By a simple calculation, (7.16) can be written as

$$\begin{aligned}(M_L^1 - C_1\omega_1)e^{-(\delta_R-\delta_L)t} &= -(M_R^1 - C_1\omega_2),\\ (M_L^2 - C_2\omega_1)e^{-(\delta_R-\delta_L)t} &= -(M_R^2 - C_2\omega_2).\end{aligned} \tag{7.18}$$

The next step is to discuss two cases:
Case 1. If $M_L^1 \neq C_1\omega_1$ and $M_L^2 \neq C_2\omega_1$, then (7.18) is equivalent to

$$(M_L^1 - C_1\omega_1)(M_R^2 - C_2\omega_2) = (M_L^2 - C_2\omega_1)(M_R^1 - C_1\omega_2). \tag{7.19}$$

Case 2. If $M_L^1 = C_1\omega_1$ or $M_L^2 = C_2\omega_1$, one of the following equations is true

$$M_L^1 - C_1\omega_1 = M_R^1 - C_1\omega_2 = 0, \text{ or } M_L^2 - C_2\omega_1 = M_R^2 - C_2\omega_2 = 0. \tag{7.19$'$}$$

On account of the method in [35], differentiating both sides of the equations in (7.15) with respect to t, one has

$$\begin{aligned}\frac{d\lambda_1}{dt} =&(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})\Big(r_{1l}^{1-p} r_{1u}^{p} - \frac{r_{1l}^{p} r_{1u}^{1-p}}{k_1} x_1 - \alpha_{1l}^{p}\alpha_{1u}^{1-p} x_2 - c_{1l}^{p} c_{1u}^{1-p}(1-m_1)y - \gamma_{1l}^{p}\gamma_{1u}^{1-p} x_1^2 - q_1 E_1\Big)\frac{C_1}{q_1 x_1}\\ &- (\delta_R\omega_1 e^{-\delta_R t} + \delta_L\omega_2 e^{-\delta_L t})\Big(p_1 - \frac{C_1}{q_1 x_1}\Big),\\ \frac{d\lambda_2}{dt} =&(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})\Big(r_{2l}^{1-p} r_{2u}^{p} - \frac{r_{2l}^{p} r_{2u}^{1-p}}{k_2} x_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p} x_1 - c_{2l}^{p} c_{2u}^{1-p}(1-m_2)y - q_2 E_2\Big)\frac{C_2}{q_2 x_2}\\ &- (\delta_R\omega_1 e^{-\delta_R t} + \delta_L\omega_2 e^{-\delta_L t})\Big(p_2 - \frac{C_2}{q_2 x_2}\Big).\end{aligned} \tag{7.20}$$

Substituting (7.15) into (7.6) and eliminating λ_3 yield

$$\begin{aligned}\frac{d\lambda_1}{dt} =&(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})\Big(p_2 - \frac{C_2}{q_2 x_2}\Big)\alpha_{2l}^{p}\alpha_{2u}^{1-p} x_2 - p_1 q_1(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})E_1\\ &- (\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})\Big(p_1 - \frac{C_1}{q_1 x_1}\Big)\Big[r_{1l}^{1-p} r_{1u}^{p} - 2\frac{r_{1l}^{p} r_{1u}^{1-p}}{k_1} x_1 - \alpha_{1l}^{p}\alpha_{1u}^{1-p} x_2\\ &- c_{1l}^{p} c_{1u}^{1-p}(1-m_1)y - 3\gamma_{1l}^{p}\gamma_{1u}^{1-p} {x_1}^2 - q_1 E_1\Big] - \Big(\frac{M_{3L}}{N_L}e^{-\delta_R t} + \frac{M_{3R}}{N_R}e^{-\delta_L t}\Big)e_{1l}^{1-p} e_{1u}^{p}(1-m_1)y,\\ \frac{d\lambda_2}{dt} =&(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})\Big(p_1 - \frac{C_1}{q_1 x_1}\Big)\alpha_{1l}^{p}\alpha_{1u}^{1-p} x_1 - p_2 q_2(\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})E_2\\ &- (\omega_1 e^{-\delta_R t} + \omega_2 e^{-\delta_L t})\Big(p_2 - \frac{C_2}{q_2 x_2}\Big)\Big[r_{2l}^{1-p} r_{2u}^{p} - 2\frac{r_{2l}^{p} r_{2u}^{1-p}}{k_2} x_2 - \alpha_{2l}^{p}\alpha_{2u}^{1-p} x_1\\ &- c_{2l}^{p} c_{2u}^{1-p}(1-m_2)y - q_2 E_2\Big] - \Big(\frac{M_{3L}}{N_L}e^{-\delta_R t} + \frac{M_{3R}}{N_R}e^{-\delta_L t}\Big)e_{2l}^{1-p} e_{2u}^{p}(1-m_2)y.\end{aligned} \tag{7.21}$$

Consider $\pi_1 = (p_1 q_1 x_1 - C_1)E_1 > 0$ and $\pi_2 = (p_2 q_2 x_2 - C_2)E_2 > 0$. The simultaneous Eqs (7.20) and

(7.21) with E_1 and E_2 omitted get

$$
\begin{aligned}
&-(\delta_R\omega_1 e^{-\delta_R t}+\delta_L\omega_2 e^{-\delta_L t})\Big(p_1-\frac{C_1}{q_1x_1}\Big)\\
&=(\omega_1 e^{-\delta_R t}+\omega_2 e^{-\delta_L t})\Big(p_2-\frac{C_2}{q_2x_2}\Big)\alpha_{2l}^{p}\alpha_{2u}^{1-p}x_2-\frac{C_1}{q_1}\Big(\omega_1 e^{-\delta_R t}+\omega_2 e^{-\delta_L t}\Big)\Big(\frac{r_{1l}^{p}r_{1u}^{1-p}}{k_1}+2\gamma_{1l}^{p}\gamma_{1u}^{1-p}x_1\Big)\\
&-p_1(\omega_1 e^{-\delta_R t}+\omega_2 e^{-\delta_L t})\Big[r_{1l}^{1-p}r_{1u}^{p}-2\frac{r_{1l}^{p}r_{1u}^{1-p}}{k_1}x_1-\alpha_{1l}^{p}\alpha_{1u}^{1-p}x_2-c_{1l}^{p}c_{1u}^{1-p}(1-m_1)y-3\gamma_{1l}^{p}\gamma_{1u}^{1-p}{x_1}^2\Big]\\
&-\Big(\frac{M_{3L}}{N_L}e^{-\delta_R t}+\frac{M_{3R}}{N_R}e^{-\delta_L t}\Big)e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y,\\
&-(\delta_R\omega_1 e^{-\delta_R t}+\delta_L\omega_2 e^{-\delta_L t})\Big(p_2-\frac{C_2}{q_2x_2}\Big)\\
&=(\omega_1 e^{-\delta_R t}+\omega_2 e^{-\delta_L t})\Big(p_1-\frac{C_1}{q_1x_1}\Big)\alpha_{1l}^{p}\alpha_{1u}^{1-p}x_1-\frac{C_2}{q_2}(\omega_1 e^{-\delta_R t}+\omega_2 e^{-\delta_L t})\frac{r_{2l}^{p}r_{2u}^{1-p}}{k_2}\\
&-p_2(\omega_1 e^{-\delta_R t}+\omega_2 e^{-\delta_L t})\Big[r_{2l}^{1-p}r_{2u}^{p}-2\frac{r_{2l}^{p}r_{2u}^{1-p}}{k_2}x_2-\alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1-c_{2l}^{p}c_{2u}^{1-p}(1-m_2)y\Big]\\
&-\Big(\frac{M_{3L}}{N_L}e^{-\delta_R t}+\frac{M_{3R}}{N_R}e^{-\delta_L t}\Big)e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y.
\end{aligned}
\tag{7.22}
$$

Dividing $e^{-\delta_L t}$ from both sides of the above two equations and merging similar terms in (7.22) yield

$$
\begin{aligned}
&e^{-(\delta_R-\delta_L)t}\Big\{-\delta_R\omega_1A_1-\omega_1A_2+\omega_1A_3+\omega_1A_4+\frac{M_{3L}}{N_L}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y\Big\}\\
&\qquad=\delta_L\omega_2A_1+\omega_2A_2-\omega_2A_3-\omega_2A_4-\frac{M_{3R}}{N_R}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y,\\
&e^{-(\delta_R-\delta_L)t}\Big\{-\delta_R\omega_1B_1-\omega_1B_2+\omega_1B_3+\omega_1B_4+\frac{M_{3L}}{N_L}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y\Big\}\\
&\qquad=\delta_L\omega_2B_1+\omega_2B_2-\omega_2B_3-\omega_2B_4-\frac{M_{3R}}{N_R}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y,
\end{aligned}
\tag{7.23}
$$

where

$$
\begin{aligned}
&A_1=p_1-\tfrac{C_1}{q_1x_1},\ \ A_2=\Big(p_2-\tfrac{C_2}{q_2x_2}\Big)\alpha_{2l}^{p}\alpha_{2u}^{1-p}x_2,\ \ A_3=\tfrac{C_1}{q_1}\Big(\tfrac{r_{1l}^{p}r_{1u}^{1-p}}{k_1}+2\gamma_{1l}^{p}\gamma_{1u}^{1-p}x_1\Big),\\
&A_4=p_1\Big[r_{1l}^{1-p}r_{1u}^{p}-2\tfrac{r_{1l}^{p}r_{1u}^{1-p}}{k_1}x_1-\alpha_{1l}^{p}\alpha_{1u}^{1-p}x_2-c_{1l}^{p}c_{1u}^{1-p}(1-m_1)y-3\gamma_{1l}^{p}\gamma_{1u}^{1-p}{x_1}^2\Big],\\
&B_1=p_2-\tfrac{C_2}{q_2x_2},B_2=\Big(p_1-\tfrac{C_1}{q_1x_1}\Big)\alpha_{1l}^{p}\alpha_{1u}^{1-p}x_1,B_3=\tfrac{C_2}{q_2}\tfrac{r_{2l}^{p}r_{2u}^{1-p}}{k_2},\\
&B_4=p_2\Big[r_{2l}^{1-p}r_{2u}^{p}-2\tfrac{r_{2l}^{p}r_{2u}^{1-p}}{k_2}x_2-\alpha_{2l}^{p}\alpha_{2u}^{1-p}x_1-c_{2l}^{p}c_{2u}^{1-p}(1-m_2)y\Big].
\end{aligned}
$$

Analogously, consider two cases as follows:

Case 1. Suppose that

$$\delta_R\omega_1A_1+\omega_1A_2-\omega_1A_3-\omega_1A_4\neq\frac{M_{3L}}{N_L}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y \tag{7.24}$$

and

$$\delta_R\omega_1B_1+\omega_1B_2-\omega_1B_3-\omega_1B_4\neq\frac{M_{3L}}{N_L}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y, \tag{7.25}$$

then let us divide both sides of the first equation by that of the second equation of (7.23)

$$\frac{\delta_R\omega_1A_1+\omega_1A_2-\omega_1A_3-\omega_1A_4-\frac{M_{3L}}{N_L}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y}{\delta_R\omega_1B_1+\omega_1B_2-\omega_1B_3-\omega_1B_4-\frac{M_{3L}}{N_L}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y}=\frac{\delta_L\omega_2A_1+\omega_2A_2-\omega_2A_3-\omega_2A_4-\frac{M_{3R}}{N_R}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y}{\delta_L\omega_2B_1+\omega_2B_2-\omega_2B_3-\omega_2B_4-\frac{M_{3R}}{N_R}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y}. \tag{7.26}$$

Case 2. Assume that

$$\delta_R\omega_1A_1+\omega_1A_2-\omega_1A_3-\omega_1A_4=\frac{M_{3L}}{N_L}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y \tag{7.27}$$

or

$$\delta_R\omega_1B_1+\omega_1B_2-\omega_1B_3-\omega_1B_4=\frac{M_{3L}}{N_L}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y, \tag{7.28}$$

then we have

$$\begin{aligned}&\delta_R\omega_1A_1+\omega_1A_2-\omega_1A_3-\omega_1A_4-\frac{M_{3L}}{N_L}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y\\&\quad=\delta_L\omega_2A_1+\omega_2A_2-\omega_2A_3-\omega_2A_4-\frac{M_{3R}}{N_R}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y=0,\end{aligned} \tag{7.29}$$

or

$$\begin{aligned}&\delta_L\omega_2B_1+\omega_2B_2-\omega_2B_3-\omega_2B_4-\frac{M_{3L}}{N_L}e_{1l}^{1-p}e_{1u}^{p}(1-m_1)y\\&\quad=\delta_L\omega_2B_1+\omega_2B_2-\omega_2B_3-\omega_2B_4-\frac{M_{3R}}{N_R}e_{2l}^{1-p}e_{2u}^{p}(1-m_2)y=0.\end{aligned} \tag{7.29'}$$

In addition, on account of the interior equilibrium, the values of E_1 and E_2 are written as

$$\begin{aligned}E_1&=\frac{r_{1l}^{1-p}r_{1u}^{p}}{q_1}-\frac{r_{1l}^{p}r_{1u}^{1-p}x_1}{k_1q_1}-\frac{\alpha_{1l}^{p}\alpha_{1u}^{1-p}}{q_1}x_2-\frac{c_{1l}^{p}c_{1u}^{1-p}(1-m_1)}{q_1}y-\frac{\gamma_{1l}^{p}\gamma_{1u}^{1-p}}{q_1}{x_1}^2,\\E_2&=\frac{r_{2l}^{1-p}r_{2u}^{p}}{q_2}-\frac{r_{2l}^{p}r_{2u}^{1-p}x_2}{k_2q_2}-\frac{\alpha_{2l}^{p}\alpha_{2u}^{1-p}}{q_2}x_1-\frac{c_{2l}^{p}c_{2u}^{1-p}(1-m_2)}{q_2}y,\end{aligned} \tag{7.30}$$

then solving (7.19)(or (7.19)′), (7.26)((7.29) or (7.29)′), (7.30) together with the right side of the third equation equaling to 0 in (2.2), we get the optimal equilibrium solutions $x_1=x_{1\widetilde{\delta}}$, $x_2=x_{2\widetilde{\delta}}$, $y=x_{\widetilde{\delta}}$ as well as the optimal harvesting efforts $E_1=E_{1\widetilde{\delta}}, E_2=E_{2\widetilde{\delta}}$.

8. Numerical simulations

We show three numerical examples, in this section, to explain the theoretical results of model (2.2).
Example 1. Set the value of parameters in model (2.2) as follows: $[r_{1l},r_{1u}]=[9.99,10.01]$, $[r_{2l},r_{2u}]=[3.99,4.01]$, $[\alpha_{1l},\alpha_{1u}]=[0.09,0.11]$, $[\alpha_{2l},\alpha_{2u}]=[0.19,0.21]$, $[c_{1l},c_{1u}]=[0.29,0.31]$, $[c_{2l},c_{2u}]=[0.19,0.21]$, $[e_{1l},e_{1u}]=[0.29,0.31]$, $[e_{2l},e_{2u}]=[0.19,0.21]$, $[\gamma_{1l},\gamma_{1u}]=[0.19,0.21]$, $[\gamma_{2l},\gamma_{2u}]=[0.09,0.11]$, $[d_l,d_u]=[0.19,0.21]$, $[s_l,s_u]=[0.09,0.11]$, $K_1=100$, $K_2=200$, $m_1=0.20$, $m_2=0.15$.

Table 2. Equilibrium points for different p.

Equilibrium	0	0.2	0.5	0.8	1	(q_1, q_2, E_1, E_2)
P_1	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	0.7000,0.5000,20.00,10.00
P_2	(0.4584,0,0)	(0.4751,0,0)	(0.5002,0,0)	(0.5253,0,0)	(0.5421,0,0)	0.9900,0.3000,10.00,20.00
P_3	(0,0.0499,0)	(0,0.2492,0)	(0,0.5494,0)	(0,0.8509,0)	(0,1.0526,0)	0.1000,0.1000,99.90,39.89
P_4	(5.4569,0,4.8000)	(5.4841,0,5.1269)	(5.5224,0,5.6528)	(5.5576,0,6.2239)	(5.5792,0,6.6313)	0.2000,0.2000,10.00,10.00
P_5	(0,20.9179,14.4011)	(0,20.2287,14.7964)	(0,19.2304,15.4039)	(0,18.2740,16.0292)	(0,17.6591,16.4564)	0.5000,0.1000,10.00,10.00
P_6	(4.5474,1.7482,0)	(4.5209,3.1658,0)	(4.5022,5.0333,0)	(4.5040,6.6567,0)	(4.5143,7.6328,0)	0.5000,0.3000,10.00,10.00
P_7	(0.7214,11.8954,0)	(0.6933,12.5451,0)	(0.6396,13.6114,0)	(0.5704,14.7933,0)	(0.5151,15.6460,0)	0.8500,0.3600,10.00,10.00
P_8	(4.5899,2.2009,5.5014)	(4.6354,1.9690,5.7178)	(4.7012,1.6303,6.0562)	(4.7642,1.3101,6.4111)	(4.8048,1.1017,6.6570)	0.3500,0.2000,10.00,10.00
P_9	(0.7764,6.2295,4.4373)	(0.5169,6.6954,4.8209)	(0.2811,6.9559,5.2697)	(0.1063,7.0356,5.6740)	(0.0093,7.0328,5.9315)	0.8000,0.2910,10.00,10.00

We show the values of nine equilibria with different $p \in [0, 1]$ in Table 2, respectively. From Table 2, P_1 is fixed at (0,0,0) with variable $p \in [0, 1]$. For P_2, P_3 and P_4, the values of preys x_1, x_2 or predator y increases and another is invariant in zero with increasing p; For P_5, prey x_2 is decreasing while predator y is increasing and prey x_1 always stays at zero with increasing p; For P_6, prey x_1 decreases and then increases while prey x_2 increases and predator y keeps in zero with increasing p; For P_7, the prey x_1 maintains decreasing while prey x_2 increases and predator y keeps in zero with increasing p; For P_8, prey x_1 and predator y are increasing while prey x_2 decreases with increasing p; For P_9, prey x_1 is decreasing while prey x_2 increases and then decreases, predator y always maintains increasing.

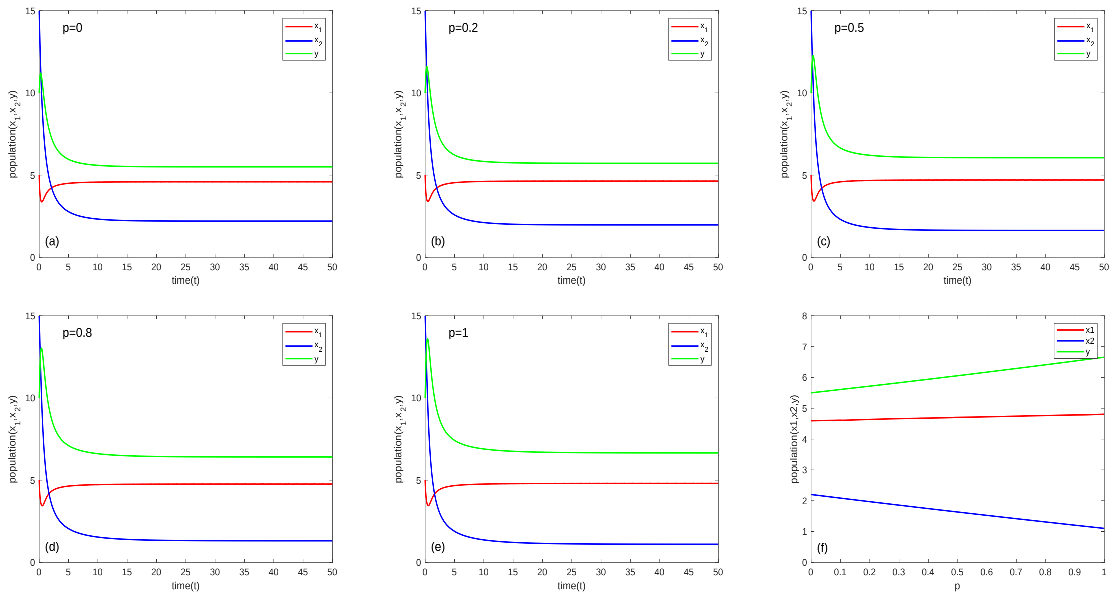

Figure 1. (a)–(e) Time series diagram of three species (x_1, x_2, y) with initial values (5,15,10) and $q_1 = 0.5, q_2 = 0.2, E_1 = 7, E_2 = 10$ for $p = 0, p = 0.2, p = 0.5, p = 0.8$ and $p = 1$, respectively, $t \in [0, 50]$. (f) Variation of interior equilibrium $P_8(x_1^\vartheta, x_2^\vartheta, y^\vartheta)$ with respect to p.

From (a)–(e) of Figure 1, we display time series of three species (x_1, x_2, y) with initial values $(5, 15, 10)$ for different p. The initial fluctuates for all species gradually trend to a stable condition level P_8 with time. Also the variation of interior equilibrium P_8 with respect to p is shown in Figure 1(f), respectively. It is easily recognize that the interior equilibrium changes for different p. As p

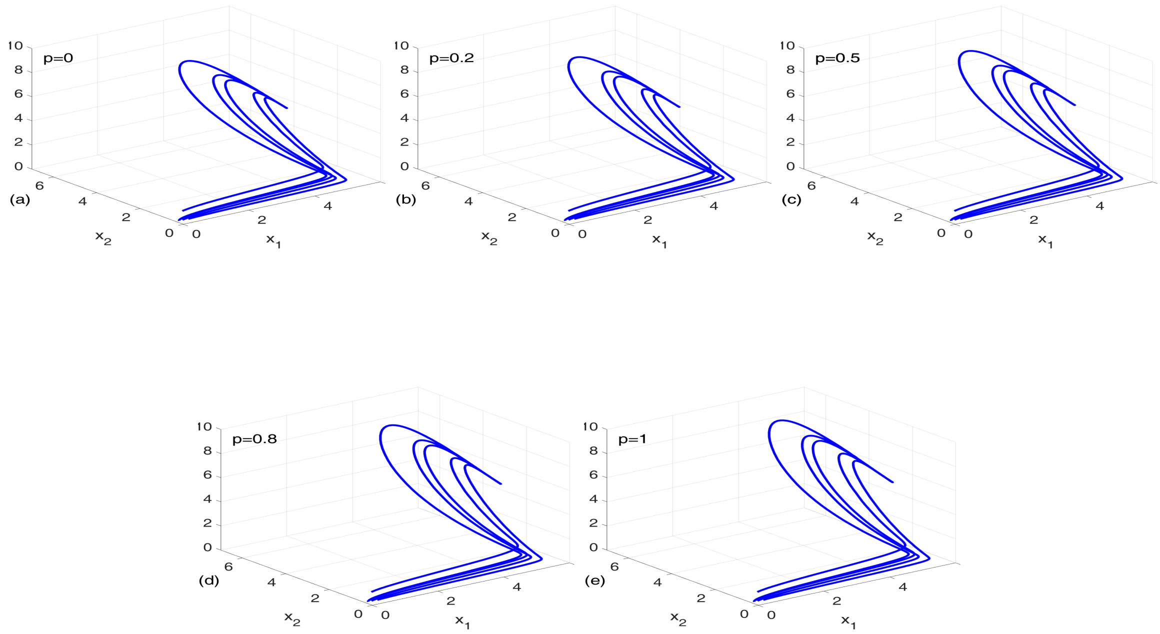

Figure 2. (a)–(e) Phase trajectories of preys x_1, x_2 and predator y with different initial values and $q_1 = 0.5$, $q_2 = 0.2$, $E_1 = 7$, $E_2 = 10$ for $p = 0$, $p = 0.2$, $p = 0.5$, $p = 0.8$ and $p = 1$, respectively.

increases, prey x_1 and predator y increase and prey x_2 decreases.

The phase trajectories of preys x_1, x_2 and predator y corresponding to interior equilibrium P_8 with different p are shown in Figure 2, respectively. Meanwhile, the interior equilibrium P_8 is also stable under different initial conditions.

Example 2. Assign the value of parameters in the model (2.2) as follows: $[r_{1l}, r_{1u}] = [4.99, 5.01]$, $[r_{2l}, r_{2u}] = [7.99, 8.01]$, $[\alpha_{1l}, \alpha_{1u}] = [0.29, 0.31]$, $[\alpha_{2l}, \alpha_{2u}] = [0.29, 0.31]$, $[c_{1l}, c_{1u}] = [0.09, 0.11]$, $[c_{2l}, c_{2u}] = [0.19, 0.21]$, $[e_{1l}, e_{1u}] = [0.19, 0.21]$, $[e_{2l}, e_{2u}] = [0.19, 0.21]$, $[\gamma_{1l}, \gamma_{1u}] = [0.09, 0.11]$, $[\gamma_{2l}, \gamma_{2u}] = [0.19, 0.21]$, $[d_l, d_u] = [0.29, 0.31]$, $[s_l, s_u] = [0.19, 0.21]$, $K_1 = 300$, $K_2 = 100$, $m_1 = 0.30$, $m_2 = 0.10$, $p_1 = 15$, $p_2 = 20$, $C_1 = 30$, $C_2 = 25$, $q_1 = 0.8$, $q_2 = 0.5$.

Table 3. Nontrivial bionomic equilibrium for different p.

p	Nontrivial bionomic equilibrium $(x_{1\infty}, x_{2\infty}, y_\infty, E_{1\infty}, E_{2\infty})$
0	(2.50,2.50,1.07,4.25,13.62)
0.2	(2.50,2.50,1.14,4.30,13.64)
0.5	(2.50,2.50,1.25,4.37,13.65)
0.8	(2.50,2.50,1.37,4.44,13.67)
1	(2.50,2.50,1.45,4.49,13.68)

We present the nontrivial bionomic equilibrium for different p in Table 3. With increasing p, $x_{1\infty}$ and $x_{2\infty}$ are invariable while y_∞, $E_{1\infty}$ and $E_{2\infty}$ increases.

Example 3. Consider the following parameter values: $[r_{1l}, r_{1u}] = [3.99, 4.01]$, $[r_{2l}, r_{2u}] = [4.99, 5.01]$, $[\alpha_{1l}, \alpha_{1u}] = [0.029, 0.031]$, $[\alpha_{2l}, \alpha_{2u}] = [0.039, 0.041]$, $[c_{1l}, c_{1u}] = [0.19, 0.21]$, $[c_{2l}, c_{2u}] = [0.39, 0.41]$, $[e_{1l}, e_{1u}] = [0.09, 0.11]$, $[e_{2l}, e_{2u}] = [0.19, 0.21]$, $[\gamma_{1l}, \gamma_{1u}] = [0.0009, 0.0011]$, $[d_l, d_u] = [0.39, 0.41]$, $[s_l, s_u] = [0.19, 0.21]$, $[\gamma_{2l}, \gamma_{2u}] = [0.0039, 0.0041]$, $K_1 = 50$, $K_2 = 45$, $m_1 = 0.4$, $m_2 = 0.5$, $p_1 = 5$, $p_2 = 10$, $C_1 = 15$, $C_2 = 20$, $q_1 = 0.4$, $q_2 = 0.5$, $\delta_1 = 0.07$, $\delta_2 = 0.08$, $\delta_3 = 0.09$, $\delta_4 = 0.1$.

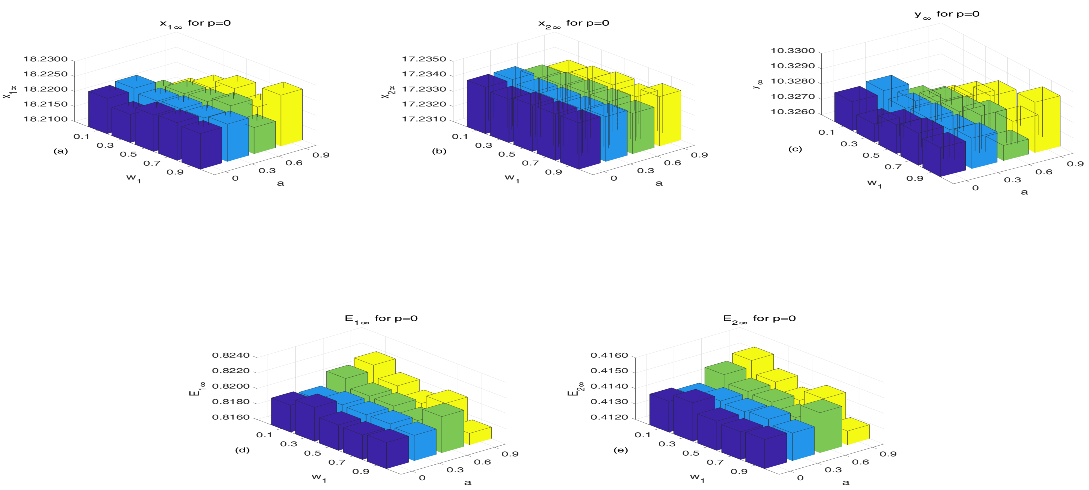

Figure 3. (a)–(e) Three-dimensional histogram of the optimal equilibrium and optimal harvesting efforts for fixed $p = 0$.

Table 4. Optimal equilibrium and optimal harvesting effort for $p = 0$.

w_1	w_2	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$
0.1	0.9	(18.2218,17.2341,10.3279)	(0.8195,0.4140)	(18.2226,17.2341,10.3281)	(0.8192,0.4138)	(18.2184,17.2337,10.3269)	(0.8209,0.4147)	(18.2165,17.2335,10.3264)	(0.8216,0.4151)
0.3	0.7	(18.2194,17.2339,10.3272)	(0.8204,0.4145)	(18.2213,17.2339,10.3277)	(0.8197,0.4141)	(18.2212,17.2338,10.3276)	(0.8198,0.4142)	(18.2205,17.2337,10.3274)	(0.8201,0.4143)
0.5	0.5	(18.2222,17.2341,10.3279)	(0.8193,0.4139)	(18.2217,17.2339,10.3278)	(0.8196,0.4140)	(18.2215,17.2339,10.3277)	(0.8197,0.4141)	(18.2233,17.2339,10.3282)	(0.8190,0.4137)
0.7	0.3	(18.2226,17.2341,10.3281)	(0.8192,0.4138)	(18.2216,17.2339,10.3278)	(0.8196,0.4141)	(18.2231,17.2339,10.3282)	(0.8190,0.4137)	(18.2201,17.2337,10.3273)	(0.8202,0.4144)
0.9	0.1	(18.2219,17.2341,10.3279)	(0.8194,0.4139)	(18.2225,17.2340,10.3280)	(0.8193,0.4138)	(18.2188,17.2337,10.3270)	(0.8207,0.4146)	(18.2270,17.2343,10.3293)	(0.8175,0.4129)

Table 5. Optimal equilibrium and optimal harvesting effort for $p = 0.2$.

w_1	w_2	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$
0.1	0.9	(17.6694,16.7450,10.5300)	(1.0892,0.5537)	(17.6714,16.7450,10.5306)	(1.0885,0.5532)	(17.6670,16.7447,10.5293)	(1.0902,0.5542)	(17.6703,16.7449,10.5302)	(1.0889,0.5535)
0.3	0.7	(17.6715,16.7451,10.5306)	(1.0884,0.5532)	(17.6714,16.7450,10.5306)	(1.0885,0.5533)	(17.6729,16.7450,10.5310)	(1.0879,0.5530)	(17.6724,16.7449,10.5308)	(1.0881,0.5531)
0.5	0.5	(17.6702,16.7450,10.5302)	(1.0889,0.5535)	(17.6696,16.7448,10.5300)	(1.0892,0.5537)	(17.6708,16.7449,10.5304)	(1.0887,0.5534)	(17.6655,16.7445,10.5288)	(1.0908,0.5546)
0.7	0.3	(17.6717,16.7451,10.5307)	(1.0883,0.5532)	(17.6712,16.7450,10.5305)	(1.0885,0.5533)	(17.6708,16.7449,10.5304)	(1.0887,0.5534)	(17.6708,16.7448,10.5304)	(1.0887,0.5534)
0.9	0.1	(17.6736,16.7453,10.5313)	(1.0876,0.5527)	(17.6719,16.7451,10.5308)	(1.0883,0.5531)	(17.6725,16.7451,10.5309)	(1.0880,0.5530)	(17.6741,16.7451,10.5314)	(1.0874,0.5526)

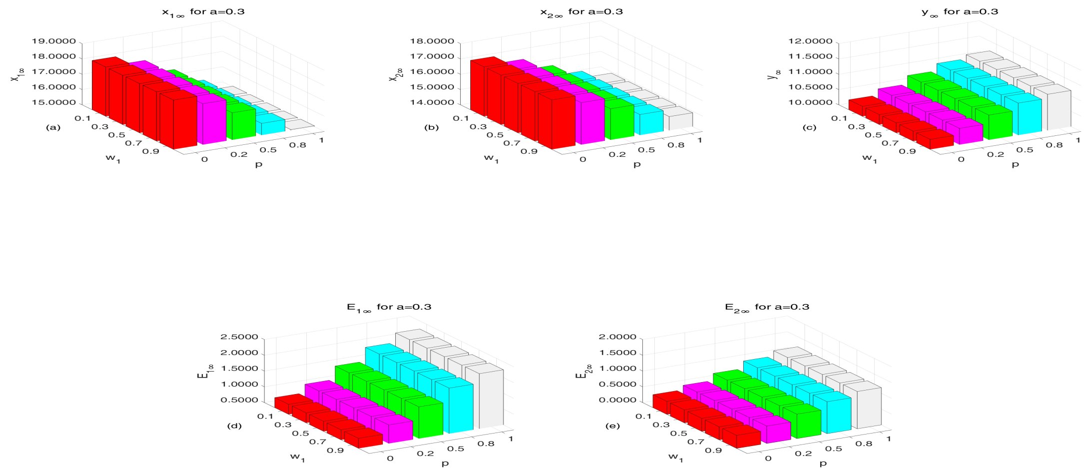

Figure 4. (a)–(e) Three-dimensional histogram of the optimal equilibrium and optimal harvesting efforts for fixed $\alpha = 0.3$.

Table 6. Optimal equilibrium and optimal harvesting effort for $p = 0.5$.

w_1	w_2	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$
0.1	0.9	(16.7656,16.0318,10.8083)	(1.5075,0.7747)	(16.7637,16.0315,10.8076)	(1.5083,0.7751)	(16.7640,16.0315,10.8077)	(1.5082,0.7750)	(16.7703,16.0318,10.8097)	(1.5057,0.7736)
0.3	0.7	(16.7671,16.0318,10.8087)	(1.5069,0.7744)	(16.7666,16.0316,10.8085)	(1.5072,0.7745)	(16.7666,16.0315,10.8085)	(1.5072,0.7745)	(16.7669,16.0315,10.8086)	(1.5071,0.7744)
0.5	0.5	(16.7662,16.0317,10.8084)	(1.5073,0.7746)	(16.7661,16.0316,10.8083)	(1.5074,0.7746)	(16.7670,16.0315,10.8086)	(1.5070,0.7744)	(16.7659,16.0314,10.8083)	(1.5076,0.7747)
0.7	0.3	(16.7666,16.0317,10.8085)	(1.5071,0.7745)	(16.7662,16.0316,10.8084)	(1.5073,0.7746)	(16.7636,16.0313,10.8075)	(1.5084,0.7752)	(16.7681,16.0316,10.8090)	(1.5066,0.7742)
0.9	0.1	(16.7659,16.0318,10.8084)	(1.5074,0.7746)	(16.7687,16.0318,10.8092)	(1.5063,0.7740)	(16.7651,16.0315,10.8081)	(1.5077,0.7748)	(16.7656,16.0315,10.8082)	(1.5076,0.7747)

Consider different combinations of ω_1,ω_2,α and p, the optimal equilibrium and optimal harvesting effort are displayed in Tables 4–8, respectively. From Tables 4–8, when p fixed, the values $x_{1\tilde{\delta}}$, $x_{2\tilde{\delta}}$, $y_{\tilde{\delta}}$ and $E_{1\tilde{\delta}}$, $E_{2\tilde{\delta}}$ fluctuate in a small range with respect to w_1, w_2 and α. Besides, if we fix some variables w_1, w_2 and α, prey $x_{1\tilde{\delta}}$ and $x_{2\tilde{\delta}}$ are decreasing while predator $y_{\tilde{\delta}}$ and optimal harvesting efforts of two preys $E_{1\tilde{\delta}}$, $E_{2\tilde{\delta}}$ are increasing with the increase of p. To better support our results, we select part of the data in Tables 4–8 to draw Figures 3 and 4. In Figure 3, p is fixed at zero while ω_1 and α are wandering from 0 to 1, respectively. The values of $x_{1\tilde{\delta}}$, $x_{2\tilde{\delta}}$, $y_{\tilde{\delta}}$ and $E_{1\tilde{\delta}}$, $E_{2\tilde{\delta}}$ oscillate on a small scale. Considering $\alpha = 0.3$ in Figure 4, $x_{1\tilde{\delta}}$, $x_{2\tilde{\delta}}$, $y_{\tilde{\delta}}$ and $E_{1\tilde{\delta}}$, $E_{2\tilde{\delta}}$ fluctuate in a small range when we fix p and adjust ω_1; $x_{1\tilde{\delta}}$ and $x_{2\tilde{\delta}}$ are decreasing, but $y_{\tilde{\delta}}$, $E_{1\tilde{\delta}}$, $E_{2\tilde{\delta}}$ are increasing with the development of p and ω_1 fixed.

Table 7. Optimal equilibrium and optimal harvesting effort for $p = 0.8$.

w_1	w_2	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$
0.1	0.9	(15.7668,15.3436,11.0469)	(1.9447,1.0127)	(15.7635,15.3432,11.0458)	(1.9460,1.0135)	(15.7657,15.3433,11.0465)	(1.9452,1.0130)	(15.7681,15.3434,11.0474)	(1.9443,1.0124)
0.3	0.7	(15.7662,15.3434,11.0467)	(1.9450,1.0129)	(15.7647,15.3432,11.0461)	(1.9456,1.0132)	(15.7640,15.3430,11.0458)	(1.9459,1.0134)	(15.7689,15.3433,11.0476)	(1.9440,1.0123)
0.5	0.5	(15.7661,15.3434,11.0466)	(1.9450,1.0129)	(15.7659,15.3433,11.0466)	(1.9451,1.0129)	(15.7653,15.3431,11.0463)	(1.9454,1.0131)	(15.7623,15.3429,11.0452)	(1.9465,1.0138)
0.7	0.3	(15.7679,15.3436,11.0473)	(1.9443,1.0125)	(15.7654,15.3433,11.0464)	(1.9453,1.0131)	(15.7657,15.3432,11.0465)	(1.9452,1.0130)	(15.7645,15.3430,11.0460)	(1.9457,1.0133)
0.9	0.1	(15.7686,15.3437,11.0476)	(1.9440,1.0123)	(15.7649,15.3433,11.0463)	(1.9455,1.0131)	(15.7690,15.3435,11.0477)	(1.9439,1.0122)	(15.7625,15.3430,11.0454)	(1.9464,1.0137)

Table 8. Optimal equilibrium and optimal harvesting effort for $p = 1$.

w_1	w_2	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$
0.1	0.9	(15.0484,14.8987,11.1786)	(2.2472,1.1825)	(15.0471,14.8984,11.1781)	(2.2477,1.1828)	(15.0500,14.8985,11.1791)	(2.2467,1.1821)	(15.0440,14.8980,11.1768)	(2.2490,1.1836)
0.3	0.7	(15.0490,14.8986,11.1787)	(2.2470,1.1824)	(15.0483,14.8984,11.1784)	(2.2473,1.1826)	(15.0476,14.8982,11.1781)	(2.2476,1.1828)	(15.0494,14.8983,11.1788)	(2.2470,1.1823)
0.5	0.5	(15.0477,14.8985,11.1783)	(2.2475,1.1827)	(15.0474,14.8983,11.1781)	(2.2477,1.1828)	(15.0497,14.8984,11.1789)	(2.2468,1.1823)	(15.0481,14.8982,11.1783)	(2.2475,1.1826)
0.7	0.3	(15.0484,14.8986,11.1785)	(2.2472,1.1825)	(15.0488,14.8984,11.1786)	(2.2472,1.1825)	(15.0488,14.8983,11.1786)	(2.2472,1.1825)	(15.0477,14.8981,11.1781)	(2.2476,1.1828)
0.9	0.1	(15.0459,14.8985,11.1777)	(2.2482,1.1831)	(15.0466,14.8984,11.1779)	(2.2479,1.1829)	(15.0476,14.8983,11.1782)	(2.2476,1.1827)	(15.0443,14.8980,11.1770)	(2.2489,1.1835)

9. Conclusions

Biological parameters, in the ecosystem, may oscillate simultaneously with the environment. Thus some biological parameters such as intrinsic growth rate of prey (r), interspecific competition (α), predation coefficient (c), mortality rate (d), intra-specific competition rate (s) can be regarded as imprecise parameters. Furthermore, refuge may increase the survival rate of prey while toxicity often reduces the amount of population. In view of these points, we have introduced interval-valued function into a prey-predator model with prey refuges and toxicity.

Then we have researched the boundedness and positivity of the model (Theorem 3.1). Also the existence and stability of equilibria have been studied (Theorem 5.1). Table 1 has shown the existence conditions of bionomic equilibria in four cases. The highlight part in the paper is to consider the inflation net discount rate as trapezoidal fuzzy number, and solve the fuzzy optimal harvesting problem.

Numerical simulations are good methods to provide visual conclusions for the dynamic behaviors of the model. Table 2 shows the nine equilibria for different p. Figure 1 reflects local stability of interior equilibrium with different p, and the corresponding phase portraits of interior equilibrium of the model are also presented in Figure 2. Table 3 clearly displays the bioeconomic equilibria in four cases. Last but not least, Tables 4–8 and Figures 3 and 4 present the optimal equilibrium and optimal harvesting effort on the different combinations of ω_1, ω_2, α and p.

In this paper, we just consider the impact of present time depicted by ordinary differential equation, while ignoring the impact of past time, the subsequent work will introduce time delay into our model to establish delay differential equation. This paper employs α-cut of trapezoidal fuzzy number to describe the inflation net discount rate. But other forms of fuzzy numbers, for instance, triangular fuzzy number and normal fuzzy number, are also significant in fuzzy set theory. Future work will focus on applying these kinds of fuzzy numbers to describe imprecise parameters.

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Conflict of interest

The authors declare that there is no conflict of interest.

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Appendix

A. Basic concept of interval number

Definition A.1. *[31] Interval number: An interval number A is expressed as closed interval $[a^n, a^m]$ and defined by $A = [a^n, a^m] = \{x | a^n < x < a^m, x \in R\}$, where R is the set of real numbers and a^n, a^m denote the left and right limits of the interval number, respectively. Also, every real number can be represented by the interval number $[a, a]$, for all $a \in R$.*

Definition A.2. *[31] Interval-valued function: Let $a > 0$, $b > 0$ and consider the interval $[a, b]$. From a mathematical point of view, any real number can be represented on a line. Similarly, we can represent an interval by a function. If the interval is of the form $[a, b]$, the interval-valued function is taken as $h(p) = a^{1-p} b^p$ for $p \in [0, 1]$.*

For any two interval numbers $A = [a^n, a^m]$ and $B = [b^n, b^m]$, we define arithmetic operations on interval-valued functions as follows:

Addition: $A + B = [a^n, a^m] + [b^n, b^m] = [a^n + b^n, a^m + b^m]$ if $a^n + b^n > 0$. The interval-valued function for the interval $A + B$ is considered as $h(p) = (a^N)^{1-p}(a^M)^p$ where $a^N = a^n + b^n$ and $a^M = a^m + b^m$.

Subtraction: $A - B = [a^n, a^m] - [b^n, b^m] = [a^n - b^m, a^m - b^n]$ if $a^n - b^m > 0$. The interval-valued function for the interval $A - B$ is taken for $h(p) = (b^N)^{1-p}(b^M)^p$ where $b^N = a^n - b^m$ and $b^M = a^m - b^n$.

Scalar multiplication:

$$\alpha A = \alpha[a^n, a^m] = \begin{cases} [\alpha a^n, \alpha a^m] & if \alpha \geq 0 \\ [\alpha a^m, \alpha a^n] & if \alpha < 0 \end{cases} \; if \;\; a^n > 0.$$

The interval-valued function interval αA is known as:

$$h(p) = (v^N)^{1-p}(v^M)^p \;\; if \alpha \geq 0 \;\; and \;\; h(p) = -(w^M)^{1-p}(w^N)^p \;\; if \alpha < 0,$$

where $v^N = \alpha a^n, v^M = \alpha a^m, w^M = |\alpha| a^m$ and $w^N = |\alpha| a^n$.

B. Basic concept of fuzzy set

Definition B.1. *[39] Fuzzy set: A fuzzy set $\widetilde{A}$ in a universe of discourse X is defined as the following set of pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}$. The mapping $\mu_{\widetilde{A}} : X \to [0, 1]$ is called the membership function of the fuzzy set $\widetilde{A}$ and $\mu_{\widetilde{A}}$ is called the membership value or degree of membership of $x \in X$ in the fuzzy set $\widetilde{A}$.*

Definition B.2. *[40] α-cut of fuzzy set: The α-cut of a fuzzy set $\widetilde{A}$ is a crisp set which is defined by $A_\alpha = \{x : \mu_{\widetilde{A}}(x) \geq \alpha\}, \alpha \in (0, 1]$. For $\alpha = 0$ the support of $\widetilde{A}$ is defined as $A_0 = Supp(\widetilde{A}) = \overline{\{x \in R, \mu_{\widetilde{A}}(x) > 0\}}$.*

Definition B.3. *[39] Convex fuzzy set: A convex fuzzy set $\tilde{A}$ is a fuzzy set on a continuous universe satisfying that A_α is a convex classical set for all α.*

Definition B.4. *[41] Fuzzy number: A fuzzy number is a convex fuzzy set with X = R.*

Definition B.5. *[42] Trapezoidal fuzzy number: A fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is defined as a trapezoidal fuzzy number if its membership function satisfy*

$$\mu_{\widetilde{A}} = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 < x < a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & x > a_4 \end{cases}$$

The pictorial form of trapezoidal fuzzy number is presented by the Figure B1 given below

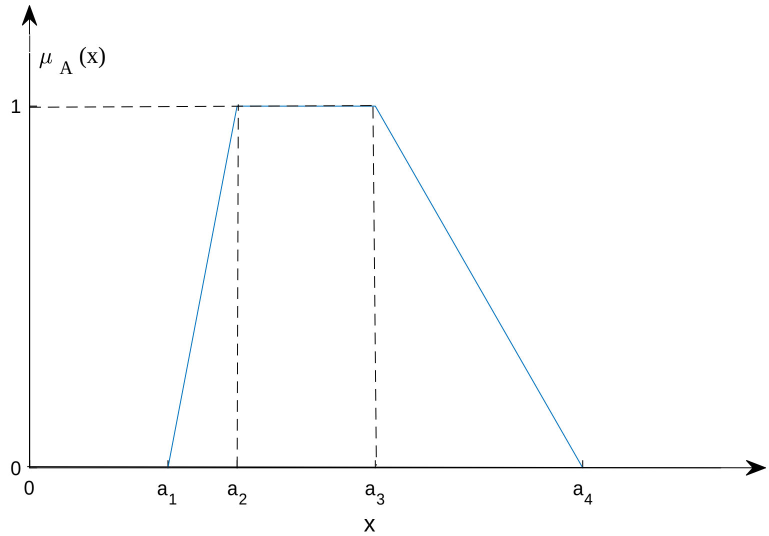

Figure B1. Trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$.

As per definition of trapezoidal fuzzy number the α-cut is a bounded closed internal $[A_L(\alpha), A_R(\alpha)]$, where $A_L(\alpha) = \inf\{x : \mu_{\widetilde{A}}(x) \geq \alpha\} = a_1 + \alpha(a_2 - a_1)$ and $A_R(\alpha) = \sup\{x : \mu_{\widetilde{A}}(x) \geq \alpha\} = a_4 - \alpha(a_4 - a_3)$.

C. Weighted sum method

Utility functions $Y_i(J_i)$, in weighted sum method [43], are defined for each objective according to the significance of J_i relative to the other objective functions. Then define a total or overall utility function Y as listed below:

$$p = \sum_{i=L,R} Y_i(J_i(x)). \tag{C.1}$$

The solution vector x^* is obtained through maximizing the total utility $Y(x)$ subject to constraint conditions.

Take a proper form of the equation (C.1) for maximization formulation as follows:

$$Y(x) = \sum_{i=L,R} \omega_i J_i(x), \ \text{subject to} \sum_{i=L,R} \omega_i = 1 \ \text{and} \ 0 < \omega_L, \omega_R < 1. \tag{C.2}$$

Here ω_L and ω_R stand for the weights of the objective functions. And we choose weights and guarantee that their sum is equal to one.