Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 19(1): 612–623.
DOI: 10.3934/mbe.2022027
Received: 27 August 2021
Accepted: 19 October 2021
Published: 18 November 2021

Research article

Quasilinearization method for an impulsive integro-differential system with delay

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Abstract: In this paper, we obtain solution sequences converging uniformly and quadratically to extremal solutions of an impulsive integro-differential system with delay. The main tools are the method of quasilinearization and the monotone iterative. The results obtained are more general and applicable than previous studies, especially the quadratic convergence of the solution for a class of integro-differential equations, which have been involved little by now.

Keywords: impulsive integro-differential system; delay; quasilinearization; uniform convergence; quadratic convergence

1. Introduction

In this paper, we employ the monotone iterative [1,2] and quasilinearization method [3,4] to discuss the existence, uniform and quadratic convergence of solution sequences for an antiperiodic boundary value problem (BVP) of impulsive integro-differential system with delay [5]:

$$\begin{cases} y'(h) = f(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h))) & h \neq h_k, h \in I \\ \triangle y(h_k) = I_k(y(h_k)) & k = 1, 2, \cdots, m \\ y(0) = -y(T) \\ y(h) = y(0) & h \in [-r, 0], \end{cases} \tag{1.1}$$

where $f \in C(I \times R^4, R), I = [0, T], I^+ = [-r, T], r > 0, h - r \leq \tau(h) \leq h, h_0 = 0 < h_1 < h_2 < \cdots < h_m < T = h_{m+1}, I_k \in C(R, R), \triangle y(h_k) = y(h_k^+) - y(h_k^-)$,

$$[\Gamma y](h) = \int_0^h K(h, s)y(s)ds, \qquad [\delta y](h) = \int_0^T H(h, s)y(s)ds,$$

$K \in C(L, R^+), L = \{(h, s) \in I \times I : h \geq s\}, H \in C(I \times I, R^+), R^+ = [0, \infty)$. We denote $k_0 = \max\{K(h, s) : (h, s) \in L\}, h_0 = \max\{H(h, s) : (h, s) \in I \times I\}, \rho = \max\{h_{u+1} - h_u\}, u = 0, 1, ..., m.$

Impulsive differential equation is a basic mathematical model to describe real world phenomena which suddenly alter states at some moments [6,7]. It is widely used in physics, population dynamics, ecology, industrial robotic, etc [8–10]. Note that, in recent years, there are many authors interest in impulsive integro-differential equations [11–13]. And, the existence and approximate controllability for neutral differential equations with delay have been widely concerned [14–17]. Nisar and Vijayakumar discussed approximate controllability for a class of Sobolev-type Hilfer fractional neutral delay differential equations [18]. The controllability result for a fuzzy delay differential system can refer to [19]. The existence and controllability for fractional integro-differential delay equations of order $1 < r < 2$ have been considered in [20] and [21].

The monotone iterative method is effective to get solution sequences, which uniformly converge to extreme solutions of equations [1]. Moreover, the quasilinearization(QSL) method is often used to get solution sequences, which are square convergent [3, 22–24]. The QSL method, whose iterations are constructed to yield rapid convergence, has been used for solving a series of problems and obtained many excellent results [25–27]. The application of the QSL method in functional differential equations, can see [3, 28, 29]. However, the application of the QSL method in impulsive integro-differential systems with delay has been little discussed.

Similar to previous studies [1,3,5], we introduce some spaces for the following use:

Letting $I^- = I^+\backslash\{h_1, h_2, \cdots, h_m\}$, $PC(I^+, R) = \{y : I^+ \to R; y(h)$ is a continuous function in I^-, $h_k(k = 1, \cdots, m)$ are some jumping points, $y(h_k^+)$ and $y(h_k^-)$ exist at h_k, and $y(h_k^-) = y(h_k)\Big\}$;

$PC'(I^+, R)$=$\{y \in PC(I^+, R); y'$ is continuous in I^-, $y'(h_k^+)$, $y'(h_k^-)$, $y'(0^+)$ and $y'(T^-)$ exist$\}$;

$E_0 = \{y \in PC(I^+, R) : y(h) = y(0), h \in [-r, 0]\}$, then the norm of E_0 is defined as $\|y\|_{E_0} = \sup\limits_{h\in I^+} |y(h)|$;

$E = PC(I^+, R) \bigcap PC'(I^+, R)$. Then $y \in E$ is a solution of system (1.1) if and only if y satisfies system (1.1).

2. Some basic concepts and conclusions

Firstly, we give the definition of upper and lower solutions.

Concept 2.1. 1001[5] A function $\phi_0 \in E \bigcap E_0$ is a lower solution of system (1.1) if and only if

$$\begin{cases} \phi_0'(h) \le f(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h))) \qquad h \neq h_k, h \in I = [0, T] \\ \triangle\phi_0(h_k) \le I_k(\phi_0(h_k)) \qquad k = 1, 2, \cdots, m \\ \phi_0(0) \le -\varphi_0(T) \\ \phi_0(h) \le \phi_0(0) \qquad h \in [-r, 0]. \end{cases}$$

Concept 2.2. 1001[5] A function $\varphi_0 \in E \bigcap E_0$ is an upper solution of system (1.1) if and only if

$$\begin{cases} \varphi_0'(h) \ge f(h, \varphi_0(h), [\Gamma\varphi_0](h), [\delta\varphi_0](h), \varphi_0(\tau(h))) \qquad h \neq h_k, h \in I = [0, T] \\ \triangle\varphi_0(h_k) \ge I_k(\varphi_0(h_k)) \qquad k = 1, 2, \cdots, m \\ \varphi_0(0) \ge -\phi_0(T) \\ \varphi_0(h) \ge \varphi_0(0) \qquad h \in [-r, 0]. \end{cases}$$

Then, we give some lemmas [5] to the following problem:

$$\begin{cases} y'(h) + Ay(h) = \sigma(h) - B_1[\Gamma y](h) - B_2[\delta y](h) - By(\tau(h)), & h \neq h_k \quad h \in I \\ \triangle y(h_k) = -L_k y(h_k) + I_k(\eta(h_k)) + L_k(\eta(h_k)), & k = 1, 2, \cdots, m \\ y(0) = -y(T) \\ y(h) = y(0) \qquad h \in [-r, 0], \end{cases} \tag{2.1}$$

where $A > 0, B_1, B_2, B \geq 0, 0 \leq L_k < 1$ and $\sigma(h) \in PC(I, R), \eta(h) \in PC'(I^+, R)$.

Lemma 2.1. [1001[5]] Let $A > 0, B_1, B_2, B \geq 0$ and $0 \leq L_k < 1$ satisfy the inequality:

$$\frac{(B_1 k_0 T + B_2 h_0 T + B)(e^{AT} - 1)}{A(e^{AT} + 1)} + \frac{e^{AT}}{e^{AT} + 1} \sum_{k=1}^{m} L_k < 1. \tag{2.2}$$

Then $y \in E$ is the unique solution of system (2.1) if $y \in E_0$ satisfies:

$$y(h) = \begin{cases} \int_0^T F(h, s)[\sigma(s) - B_1[\Gamma y](s) - B_2[\delta y](s) - By(\tau(s))]ds \\ + \sum\limits_{k=1}^{m} F(h, h_k)[-L_k y(h_k) + I_k(\eta(h_k)) + L_k \eta(h_k)] & h \in I \\ \\ \int_0^T F(0, s)[\sigma(s) - B_1[\Gamma y](s) - B_2[\delta y](s) - By(\tau(s))]ds \\ + \sum\limits_{k=1}^{m} F(0, h_k)[-L_k y(h_k) + I_k(\eta(h_k)) + L_k \eta(h_k)] & h \in [-r, 0], \end{cases} \tag{2.3}$$

where

$$F(h, s) = \frac{1}{e^{AT} + 1} \begin{cases} e^{A(T-h+s)}, & 0 \leq s \leq h \leq T, \\ -e^{A(s-h)}, & 0 \leq h < s \leq T. \end{cases}$$

Lemma 2.2. [1001[5]] Suppose that $y \in E$ satisfies

$$\begin{cases} y'(h) + Ay(h) + B_1[\Gamma y](h) + B_2[\delta y](h) + By(\tau(h)) \leq 0, & h \neq h_k, \qquad h \in I \\ \triangle y(h_k) \leq -L_k y(h_k) \quad k = 1, 2, \cdots, m \\ y(0) \leq 0 \\ y(h) = y(0), \qquad h \in [-r, 0], \end{cases}$$

where constants $A > 0, B_1, B_2, B \geq 0, 0 \leq L_k < 1$, with

$$\sum_{k=1}^{m} L_k + \rho(m+1)(A + B_1 k_0 T + B_2 h_0 T + B) \leq 1.$$

Then $y(h) \leq 0$ on I^+.

Lemma 2.3. [1001[5]] If the functions ϕ_0, φ_0 are lower and upper solutions of BVP system (1.1), which satisfy $\phi_0(h) \leq \varphi_0(h)$ in I^+, and both the f and I_k satisfy one-sided Lipschitz condition, then we can find sequences $\phi_n, \varphi_n \subset [\phi_0, \varphi_0]$ that uniformly converge to minimal and maximal solutions of the BVP system (1.1).

3. A main theorem

Theorem 3.1. Suppose that the following assumptions are true:
(S_1) Functions $\phi_0(h), \varphi_0(h)$ are lower and upper solutions of the BVP system (1.1), which satisfy $\phi_0(h) \leq \varphi_0(h)$ in I^+;
(S_2) f satisfies $f_y(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h))) < 0, f_{\Gamma y}(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h))) \leq 0$, $f_{\delta y}(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h))) \leq 0$, and $f_{y\tau}(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h))) \leq 0$. And, the quadratic form $K(f(h, y, k, z, p))$ is

$$\begin{aligned}K(f) &= (y-v)^2 f_{yy}(h, y_1, y_2, y_3, y_4) + (k-u)^2 f_{\Gamma y \Gamma y}(h, y_1, y_2, y_3, y_4) + (z-w)^2 f_{\delta y \delta y}(h, y_1, y_2, y_3, y_4)\\ &+ (p-q)^2 f_{y\tau y\tau}(h, y_1, y_2, y_3, y_4) + 2(y-v)(k-u) f_{y\Gamma y}(h, y_1, y_2, y_3, y_4)\\ &+ 2(y-v)(z-w) f_{y\delta y}(h, y_1, y_2, y_3, y_4) + 2(y-v)(p-q) f_{yy\tau}(h, y_1, y_2, y_3, y_4)\\ &+ 2(k-u)(z-w) f_{\Gamma y\delta y}(h, y_1, y_2, y_3, y_4) + 2(k-u)(p-q) f_{\Gamma yy\tau}(h, y_1, y_2, y_3, y_4)\\ &+ 2(z-w)(p-q) f_{\delta yy\tau}(h, y_1, y_2, y_3, y_4),\end{aligned}$$

and $K(f) \leq 0$ on $I \times R^4$, where $\phi_0 \leq v \leq y_1 \leq y \leq \varphi_0, \phi_0 \leq u \leq y_2 \leq k \leq \varphi_0, \phi_0 \leq w \leq y_3 \leq z \leq \varphi_0, \phi_0 \leq q \leq y_4 \leq p \leq \varphi_0, \quad h \neq h_k, h \in I$;
(S_3) The functions I_k satisfy $-1 \leq I_k'(.) \leq 0$ and $I_k''(.) \geq 0$, $k = 1, 2, \cdots, m$.

Then we can find monotone solution sequences $\{\phi_n(h)\}$ and $\{\varphi_n(h)\}$, which quadratically and uniformly converge to extremal solutions of system (1.1) on $[\phi_0, \varphi_0]$.

Proof: By using (S_2) and the Taylor's formula, we can obtain

$$f(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h))) \leq Q(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h)); v(h)),$$

where

$$\begin{aligned}&Q\ (h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h)); v(h)) = f(h, v(h), [\Gamma v](h), [\delta v](h), v(\tau(h)))\\ &+\ f_y(h, v(h), [\Gamma v](h), [\delta v](h), v(\tau(h)))(y(h) - v(h)) + f_{\Gamma y}(h, v(h), [\Gamma v](h), [\delta v](h), v(\tau(h)))([\Gamma y](h) - [\Gamma v](h))\\ &+\ f_{\delta y}(h, v(h), [\Gamma v](h), [\delta v](h), v(\tau(h)))([\delta y](h) - [\delta v](h))\\ &+\ f_{y\tau}(h, v(h), [\Gamma v](h), [\delta v](h), v(\tau(h)))(y(\tau(h)) - v(\tau(h))),\end{aligned}$$

and employing the Taylor's formula together with (S_3), we obtain

$$\Delta a(h_k) \geq I_k(b(h_k)) + I_k'(b(h_k))(a(h_k) - b(h_k)),$$

where $\phi_0(h_k) \leq b(h_k) \leq a(h_k) \leq \varphi_0(h_k)$.

Now, solution sequences $\phi_i(h)$ and $\varphi_i(h)$ are constructed to satisfy:

$$\begin{cases} \phi_i'(h) - f_y(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\phi_i(h) \\ -f_{\Gamma y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\Gamma\phi_i](h) \\ -f_{\delta y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\delta\phi_i](h) \\ -f_{y\tau}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\phi_i(\tau(h)) \\ = f(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h))) \\ -f_y(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\phi_{i-1}(h) \\ -f_{\Gamma y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\Gamma\phi_{i-1}](h) \\ -f_{\delta y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\delta\phi_{i-1}](h) \\ -f_{y\tau}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\phi_{i-1}(\tau(h)) \quad h \neq h_k, h \in I \\ \Delta\phi_i(h_k) = I_k(\phi_{i-1}(h_k)) + I_k'(\phi_{i-1}(h_k))(\phi_i(h_k) - \phi_{i-1}(h_k)) \qquad k = 1, 2, \cdots, m \\ \phi_i(0) = -\varphi_{i-1}(T) \\ \phi_i(h) = \phi_i(0) \qquad h \in [-r, 0], \end{cases} \tag{3.1}$$

$$\begin{cases} \varphi_i'(h) - f_y(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\varphi_i(h) \\ -f_{\Gamma y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\Gamma\varphi_i](h) \\ -f_{\delta y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\delta\varphi_i](h) \\ -f_{y\tau}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\varphi_i(\tau(h)) \\ = f(h, \varphi_{i-1}(h), [\Gamma\varphi_{i-1}](h), [\delta\varphi_{i-1}](h), \varphi_{i-1}(\tau(h))) \\ -f_y(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\varphi_{i-1}(h) \\ -f_{\Gamma y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\Gamma\varphi_{i-1}](h) \\ -f_{\delta y}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))[\delta\varphi_{i-1}](h) \\ -f_{y\tau}(h, \phi_{i-1}(h), [\Gamma\phi_{i-1}](h), [\delta\phi_{i-1}](h), \phi_{i-1}(\tau(h)))\varphi_{i-1}(\tau(h)) \quad h \neq h_k, h \in I \\ \Delta\varphi_i(h_k) = I_k(\varphi_{i-1}(h_k)) + I_k'(\phi_{i-1}(h_k))(\varphi_i(h_k) - \varphi_{i-1}(h_k)) \qquad k = 1, 2, \cdots, m \\ \varphi_i(0) = -\phi_{i-1}(T) \\ \varphi_i(h) = \varphi_i(0) \qquad h \in [-r, 0]. \end{cases} \tag{3.2}$$

Obviously, by Lemma 2.1, we know that system (3.1) or (3.2) has an unique solution, respectively. We will finish our proof in four steps:

1. We prove that $\phi_0 \leq \phi_1$ and $\varphi_1 \leq \varphi_0$.

Let $i = 1$, then by system (3.1), we have

$$\begin{cases} \phi_1'(h) - f_y(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))\phi_1(h) - f_{\Gamma y}(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))[\Gamma\phi_1](h) \\ -f_{\delta y}(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))[\delta\phi_1](h) - f_{y\tau}(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))\phi_1(\tau(h)) \\ = f(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h))) - f_y(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))\phi_0(h) \\ -f_{\Gamma y}(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))[\Gamma\phi_0](h) - f_{\delta y}(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))[\delta\phi_0](h) \\ -f_{y\tau}(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))\phi_0(\tau(h)) \quad h \neq h_k, h \in I \\ \Delta\phi_1(h_k) = I_k(\phi_0(h_k)) + I_k'(\phi_0(h_k))(\phi_1(h_k) - \phi_0(h_k)) \\ \phi_1(0) = -\varphi_0(T) \\ \phi_1(h) = \phi_1(0) \qquad h \in [-r, 0]. \end{cases}$$

Setting $\varpi(h) = \phi_0(h) - \phi_1(h)$, we get

$\varpi'(h) - f_y(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))\varpi(h) - f_{\Gamma y}(h, \phi_0(h), [\Gamma\phi_0](h), [\delta\phi_0](h), \phi_0(\tau(h)))[\Gamma\varpi](h)$

$$
\begin{aligned}
&- f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\varpi](h) - f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\varpi(\tau(h))\\
&= \phi_0^{'}(h) - \phi_1^{'}(h) - f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_0(h)\\
&+ f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_1(h) - f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\Gamma\phi_0](h)\\
&+ f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\Gamma\phi_1](h) - f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\phi_0](h)\\
&+ f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\phi_1](h) - f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_0(\tau(h))\\
&+ f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_1(\tau(h)) \le f(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\\
&- f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_1(h) - f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\Gamma\phi_1](h)\\
&- f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\phi_1](h) - f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_1(\tau(h))\\
&- f(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h))) + f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_0(h)\\
&+ f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\Gamma\phi_0](h) + f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\phi_0](h)\\
&+ f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_0(\tau(h)) - f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_0(h)\\
&+ f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_1(h) - f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\Gamma\phi_0](h)\\
&+ f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\Gamma\phi_1](h) - f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\phi_0](h))\\
&+ f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\phi_1](h) - f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_0(\tau(h))\\
&+ f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\phi_1(\tau(h)) = 0.
\end{aligned}
$$

We can easily prove that

$$\Delta\varpi(h_k) \le I_k^{'}(\phi_0(h_k))\varpi(h_k), \quad \varpi(0) \le 0, \quad \varpi(h) = \varpi(0), h \in [-r, 0].$$

So it is clear that $\varpi(h) \le 0$ (from Lemma 2.2), i.e., $\phi_0 \le \phi_1$. Similarly, we can get that $\varphi_1 \le \varphi_0$ for all $h \in I^+$.

2. We show that $\phi_1 \le \varphi_1$ on I^+.

Letting $\varpi(h) = \phi_1 - \varphi_1$ and by $(S_1) - (S_3)$, we have

$$
\begin{aligned}
\varpi'(h) &- f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\varpi(h) - f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\Gamma\varpi](h)\\
&- f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))[\delta\varpi](h) - f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))\varpi(\tau(h))\\
&= f(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h))) - f(h,\varphi_0(h),[\Gamma\varphi_0](h),[\delta\varphi_0](h),\varphi_0(\tau(h)))\\
&- f_y(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))(\phi_0(h) - \varphi_0(h))\\
&- f_{\Gamma y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))([\Gamma\phi_0](h) - [\Gamma\varphi_0](h))\\
&- f_{\delta y}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))([\delta\phi_0](h) - [\delta\varphi_0](h))\\
&- f_{y\tau}(h,\phi_0(h),[\Gamma\phi_0](h),[\delta\phi_0](h),\phi_0(\tau(h)))(\phi_0(\tau(h)) - \varphi_0(\tau(h))) \le 0,\\
\Delta\varpi(h_k) &= I_k(\phi_0(h_k)) - I_k(\varphi_0(h_k)) + I_k^{'}(\phi_0(h_k))\varpi(h_k) - I_k^{'}(\phi_0(h_k))(\phi_0(h_k) - \varphi_0(h_k))\\
&\le I_k^{'}(\phi_0(h_k))\varpi(h_k),\\
&\quad \varpi(0) \le 0, \qquad \varpi(h) = \varpi(0) \qquad h \in [-r, 0].
\end{aligned}
$$

From Lemma 2.2, we get $\varpi(h) \le 0$, i.e., $\phi_1 \le \varphi_1$ for all $h \in I^+$. So we have $\phi_0(h) \le \phi_1(h) \le \varphi_1(h) \le \varphi_0(h)$ in I^+. Then, by mathematical induction, we obtain $\phi_n(h)$ and $\varphi_n(h)$ satisfying

$$\phi_0(h) \le \phi_1(h) \le \cdots \le \phi_n(h) \le \cdots \varphi_n(h) \le \cdots \varphi_1(h) \le \varphi_0(h), \quad h \in I^+,$$

and each $\phi_i(h), \varphi_i(h) \in E \bigcap E_0 (i = 1, 2, \cdots)$ satisfy system (3.1) or (3.2), respectively. We can easily prove that the sequences $\phi_n(h)$ and $\varphi_n(h)$ are uniformly bounded and equi-continuous, then by Ascoli-Arzela criterion [6], they uniformly converge to two solutions of system (1.1):

$$\lim_{n\to\infty}\phi_n(h) = \varsigma(h), \qquad \lim_{n\to\infty}\varphi_n(h) = q(h).$$

3. We verify that $\varsigma(h)$ and q(h) are minimum and maximum solutions of system (1.1) in $[\phi_0, \varphi_0]$, respectively.

Suppose y(h) is an arbitrary solution of system (1.1), which satisfies $\phi_0(h) \leq y(h) \leq \varphi_0(h)$ in I^+. Now, we assume that $\phi_n(h) \leq y(h) \leq \varphi_n(h)$ is hold for a positive integer n, in what follows we prove that $\phi_{n+1}(h) \leq y(h) \leq \varphi_{n+1}(h)$.
Letting $\varpi(h) = \phi_{n+1}(h) - y(h)$, we obtain

$$\begin{aligned}
\varpi'(h) &= \phi'_{n+1}(h) - y'(h)\\
&= f_y(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\phi_{n+1}(h)\\
&+ f_{\Gamma y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\Gamma\phi_{n+1}](h)\\
&+ f_{\delta y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\delta\phi_{n+1}](h)\\
&+ f_{y\tau}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\phi_{n+1}(\tau(h)) + f(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\\
&- f_y(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\phi_n(h) - f_{\Gamma y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\Gamma\phi_n](h)\\
&- f_{\delta y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\delta\phi_n](h)\\
&- f_{y\tau}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\phi_n(\tau(h)) - f(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h)))\\
&= f_y(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\varpi(h) + f_{\Gamma y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\Gamma\varpi](h)\\
&+ f_{\delta y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\delta\varpi](h) + f_{y\tau}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\varpi(\tau(h))\\
&+ f(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h))) - f_y(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))(\phi_n(h) - y(h))\\
&- f_{\Gamma y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))([\Gamma\phi_n](h) - [\Gamma y](h))\\
&- f_{\delta y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))([\delta\phi_n](h) - [\delta y](h))\\
&- f_{y\tau}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))(\phi_n(\tau(h)) - y(\tau(h))) - f(h, y(h), [\Gamma y](h), [\delta y](h), y(\tau(h)))\\
&\leq f_y(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\varpi(h) + f_{\Gamma y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\Gamma\varpi](h)\\
&+ f_{\delta y}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))[\delta\varpi](h)\\
&+ f_{y\tau}(h, \phi_n(h), [\Gamma\phi_n](h), [\delta\phi_n](h), \phi_n(\tau(h)))\varpi(\tau(h)) \qquad h \neq h_k, \quad h \in I,\\
\Delta\varpi(h_k) &= I_k(\phi_n(h_k)) - I_k(y(h_k)) + I'_k(\phi_n(h_k))\varpi(h_k) - I'_k(\phi_n(h_k))(\phi_n(h_k) - y(h_k))\\
&\leq I'_k(\phi_n(h_k))\varpi(h_k),\\
&\quad \varpi(0) \leq 0, \qquad \varpi(h) = \varpi(0) \qquad h \in [-r, 0].
\end{aligned}$$

Clearly, by Lemma 2.2, we have $\varpi(h) \leq 0$, i.e., $\phi_{n+1}(h) \leq y(h)$ on I^+. Similarly, it can be proved that $y(h) \leq \varphi_{n+1}(h)$ on I^+. So $\phi_{n+1}(h) \leq y(h) \leq \varphi_{n+1}(h)$. Then, by taking $n \to \infty$, it is clear that $\varsigma(h) \leq y(h) \leq q(h)$.
4. Finally, we prove that the quadratic convergence of ϕ_n and φ_n.

First of all, letting $\varpi_n(h) = \varsigma(h) - \phi_n(h) \geq 0$, then

$$\varpi_n{}'(h) = \varsigma'(h) - \phi'_n(h)$$

$$
\begin{aligned}
&= f(h,\varsigma(h),[\Gamma\varsigma](h),[\delta\varsigma](h),\varsigma(\tau(h))) - f_y(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\phi_n(h)\\
&- f_{\Gamma y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\Gamma\phi_n](h)\\
&- f_{\delta y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\delta\phi_n](h)\\
&- f_{y\tau}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\phi_n(\tau(h))\\
&- f(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\\
&+ f_y(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\phi_{n-1}(h)\\
&+ f_{\Gamma y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\Gamma\phi_{n-1}](h)\\
&+ f_{\delta y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\delta\phi_{n-1}](h)\\
&+ f_{y\tau}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\phi_{n-1}(\tau(h))\\
&= f_y(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\varpi_n(h)\\
&+ f_{\Gamma y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\Gamma\varpi_n](h)\\
&+ f_{\delta y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\delta\varpi_n](h)\\
&+ f_{y\tau}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\varpi_n(\tau(h))\\
&+ \frac{1}{2}[\varpi_{n-1}^2(h)f_{yy}(h,y_1,y_2,y_3,y_4) + [\Gamma\varpi_{n-1}]^2(h)f_{\Gamma y\Gamma y}(h,y_1,y_2,y_3,y_4)\\
&+ [\delta\varpi_{n-1}]^2(h)f_{\delta y\delta y}(h,y_1,y_2,y_3,y_4) + (\varpi_{n-1}(\tau(h)))^2 f_{y\tau y\tau}(h,y_1,y_2,y_3,y_4)\\
&+ 2\varpi_{n-1}(h)[\Gamma\varpi_{n-1}](h)f_{y\Gamma y}(h,y_1,y_2,y_3,y_4) + 2\varpi_{n-1}(h)[\delta\varpi_{n-1}](h)f_{y\delta y}(h,y_1,y_2,y_3,y_4)\\
&+ 2\varpi_{n-1}(h)(\varpi_{n-1}(\tau(h)))f_{yy\tau}(h,y_1,y_2,y_3,y_4) + 2[\Gamma\varpi_{n-1}](h)[\delta\varpi_{n-1}](h)f_{\Gamma y\delta y}(h,y_1,y_2,y_3,y_4)\\
&+ 2[\Gamma\varpi_{n-1}](h)(\varpi_{n-1}(\tau(h)))f_{\Gamma yy\tau}(h,y_1,y_2,y_3,y_4) + 2[\delta\varpi_{n-1}](h)(\varpi_{n-1}(\tau(h)))f_{\delta yy\tau}(h,y_1,y_2,y_3,y_4)].
\end{aligned}
$$

We write the above expression as follows:

$$
\begin{aligned}
\varpi_n{}'(h) &- f_y(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\varpi_n(h)\\
&= f_{\Gamma y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\Gamma\varpi_n](h)\\
&+ f_{\delta y}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))[\delta\varpi_n](h)\\
&+ f_{y\tau}(h,\phi_{n-1}(h),[\Gamma\phi_{n-1}](h),[\delta\phi_{n-1}](h),\phi_{n-1}(\tau(h)))\varpi_n(\tau(h))\\
&+ \sigma(\varpi_{n-1}(h),[\Gamma\varpi_{n-1}](h),[\delta\varpi_{n-1}](h),\varpi_{n-1}(\tau(h))), \qquad h \neq h_k \quad h \in I,
\end{aligned}
$$

where

$$
\phi_{n-1}(h) \leq y_1 \leq \varsigma(h), [\Gamma\phi_{n-1}](h) \leq y_2 \leq [\Gamma\varsigma](h), [\delta\phi_{n-1}](h) \leq y_3 \leq [\delta\varsigma](h), \phi_{n-1}(\tau(h)) \leq y_4 \leq \varsigma(\tau(h)),
$$

and

$$
\begin{aligned}
\sigma(\varpi_{n-1}(h) &, [\Gamma\varpi_{n-1}](h),[\delta\varpi_{n-1}](h),\varpi_{n-1}(\tau(h)))\\
&= \frac{1}{2}[\varpi_{n-1}^2(h)f_{yy}(h,y_1,y_2,y_3,y_4) + [\Gamma\varpi_{n-1}]^2(h)f_{\Gamma y\Gamma y}(h,y_1,y_2,y_3,y_4)\\
&+ [\delta\varpi_{n-1}]^2(h)f_{\delta y\delta y}(h,y_1,y_2,y_3,y_4) + (\varpi_{n-1}(\tau(h)))^2 f_{y\tau y\tau}(h,y_1,y_2,y_3,y_4)\\
&+ 2\varpi_{n-1}(h)[\Gamma\varpi_{n-1}](h)f_{y\Gamma y}(h,y_1,y_2,y_3,y_4) + 2\varpi_{n-1}(h)[\delta\varpi_{n-1}](h)f_{y\delta y}(h,y_1,y_2,y_3,y_4)\\
&+ 2\varpi_{n-1}(h)(\varpi_{n-1}(\tau(h)))f_{yy\tau}(h,y_1,y_2,y_3,y_4) + 2[\Gamma\varpi_{n-1}](h)[\delta\varpi_{n-1}](h)f_{\Gamma y\delta y}(h,y_1,y_2,y_3,y_4)\\
&+ 2[\Gamma\varpi_{n-1}](h)(\varpi_{n-1}(\tau(h)))f_{\Gamma yy\tau}(h,y_1,y_2,y_3,y_4)
\end{aligned}
$$

$$+\ 2[\delta\varpi_{n-1}](h)(\varpi_{n-1}(\tau(h)))f_{\delta yy\tau}(h,y_1,y_2,y_3,y_4)],$$

$$\begin{aligned}
\Delta\varpi_n(h_k) &= \Delta\varsigma(h_k)-\Delta\phi_n(h_k)\\
&= I_k(\varsigma(h_k))-I_k(\phi_{n-1}(h_k))-I_k{}'(\phi_{n-1}(h_k))[\phi_n(h_k)-\phi_{n-1}(h_k)]\\
&= I_k(\phi_{n-1}(h_k))+I_k{}'(\phi_{n-1}(h_k))[\varsigma(h_k)-\phi_{n-1}(h_k)]+\frac{1}{2}I_k{}''(\xi)[\varsigma(h_k)-\phi_{n-1}(h_k)]^2\\
&- I_k(\phi_{n-1}(h_k))-I_k{}'(\phi_{n-1}(h_k))[\phi_n(h_k)-\phi_{n-1}(h_k)]\\
&= I_k{}'(\phi_{n-1}(h_k))\varpi_n(h_k)+\frac{1}{2}I_k{}''(\xi)\varpi_{n-1}^2(h_k),k=1,2,\cdots,m,\\
\varpi_n(0) &= \varsigma(0)-\phi_n(0)=-\varpi_n(T)+\eta,\\
\varpi_n(h) &= \varsigma(h)-\phi_n(h)=\varsigma(0)-\phi_n(0)=\varpi_n(0), \qquad h\in[-r,0],
\end{aligned}$$

where $\phi_{n-1}(h_k)\le\xi\le\varsigma(h_k),\eta=\varphi_{n-1}(T)-\phi_n(T)$. By Lemma 2.1 , the solution of the above system is

$$\varpi_n(h)=\begin{cases}
\int_0^T F(h,s)[\sigma(\varpi_{n-1}(s),[\Gamma\varpi_{n-1}](s),[\delta\varpi_{n-1}](s),\varpi_{n-1}(\tau(s)))\\
+f_{\Gamma y}(s,\phi_{n-1}(s),[\Gamma\phi_{n-1}](s),[\delta\phi_{n-1}](s),\phi_{n-1}(\tau(s)))[\Gamma\varpi_n](s)\\
+f_{\delta y}(s,\phi_{n-1}(s),[\Gamma\phi_{n-1}](s),[\delta\phi_{n-1}](s),\phi_{n-1}(\tau(s)))[\delta\varpi_n](s)\\
+f_{y\tau}(s,\phi_{n-1}(s),[\Gamma\phi_{n-1}](s),[\delta\phi_{n-1}](s),\phi_{n-1}(\tau(s)))\varpi_n(\tau(s))]ds+\frac{e^{M(T)-M(h)}}{1+e^{M(T)}}\eta\\
+\sum\limits_{k=1}^{m}F(h,h_k)[I_k'(\phi_{n-1}(h_k))\varpi_n(h_k)+\frac{1}{2}I_k{}''(\xi)\varpi_{n-1}^2(h_k)] \qquad h\in I,\\
\int_0^T F(0,s)[\sigma(\varpi_{n-1}(s),[\Gamma\varpi_{n-1}](s),[\delta\varpi_{n-1}](s),\varpi_{n-1}(\tau(s)))\\
+f_{\Gamma y}(s,\phi_{n-1}(s),[\Gamma\phi_{n-1}](s),[\delta\phi_{n-1}](s),\phi_{n-1}(\tau(s)))[\Gamma\varpi_n](s)\\
+f_{\delta y}(s,\phi_{n-1}(s),[\Gamma\phi_{n-1}](s),[\delta\phi_{n-1}](s),\phi_{n-1}(\tau(s)))[\delta\varpi_n](s)\\
+f_{y\tau}(s,\phi_{n-1}(s),[\Gamma\phi_{n-1}](s),[\delta\phi_{n-1}](s),\phi_{n-1}(\tau(s)))\varpi_n(\tau(s))]ds+\frac{e^{M(T)}}{1+e^{M(T)}}\eta\\
+\sum\limits_{k=1}^{m}F(0,h_k)[I_k'(\phi_{n-1}(h_k))\varpi_n(h_k)+\frac{1}{2}I_k{}''(\xi)\varpi_{n-1}^2(h_k)] \qquad h\in[-r,0],
\end{cases}$$

where $M(h)=-\int_0^h f_y(v,\phi_{n-1}(v),[\Gamma\phi_{n-1}](v),[\delta\phi_{n-1}](v),\phi_{n-1}(\tau(v)))dv$. Letting $|f_{yy}|\le\delta_1,|f_{\Gamma y\Gamma y}|\le\delta_2,|f_{\delta y\delta y}|\le\delta_3,|f_{y\tau y\tau}|\le\delta_4,|f_{y\Gamma y}|\le\delta_5,|f_{y\delta y}|\le\delta_6,|f_{yy\tau}|\le\delta_7,|f_{\Gamma y\delta y}|\le\delta_8,|f_{\Gamma yy\tau}|\le\delta_9,|f_{\delta yy\tau}|\le\delta_{10}$, we have

$$\begin{aligned}
\sigma(\varpi_{n-1}(h) &,\ [\Gamma\varpi_{n-1}](h),[\delta\varpi_{n-1}](h),\varpi_{n-1}(\tau(h)))\\
&\le \frac{1}{2}\delta_1\varpi_{n-1}^2(h)+\frac{1}{2}\delta_2[\Gamma\varpi_{n-1}]^2(h)+\frac{1}{2}\delta_3[\delta\varpi_{n-1}]^2(h)+\frac{1}{2}\delta_4(\varpi_{n-1}(\tau(h)))^2\\
&+ \delta_5\varpi_{n-1}(h)[\Gamma\varpi_{n-1}](h)+\delta_6\varpi_{n-1}(h)[\delta\varpi_{n-1}](h)+\delta_7\varpi_{n-1}(h)(\varpi_{n-1}(\tau(h)))\\
&+ \delta_8[\Gamma\varpi_{n-1}](h)[\delta\varpi_{n-1}](h)+\delta_9[\Gamma\varpi_{n-1}](h)(\varpi_{n-1}(\tau(h)))+\delta_{10}[\delta\varpi_{n-1}](h)(\varpi_{n-1}(\tau(h)))\\
&\le (\frac{1}{2}\delta_1+\frac{1}{2}\delta_5+\frac{1}{2}\delta_6+\frac{1}{2}\delta_7)\varpi_{n-1}^2(h)+(\frac{1}{2}\delta_2+\frac{1}{2}\delta_5+\frac{1}{2}\delta_8+\frac{1}{2}\delta_9)[\Gamma\varpi_{n-1}]^2(h)\\
&+ (\frac{1}{2}\delta_3+\frac{1}{2}\delta_6+\frac{1}{2}\delta_8+\frac{1}{2}\delta_{10})[\delta\varpi_{n-1}]^2(h)\\
&+ (\frac{1}{2}\delta_4+\frac{1}{2}\delta_7+\frac{1}{2}\delta_9+\frac{1}{2}\delta_{10})(\varpi_{n-1}(\tau(h)))^2.
\end{aligned}$$

Taking the norm of ϖ_{n-1} on I^+ is $\|\varpi_{n-1}\|_{E_0} = \max_{I^+}\{\varpi_{n-1}(h), [\Gamma\varpi_{n-1}](h), [\delta\varpi_{n-1}](h), \varpi_{n-1}(\tau(h))\}$. Obviously, from the expression of $\varpi_n(h)$ we know that the follow formula is established for a constant ζ:

$$\|\varpi_n\|_{E_0} \le \zeta\|\varpi_{n-1}\|^2_{E_0}.$$

Therefore, the ϖ_n is quadratic convergent.

4. Conclusions

In this paper, we mainly use the monotone iterative and quasilinearization method to study the quadratic convergence of the extremal solution for a class of integro-differential equations with delay. The results obtained are new and more general than previous studies, which can be applied to several special cases: 1) If $\tau(h) = h$, then the Bvp(1.1) is an impulsive integro-differential system with anti-periodic boundary value conditions; 2) If $\tau(h) = h + \theta$, $\theta \in [-r, 0]$, then the Bvp(1.1) becomes a delay impulsive integro-differential equation; 3) If $I_k(y(h_k)) = 0$, then the Bvp(1.1) is a non-impulsive integro-differential equation; 4) If $K(h, s) = 0, H(h, s) = 0$, then the Bvp(1.1) is reduced to an impulsive functional differential equation with delay; 5) If $K(h, s) = 0, H(h, s) = 0, \tau(h) = h$, then the Bvp(1.1) becomes an impulsive ordinary differential equation.

Acknowledgments

This research was supported by the National Science Foundation of China (No. 11602092); the China Postdoctoral Science Foundation (No. 2018M632184).

Conflict of interest

The authors declare there is no conflict of interest.

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