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Research article

Lie symmetries of Benjamin-Ono equation

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Abstract: Lie Symmetry analysis is often used to exploit the conservative laws of nature and solve or at least reduce the order of differential equation. One dimension internal waves are best described by Benjamin-Ono equation which is a nonlinear partial integro-differential equation. Present article focuses on the Lie symmetry analysis of this equation because of its importance. Lie symmetry analysis of this equation has been done but there are still some gaps and errors in the recent work. We claim that the symmetry algebra is of five dimensional. We reduce the model and solve it. We give its solution and analyze them graphically.

Keywords: Benjamin-Ono equation; Lie symmetry; one parameter Lie group; similarity reduction; invariant solutions; optimal system

1. Introduction

Symmetry has been a source of inspiration as powerful tool in formulation of the laws of the universe. This connection between continuous symmetries and conservation laws brought to light by famous mathematician Emmy Noether in 1918. Although symmetry has been in action in some form throughout the history, but the major landmark was achieved when 19th century mathematician Sophus Lie investigated the continuous group of transformations which leave differential equation invariant. Thus Lie succeeded to develop a general theory of integration of ODEs in parallel to the theory developed by Galois and Abel for algebraic equations. Now this theory provides us enough skills to derive solutions of DEs algorithmically. This theory could not attract much attention for the first fifty years after its development and only abstract theory of Lie groups developed. L. V. Ovsiannikov, started to use this theory in 1960 on difficult problems even emerging from Mathematical Physics. A considerable revival of interest in Lie's theory has been in progress since last two decades and a comprehensive literature is at hands now.

A great number of physical phenomenons is transformed into differential equations . Lie symmetry analysis can change the given differential equation into an equivalent form which is easier to solve. *L*ie groups were first developed as a tool for solving and simplifying ordinary and partial differential equations (PDEs). Galois's use of finite groups to solve algebraic equations of degrees two, three, and four, as well as to prove that the general polynomial equation of degrees larger than four could not be solved by radicals, served as the paradigm for this application [1]. Lie established that a two-dimensional linear second-order PDE can accept only a three-parameter invariance group in his first publication on the subject [2, 3], (Except for the trivial infinite parameter symmetry group, which arises from linearity). He calculated the one-dimensional heat conductivity equation's maximum invariance group and used it to provide explicit solutions, .

Lie system theory is a branch of mathematics that deals with the study of Lie systems By using a time-independent function [4, 5], deals with non-autonomous systems of first-order ordinary differential equations [6] and then partial differential equations [7], such that all of their solutions may be stated in terms of generic sets of specific solutions and certain constants. Superposition rules are such functions, and Lie systems are systems that admit this mathematical characteristic.

The theory of differential equations and mathematical modelling provide a planform to create a relation between environment and mathematics. Some differential equations describe physical phenomena relating to nature. One such equation is the Benjamin-Ono, BO, equation which governs some forms of internal waves as a nonlinear evolution equation. Most Waves in Real-Life do not travel with a permanent profile, that is, dispersion continuously changes the profile of the wave. Since most media are dispersive, they cause waves to change form (disperse). For example, waves on the surface of water. Physicists had long believed that it was impossible to observe such special waves of permanent wave form in any dispersive medium.

In 1834, Sir J.Scott Russel accidentally observed such a wave, that traveled several miles without change of any form, set off by a boat in a canal near Edinburgh. That discovery initiated a series of studies by Russel himself and then by many other physicists who finally confirmed the existence of such waves that are now called Solitons. With advancements in the field of nonlinear analysis, the study of solitons has grown tremendously. They have recently become of great interest to Physicists and Mathematicians due to its observance in many fields such as Optics, Quantum Mechanics, and others.

In mathematical literature, there are many nonlinear equations which are known to have soliton solutions. The Korteweg-deVries Equation, which describes moderately small, shallow-water waves, exhibits Soliton solutions. The Benjamin-Ono equation describes internal waves of deep stratified fluids and it also has Soliton solutions.

The Lie symmetry group theory aids in the discovery of invariant solutions, conservation rules, and symmetry in partial differential equations. Thomas Brook presented the Benjamin-Ono equation as a nonlinear form of a nonlinear partial integro-differential equation. H. Ono in 1975 and Benjamin in 1967 [9, 10] given by,

$$u_n + Hu_{mm} + uu_m = 0, \qquad u = u(m,n), \tag{1.1}$$

where H is the Hilbert transformation operator described by

$$H(u(m,n)) = \frac{p.v.}{\pi}\int_{-\infty}^{\infty}\frac{u(v,n)}{n-v}dv.$$

The symbols *P.V.* is the principal value of the integral. Because of the soliton solution, BO equation has been studied extensively. Recently J. B. Hong et al. in 2020 computed the symmetry algebra of the equation and discovered that four-vector fields with not closed symmetry [11]. This equation comprises five generators that keep the commutator table closed. We must discover the symmetry of the Benjamin-Ono equation and construct the adjoint table in this work.

Because of the equation's importance in describing wave motion in deep water, the Lie symmetry approach was used to solve the equation to find additional solutions, allowing the characteristics of the wave to be seen. It can be seen that the attained results are established to be significant for the expression of some physical displays of problems in mathematical physics and deep engineering. In deep engineering, BO equations are used in computer simulation for the water waves in far-off water and open wave.

Chetverikov et al. in [12] discussed general theory to solve integro-differential equation. Chetverikov also discussed symmetry algebra of the this equation in [13]. $CASE$ reduces the problem of finding the N-soliton solution of the Benjaign-Ono equation to that of solving an algebraic equation of degree N [14]. Recently J. B. Hong et al. in 2020 computed the symmetry algebra of the equation and discovered that four-vector fields but we have investigated that the sysltem of algebra given by them is not closed, [11]. We in fact claim that symmetry algebra is of five dimensional. We further discover the symmetries of the Benjamin-Ono equation and construct the adjoint table. We also compute optimal system of Lie symmetries and construct invariant solutions. In the end we analyze these solutions graphically.

2. Lie symmetries of Benjamin-Ono equation

One of the most helpful approaches for dealing with partial differential equations (PDE) and ordinary differential equations (ODEs) is symmetry analysis [6]. Previous research on applied symmetry analysis to solve differential equations [15, 16] demonstrates this. It is often used to solve a difficult problem because of its effectiveness in solving an equation. Kumar [17], for example, employed Lie symmetry to solve shallow wave equations, while Zhang [18] utilized it to evaluate longitudinal wave motion equations.

Consider a point transformation using a local one-parameter Lie group:

$$\begin{aligned} m^* &= m + \epsilon\kappa(m,n,u) + O(\epsilon^2), \\ n^* &= n + \epsilon\lambda(m,n,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\rho(m,n,u) + O(\epsilon^2), \end{aligned} \tag{2.1}$$

where the group parameter is $\epsilon \in R$.

2.1. Main results

We want to determine the symmetry algebras of $u_n + Hu_{mm} + uu_m = 0$. The infinitesimal operator in general is defined as,

$$J = \kappa\frac{\partial}{\partial m} + \lambda\frac{\partial}{\partial n} + \rho\frac{\partial}{\partial u}.$$

The derivation of first order prolongation of infinitesimal operator,

$$J^{[1]} = J + \rho^{m}\frac{\partial}{\partial u_m} + \rho^{n}\frac{\partial}{\partial u_n},$$

where as second order prolongation is,

$$J^{[2]} = J^{[1]} + \rho^{mm}\frac{\partial}{\partial u_{mm}} + \rho^{nm}\frac{\partial}{\partial u_{nm}} + \rho^{nn}\frac{\partial}{\partial u_{nn}},$$

with coefficients ρ^m, ρ^n & ρ^{mm} which is determined by [6, 15],

$$\begin{aligned}
\rho^m &= D_m(\rho - \kappa u_m - \lambda u_n) + \kappa u_{mm} + \lambda u_{mn},\\
\rho^n &= D_n(\rho - \kappa u_m - \lambda u_n) + \kappa u_{mn} + \lambda u_{nn},\\
\rho^{mm} &= D_m D_m(\rho - \kappa u_m - \lambda u_n) + \kappa u_{mmm} + \lambda u_{nmm}.
\end{aligned}$$

To calculate symmetry of Eq (1.1), we apply,

$$J^{[2]}(u_n + Hu_{mm} + uu_m) = 0. \tag{2.2}$$

After simplifying, one can get Eq (2.2) as

$$\rho^n + H\rho^{mm} + \rho^m u + \rho u_m = 0. \tag{2.3}$$

Since ρ^n, $\rho^{mm}$and ρ^m are present, these three terms are to be derived as follows:

$$\begin{aligned}
&\rho^n = \rho_n - \kappa_n u_m - \lambda_n u_n + \rho_u u_n - \kappa_u u_n u_m - \lambda_u u_n^2\\
&\rho^m = \rho_m - \kappa_m u_m - \lambda_m u_n + \rho_u u_m - \kappa_u u_m^2 - \lambda_u u_n u_m\\
&\rho^{mm} = \rho_{mm} + u_m[-\kappa_{mm} + 2\rho_{um}] + u_{mm}[\rho_u - 2\kappa_m] + u_{nm}[-2\lambda_m] + u_m^2[\rho_{uu} - 2\kappa_{um}]\\
&-3\kappa_u u_m u_{mm} - 2\lambda_{um} u_m u_n - \lambda_{uu} u_n u_m^2 - \lambda_u u_n u_{mm} - 2\lambda_u u_m u_{nm} - \lambda_{mm} u_n - \kappa_{uu} u_m^3.
\end{aligned}$$

Substitute ρ^n, ρ^m, ρ^{mm} and Eq (1.1) into Eq (2.3), we get

$$\begin{aligned}
&\rho_n - \kappa_n u_m - \lambda_n[-Hu_{mm} - uu_m] + [-Hu_{mm} - uu_m]\rho_u - \kappa_u u_m[-Hu_{mm} - uu_m] - \lambda_u[-Hu_{mm} - uu_m]^2\\
&+H[\rho_{mm} + u_m[2\rho_{um} - \kappa_{mm}] - \lambda_{mm}[-Hu_{mm} - uu_m] + u_m^m[-2\kappa_{um} + \rho_{uu}] - 2\lambda_{mu} u_m[-Hu_{mm} - uu_m]\\
&-\kappa_{uu} u_m^3 + u_{mm}[\rho_u - 2\kappa_m] - \lambda_m u_{nm} - 3\kappa_u u_m u_{mm} - \lambda_u u_{mm}[-Hu_{mm} - uu_m] - 2\lambda_u u_m u_{nm} -\\
&\lambda_{uu} u_m^2[-Hu_{mm} - uu_m]] + u[\rho_m - u_2[\kappa_m - \rho_u] - \kappa_u u_m^2 - \lambda_m[-Hu_{mm} - uu_m] -\\
&\lambda_u u_m[-Hu_{mm} - uu_m]] + \rho u_m = 0.
\end{aligned}$$

The equation is then expanded and categorized by the derivatives of u.

$$\begin{cases}
u_m: & -\kappa_n + u\lambda_n + 2H\rho_{mu} - H\kappa_{mm} - u\kappa_m + Hu\lambda_{mm} + u^2\lambda_m + \rho = 0,\\
u_{mm}: & \lambda_n + H^2\lambda_{mm} - 2H\kappa_m + Hu\lambda_m = 0,\\
u_m u_{mm}: & -2\kappa_u + 2H\lambda_{mu} = 0,\\
u_m u_{mn}: & -2H\lambda_u = 0,\\
u_m^2: & -2H\kappa_{mu} + H\rho_{uu} + 2Hu\lambda_{mu} = 0,\\
u_m^2 u_{mm}: & H^2\lambda_{uu} = 0,\\
u_{mn}: & -2H\lambda_m = 0\\
u_m^3: & -H\kappa_{uu} + Hu\lambda_{uu} = 0,\\
Rest: \rho_n + H\rho_{mm} + u\rho_m = 0. &
\end{cases} \tag{2.4}$$

After simplifying Eq (2.4), we obtain

$$\kappa(m,n,u) = (B_1 m + 2B_4)n + B_2 m + 2B_5, \quad \lambda(m,n,u) = B_1 n^2 + 2B_2 n + 2B_3$$
$$\rho(m,n,u) = (m - un)B_1 - B_2 u + 2B_4,$$

where B_1, B_2, B_3, B_4 and B_5 are arbitrary constants. As a result, the infinitesimal generators allowed by Eq (1.1) are as follows:

$$\begin{cases} J_1 = \frac{\partial}{\partial m}, \\ J_2 = \frac{\partial}{\partial n}, \\ J_3 = n\frac{\partial}{\partial m} + \frac{\partial}{\partial u}, \\ J_4 = -u\frac{\partial}{\partial u} + 2n\frac{\partial}{\partial n} + m\frac{\partial}{\partial m}, \\ J_5 = mn\frac{\partial}{\partial m} + n^2\frac{\partial}{\partial n} + (m - un)\frac{\partial}{\partial u}. \end{cases} \tag{2.5}$$

If Eq (2.5) generates the one parameter group $G'i(m,n,u)$, then

$$G'_1 = (m,n,u) \to (m + \epsilon, n, u)$$
$$G'_2 = (m,n,u) \to (m, n + \epsilon, u)$$
$$G'_3 = (m,n,u) \to (m + n\epsilon, n, u + \epsilon)$$
$$G'_4 = (m,n,u) \to (me^{\epsilon}, ne^{2\epsilon}, ue^{-\epsilon})$$
$$G'_5 = (m,n,u) \to (\frac{m}{1 - n\epsilon}, \frac{n}{1 - n\epsilon}, u + \epsilon(m - un)),$$

where the group parameter is $\epsilon \in R$.
There will be a family of solutions, known as group invariant solutions, for each parameter subgroup of the entire symmetry group of a system.
That is to say, if $g = (m,n)$ satisfies Eq (1.1), then $g_i (i = 1,2,3,4,5)$ are also solutions of Eq (1.1):

$$g_1 = \vartheta(m - \epsilon, n)$$
$$g_2 = \vartheta(m, n - \epsilon)$$
$$g_3 = \vartheta(m - n\epsilon, n)$$
$$g_4 = e^{-\epsilon}\vartheta(me^{-\epsilon}, ne^{-2\epsilon})$$
$$g_5 = \epsilon(m - un)\vartheta(\frac{m}{1 + n\epsilon}, \frac{n}{1 + n\epsilon}),$$

where $g_i = G_i * \vartheta$, $i = 1,2,3,4,5$ and ϵ is any positive number.

2.2. *Commutator table*

We give commutator table for $J_i (i = 1, ..., 5)$ given in Table 1,
where the Lie bracket is defined as usually given by,

$$[J_i, J_j] = J_i J_j - J_j J_i.$$

Table 1. Commutator table.

[.., ..]	J_1	J_2	J_3	J_4	J_5
J_1	0	0	0	J_1	J_3
J_2	0	0	J_1	$2J_2$	J_4
J_3	0	$-J_1$	0	$-J_3$	0
J_4	$-J_1$	$-2J_4$	J_3	0	$2J_5$
J_5	$-J_3$	$-J_4$	0	$-2J_5$	0

Table 2. The adjoint representation of J_i.

Ad	J_1	J_2	J_3	J_4	J_5
J_1	J_1	J_2	J_3	$J_4 - \epsilon J_1$	$J_5 - \epsilon J_3$
J_2	J_1	J_2	$J_3 - \epsilon J_1$	$J_4 - 2\epsilon J_2$	$J_5 - \epsilon J_4 + \epsilon^2 J_2$
J_3	J_1	$J_2 + \epsilon J_1$	J_3	$J_4 + \epsilon J_3$	J_5
J_4	$J_1 e^{\epsilon}$	$J_2 + 2\epsilon J_4$	$J_3 e^{-\epsilon}$	J_4	$J_5 e^{-2\epsilon}$
J_5	$J_1 + J_3\epsilon$	$J_2 + \epsilon J_4 + \epsilon^2 J_5$	J_3	$J_4 + 2\epsilon J_5$	J_5

2.3. Adjoint table

The adjoint representation of J_i is given in Table 2 constructed using formula, $Ad(e^{\epsilon J_i})J_j$ defined by [6], given by

$$Ad(e^{\epsilon J_i})J_j = J_j - \epsilon[J_i, J_j] + \frac{\epsilon^2}{2}[J_i, [J_i, J_j]] - ... \tag{2.6}$$

Using this definition in conjunction with Table 1, we get Table 2 where the $(i, j)-th$ entry represents $Ad(e^{\epsilon J_i})J_j$.

3. Optimal system of Benjamin-Ono equation

In this part we derive optimal system of one-dimensional Lie subalgebras. **Preposition 1:** Eq (1.1) permits an optimal system of one-dimensional subalgebras involving J_i is given as,

$$\{J_1, J_2, J_4, aJ_2 + J_3, aJ_2 + J_5\}, \tag{3.1}$$

where a is an arbitrary constant.
Proof: Consider an arbitrary element spanned by J,

$$J = a_1 J_1 + a_2 J_2 + a_3 J_3 + a_4 J_4 + a_5 J_5, \tag{3.2}$$

where $a_i(i = 1, 2, 3, 4, 5)$ and $a_i(i = 1, 2, 3, 4, 5)$ are arbitrary constants. Our goal is to use the adjoint map of J to simplify as many of the coefficients a_i as feasible. We begin with the J_5 coefficient and explore three scenarios involving a_5.
Case 1: Let $a_5 \neq 0$, and

$$J' = Ad(e^{-a_3 J_1})J = a'_1 J_1 + a_4 J_4 + a_2 J_2 + J_5$$

$a'_1 = a_1 + a_4 a_3$ is the formula. The J_3 coefficient vanishes when the adjoint action is used. In order to disappearance coefficient a_4, we apply $Ad(e^{a_4 J_2)}$ to J', resulting in

$$Ad(e^{a_4 J_2})(J) = a'_1 J_1 + a'_2 J_2 + J_5 = J''$$

where $a'_1 = a_1 + a_4 a_3$ and $a'_2 = a_2 + a_4^2 - 2$. In order to cancel coefficient a_1, we further act on J'' by $Ad(e^{-\frac{a'_1}{a'_2} J_1 2})$ we get $J''' = a'_2 J_2 + J_5$.
Case 2: Let $a_5 = 0$ and $a_4 \neq 0$. Let$J = a_1 J_1 + a_2 J_2 + a_3 J_3 + a_4$. Following the above procedure, we act on J by $Ad(e^{-a_3 J_3})$
and J by $Ad(e^{-a_1 J_1})$ and J by $Ad(e^{\frac{a_2}{2} J_2})$ successively to make the coefficient a_1, a_2 and a_3 zero. Thus, every one-dimensional subalgebra generated by J with $a_5 = 0$, $a_4 \neq 0$ is equivalent to the subalgebra spanned by J_4.
Case 3: Now let $J = a_1 J_1 + a_2 J_2 + J_3$. Let $a_5 = 0$, $a_4 = 0$ and $a_3 \neq 0$, Following the process outlined above, we apply $Ad(e^{a_1 J_2})$ to J. Successively to make the coefficient J_1 zero. Thus every one dimensional subalgebra generated by J with $a_5 = 0$, $a_4 = 0$, $a_3 = 1$ is equivalent to subalgebra spanned by $a_2 J_2 + J_3$.
Case 4: Let $J = a_1 J_1 + J_2$ and $a_5 = 0$, $a_4 = 0$, $a_3 = 0$ and $a_2 \neq 0$. Now we make coefficient $a_1 = 0$. We act on J by $Ad(e^{-a_1 J_3})$.
We get $J' = J_2$. Thus every one dimensional subalgebra generated by J with $a_5 = 0$, $a_4 = 0$, $a_3 = 0$, $a_2 = 1$ is equivalent to subalgebra spanned by J_2.
Case 5: Let $J = J_1$ with $a_i = 0, i = 2, 3, 4, 5$ and $a_1 \neq 0$. Thus every one dimensional subalgebra generated by J with $a_5 = 0$, $a_4 = 0$, $a_3 = 0$, $a_2 = 0$ and $a_1 = 1$ is equivalent to subalgebra spanned by J_1. So Eq (1.1) admits an optimum system of one-dimensional subalgebras, which is determined by

$$\{J_1, J_2, J_4, aJ_2 + J_3, aJ_2 + J_5\}, \tag{3.3}$$

where a is an arbitrary constant. So it completes the proof.

4. Similarity reductions, invariant solutions and their graphical representation

Based on the optimum system computed in the previous part, we conduct similarity reductions and find invariant solutions for Eq (1) in this section.

Case 1: Reduction by J_1
The characteristic equation of $J_1 = \frac{\partial}{\partial m}$ is

$$\frac{dm}{1} = \frac{dn}{0} = \frac{du}{0}$$

$$dn = 0, du = 0$$

$u = w, n = r$ and $u_n = \frac{dw}{dr} = w'$ put in PDE Eq (1.1). $u(m, n) = c$ where c is constant. This is the invariant solution

Case 2: Reduction by J_2
Taking $J_2 = \frac{\partial}{\partial n}$ we get the characteristic equation,

$$\frac{dm}{0} = \frac{dn}{1} = \frac{du}{0}$$
$$\frac{dm}{0} = \frac{dn}{1}, \frac{dn}{1} = \frac{du}{0}$$
$$m = r, u = w, u_m = \frac{dw}{dr} = w' \quad and \quad u_{mm} = w''$$

So, putting the values of u_m, u_{mm} put in PDE Eq (1.1) we get,

$$u\,(m, n) = \tanh\left(1/2\,\frac{\sqrt{B_1\,H}\,(m + B_2)\,\sqrt{2}}{H}\right)\sqrt{B_1\,H}\,\sqrt{2},$$

which is the invariant solution of PDE Eq (1.1).

Case 3: Reduction by $J_1 + J_2$

$$Taking J_1 + J_2 = \frac{\partial}{\partial m} + \frac{\partial}{\partial n},$$

we get the characteristic equation

$$\frac{dm}{1} = \frac{dn}{1} = \frac{du}{0}.$$

Solving this we get $r = -m + n$ and $u = w$, where r and w are constant. So solutions is $w = f(r)$. After computing the value of u_n, u_m and u_{mm} and putting in Eq (1.1) we get the invariant solution given as

$$u\,(m, n) = \sqrt{2}\tan\left(1/2\,\frac{\sqrt{B_1\,H}\,(B_2 - m + n)\,\sqrt{2}}{H}\right)\sqrt{B_1\,H} + 1 \tag{4.1}$$

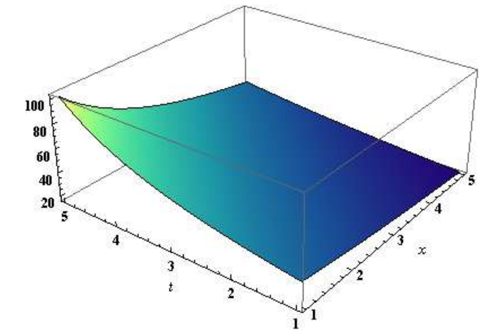

Figure 1. Graph of u for $B_1 = 3$, $B_2 = 1112$, $H = 90$.

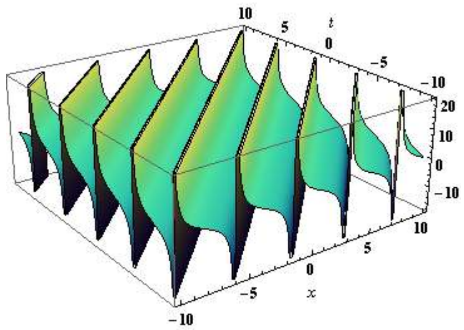

Figure 2. Graph of u for $B_1 = 3$, $B_2 = 6$ and $H = 1$.

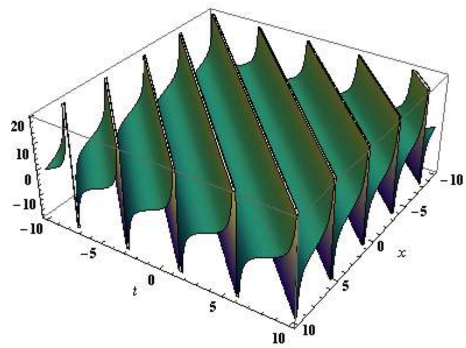

Figure 3. Graph of u for $B_1 = 12$, $B_2 = 6$ and $H = 1$.

Figures 1–3 are the graphs of above equation obtained for different values of the involved constants.

Case 4: Reduction by J_3

$$Taking J_3 = n\frac{\partial}{\partial m} + 1\frac{\partial}{\partial u}$$

we get the characteristic equation as

$$\frac{dm}{n} = \frac{dn}{0} + \frac{du}{1}.$$

The characteristic Q invariants of J_3 is

$$r = n, \qquad u = w + \frac{m}{n}$$

where r and w are constant. The invariant solution obtained is,
$u(m, n) = \frac{m+B_1}{n}$.
Case 5: Reduction by J_5 Taking

$$J_5 = mn\frac{\partial}{\partial m} + n^2\frac{\partial}{\partial n} + (m - un)\frac{\partial}{\partial u},$$

we get the characteristic equation as

$$\frac{dm}{mn} = \frac{dn}{n^2} = \frac{du}{m - un}.$$

The characteristics Q invariants is

$$r = \frac{n}{m}, \qquad w = \frac{(tu - m)m}{n} \tag{4.2}$$

The invariants solution is

$$u\,(m, n) = \frac{1}{B_1\,n}\left(B_1\,m - \tan\left(1/2\,(\frac{B_2\,n + m}{HB_1\,n})\right)\right). \tag{4.3}$$

If we take $H = 1$ then the following Figures 4-6 represent invariant solution of the above equation,

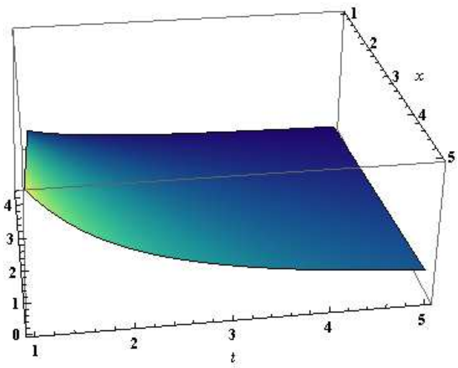

Figure 4. Graph of u for $B_1 = 6$, $B_2 = 200$.

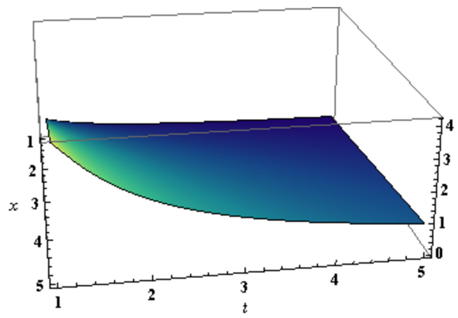

Figure 5. Graph of u for $B_1 = 12$, $B_2 = 200$.

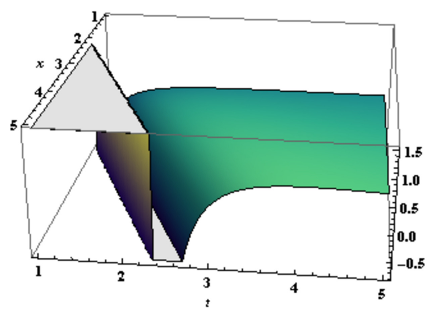

Figure 6. Graph of u for $B_1 = 1$, $B_2 = 1$.

Case: 6 Reduction by J_4.

$$J_4 = m\frac{\partial}{\partial m} + 2n\frac{\partial}{\partial n} - u\frac{\partial}{\partial u}$$

The characteristic equation is

$$\begin{aligned}
&\frac{dm}{m} = \frac{dn}{2n} = -\frac{du}{u}\\
&\frac{dm}{m} = -\frac{du}{u} \& \frac{dm}{m} = \frac{dn}{2n}\\
&\frac{dm}{m} = -\frac{du}{u}\\
&ln(m) + ln(\gimel) = -ln(u), \Rightarrow \gimel = um
\end{aligned}$$

Now

$$\frac{dm}{m} = \frac{dn}{2n} \& \beth = \frac{m}{\sqrt{n}}$$

Derivative below

$$\begin{aligned}
u_n &= -\frac{1}{2}\gimel_\beth n^{\frac{-3}{2}}, u_m = \frac{-\gimel}{m^2} + \frac{\gimel_\beth}{m\sqrt{mn}}\\
u_{mm} &= \frac{2\gimel}{m^3} + \frac{\gimel_{\beth\beth}}{mn} - \frac{2\gimel_\beth}{m^2\sqrt{n}}
\end{aligned}$$

Substitute all the derivatives of u into Eq (1.1) to obtain a first order ODE after second order derivatives have been cancelled off

$$-\frac{\gimel_\beth}{2} + \frac{2H\gimel}{\beth^3} + \frac{\gimel_{\beth\beth}}{\beth} - \frac{2\gimel_\beth}{\beth^2} + \frac{\gimel^2}{\beth^3} + \frac{\gimel\gimel_\beth}{\beth^2} = 0$$

This is a second order ODE. So, let $Q = \beth$, $P = \beth_\gimel$, and $P\frac{dP}{dQ} = \beth_{\gimel\gimel}$. After simplification,

$$P' = \frac{\beth}{2} - \frac{2HQ}{\beth^2 P} + \frac{2}{\beth} - \frac{Q^2}{\beth^2} - \frac{Q}{\beth}$$

This is a Chini Eq [11], which only admits under certain conditions. The equation we obtain does not lie under the condition, hence for this moment, no possible solutions were obtained here. However, from a non-linear PDE to first-order ODE.

Case: 7 Reduction by $J_2 + aY_3$

$$J_2 + a.J_3 = n\frac{\partial}{\partial m} + a\frac{\partial}{\partial n} - \frac{\partial}{\partial u}$$

The method of characteristic is

$$\begin{aligned}
&\frac{dm}{n} = \frac{dn}{a} = \frac{du}{1}\\
&\frac{dm}{n} = \frac{dn}{a} \& \quad \frac{dn}{a} = -\frac{du}{1}
\end{aligned}$$

$$ax - \frac{n^2}{2} = r \quad \&u = \frac{1}{a}(-n + w)$$

where r and w are new invariants. The solution is $w = f(r)$. Now we have to find u_n, u_m, $u_m m$ and put in Eq (1.1). After simplification, we get the ODE which is

$$1 + Hw^{''} + ww^{'} = 0.$$

It is a non-linear ordinary differential equation. Analytically it is very difficult to solve. We suggest to solve this equation numerically.

5. Discussion and conclusions

Our major goal was to find an optimal system of vector fields. An optimal system is a powerful instrument that reduces the problem of obtaining all the PDEs solutions. This is performed by categorizing the solutions into equivalence classes using the symmetry generators found in the optimal system. We obtained adjoint table by using commutator table. With the help of adjoint table, we performed the optimal system of Benjamin-Ono equation. All the invariant solutions are obtained through reductions of generators in optimal classes and the graphical picture of invariant solutions is also presented. In this article, we examined the Benjamin-Ono equation that regulates multiple kinds of internal waves as a nonlinear evolution equation. Internal waves are disturbances at a point where two immiscible fluids meet differing densities that are caused by gravity. The resultant partial differential equation has been analyzed using Lie symmetry. We proved that the Lie symmetry algebra is in fact five dimension. We have also computed the optimal system of solution.

Conflict of interest

Authors declare that they have no conflict of interest.

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