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Research article

Expected Bayesian estimation for exponential model based on simple step stress with Type-I hybrid censored data

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Abstract: The procedure of selecting the values of hyper-parameters for prior distributions in Bayesian estimate has produced many problems and has drawn the attention of many authors, therefore the expected Bayesian (E-Bayesian) estimation method to overcome these problems. These approaches are used based on the step-stress acceleration model under the Exponential Type-I hybrid censored data in this study. The values of the distribution parameters are derived. To compare the E-Bayesian estimates to the other estimates, a comparative study was conducted using the simulation research. Four different loss functions are used to generate the Bayesian and E-Bayesian estimators. In addition, three alternative hyper-parameter distributions were used in E-Bayesian estimation. Finally, a real-world data example is examined for demonstration and comparative purposes.

Keywords: E-Bayesian estimation; Bayesian estimation; simple step stress; Type-I hybrid censoring; exponential distribution

1. Introduction

In statistical inference, the prior distribution and loss function must be chosen carefully. However, the hyper parameters may influence the prior distribution parameters. We frequently employ the hierarchical Bayesian technique in this case. The concept of hierarchical prior distribution was initially proposed by Lindley and Smith [1]. The hierarchical Bayesian technique requires two steps to complete the prior distribution setting, making it more resilient than the Bayesian method. The method for constructing hierarchical prior distribution was developed by Han [2]. Data analysis has recently employed hierarchical Bayesian techniques; for further information, see Ando and Zellner [3], Han [4], Kzlaslan [5], and Han [6]. For testing data from products with exponential distributions and the quadratic loss function, from Han [7], the reliability parameter was estimated using E-Bayes and hierarchical Bayes methods. With the help of simulation studies, he proved that the E-Bayesian estimator is both efficient and simple to use. For estimating the dependability parameter of the geometric distribution based on scaled squared loss function in complete samples, Yin and Liu [8] built the E-Bayesian estimation and hierarchical Bayesian estimation algorithms. In terms of calculation complexity, the E-Bayes technique is more stable and convenient than the hierarchical Bayes method, they concluded. For additional information on related studies of the E-Bayesian estimation approach, see Jaheen and Okasha [9], Cai et al. [10], Okasha [11], and Azimi et al. [12]. Because of the rapid development of advanced technology, products and devices are becoming more and more reliable, and product life is increasing. Under normal conditions, obtaining failure information for such highly reliable products is difficult, if not impossible. As a result, accelerated life testing (ALT) is the most common method for obtaining sufficient failure time data in a short period of time. In such test conditions, products are subjected to higher-than-usual levels of stress in order to induce early failures. Failure time data from such accelerated tests are analysed and extrapolated to estimate life characteristics under normal operating conditions. One of the most important types of ALT is the step-stress life testing (SSALT) in which the experimenter can choose one or more stress factors in the experiment, such as temperature, vibration, or humidity that may affect the product's life. A set of identical experimental units, say n, are examined in an appropriate testing experiment under a starting stress level of s_1, and then the stress levels are increased to s_2, s_3,...,s_j at predetermined times, say $\tau_1, \tau_2, ..., \tau_j$ respectively. In SSLT if the experiment is performed depending on two stress levels say s_0, s_1, then this type is reduced to the simple step-stress life testing (SSSLT).

The loss function, is crucial in Bayesian approaches. The squared error loss function is the most often used loss function in Bayesian inference (SELF). This loss function is symmetrical, meaning that overestimation and underestimation are given equal weight. The following is the definition of the square error loss function (SELF):

$$L_{BS}\left(\widehat{\theta}, \theta\right) = \left(\widehat{\theta} - \theta\right)^2, \tag{1.1}$$

where $\widehat{\theta}$ is an estimator of θ. The Bayes estimator of θ SELF denoted by $\widehat{\theta}_{BS}$ can be obtained as

$$\widehat{\theta}_{BS} = E_\theta\left[\theta\right]. \tag{1.2}$$

where $E_\theta\left[\theta\right]$ is the expected value is determined with respect to the posterior distribution. Bayesian estimation is derived by using the Degroot loss function (DLF) which is defined by Degroot [13] as follows:

$$L_{BD}\left(\widehat{\theta}, \theta\right) = \left(\frac{\theta - \widehat{\theta}}{\widehat{\theta}}\right)^2, \tag{1.3}$$

the Bayesian estimator based on DLF is denoted by $\widehat{\theta}_{BD}$ and can be expressed as

$$\widehat{\theta}_{BD} = \frac{E_\theta\left[\theta^2\right]}{E_\theta\left[\theta\right]}. \tag{1.4}$$

The quadratic loss function (QLF) was defined as follows:

$$L_{BQ}\left(\widehat{\theta}, \theta\right) = \left(\theta - \widehat{\theta}\right)^2, \tag{1.5}$$

we can get bayesian estimation under quadratic loss denoted by $\widehat{\theta}_{BQ}$ can be obtained as

$$\widehat{\theta}_{BQ} = \frac{E_{\theta}\left[\theta^{-1}\right]}{E_{\theta}\left[\theta^{-2}\right]}. \tag{1.6}$$

Under the assumption that the minimal loss occurs at $\widehat{\theta} = \theta$, the LINEX loss function (LLF) can be expressed as

$$L_{BL}\left(\widehat{\theta}, \theta\right) = \exp\left[\upsilon\left(\widehat{\theta} - \theta\right)\right] - \upsilon\left(\widehat{\theta} - \theta\right) - 1 \tag{1.7}$$

where $\upsilon \neq 0$. The Bayesian estimator of θ, denoted by $\widehat{\theta}_L$ under LLF, the value $\widehat{\theta}_{BL}$ which minimizes $E_{\theta}\left[L_{BL}\left(\widehat{\theta}, \theta\right)\right]$ is given by,

$$\widehat{\theta}_{BL} = \frac{-1}{\upsilon} \ln\left\{E_{\theta}\left[\exp\left(-\upsilon\theta\right)\right]\right\}, \tag{1.8}$$

where $E_{\theta}\left[\exp\left(-\upsilon\theta\right)\right]$ is finite. The maximum likelihood and Bayesian estimation methods are regarded as the inferential features in these investigations. Studying the E-Bayesian estimators and the accompanying properties in the presence of the SSLT model based on Type-I hybrid censoring, however, has not received much attention. Additionally, we present a set of guidelines that we believe applied statisticians and reliability engineers will find extremely useful when selecting the appropriate estimation method to estimate the unknown parameters of the exponential distribution under the SSLT model. Furthermore, a simulation study and analysis of both simulated and real data sets demonstrate that E-Bayesian estimators outperform alternative estimators based on maximum likelihood and Bayesian approaches, encouraging their application in practical contexts. The resulting estimators are obtained based on four different loss functions. by using SEF, DLF, QLF and LLF. This article is organized as follows: Section 2 provides an overview of the step-stress acceleration model depending on the Type-I hybrid censored data. In Section 3, determines the maximum likelihood estimates (ML) of unknown parameters, In Section 4, Bayesian estimation of unknown parameters under different prior distributions and different loss functions are computed. In Section 5, the formulas of E- Bayesian are discussed. Comparison between Bayes and E-Bayes estimates have been made using simulation study in Section 6. A real data set is analyzed in Section 7. Finally, the paper is concluded in Section 8.

2. Description of the model

In this section, we assume that the data are drawn from a cumulative exposure model, by applying a simple step-stress technique with Type-I HCS using two stress levels s_0 and s_1. The lifespan distributions at s_0 and s_1 are following the exponential distribution with failure rates of 1 and 2, correspondingly. The probability density function (PDF) and cumulative distribution function (CDF) are presented by

$$f_i\left(y; \lambda_j\right) = \lambda_j \exp(-\lambda_j y), y \geq 0, \lambda_j > 0, j = 1, 2 \tag{2.1}$$

and

$$F_j\left(y; \lambda_j\right) = 1 - \exp(-\lambda_j y), y \geq 0, \lambda_j > 0, j = 1, 2 \tag{2.2}$$

respectively. As a result, the cumulative exposure distribution (CED) $G(y)$ is given as

$$G(y)=\begin{cases} G_1(y)=F_1\left(y;\lambda_1\right) & \text{if } 0<y<\tau, \\ G_2(y)=F_2\left(y-\left(1-\frac{\lambda_1}{\lambda_2}\right)\tau;\lambda_2\right) & \text{if } \tau\leq y<\infty, \end{cases} \tag{2.3}$$

where $F_j(.)$ is as given in (2.2). The corresponding PDF is:

$$g(y)=\begin{cases} g_1(y)=f_1\left(y;\lambda_1\right) & \text{if } 0<y<\tau, \\ g_2(y)=\lambda_2\exp\left[-\lambda_2\left(y-\tau\right)-\lambda_1\tau\right] & \text{if } \tau\leq y<\infty, \end{cases} \tag{2.4}$$

Based on the Type-I HCS, we have n units under s_0 stress level. At time τ, the stress level is raised to s_1, and the life-testing test is finished at T^*. Here, $T^*=\min\{Y_{r:n},T\}$, we will observe the instances below:

- $r\leq n$ and $0<\tau<T<\infty$ are Predetermined in ahead of time;
- $Y_{1:n}<...<Y_{n:n}$ display the n units' failure times in order;
- T represents a certain period when the stress level shifts from s_0 to s_1;
- $Y_{r:n}$ indicates the time at which the rth fails;
- T stands for the experiment's maximum time limit;
- d indicates the number of units that fail prior to time T;
- T^* is the random moment at which the life-testing experiment comes to an end;
- D^* stands for the number of components that break before T.

Let m_1 represent the number of units that fail before time τ, m_2 be the number of units that fail after the time τ and before time T^* at stress level s_1, where T^* is termination time of the experiment, it is given by,

$$T^*=\begin{cases} T, & \text{if } T<Y_{r:n}, \\ Y_{r:n}, & \text{if } Y_{r:n}\leq T, \end{cases} \tag{2.5}$$

Using this notation, we will notice one of the following three cases:

1) Case 1: Suppose $Y_{r:n}\leq\tau<T$, we will observe $\{y_{1:n}<...<y_{r:n}\leq\tau<T\}$.
2) Case 2: Suppose $\tau<Y_{r:n}\leq T$, we will observe $\{y_{1:n}<...<y_{m_1:n}\leq\tau<y_{m_1+1:n}<...<y_{r:n}\leq T\}$.
3) Case 3: Suppose $T<Y_{r:n}$, we will observe $\{y_{1:n}<...<y_{m_1:n}\leq\tau<y_{m_1+1:n}<...<y_{m_1+m_2:n}\leq T^*=T\}$.

We can write the likelihood function of λ_1 and λ_2 based on the Type-I hybrid censored sample using (2.3) and (2.4), as follows:

$$L\left(\lambda_1,\lambda_2|\underline{\mathbf{x}}\right)=\begin{cases} \frac{n!}{(n-r)!}\left\{\prod_{j=1}^{r}g_1\left(y_{j:n}\right)\right\}\left\{1-G_1\left(y_{r:n}\right)\right\}^{n-r}, & \text{Case1} \\ \frac{n!}{(n-D^*)!}\left\{\prod_{j=1}^{m_1}g_1\left(y_{j:n}\right)\right\}\left\{\prod_{j=m_1+1}^{D^*}g_2\left(y_{j:n}\right)\right\}\left\{1-G_2\left(T^*\right)\right\}^{n-D^*}, & \text{Cases 1 and 2} \end{cases} \tag{2.6}$$

where is the total number of failures and is given by,

$$D^*=m_1+m_2=\begin{cases} d, & \text{if } T<Y_{r:n}, \\ r, & \text{if } Y_{r:n}\leq T, \end{cases} \tag{2.7}$$

3. Maximum likelihood estimation

We must maximize the likelihood with regard to λ_1 and λ_2 when computing the ML estimates. Using (2.3), (2.4) and (2.6), then the appropriate likelihood function, which is as follows:

$$L(\lambda_1, \lambda_2|\underline{\mathbf{x}}) = \begin{cases} \frac{n!}{(n-r)!}\lambda_1^r \exp\left\{-\lambda_1\left[\sum\limits_{j=1}^{r} y_{j:n} + (n-r)\, y_{j:n}\right]\right\}, & \text{case1}, \\ \frac{n!}{(n-D^*)!}\lambda_1^{m_1}\lambda_2^{m_2}\exp\left\{-\lambda_1 W_1(\underline{\mathbf{x}}) - \lambda_2 W_2(\underline{\mathbf{x}})\right\} & \text{case2, 3}, \end{cases} \tag{3.1}$$

where

$$W_1(\underline{\mathbf{x}}) = \sum_{j=1}^{m_1} y_{j:n} + (n-m_1)\,\tau, \tag{3.2}$$

$$\begin{aligned} W_2(\underline{\mathbf{x}}) &= \sum_{j=m_1+1}^{D^*}\left(y_{j:n}-\tau\right) + (n-D^*)\,(T^*-\tau)\,. \\ &= \begin{cases} \sum\limits_{j=m_1+1}^{d}\left(y_{j:n}-\tau\right) + (n-d)\,(T-\tau)\,, & \text{if } T < Y_{r:n}, \\ \sum\limits_{j=m_1+1}^{r}\left(y_{j:n}-\tau\right) + (n-r)\,(y_{r:n}-\tau) & \text{if } \tau < Y_{r:n} \le T \end{cases} \end{aligned} \tag{3.3}$$

From Eq (3.1), we can deduce the following.

1) In Case 3, when $m_1 = 0$ and $m_2 = 0$, the MLEs of λ_1 and λ_2 do not exist.

2) In Cases 1 and 3, when $m_1 \neq 0$, $m_2 = 0$, the MLE of λ_2 does not exist, and $W_1(\underline{\mathbf{x}})$ is a complete sufficient statistic for λ_1.

3) If $m_1 = 0$, $m_2 \neq 0$ in Cases 2 and 3, the MLE of λ_1 does not exist, and $W_2(\underline{\mathbf{x}})$ is a complete sufficient statistic for λ_2.

4) If at least one failure happens before τ and between τ and T in Cases 2 and 3, the MLEs of λ_1 and λ_2 do exist, and $(W_1(\underline{\mathbf{x}}), W_2(\underline{\mathbf{x}}))$ is a joint complete sufficient statistic for (λ_1, λ_2). In this situation, the log-likelihood function of λ_1 and λ_2 is given by,

$$\log(L(\lambda_1, \lambda_2|\underline{\mathbf{x}})) = \log\frac{n!}{(n-D^*)!} + m_1\log(\lambda_1) + m_2\log(\lambda_2) - \lambda_1 W_1(\underline{\mathbf{x}}) - \lambda_2 W_2(\underline{\mathbf{x}}) \tag{3.4}$$

From (3.4), the MLEs of λ_1 and λ_2 are easily determined as

$$\widehat{\lambda}_{1ML} = \frac{m_1}{W_1(\underline{\mathbf{x}})}, \tag{3.5}$$

$$\widehat{\lambda}_{2ML} = \frac{m_2}{W_2(\underline{\mathbf{x}})}. \tag{3.6}$$

4. Bayesian estimation

In this Section, the Bayes estimators for the parameters λ_1 and λ_2 using SEF, DLF, QLF and LLF are derived. For creating the Bayesian estimation, we suppose that the parameters λ_1 and λ_2 are independently distributed and following gamma distribution. Let λ_1, λ_2, have gamma priors with scale parameters b_i and shape parameters a_i, $i = 1,2$. The joint prior density of λ_1 and λ_2 can be expressed as follows

$$\pi(\lambda_1, \lambda_2) \propto \prod_{i=1}^{2} \lambda_i^{a_i-1} \exp(-b_i\lambda_i), b_i, a_i > 0, \text{ for } i = 1,2. \tag{4.1}$$

The posterior PDF of λ_1 and λ_2 is given from (2.6), (4.1), as follows:

$$\pi^*(\lambda_1, \lambda_2|\underline{\mathbf{x}}) = I^{-1} \prod_{i=1}^{2} \lambda_i^{m_i+a_i-1} \exp\{-\lambda_i [W_i(\underline{\mathbf{x}}) + b_i]\}, \text{ for } i = 1,2, \tag{4.2}$$

where I is the normalizing constant given as

$$\begin{aligned} I &= \int_0^\infty \pi^*(\lambda_1, \lambda_2|\underline{\mathbf{x}})\, d\lambda_1 d\lambda_2 \\ &= \prod_{i=1}^{2} \frac{\Gamma(m_i + a_i)}{[W_i(\underline{\mathbf{x}}) + b_i]^{(m_i+a_i)}} \end{aligned} \tag{4.3}$$

From (4.2), it is worth noting that the posterior density functions of λ_i for $i = 1,2$ are similar to $gamma(n_i + a_i, W_i(\underline{\mathbf{x}}) + b_i)$. Based on the SELF, the Bayes estimators of λ_i with $i = 1,2$ are given by,

$$\begin{aligned} \widehat{\lambda}_{iBS} &= E[\lambda_i] \\ &= I^{-1} \int_0^\infty \int_0^\infty \lambda_i \prod_{i=1}^{2} \lambda_i^{m_i+a_i-1} \exp[-\lambda_i (W_i(\underline{\mathbf{x}}) + b_i)]\, d\lambda_1 d\lambda_2 \\ &= \frac{m_i + a_i}{W_i(\underline{\mathbf{x}}) + b_i}, \text{ for } i = 1,2. \end{aligned} \tag{4.4}$$

The Bayesian estimate of λ_i for $i = 1,2$ under DLF loss function is given by,

$$\begin{aligned} \widehat{\lambda}_{iBD} &= \frac{E\left[\lambda_i^2\right]}{E[\lambda_i]} \\ &= \frac{I^{-1} \int_0^\infty \int_0^\infty \lambda_i^2 \prod_{i=1}^{2} \lambda_i^{m_i+a_i-1} \exp[-\lambda_i (W_i(\underline{\mathbf{x}}) + b_i)]\, d\lambda_1 d\lambda_2}{I^{-1} \int_0^\infty \int_0^\infty \lambda_i \prod_{i=1}^{2} \lambda_i^{m_i+a_i-1} \exp[-\lambda_i (W_i(\underline{\mathbf{x}}) + b_i)]\, d\lambda_1 d\lambda_2} \\ &= \frac{m_i + a_i + 1}{W_i(\underline{\mathbf{x}}) + b_i}, \text{ for } i = 1,2. \end{aligned} \tag{4.5}$$

The Bayesian estimate of λ_i for $i = 1, 2$ under QLF is given by,

$$
\begin{aligned}
\widehat{\lambda}_{iBQ} &= \frac{E\left[\lambda_i^{-1}\right]}{E\left[\lambda_i^{-2}\right]} \\
&= \frac{I^{-1}\int_0^\infty\int_0^\infty \lambda_i^{-1}\prod_{i=1}^{2}\lambda_i^{m_i+a_i-1}\exp\left[-\lambda_i\left(W_i\left(\underline{\mathbf{x}}\right)+b_i\right)\right]d\lambda_1 d\lambda_2}{I^{-1}\int_0^\infty\int_0^\infty \lambda_i^{-2}\prod_{i=1}^{2}\lambda_i^{m_i+a_i-1}\exp\left[-\lambda_i\left(W_i\left(\underline{\mathbf{x}}\right)+b_i\right)\right]d\lambda_1 d\lambda_2} \\
&= \frac{m_i+a_i-2}{W_i\left(\underline{\mathbf{x}}\right)+b_i}, \text{ for } i = 1, 2. \qquad (4.6)
\end{aligned}
$$

The Bayesian estimate of λ_i for $i = 1, 2$ under LLF is given by,

$$
\begin{aligned}
\widehat{\lambda}_{iBL} &= \frac{-1}{\upsilon}\ln\left\{E\left[\exp\left(-\upsilon\lambda_i\right)\right]\right\} \\
&= \frac{-1}{\upsilon}\ln\left\{I^{-1}\int_0^\infty\int_0^\infty \exp\left(-\upsilon\lambda_i\right)\prod_{i=1}^{2}\lambda_i^{m_i+a_i-1}\exp\left[-\lambda_i\left(W_i\left(\underline{\mathbf{x}}\right)+b_i\right)\right]d\lambda_1 d\lambda_2\right\} \qquad (4.7) \\
&= \frac{-1}{\upsilon}\ln\left\{\left[\frac{W_i\left(\underline{\mathbf{x}}\right)+b}{W_i\left(\underline{\mathbf{x}}\right)+b+\upsilon}\right]^{(a_i+m_i)}\right\}, \text{ for } i = 1, 2. \qquad (4.8)
\end{aligned}
$$

5. E-Bayesian estimation method

Here, three different prior distributions of hyper-parameters are investigated in this section to see how they affect the E-Bayesian estimates of λ_i for $i = 1, 2$. We select the hyper-parameters a_i and b_i for $i = 1, 2$ to prove that $\pi(\lambda)$ is a decreasing function of λ_i. The first derivative of $\pi(\lambda_i)$ regarding λ_i for $i = 1, 2$ is as follows:

$$
\frac{\partial\pi(\lambda_i)}{\partial\lambda_i} \propto \lambda_i^{a_i-1}e^{-b_i\lambda_i}\left[(\lambda_i-1)-b_i\lambda_i\right]. \qquad (5.1)
$$

Thus, for $0 < a_i < 1$ and $b_i > 0$, the prior PDF $\pi(\lambda_i)$ is a decreasing function of λ_i for $i = 1, 2$. Suppose that a_i and b_i, $i = 1, 2$ are independent with bivariate PDF given by,

$$
p(a_i, b_i) = p_i(a_i)p_i(b_i), \text{ for } i = 1, 2 \qquad (5.2)
$$

the E-Bayesian (EB) estimates of the parameter λ_i are expectation of the Bayesian estimate of λ_i for $i = 1, 2$ and can be obtained as follows:

$$
\widehat{\lambda}_{iEB} = E\left[\widehat{\lambda}_B|\underline{\mathbf{x}}\right] = \int_A \widehat{\lambda}_B\left(a_i, b_i\right)p\left(a_i, b_i\right)da_i db_i, \qquad (5.3)
$$

According to three various prior PDF of the hyper-parameters a_i and b_i, the E-Bayesian estimates of the parameter λ_i for $i = 1, 2$, can be derived. As a result, prior distributions chosen to show how different prior distributions affect the estimation of the E-Bayesian of λ_i for $i = 1, 2$. We suggest the following prior PDFs

$$\begin{array}{ll} p_1(a_i, b_i) = \frac{1}{c_i}, & 0 < a_i < 1, 0 < b_i < c_i, \\ p_2(a_i, b_i) = \frac{2b_i}{c_i^2}, & 0 < a_i < 1, 0 < b_i < c_i, \\ p_3(a_i, b_i) = \frac{2(c-b_i)}{c_i^2}, & 0 < a_i < 1, 0 < b_i < c_i, \end{array} \tag{5.4}$$

For more details, one can refer to Rabie and Li [14–16], and Rabie [17].

5.1. E-Bayesian estimation using SELF

The E-Bayesian estimate of λ_i for $i = 1, 2$, under the SEL based on $p_1(a_i, b_i)$, $p_2(a_i, b_i)$, and $p_3(a_i, b_i)$ are computed from (4.4), (5.3) and (5.4), respectively, as follows:

$$\begin{aligned} \widehat{\lambda}^1_{i\,EBS} &= \int_0^1 \int_0^{c_i} \frac{1}{c_i} \left[\frac{m_i + a_i}{W_i(\underline{\mathbf{x}}) + b_i} \right] db_i da_i \\ &= \frac{2m_i + 1}{2c_i} \ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right), \text{ for } i = 1, 2, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \widehat{\lambda}^2_{i\,EBS} &= \int_0^1 \int_0^{c_i} \frac{2b_i}{c_i^2} \left[\frac{m_i + a_i}{W_i(\underline{\mathbf{x}}) + b_i} \right] db_i da_i \\ &= \frac{2m_i + 1}{c_i} \left[1 - \frac{W_i(\underline{\mathbf{x}})}{c_i} \ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right) \right], \text{ for } i = 1, 2, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \widehat{\lambda}^3_{i\,EBS} &= \int_0^1 \int_0^{c_i} \frac{2(c - b_i)}{c_i^2} \left[\frac{m_i + a_i}{W_i(\underline{\mathbf{x}}) + b_i} \right] db_i da_i \\ &= \frac{2m_i + 1}{c_i} \left[\left(1 + \frac{W_i(\underline{\mathbf{x}})}{c_i}\right) \ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right) - 1 \right], \text{ for } i = 1, 2. \end{aligned} \tag{5.7}$$

5.2. E-Bayesian estimation using DLF

Based on $p_1(a_i, b_i)$, $p_2(a_i, b_i)$, and $p_3(a_i, b_i)$, under the DLF, the E-Bayesian estimate of λ, can be derived from (4.5), (5.3) and (5.4), respectively as follows:

$$\begin{aligned} \widehat{\lambda}^1_{i\,EBD} &= \int_0^1 \int_0^{c_i} \frac{1}{c_i} \left[\frac{m_i + a_i + 1}{W_i(\underline{\mathbf{x}}) + b_i} \right] db_i da_i \\ &= \frac{2m_i + 3}{2c_i} \ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right), \text{ for } i = 1, 2, \end{aligned} \tag{5.8}$$

$$\widehat{\lambda}^2_{i\,EBD} = \int_0^1 \int_0^{c_i} \frac{2b_i}{c_i^2} \left[\frac{m_i + a_i + 1}{W_i(\underline{\mathbf{x}}) + b_i} \right] db_i da_i$$

$$= \frac{2m_i + 3}{c_i}\left[1 - \frac{W_i(\underline{\mathbf{x}})}{c_i}\ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right)\right], \text{ for } i = 1, 2, \tag{5.9}$$

$$\begin{aligned} \widehat{\lambda}^3_{i\,EBD} &= \int_0^1 \int_0^{c_i} \frac{2(c - b_i)}{c_i^2}\left[\frac{m_i + a_i + 1}{W_i(\underline{\mathbf{x}}) + b_i}\right] db_i da_i \\ &= \frac{2m_i + 3}{c_i}\left[\left(1 + \frac{W_i(\underline{\mathbf{x}})}{c_i}\right)\ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right) - 1\right], \text{ for } i = 1, 2. \end{aligned} \tag{5.10}$$

5.3. E-Bayesian estimation using QLF

The E-Bayesian estimate of λ_i for $i = 1, 2$, under the QLF based on $p_1(a_i, b_i)$, $p_2(a_i, b_i)$, and $p_3(a_i, b_i)$ are computed from (4.6), (5.3) and (5.4) ,respectively, as follows:

$$\begin{aligned} \widehat{\lambda}^1_{i\,EBQ} &= \int_0^1 \int_0^{c_i} \frac{1}{c_i}\left[\frac{m_i + a_i - 2}{W_i(\underline{\mathbf{x}}) + b_i}\right] db_i da_i \\ &= \frac{2m_i - 3}{2c_i}\ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right), \text{ for } i = 1, 2, \end{aligned} \tag{5.11}$$

$$\begin{aligned} \widehat{\lambda}^2_{i\,EBQ} &= \int_0^1 \int_0^{c_i} \frac{2b_i}{c_i^2}\left[\frac{m_i + a_i - 2}{W_i(\underline{\mathbf{x}}) + b_i}\right] db_i da_i \\ &= \frac{2m_i - 3}{c_i}\left[1 - \frac{W_i(\underline{\mathbf{x}})}{c_i}\ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right)\right], \text{ for } i = 1, 2, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \widehat{\lambda}^3_{i\,EBQ} &= \int_0^1 \int_0^{c_i} \frac{2(c - b_i)}{c_i^2}\left[\frac{m_i + a_i - 2}{W_i(\underline{\mathbf{x}}) + b_i}\right] db_i da_i \\ &= \frac{2m_i - 3}{c_i}\left[\left(1 + \frac{W_i(\underline{\mathbf{x}})}{c_i}\right)\ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}})}\right) - 1\right], \text{ for } i = 1, 2. \end{aligned} \tag{5.13}$$

5.4. E-Bayesian estimation using LLF

Also, based on $p_1(a_i, b_i)$, $p_2(a_i, b_i)$, and $p_3(a_i, b_i)$, under the LINEX loss function, the E-Bayesian estimate of λ, can be derived from (4.8), (5.3) and (5.4), respectively as follows:

$$\begin{aligned} \widehat{\lambda}^1_{i\,EBL} &= \frac{(m_i + a_i)}{\upsilon}\int_0^1 \int_0^{c_i} \frac{1}{c_i}\ln\left[\frac{W_i(\underline{\mathbf{x}}) + b_i + \upsilon}{W_i(\underline{\mathbf{x}}) + b_i}\right] db_i da_i \\ &= \frac{2m_i + 1}{2\upsilon c_i}\left\{(c_i + W_i(\underline{\mathbf{x}}))\ln\left(1 + \frac{\upsilon}{W_i(\underline{\mathbf{x}}) + c_i}\right) + \upsilon \right. \\ &\quad \left. \times \ln\left(1 + \frac{c_i}{W_i(\underline{\mathbf{x}}) + \upsilon}\right) - W_i(\underline{\mathbf{x}})\ln\left(1 + \frac{\upsilon}{W_i(\underline{\mathbf{x}})}\right)\right\}, \end{aligned} \tag{5.14}$$

$$\widehat{\lambda^2_{i}}_{EBL} = \frac{(m_i+a_i)}{\upsilon}\int_0^1\int_0^{c_i}\frac{2b_i}{c_i^2}\ln\left[\frac{W_i(\underline{\mathbf{x}})+b_i+\upsilon}{W_i(\underline{\mathbf{x}})+b_i}\right]db_i da_i$$
$$= \frac{2m_i+1}{2\upsilon c_i^2}\left\{\upsilon c_i - \upsilon\left(2W_i(\underline{\mathbf{x}})+\upsilon\right)\ln\left(1+\frac{c_i}{W_i(\underline{\mathbf{x}})+\upsilon}\right)+\left[W_i(\underline{\mathbf{x}})\right]^2\right.$$
$$\left.\times\ln\left(1+\frac{\upsilon}{W_i(\underline{\mathbf{x}})}\right)-\left(c_i^2-\left[W_i(\underline{\mathbf{x}})\right]^2\right)\ln\left(1+\frac{\upsilon}{W_i(\underline{\mathbf{x}})+c_i}\right)\right\} \tag{5.15}$$

$$\widehat{\lambda^3_{i}}_{EBL} = \frac{(m_i+a_i)}{\upsilon}\int_0^1\int_0^{c_i}\frac{2(c-b_i)}{c_i^2}\ln\left[\frac{W_i(\underline{\mathbf{x}})+b_i+\upsilon}{W_i(\underline{\mathbf{x}})+b_i}\right]db_i da_i$$
$$\frac{2m_i+1}{2c_i^2\upsilon}\left\{-c_i\upsilon + W_i(\underline{\mathbf{x}})\left(W_i(\underline{\mathbf{x}})+2c_i\right)\ln\left(1+\frac{\upsilon}{W_i(\underline{\mathbf{x}})}\right)+\upsilon\left(2\left(W_i(\underline{\mathbf{x}})+c_i\right)+\upsilon\right)\right.$$
$$\left.\times\ln\left(1+\frac{c_i}{W_i(\underline{\mathbf{x}})+\upsilon}\right)+\left(c_i+\left[W_i(\underline{\mathbf{x}})\right]^2\right)\ln\left(1+\frac{\upsilon}{W_i(\underline{\mathbf{x}})+c_i}\right)\right\} \tag{5.16}$$

For recent work of Bayesian estimation and loss functions, see for example, Nagy et al. in [18], Nagy and Alrasheedi in [19,20], and [21], and Raheem et al. in [22].

6. Simulation studies

In this section, we provide some simulation results for various choices of (n, m, τ, T), where, $n = 50, 80, 100$ and $m = 30, 64, 80$ with two values of both $\tau = 0.5, 0.8$ and $T = 1.6, 2.5$. The values of a_i and b_i, $i = 1, 2$ are generated from Eq (5.4). Its chosen to be $(a_1, b_1) = (0.6, 0.7)$ for λ_1 with $c_1 = 0.75$, where $(a_2, b_2) = (0.4, 0.8)$ for λ_2 with $c_2 = 0.85$. For a given values of a_i and b_i, $i = 1, 2$, values of λ_1, λ_2 are generated from Gamma(a_i, b_i). By trying and error, the values of parameters have been chosen randomly to be $(\lambda_1, \lambda_2) = (0.85, 0.5)$. In the same way, these values provide short lifetimes and the least mean square error. These values of λ_1, λ_2 are used to generate Type-I hybrid censored sample from Exponential distribution as follows:

$$X = \frac{-1}{\lambda_k}(\ln(1-U)), k = 1, 2, \tag{6.1}$$

where, U is generated from $U(0, 1)$. All estimators are obtained in an explicit form. The maximum likelihood estimates of (λ_1, λ_2) are given from Eqs (3.5) and (3.6), respectively. The Bayesian estimates under SELF, DLF, QLF and LLF are obtained from Eqs (4.4), (4.6), (4.5) and (1.8), respectively. The E-Bayesian estimates based on SELF, DLF, QLF and LLF are obtained from Eqs (5.5–5.7), (5.8–5.10), (5.11–5.13) and (5.14–5.16), respectively. All results are listed in Table 1, for λ_1 and in Table 2, for λ_2. The real data example is performed based on the same procedures and by using the four loss functions and listed the results in Tables 3 and 4.

Figures 1 and 2 were created to demonstrate the differences between the Bayesian and E-Bayesian estimates based on the three prior distributions of the hyperparameters a and b for each loss function in order to examine the pertinent aspects of E-Bayesian estimation. In Figure 1(a)–(d), we compared the

Table 1. The average estimated values (AE) and the mean square error (MSE) for λ_1 when $\lambda_1 = 0.8571$, $\lambda_2 = 0.5$, $a_1 = 0.6$, $b_1 = 0.7$, $c_1 = 0.75$, $a_2 = 0.4$, $b_2 = 0.8$, $c_2 = 0.85$.

Method		$\tau = 0.5, T = 1.6$			$\tau = 0.8, T = 2.5$		
		(n,m) (50,30)	(n,m) (80,64)	(n,m) (100,80)	(n,m) (50,30)	(n,m) (80,64)	(n,m) (100,80)
MLE	AE	0.6624	0.6563	0.6543	0.6374	0.6437	0.6390
	MSE	0.0679	0.0581	0.0548	0.0659	0.0566	0.0567
BSEL	AE	0.6684	0.6603	0.6575	0.6421	0.6466	0.6414
	MSE	0.0638	0.0558	0.0531	0.0631	0.0551	0.0555
BDLF	AE	0.7140	0.6891	0.6806	0.6735	0.6664	0.6573
	MSE	0.0494	0.0455	0.0445	0.0510	0.0473	0.04904
BQLF	AE	0.5772	0.6026	0.6113	0.5793	0.6069	0.6096
	MSE	0.1053	0.0814	0.0734	0.0933	0.0730	0.0700
BLLF	AE	0.6608	0.6555	0.6537	0.6370	0.6434	0.6389
	MSE	0.0660	0.0574	0.0544	0.0650	0.0563	0.0565
EBSEL1	AE	0.6740	0.6637	0.6602	0.6456	0.6488	0.6432
	MSE	0.0627	0.0549	0.0522	0.0620	0.0543	0.0548
EBSEL2	AE	0.6701	0.6612	0.6583	0.6430	0.6472	0.6419
	MSE	0.0638	0.0556	0.0529	0.0629	0.0549	0.0554
EBSEL3	AE	0.6779	0.6661	0.6621	0.6482	0.6505	0.6445
	MSE	0.0617	0.0541	0.0516	0.0611	0.0537	0.0543
EBDLF1	AE	0.7203	0.6928	0.6835	0.6774	0.6688	0.6592
	MSE	0.0486	0.0447	0.0438	0.0500	0.0465	0.0484
EBDLF2	AE	0.7161	0.6902	0.6815	0.6746	0.6671	0.6578
	MSE	0.0494	0.0454	0.0444	0.0508	0.0471	0.0489
EBDLF3	AE	0.7245	0.6953	0.6855	0.6801	0.6705	0.6605
	MSE	0.0478	0.0440	0.0432	0.0493	0.0460	0.0479
EBQLF1	AE	0.5814	0.6055	0.6136	0.5822	0.6089	0.6112
	MSE	0.1038	0.0803	0.0724	0.0921	0.0722	0.0693
EBQLF2	AE	0.5781	0.6033	0.6118	0.5798	0.6074	0.6100
	MSE	0.1053	0.0812	0.0732	0.0932	0.0729	0.0699
EBQLF3	AE	0.5848	0.6077	0.6154	0.5845	0.6104	0.6125
	MSE	0.1023	0.0793	0.0717	0.0909	0.0715	0.0688
EBLLF1	AE	0.6663	0.6589	0.6564	0.6405	0.6456	0.6406
	MSE	0.0648	0.0564	0.0536	0.0639	0.0555	0.0559
EBLLF2	AE	0.6624	0.6565	0.6545	0.6379	0.6440	0.6393
	MSE	0.0659	0.0573	0.0543	0.0648	0.0561	0.0564
EBLLF3	AE	0.6701	0.6612	0.6583	0.6431	0.6472	0.6419
	MSE	0.0637	0.0556	0.0529	0.0629	0.0549	0.0553

Table 2. The average estimated values (AE) and the mean square error (MSE) for λ_2 when $\lambda_1 = 0.8571$, $\lambda_2 = 0.5$, $a_1 = 0.6$, $b_1 = 0.7$, $c_1 = 0.75$, $a_2 = 0.4$, $b_2 = 0.8$, $c_2 = 0.85$.

Method		$\tau = 0.5, T = 1.6$			$\tau = 0.8, T = 2.5$		
		(n,m) (50,30)	(n,m) (80,64)	(n,m) (100,80)	(n,m) (50,30)	(n,m) (80,64)	(n,m) (100,80)
MLE	AE	0.3316	0.3370	0.3376	0.2928	0.2908	0.2852
	MSE	0.0372	0.0320	0.0309	0.0484	0.0472	0.0488
BSEL	AE	0.3355	0.3394	0.3396	0.2968	0.2934	0.2873
	MSE	0.0355	0.0311	0.0302	0.0466	0.0460	0.0479
BDLF	AE	0.3653	0.3581	0.3546	0.3215	0.3090	0.2997
	MSE	0.0268	0.0255	0.0256	0.0373	0.0399	0.0428
BQLF	AE	0.2760	0.3020	0.3095	0.2475	0.2623	0.2626
	MSE	0.0583	0.0443	0.0406	0.0688	0.0598	0.0590
BLLF	AE	0.3330	0.3378	0.3383	0.2949	0.2923	0.2864
	MSE	0.0362	0.0316	0.0305	0.0473	0.0465	0.0482
EBSEL1	AE	0.3424	0.3437	0.3430	0.3021	0.2967	0.2899
	MSE	0.0335	0.0298	0.0291	0.0446	0.0447	0.0468
EBSEL2	AE	0.3409	0.3428	0.3423	0.3010	0.2960	0.2894
	MSE	0.0339	0.0301	0.0293	0.0449	0.0450	0.0470
EBSEL3	AE	0.3438	0.3446	0.3437	0.3032	0.2974	0.2904
	MSE	0.0331	0.0296	0.0289	0.0442	0.0445	0.0466
EBDLF1	AE	0.3725	0.3625	0.3581	0.3270	0.3124	0.3023
	MSE	0.0251	0.0244	0.0247	0.0355	0.0386	0.0418
EBDLF2	AE	0.3708	0.3615	0.3573	0.3258	0.3117	0.3018
	MSE	0.0254	0.0246	0.0249	0.0358	0.0389	0.0420
EBDLF3	AE	0.3741	0.3635	0.3589	0.3282	0.3131	0.3029
	MSE	0.0248	0.0242	0.0245	0.0351	0.0384	0.0416
EBQLF1	AE	0.2822	0.3060	0.3128	0.2523	0.2654	0.2650
	MSE	0.0562	0.0429	0.0394	0.0665	0.0583	0.0578
EBQLF2	AE	0.2809	0.3052	0.3121	0.2514	0.2648	0.2646
	MSE	0.0562	0.0431	0.0396	0.0669	0.0586	0.0580
EBQLF3	AE	0.2834	0.3068	0.3135	0.2532	0.2660	0.2655
	MSE	0.0553	0.0426	0.0392	0.0661	0.0581	0.0576
EBLLF1	AE	0.3398	0.3421	0.3417	0.3002	0.2955	0.2890
	MSE	0.0342	0.0303	0.0295	0.0452	0.0453	0.0472
EBLLF2	AE	0.3383	0.3411	0.3410	0.2992	0.2949	0.2885
	MSE	0.0345	0.0305	0.0297	0.0456	0.0454	0.0474
EBLLF3	AE	0.3412	0.3430	0.3424	0.3013	0.2962	0.2895
	MSE	0.0338	0.0300	0.0293	0.0448	0.0449	0.0470

Bayesian and E-Bayesian estimates for λ_1 in case of $\tau = 0.5$ and $T = 1.6$ under the loss functions SEL, DLF, QLF and LLF respectively. Where Figure 2(a)–(d), we compared the Bayesian and E-Bayesian estimates for λ_2 in case of $\tau = 0.8$ and $T = 2.5$ under the loss functions SEL, DLF, QLF and LLF respectively. In each figure, we have compared the E-Bayesian estimates under the three proposed priors of the hyperparameters a and b. From all these graphs we found that: far all proposed loss function and for j=1,2,

1) $\hat{\lambda}_j^{\,B} < \hat{\lambda}_j^{\,EB2} < \hat{\lambda}_j^{\,EB1} < \hat{\lambda}_j^{\,EB3}$

2) $\lim_{n\to\infty} \hat{\lambda}_j^{\,EB2} = \lim_{n\to\infty} \hat{\lambda}_j^{\,EB1} = \lim_{n\to\infty} \hat{\lambda}_j^{\,EB3}$

These properties have been discussed in different situations by many authors, see for example Nassar et al. [23]

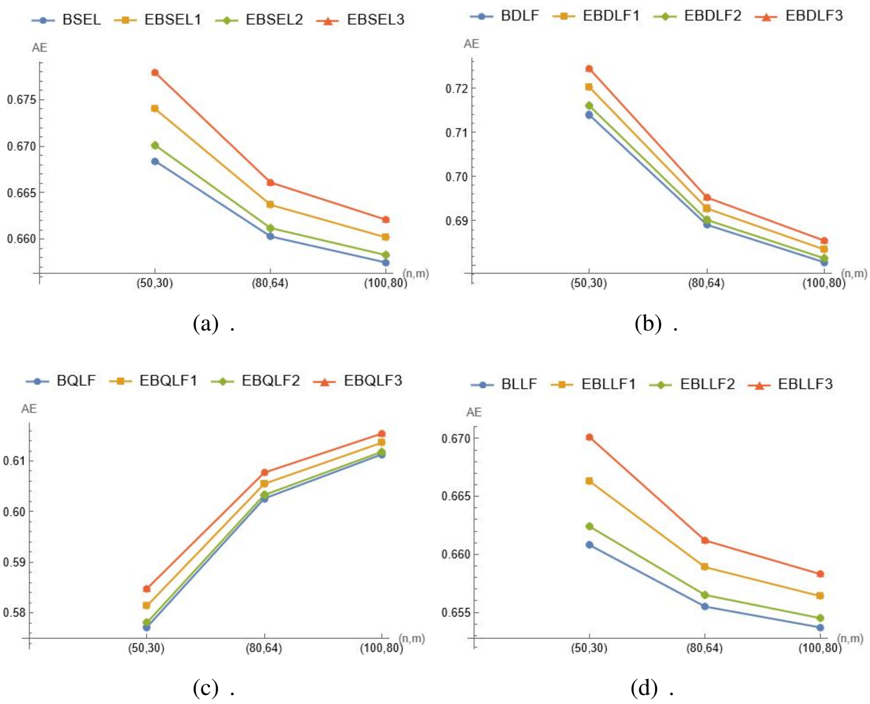

Figure 1. The Bayesian and E-Bayesian behavour for the AE of λ_1 in case of $\tau = 0.5$ and $T = 1.6$.

7. Example of real-life data

In this section, to demonstrate the performance of the offered approaches in the application, we present an example of real-world data. These data were used by Bhaumik et al. [24], representing vinyl chloride data obtained from clean upgradient monitoring wells in mg/l. The exponential distribution has been fitted on these data by Shanker et al. [25], who found that it yields a decent match to the exponential distribution. As shown in the table below, there are 34 observations in this set of data.

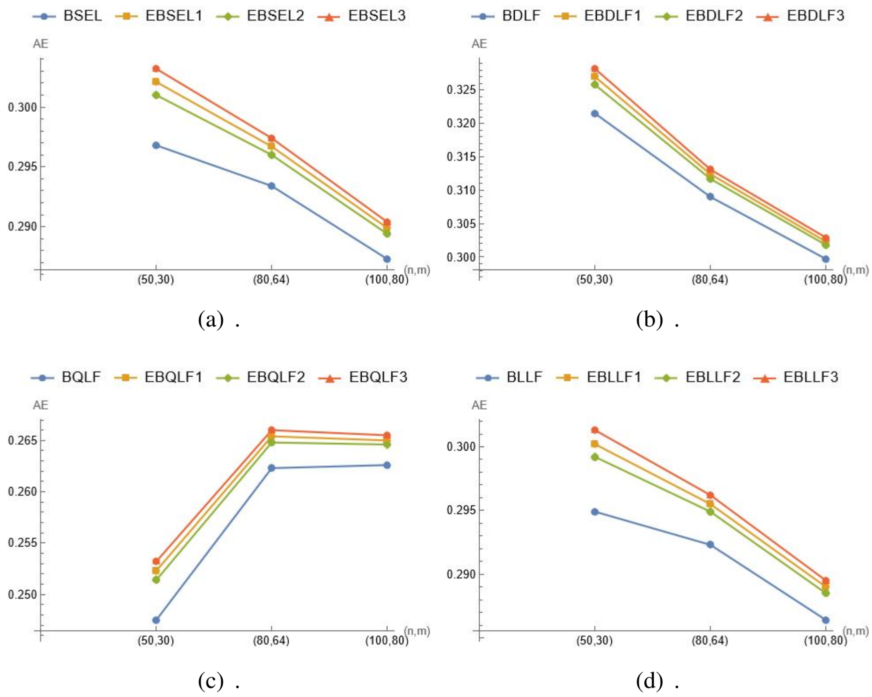

Figure 2. The Bayesian and E-Bayesian behavour for the AE of λ_2 in case of $\tau = 0.8$ and $T = 2.5$.

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8	0.8	0.4	0.6	0.1	1.8	0.9	2	4
0.9	0.4	2	0.5	5.3	3.2	2.7	2.9	2.5	2.3	1	0.2	6.8	1.2	0.4	0.2	0.1

We suppose that values of data set represent lifetime of failure observations which follow the exponential distribution. Using a step-stress approach based on Type-I HCS on these data with the same loss functions as before, we obtain estimates of λ_1 and λ_2 based on the same used techniques and showed in Tables 3 and 4.

8. Conclusions

We looked at the E-Bayesian estimation of the simple step-stress model under the cumulative exposure model for the exponential distribution with Type-I hybrid censored data in this article. The E-Bayesian estimators are derived by considering the loss functions SEL, DLF, QLF, and LINEX. To the hyperparameters, three different distributions are considered. The average estimates (AE) and mean squared error (MSE) for each of the four loss functions are also calculated. Some E-Bayesian estimator properties are illustrated graphically. A simulation study is carried out to demonstrate the effectiveness of the various estimators. According to the simulation results, E-Bayesian estimates outperform Maximum likelihood and Bayesian estimates. To estimate the parameters of the exponential distribution under the simple step-stress model based on Type-I hybrid censored data, we recommend using the E-Bayesian method. In terms of minimum MSE, E-Bayesian estimators using the prior distribution 3 outperform other estimates. The results of the simulation are confirmed by the analysis of the real data

Table 3. Real Data the estimated values for λ_1, $a_1 = 0.6$, $b_1 = 0.7$, $c_1 = 0.75$, $a_2 = 0.4$, $b_2 = 0.8$, $c_2 = 0.85$.

Method	$\tau = 0.5, T = 1.6$	$\tau = 0.8, T = 3$
	(n,m)	(n,m)
	(34,25)	(34,25)
MLE	0.7302	0.6500
BSEL	0.7357	0.6562
BDLF	0.7974	0.6993
BQLF	0.6122	0.5701
BLLF	0.7246	0.6493
EBSEL1	0.7446	0.6612
EBSEL2	0.7387	0.6576
EBSEL3	0.7504	0.6648
EBDLF1	0.8076	0.7049
EBDLF2	0.8012	0.7010
EBDLF3	0.8140	0.7087
EBQLF1	0.6186	0.5739
EBQLF2	0.6137	0.5708
EBQLF3	0.6234	0.5770
EBLLF1	0.7331	0.6541
EBLLF2	0.7274	0.6506
EBLLF3	0.7388	0.6576

Table 4. Real Data: the estimated values for λ_2, $a_1 = 0.6$, $b_1 = 0.7$, $c_1 = 0.75$, $a_2 = 0.4$, $b_2 = 0.8$, $c_2 = 0.85$.

Method	$\tau = 0.5, T = 1.6$	$\tau = 0.8, T = 3$
	(n,m)	(n,m)
	(34,25)	(34,25)
MLE	0.4370	0.4590
BSEL	0.4395	0.4604
BDLF	0.4880	0.5034
BQLF	0.3424	0.3745
BLLF	0.4343	0.4556
EBSEL1	0.4526	0.4724
EBSEL2	0.4495	0.4695
EBSEL3	0.4558	0.4753
EBDLF1	0.5021	0.5161
EBDLF2	0.4986	0.5129
EBDLF3	0.5056	0.5193
EBQLF1	0.3537	0.3850
EBQLF2	0.3513	0.3826
EBQLF3	0.3562	0.3874
EBLLF1	0.4471	0.4673
EBLLF2	0.4441	0.4644
EBLLF3	0.4502	0.4702

set. At the end, we can suggest "the proposed methods in a constant-stress partially accelerated life test model based on a generalized hybrid censoring scheme" as a future work.

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Conflicts of interest

The authors declare there is no conflict of interest.

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