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Research article

Existence of traveling wave solutions for a delayed nonlocal dispersal SIR epidemic model with the critical wave speed

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Abstract: This paper is about the existence of traveling wave solutions for a delayed nonlocal dispersal SIR epidemic model with the critical wave speed. Because of the introduction of nonlocal dispersal and the generality of incidence function, it is difficult to investigate the existence of critical traveling waves. To this end, we construct an auxiliary system and show the existence of traveling waves for the auxiliary system. Employing the results for the auxiliary system, we obtain the existence of traveling waves for the delayed nonlocal dispersal SIR epidemic model with the critical wave speed under mild conditions.

Keywords: delayed SIR model; nonlocal dispersal; nonlinear incidence; minimal wave speed; traveling waves

1. Introduction

In the biological context, to better understand the spatial spread of infectious diseases, epidemic waves in all kinds of epidemic models are attracting more and more attention, for instance, in Wu et al. [1], Wang et al. [2] and Zhang et al. [3–5]. Biologically speaking, the existence of an epidemic wave suggests that the disease can spread in the population. The traveling wave describes the epidemic wave moving out from an initial disease-free equilibrium to the endemic equilibrium with a constant speed. Various theoretical results, numerical algorithms and applications have been studied extensively for traveling waves about epidemic models in the literature; for instance, we refer the reader to [6–9]. More precisely, Hosono and Ilyas [10] studied the existence of traveling wave solutions for a reaction-diffusion model. In view of the fact that individuals can move freely and randomly and can be exposed to the infection from contact with infected individuals in different spatial location, Wang and Wu [11] investigated the existence and nonexistence of non-trivial traveling wave solutions of a general class of diffusive Kermack-Mckendrick SIR models with nonlocal and delayed transmission, see also [12]. Incorporating random diffusion into epidemic model, then the dynamics of disease transmission between species in a heterogeneous habitat can be described by a variety of reaction-diffusion models (see, for example, [13–15] and the references therein). Random diffusion is essentially a local behavior, which depicts the individuals at the location x can only be influenced by the individuals in the neighborhood of the location x. In real life, individuals can move freely. One way to solve such problems is to introduce nonlocal dispersal, which is the standard convolution with space variable. Recently, Yang et al. [16] studied a nonlocal dispersal Kermack-McKendrick epidemic model. Cheng and Yuan [17] investigated the traveling waves of a nonlocal dispersal Kermack-McKendrick epidemic model with delayed transmission, Zhang et al. [18] discussed the traveling waves for a delayed SIR model with nonlocal dispersal and nonlinear incidence, and Zhou et al. [19] proved the existence and non-existence of traveling wave solutions for a nonlocal dispersal SIR epidemic model with nonlinear incidence rate. As we know, there are many existence of traveling wave solutions for reaction-diffusion models when the wave speed is greater than the minimum wave speed (see, e.g. [20–22]). However, there are few discussions on the existence of traveling wave solutions when the wave speed is equal to the minimum wave speed (the critical wave speed), see [23–26].

In this paper, we focus on the delayed SIR model with the nonlocal dispersal and nonlinear incidence which proposed by Zhang et al. [18] as follows:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1(J * S(x,t) - S(x,t)) - f(S(x,t))g(I(x,t-\tau)), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2(J * I(x,t) - I(x,t)) + f(S(x,t))g(I(x,t-\tau)) - \gamma I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3(J * R(x,t) - R(x,t)) + \gamma I(x,t), \end{cases} \tag{1.1}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ denote the densities of susceptible, infective and removal individuals at time t and location x, respectively. The parameters $d_i > 0 (i = 1,2,3)$ are diffusion rates for susceptible, infected and removal individuals, respectively. The removal rate γ is positive number and $\tau > 0$ is a given constant. Moreover, $J * S(x,t)$, $J * I(x,t)$ and $J * R(x,t)$ represent the standard convolution with space variable x, namely,

$$J * u(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)dy = \int_{\mathbb{R}} J(y)u(x-y,t)dy,$$

where u can be either S, I or R. Throughout this paper, assume that the nonlinear functions f and g, and the dispersal kernel J satisfy the following assumptions:

- (A1) $f(S)$ is positive and continuous for all $S > 0$ with $f(0) = 0$ and $f'(S)$ is positive and bounded for all $S \geq 0$ with $L := \max_{S\in[0,\infty)} f'(S)$;
- (A2) $g(I)$ is positive and continuous for all $I > 0$ with $g(0) = 0, g'(I) > 0$ and $g''(I) \leq 0$ for all $I \geq 0$;
- (A3) $J \in C^1(\mathbb{R}), J(y) = J(-y) \geq 0, \int_{\mathbb{R}} J(y)dy = 1$ and J is compactly supported.

Since the third equation in (1.1) is decoupled with the first two equations, it is enough to consider the following subsystem of (1.1):

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1(J * S(x,t) - S(x,t)) - f(S(x,t))g(I(x,t-\tau)), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2(J * I(x,t) - I(x,t)) + f(S(x,t))g(I(x,t-\tau)) - \gamma I(x,t). \end{cases} \tag{1.2}$$

We recall that, a traveling wave solution of system (1.2) is a solution of form $(S(\xi), I(\xi))$ for system (1.2), where $\xi = x + ct$. Substituting $(S(\xi), I(\xi))$ with $\xi = x + ct$ into system (1.2) yields the following system:

$$\begin{cases} cS'(\xi) = d_1(J * S(\xi) - S(\xi)) - f(S(\xi))g(I(\xi - c\tau)), \\ cI'(\xi) = d_2(J * I(\xi) - I(\xi)) + f(S(\xi))g(I(\xi - c\tau)) - \gamma I(\xi). \end{cases} \tag{1.3}$$

Clearly, if $\tau = 0$, then system (1.2) becomes

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1(J * S(x,t) - S(x,t)) - f(S(x,t))g(I(x,t)), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2(J * I(x,t) - I(x,t)) + f(S(x,t))g(I(x,t)) - \gamma I(x,t), \end{cases} \tag{1.4}$$

which was considered by Zhou et al. [19]. Combining the method of auxiliary system, Schauder's fixed point theorem and three limiting arguments, they proved the following result.

Theorem 1.1. *([19, Theorem 2.3]) Assume that (A1)-(A3) hold. If $\mathcal{R}_0 > 1$ and $c \geq c^*$, where $c^* > 0$ is the minimal wave speed and $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma}$ is the reproduction number of* (1.4)*, then system* (1.4) *admits a nontrivial and nonnegative traveling wave solution* $(S(x+ct), I(x+ct))$ *satisfying the following asymptotic boundary conditions:*

$$S(-\infty) = S_0,\ S(+\infty) = S_\infty < S_0,\ I(\pm\infty) = 0, \tag{1.5}$$

where $S_0 > 0$ is a constant representing the size of the susceptible individuals before being infected.

For (1.3) satisfying (1.5), Zhang et al. [18] obtained the following result.

Theorem 1.2. *([18, Theorem 2.7]) Assume that (A1)-(A3) hold. In addition, suppose that*

(H) there exists $I_0 > 0$ such that $f(S_0)g(I_0) - \gamma I_0 \leq 0$.

If $\mathcal{R}_0 > 1$ and $c > c^*$, where $c^* > 0$ is the minimal wave speed and $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma}$ is the reproduction number of (1.3)*, then system* (1.3) *admits a traveling wave solution* $(S(\xi), I(\xi))$ *satisfying* (1.5)*.*

We note that the assumption (H) plays a key role in the proof of Theorem 1.2 ([18, Theorem 2.7]). However, we should pointed out here that (H) cannot be applied for some incidence, such as bilinear incidence, see [27]. Therefore, one natural question is: can we obtain the existence of traveling wave solutions for system (1.2) without assumption (H)? This constitutes our first motivation of the present paper. In addition, as was pointed out in [28] that, epidemic waves with the minimal/critical speed play a significant role in the study of epidemic spread. However, it is very challenging to investigate traveling waves with the critical wave speed. Herein, we should point out that Zhang et al. [29] defined a minimal wave speed $c^* := \inf_{\lambda>0} \frac{d_2 \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - d_2 + f(S_0)g'(0)e^{-\lambda c\tau} - \gamma}{\lambda}$ and then studied the existence of critical traveling waves for system (1.1). They took a bit lengthy analysis to derive the boundedness of the density of infective individual I. Unlike [29], we will apply the auxiliary system to obtain the existence of critical traveling waves, since the method is first applied in nonlocal dispersal epidemic model in 2018, see [19] for more details. Our second motivation is to make an attempt in this direction.

The rest of this paper is organized as follows. In Section 2, we propose an auxiliary system and establish the existence of traveling wave solutions for the auxiliary system. In Section 3, we prove the existence of traveling waves under the critical wave speed. The paper ends with an application for our general results and a brief conclusion in Section 4.

2. Existence of traveling wave solutions for an auxiliary system

In this section, we will derive the existence of traveling wave solutions for the following auxiliary system on $\mathbb{R}$:

$$\begin{cases} cS'(\xi) = d_1(J * S(\xi) - S(\xi)) - f(S(\xi))g(I(\xi - c\tau)), \\ cI'(\xi) = d_2(J * I(\xi) - I(\xi)) + f(S(\xi))g(I(\xi - c\tau)) - \gamma I(\xi) - \varepsilon I^2(\xi), \end{cases} \tag{2.1}$$

where $\varepsilon > 0$ is a constant.

Clearly, (A1) and (A2) imply that $f(0) = g(0) = 0$. Thus, linearizing the second equation in (2.1) at the initial disease free point $(S_0, 0)$ yields

$$d_2 \int_{\mathbb{R}} J(y)(I(\xi - y) - I(\xi))dy - cI'(\xi) + f(S_0)g'(0)I(\xi - c\tau) - \gamma I(\xi) = 0. \tag{2.2}$$

Substituting $I(\xi) = e^{\lambda\xi}$ into (2.2) leads to the corresponding characteristic equation:

$$\Delta(\lambda, c) := d_2 \int_{\mathbb{R}} J(y)(e^{-\lambda y} - 1)dy - c\lambda + f(S_0)g'(0)e^{-\lambda c\tau} - \gamma = 0. \tag{2.3}$$

Lemma 2.1. *([18]) Suppose that $\mathcal{R}_0 := \frac{f(S_0)g'(0)}{\gamma} > 1$. Then there exist $c^* > 0$ and $\lambda^* > 0$ such that*

$$\Delta(\lambda^*, c^*) = 0 \quad and \quad \frac{\partial \Delta(\lambda, c)}{\partial \lambda}|_{(\lambda^*, c^*)} = 0.$$

Obviously, $\Delta(\lambda, c) = 0$ also has the following properties:

(i) If $c > c^*$, then $\Delta(\lambda, c) = 0$ has two different positive roots $\lambda_1 := \lambda_1(c) < \lambda_2 := \lambda_2(c)$ with

$$\Delta(\cdot, c) \begin{cases} > 0, \lambda \in [0, \lambda_1(c)) \cup (\lambda_2(c), +\infty), \\ < 0, \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

(ii) If $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda \geq 0$.

We now present some lemmas for our main results. Throughout this section, we always assume that $\mathcal{R}_0 > 1$ and $c > c^*$.

For our purpose, we define the following functions on $\mathbb{R}$:

$$\overline{S}(\xi) := S_0,\ \underline{S}(\xi) := \max\{S_0 - \rho e^{\alpha\xi}, 0\},\ \overline{I}(\xi) := \min\{e^{\lambda_1\xi}, K_\varepsilon\},\ \underline{I}(\xi) := \max\{e^{\lambda_1\xi}(1 - Me^{\eta\xi}), 0\},$$

where $K_\varepsilon = \frac{f(S_0)g'(0)-\gamma}{\varepsilon}$, λ_1 is the smallest positive real root of Eq.(2.3), and α, ρ, η, M are four positive constants to be determined in the following lemmas.

Lemma 2.2. *The function $\overline{I}(\xi)$ satisfies the following inequality:*

$$c\overline{I}'(\xi) \geq d_2 J * \overline{I}(\xi) - d_2\overline{I}(\xi) + f(\overline{S}(\xi))g(\overline{I}(\xi - c\tau)) - \gamma\overline{I}(\xi) - \varepsilon\overline{I}^2(\xi). \tag{2.4}$$

Proof. The concavity of $g(I)$ with respect to I implies that $g(I) \leq g'(0)I$ and so

$$g(I(\xi - c\tau)) \leq g'(0)I(\xi - c\tau). \tag{2.5}$$

When $\overline{I}(\xi) = e^{\lambda_1\xi}$, it follows from (2.5) that

$$\begin{aligned}
& d_2 J * \overline{I}(\xi) - d_2\overline{I}(\xi) - c\overline{I}'(\xi) + f(\overline{S}(\xi))g(\overline{I}(\xi - c\tau)) - \gamma\overline{I}(\xi) - \varepsilon\overline{I}^2(\xi) \\
\leq\ & d_2 J * \overline{I}(\xi) - d_2\overline{I}(\xi) - c\overline{I}'(\xi) + f(S_0)g'(0)\overline{I}(\xi - c\tau) - \gamma\overline{I}(\xi) - \varepsilon\overline{I}^2(\xi) \\
=\ & \left(d_2\int_{\mathbb{R}} J(y)(e^{-\lambda_1 y} - 1)dy - c\lambda_1 + f(S_0)g'(0)e^{-\lambda_1 c\tau} - \gamma\right)e^{\lambda_1\xi} - \varepsilon e^{2\lambda_1\xi} \\
=\ & e^{\lambda_1\xi}\Delta(\lambda_1, c) - \varepsilon e^{2\lambda_1\xi} \\
=\ & -\varepsilon e^{2\lambda_1\xi} \\
\leq\ & 0.
\end{aligned}$$

When $\overline{I}(\xi) = K_\varepsilon = \frac{f(S_0)g'(0)-\gamma}{\varepsilon}$, we derive from (A3) and (2.5) that

$$\begin{aligned}
& d_2 J * \overline{I}(\xi) - d_2\overline{I}(\xi) - c\overline{I}'(\xi) + f(\overline{S}(\xi))g(\overline{I}(\xi - c\tau)) - \gamma\overline{I}(\xi) - \varepsilon\overline{I}^2(\xi) \\
\leq\ & f(S_0)g'(0)\overline{I}(\xi - c\tau) - \gamma K_\varepsilon - \varepsilon K_\varepsilon^2 \\
=\ & f(S_0)g'(0)K_\varepsilon - \gamma K_\varepsilon - \varepsilon K_\varepsilon^2 \\
=\ & 0,
\end{aligned}$$

and the lemma follows. □

Lemma 2.3. *Suppose that $\alpha \in (0, \lambda_1)$ is sufficiently small. Then the function $\underline{S}(\xi)$ satisfies*

$$c\underline{S}'(\xi) \leq d_1 J * \underline{S}(\xi) - d_1\underline{S}(\xi) - f(\underline{S}(\xi))g(\overline{I}(\xi - c\tau)) \tag{2.6}$$

for any $\xi \neq \xi_1 := \frac{1}{\alpha}\ln(\frac{S_0}{\rho})$ and $\rho > S_0$ large enough.

Proof. If $\xi > \xi_1$, then $\underline{S}(\xi) = 0$ and (2.6) holds. If $\xi < \xi_1$, then $\underline{S}(\xi) = S_0 - \rho e^{\alpha\xi}$ and $\overline{I}(\xi) = e^{\lambda_1\xi}$. According to the assumptions (A1), (A2) and (2.5), one has

$$\begin{aligned}
& d_1 J * \underline{S}(\xi) - d_1\underline{S}(\xi) - c\underline{S}'(\xi) - f(\underline{S}(\xi))g(\overline{I}(\xi - c\tau)) \\
\geq\ & d_1 J * \underline{S}(\xi) - d_1\underline{S}(\xi) - c\underline{S}'(\xi) - f(S_0)g'(0)\overline{I}(\xi - c\tau) \\
=\ & e^{\alpha\xi}\left(-d_1\rho\int_{\mathbb{R}} J(y)(e^{-\alpha y} - 1)dy + c\rho\alpha - f(S_0)g'(0)e^{-\lambda_1 c\tau}e^{(\lambda_1-\alpha)\xi}\right).
\end{aligned} \tag{2.7}$$

Since $0 < \alpha < \lambda_1$ and $e^{(\lambda_1-\alpha)\xi} < (\frac{S_0}{\rho})^{\frac{\lambda_1-\alpha}{\alpha}}$ for $\xi < \xi_1$, it follows from (2.7) that

$$d_1 J * \underline{S}(\xi) - d_1\underline{S}(\xi) - c\underline{S}'(\xi) - f(\underline{S}(\xi))g(\overline{I}(\xi - c\tau))$$

$$\geq \quad e^{\alpha\xi}\left(-d_1\rho\int_{\mathbb{R}}J(y)(e^{-\alpha y}-1)dy+c\rho\alpha-f(S_0)g'(0)e^{-\lambda_1 c\tau}\left(\frac{S_0}{\rho}\right)^{\frac{\lambda_1-\alpha}{\alpha}}\right). \tag{2.8}$$

Taking $\rho=\frac{1}{\alpha}$ in (2.8) and noting that

$$\lim_{\alpha\to 0^+}(\alpha S_0)^{\frac{\lambda_1-\alpha}{\alpha}}=0, \quad \lim_{\alpha\to 0^+}\frac{1}{\alpha}\int_{\mathbb{R}}J(y)(e^{-\alpha y}-1)dy=0,$$

for sufficiently small $\alpha>0$, we have

$$-\frac{d_1}{\alpha}\int_{\mathbb{R}}J(y)(e^{-\alpha y}-1)dy+c-f(S_0)g'(0)e^{-\lambda_1 c\tau}(\alpha S_0)^{\frac{\lambda_1-\alpha}{\alpha}}>0. \tag{2.9}$$

Owing to (2.8) and (2.9), we find

$$\begin{aligned} & d_1J*\underline{S}(\xi)-d_1\underline{S}(\xi)-c\underline{S}'(\xi)-f(\underline{S}(\xi))g(\bar{I}(\xi-c\tau)) \\ \geq\ & e^{\alpha\xi}\left(-\frac{d_1}{\alpha}\int_{\mathbb{R}}J(y)(e^{-\alpha y}-1)dy+c-f(S_0)g'(0)e^{-\lambda_1 c\tau}(\alpha S_0)^{\frac{\lambda_1-\alpha}{\alpha}}\right) \\ >\ & 0. \end{aligned}$$

This completes the proof. □

Lemma 2.4. *Assume that* $0<\eta<\min\{\lambda_2-\lambda_1,\lambda_1\}$. *Then for sufficiently large* $M>1$, *the function* $\underline{I}(\xi)$ *satisfies*

$$c\underline{I}'(\xi)\leq d_2(J*\underline{I}(\xi)-\underline{I}(\xi))+f(\underline{S}(\xi))g(\underline{I}(\xi-c\tau))-\gamma\underline{I}(\xi)-\varepsilon\underline{I}^2(\xi) \tag{2.10}$$

for any $\xi\neq\xi_2:=\frac{1}{\eta}\ln\frac{1}{M}$.

Proof. If $\xi>\xi_2$, then $\underline{I}(\xi)=0$, $J*\underline{I}(\xi)\geq 0$ and $g(\underline{I}(\xi-c\tau))\geq 0$ (by (A2)), Thus (2.10) holds. If $\xi<\xi_2$, then we can take $M_1>1$ such that $\frac{1}{\eta}\ln\frac{1}{M_1}+1=\xi_1$ and choose large enough $M\geq M_1$ with $\underline{I}(\xi)=e^{\lambda_1\xi}(1-Me^{\eta\xi})$ and $\underline{S}(\xi)=S_0-\rho e^{\alpha\xi}$. Thus, (2.10) is equivalent to

$$\begin{aligned} & f(S_0)g'(0)\underline{I}(\xi-c\tau)-f(\underline{S}(\xi))g(\underline{I}(\xi-c\tau))+\varepsilon\underline{I}^2(\xi) \\ \leq\ & d_2(J*\underline{I}(\xi)-\underline{I}(\xi))-c\underline{I}'(\xi)+f(S_0)g'(0)\underline{I}(\xi-c\tau)-\gamma\underline{I}(\xi) \end{aligned}$$

and so

$$\begin{aligned} & f(S_0)g'(0)\underline{I}(\xi-c\tau)-f(\underline{S}(\xi))g(\underline{I}(\xi-c\tau))+\varepsilon\underline{I}^2(\xi) \\ \leq\ & d_2\int_{\mathbb{R}}J(y)e^{\lambda_1(\xi-y)}(1-Me^{\eta(\xi-y)})dy-d_2e^{\lambda_1\xi}(1-Me^{\eta\xi})-c\left(\lambda_1e^{\lambda_1\xi}-M(\lambda_1+\eta)e^{(\lambda_1+\eta)\xi}\right) \\ & +f(S_0)g'(0)\left(e^{\lambda_1(\xi-c\tau)}-Me^{(\lambda_1+\eta)(\xi-c\tau)}\right)-\gamma\left(e^{\lambda_1\xi}-Me^{(\lambda_1+\eta)\xi}\right) \\ =\ & d_2\left(\int_{\mathbb{R}}J(y)e^{-\lambda_1 y}dy-d_2-c\lambda_1+f(S_0)g'(0)e^{-\lambda_1 c\tau}-\gamma\right)e^{\lambda_1\xi} \\ & -Me^{(\lambda_1+\eta)\xi}\left(d_2\int_{\mathbb{R}}J(y)e^{-(\lambda_1+\eta)y}dy-d_2-c(\lambda_1+\eta)+f(S_0)g'(0)e^{-(\lambda_1+\eta)c\tau}-\gamma\right) \\ =\ & -M\Delta(\lambda_1+\eta,c)e^{(\lambda_1+\eta)\xi}. \end{aligned} \tag{2.11}$$

For any $\widehat{\varepsilon} \in (0, g'(0))$, noting that $\lim_{I \to 0^+} \frac{g(I)}{I} = g'(0)$, there exists a small positive number δ_0 such that

$$\frac{g(I)}{I} \geq g'(0) - \widehat{\varepsilon}, \quad 0 < I < \delta_0. \tag{2.12}$$

Choose M large enough such that $0 < \underline{I}(\xi) < \delta_0$. Then, it follows from (2.12) that

$$\begin{aligned}
& f(S_0)g'(0)\underline{I}(\xi - c\tau) - f(\underline{S}(\xi))g(\underline{I}(\xi - c\tau)) \\
= \ & \left(f(S_0)g'(0) - f(\underline{S}(\xi)) \frac{g(\underline{I}(\xi - c\tau))}{\underline{I}(\xi - c\tau)} \right) \underline{I}(\xi - c\tau) \\
\leq \ & \left(\frac{f(S_0)g'(0) - f(\underline{S}(\xi)) \frac{g(\underline{I}(\xi - c\tau))}{\underline{I}(\xi - c\tau)} + \underline{I}(\xi - c\tau)}{2} \right)^2 \\
\leq \ & \left(\frac{f(S_0)g'(0) - f(\underline{S}(\xi))(g'(0) - \widehat{\varepsilon}) + \underline{I}(\xi - c\tau)}{2} \right)^2 .
\end{aligned} \tag{2.13}$$

Since (2.13) holds for arbitrary sufficiently small $\widehat{\varepsilon} \in (0, g'(0))$ and $\underline{S}(\xi) \to S_0$ for sufficiently large M, one can conclude from (2.13) that

$$f(S_0)g'(0)\underline{I}(\xi - c\tau) - f(\underline{S}(\xi))g(\underline{I}(\xi - c\tau)) \leq \underline{I}^2(\xi - c\tau).$$

Then, to prove (2.11), we only need to show that

$$\underline{I}^2(\xi - c\tau) + \varepsilon \underline{I}^2(\xi) \leq -M\Delta(\lambda_1 + \eta, c)e^{(\lambda_1 + \eta)\xi}.$$

Noting $\underline{I}^2(\xi - c\tau) \leq e^{2\lambda_1 \xi}$ and $\underline{I}^2(\xi) \leq e^{2\lambda_1 \xi}$, it suffices to show that

$$(1 + \varepsilon)e^{(\lambda_1 - \eta)\xi} \leq -M\Delta(\lambda_1 + \eta, c). \tag{2.14}$$

Due to $0 < \eta < \lambda_2 - \lambda_1$, we have $\Delta(\lambda_1 + \eta, c) < 0$ (by Lemma 2.1). Then (2.14) leads to

$$M \geq \frac{(1 + \varepsilon)e^{(\lambda_1 - \eta)\xi}}{-\Delta(\lambda_1 + \eta, c)}.$$

The facts that $\xi < \xi_2 < 0$ and $0 < \eta < \lambda_1$ imply that $e^{(\lambda_1 - \eta)\xi} < 1$. To end the proof, we only need to take

$$M \geq \max\left\{ \frac{1 + \varepsilon}{-\Delta(\lambda_1 + \eta, c)} + 1, M_1 \right\}.$$

This completes the proof. □

Next we define a bounded set as follows:

$$\Gamma_{X,\tau} = \left\{ (\phi(\cdot), \varphi(\cdot)) \in C([-X - c\tau, X], \mathbb{R}^2) \left| \begin{array}{l} \phi(\xi) = \underline{S}(-X), \\ \varphi(\xi) = \underline{I}(-X), \\ \text{for any} \quad \xi \in [-X - c\tau, -X], \\ \underline{S}(\xi) \leq \phi(\xi) \leq S_0, \\ \underline{I}(\xi) \leq \varphi(\xi) \leq \overline{I}(\xi), \\ \text{for any} \quad \xi \in [-X, X], \end{array} \right. \right\}$$

where
$$X > \max\left\{\frac{1}{\eta}\ln M, \frac{1}{\alpha}\ln\frac{\rho}{S_0}\right\}.$$

For any $(\phi(\cdot), \varphi(\cdot)) \in C([-X - c\tau, X], \mathbb{R}^2)$, we define
$$\widehat{\phi}(\xi) = \begin{cases} \phi(X), & \xi > X, \\ \phi(\xi), & -X - c\tau \le \xi \le X, \\ \underline{S}(\xi + c\tau), & \xi < -X - c\tau \end{cases} \tag{2.15}$$

and
$$\widehat{\varphi}(\xi) = \begin{cases} \varphi(X), & \xi > X, \\ \varphi(\xi), & -X - c\tau \le \xi \le X, \\ \underline{I}(\xi + c\tau), & \xi < -X - c\tau. \end{cases} \tag{2.16}$$

We consider the following initial value problems:
$$cS'(\xi) = d_1\int_{\mathbb{R}} J(y)\widehat{\phi}(\xi - y)dy - d_1 S(\xi) - f(S(\xi))g(\varphi(\xi - c\tau)) \tag{2.17}$$

and
$$cI'(\xi) = d_2\int_{\mathbb{R}} J(y)\widehat{\varphi}(\xi - y)dy + f(\phi(\xi))g(\varphi(\xi - c\tau)) - (d_2 + \gamma)I(\xi) - \varepsilon I^2(\xi) \tag{2.18}$$

with
$$S(-X) = \underline{S}(-X),\ I(-X) = \underline{I}(-X). \tag{2.19}$$

By the existence theorem of ordinary differential equations, problems (2.17)-(2.19) admit a unique solution $(S_X(\cdot), I_X(\cdot))$ satisfying $S_X(\cdot) \in C^1([-X, X])$ and $I_X(\cdot) \in C^1([-X, X])$. Thus, we can define an operator $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : \Gamma_{X,\tau} \to C([-X - c\tau, X])$ by
$$\mathcal{F}_1[\phi, \varphi](\xi) = S_X(\xi),\ \mathcal{F}_2[\phi, \varphi](\xi) = I_X(\xi) \text{ for } \xi \in [-X, X]$$

and
$$\mathcal{F}_1[\phi, \varphi](\xi) = S_X(-X),\ \mathcal{F}_2[\phi, \varphi](\xi) = I_X(-X) \text{ for } \xi \in [-X - c\tau, -X].$$

Proposition 2.1. *The operator $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ maps $\Gamma_{X,\tau}$ into $\Gamma_{X,\tau}$.*

Proof. For any $(\phi(\cdot), \varphi(\cdot)) \in \Gamma_{X,\tau}$, we should show that
$$\underline{S}(\xi) \le \mathcal{F}_1[\phi, \varphi](\xi) \le S_0, \quad \underline{I}(\xi) \le \mathcal{F}_2[\phi, \varphi](\xi) \le \bar{I}(\xi), \quad \forall \xi \in [-X, X]$$

and
$$\mathcal{F}_1[\phi, \varphi](\xi) = \underline{S}(-X), \quad \mathcal{F}_2[\phi, \varphi](\xi) = \underline{I}(-X), \quad \forall \xi \in [-X - c\tau, -X].$$

By the definition of the operator $\mathcal{F}$, it is easy to see that the last two equalities hold.

For $\xi \in [-X, X]$, we first consider $\mathcal{F}_1[\phi, \varphi](\xi)$. By the definition of the operator $\mathcal{F}$, it is sufficient to prove $\underline{S}(\xi) \le S_X(\xi) \le S_0$. Note that $f(0) = 0$ (see (A1)). Then it is obvious that 0 is a sub-solution of

(2.17). It follows from the maximum principle that $S_X(\xi) \geq 0$ for $\xi \in [-X, X]$. From the definition of $\widehat{\phi}(\xi)$, (A1) and (A2), we obtain

$$\begin{aligned} & d_1 \int_{\mathbb{R}} J(y)\widehat{\phi}(\xi - y)dy - d_1\overline{S}(\xi) - f(\overline{S}(\xi))g(\varphi(\xi - c\tau)) - c\overline{S}'(\xi) \\ \leq\ & d_1 J * \overline{S}(\xi) - d_1\overline{S}(\xi) - f(\overline{S}(\xi))g(\varphi(\xi - c\tau)) - c\overline{S}'(\xi) \\ \leq\ & 0, \end{aligned}$$

which implies that $\overline{S}(\xi) = S_0$ is a super-solution of (2.17). Thus, we have $S_X(\xi) \leq S_0$ for $\xi \in [-X, X]$. Clearly, $\underline{S}(\xi) = S_0 - \rho e^{\alpha\xi}$ for $\xi \in [-X, \xi_1)$. Thus, utilizing Lemma 2.3 and (A2),

$$\begin{aligned} & c\underline{S}'(\xi) - d_1 \int_{\mathbb{R}} J(y)\widehat{\phi}(\xi - y)dy + d_1\underline{S}(\xi) + f(\underline{S}(\xi))g(\varphi(\xi - c\tau)) \\ \leq\ & c\underline{S}'(\xi) - d_1[J * \underline{S}(\xi) - \underline{S}(\xi)] + f(\underline{S}(\xi))g(\overline{I}(\xi - c\tau)) \\ \leq\ & 0, \end{aligned}$$

for any $\xi \in (-X, \xi_1)$. Since $S_X(-X) = \underline{S}(-X)$, applying the comparison principle, we have $\underline{S}(\xi) \leq S_X(\xi)$ for $\xi \in [-X, \xi_1)$ and so $\underline{S}(\xi) \leq S_X(\xi) \leq S_0$ for all $\xi \in [-X, X]$.

Next, we consider $\mathcal{F}_2[\phi, \varphi](\xi)$. Similarly, we only need to show that $\underline{I}(\xi) \leq I_X(\xi) \leq \overline{I}(\xi)$. First, from the maximum principle, we have $I_X(\xi) \geq 0$ for $\xi \in [-X, X]$. Thus, it follows from Lemma 2.4, $\underline{S}(\xi) \leq \phi(\xi)$, $\underline{I}(\xi) \leq \widehat{\varphi}(\xi)$, (A1), (A2) and $\underline{I}(\xi) = e^{\lambda_1\xi}(1 - Me^{\eta\xi})$ for $\xi \in [-X, \xi_2)$ that

$$\begin{aligned} & c\underline{I}'(\xi) - d_2 \int_{\mathbb{R}} J(y)\widehat{\varphi}(\xi - y)dy - f(\phi(\xi))g(\varphi(\xi - c\tau)) + (d_2 + \gamma)\underline{I}(\xi) + \varepsilon\underline{I}^2(\xi) \\ \leq\ & c\underline{I}'(\xi) - d_2[J * \underline{I}(\xi) - \underline{I}(\xi)] - f(\underline{S}(\xi))g(\underline{I}(\xi - c\tau)) + \gamma\underline{I}(\xi) + \varepsilon\underline{I}^2(\xi) \\ \leq\ & 0 \end{aligned}$$

for all $\xi \in [-X, \xi_2)$. Since $I_X(-X) = \underline{I}(-X)$, the comparison principle implies that $\underline{I}(\xi)$ is a sub-solution of (2.18) on $[-X, \xi_2)$. Recalling the fact that $\underline{I}(\xi) = 0$ for $\xi \in [\xi_2, X]$, it is easy to see that

$$\underline{I}(\xi) \leq I_X(\xi),\ \forall \xi \in [-X, X]. \tag{2.20}$$

Since $\phi(\xi) \leq S_0$ and $\widehat{\varphi(\xi)} \leq \overline{I}(\xi)$ for all $\xi \in [-X, X]$, from (A1), (A2) and Lemma 2.2, we deduce that

$$\begin{aligned} & c\overline{I}'(\xi) - d_2 \int_{\mathbb{R}} J(y)\widehat{\varphi}(\xi - y)dy - f(\phi(\xi))g(\varphi(\xi - c\tau)) + (d_2 + \gamma)\overline{I}(\xi) + \varepsilon\overline{I}^2(\xi) \\ \geq\ & c\overline{I}'(\xi) - d_2[J * \overline{I}(\xi) - \overline{I}(\xi)] - f(S_0)g(\overline{I}(\xi - c\tau)) + \gamma\overline{I}(\xi) + \varepsilon\overline{I}^2(\xi) \\ \geq\ & 0, \end{aligned}$$

which ensures that $\overline{I}(\xi)$ is a super-solution of (2.18) on $[-X, X]$ by the comparison principle. Combining with (2.20), we know that $\underline{I}(\xi) \leq I_X(\xi) \leq \overline{I}(\xi)$ for $\xi \in [-X, X]$. The proof is finished. □

Proposition 2.2. *The operator $\mathcal{F} : \Gamma_{X,\tau} \to \Gamma_{X,\tau}$ is completely continuous.*

Proof. We first show the compactness of $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$. We need to prove that, for any bounded subset $B \subset \Gamma_{X,\tau}$, the set $\mathcal{F}(B)$ is precompact. In view of the definition of the operator $\mathcal{F}$, for any $(S_X, I_X) \in \mathcal{F}(B)$, there exists $(\phi, \varphi) \in B$ such that $\mathcal{F}[\phi,\varphi](\xi) = (S_X, I_X)(\xi)$ for $\xi \in [-X, X]$ and $\mathcal{F}[\phi,\varphi](\xi) = (S_X, I_X)(-X)$ for $\xi \in [-X - c\tau, -X]$.

Since $(\phi, \varphi) \in B$, there exists a constant $M_1 > 0$ such that

$$|S_X(\xi)| \le M_1, \quad |I_X(\xi)| \le M_1, \quad \forall \xi \in [-X - c\tau, X].$$

Moreover, since $(\phi, \varphi) \in B$, from (2.17), (2.18) and the above inequalities, we know that there exists some constant $M_2 > 0$ such that

$$|S'_X(\xi)| \le M_2, \quad |I'_X(\xi)| \le M_2, \quad \forall \xi \in [-X - c\tau, X].$$

It follows that $\mathcal{F}(B)$ is a family of the uniformly bounded and equicontinuous functions. The compactness of $\mathcal{F}(B)$ then follows from the Arzelà-Ascoli theorem and the definition of $\Gamma_{X,\tau}$.

Next we prove the continuity of $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$. By the definition of the operator $\mathcal{F}$, we assume that $(\phi_i(\xi), \varphi_i(\xi)) \in \Gamma_{X,\tau} (i = 1, 2)$ and $S_{X,i}(\xi) = \mathcal{F}_1[\phi_i, \varphi_i](\xi), I_{X,i}(\xi) = \mathcal{F}_2[\phi_i, \varphi_i](\xi)$ for $\xi \in [-X, X]$. We will prove the continuity of $\mathcal{F}$ by the following two steps.

Step 1. The continuity of $\mathcal{F}_1$.

It follows from (2.17) that

$$\begin{aligned} & c(S'_{X,1}(\xi) - S'_{X,2}(\xi)) + d_1(S_{X,1}(\xi) - S_{X,2}(\xi)) \\ = \;& d_1 \int_{\mathbb{R}} J(\xi - y)(\widehat{\phi_1}(y) - \widehat{\phi_2}(y))dy + [f(S_{X,2}(\xi))g(\varphi_2(\xi - c\tau)) - f(S_{X,1}(\xi))g(\varphi_1(\xi - c\tau))], \end{aligned} \tag{2.21}$$

where $\widehat{\phi_i}(\xi) (i = 1, 2)$ is defined analogously as the $\widehat{\phi}(\xi)$ in (2.15).

In view of

$$\begin{aligned} & \left| d_1 \int_{\mathbb{R}} J(\xi - y)(\widehat{\phi_1}(y) - \widehat{\phi_2}(y))dy \right| \\ = \;& d_1 \bigg| \int_{-\infty}^{-X-c\tau} J(\xi - y)(\underline{S}(y + c\tau) - \underline{S}(y + c\tau))dy + \int_{-X-c\tau}^{X} J(\xi - y)(\phi_1(y) - \phi_2(y))dy \\ & + \int_{X}^{+\infty} J(\xi - y)(\phi_1(X) - \phi_2(X))dy \bigg| \\ \le \;& d_1 \int_{-X-c\tau}^{X} J(\xi - y)|\phi_1(y) - \phi_2(y)|dy + d_1 \int_{X}^{+\infty} J(\xi - y)|\phi_1(X) - \phi_2(X)|dy \\ = \;& d_1 \int_{-X-c\tau}^{-X} J(\xi - y)|\underline{S}(-X) - \underline{S}(-X)|dy + d_1 \int_{-X}^{X} J(\xi - y)|\phi_1(y) - \phi_2(y)|dy \\ & + d_1 \int_{X}^{+\infty} J(\xi - y)|\phi_1(X) - \phi_2(X)|dy \\ \le \;& 2d_1 \max_{y \in [-X,X]} |\phi_1(y) - \phi_2(y)|, \end{aligned} \tag{2.22}$$

it follows from (2.5), the mean-value theorem, (A1), (A2) and the definition of $\Gamma_{X,\tau}$ that

$$|f(S_{X,2}(\xi))g(\varphi_2(\xi - c\tau)) - f(S_{X,1}(\xi))g(\varphi_1(\xi - c\tau))|$$

$$
\begin{aligned}
&= |f(S_{X,2}(\xi))g(\varphi_2(\xi - c\tau)) - f(S_{X,2}(\xi))g(\varphi_1(\xi - c\tau)) + f(S_{X,2}(\xi))g(\varphi_1(\xi - c\tau)) \\
&\quad - f(S_{X,1}(\xi))g(\varphi_1(\xi - c\tau))| \\
&\le f(S_0)|g(\varphi_2(\xi - c\tau)) - g(\varphi_1(\xi - c\tau))| + |g(\varphi_1(\xi - c\tau))||f(S_{X,2}(\xi)) - f(S_{X,1}(\xi))| \\
&\le f(S_0)g'(0)|\varphi_2(\xi - c\tau) - \varphi_1(\xi - c\tau)| + g'(0)\varphi_1(\xi - c\tau)L|S_{X,2}(\xi) - S_{X,1}(\xi)| \\
&\le f(S_0)g'(0)\max_{\xi\in[-X,X]}|\varphi_1(\xi) - \varphi_2(\xi)| + Lg'(0)K_\varepsilon|S_{X,2}(\xi) - S_{X,1}(\xi)|. \qquad (2.23)
\end{aligned}
$$

If $S_{X,1}(\xi) - S_{X,2}(\xi) > 0$, then we deduce from (2.21)-(2.23) that

$$
\begin{aligned}
&c(S'_{X,1}(\xi) - S'_{X,2}(\xi)) + (d_1 - Lg'(0)K_\varepsilon)(S_{X,1}(\xi) - S_{X,2}(\xi)) \\
\le\ & f(S_0)g'(0)\max_{\xi\in[-X,X]}|\varphi_1(\xi) - \varphi_2(\xi)| + 2d_1\max_{y\in[-X,X]}|\phi_1(y) - \phi_2(y)|. \qquad (2.24)
\end{aligned}
$$

Applying the Gronwall inequality (see p. 90 of [31]) to (2.24), we obtain that $\mathcal{F}_1$ is continuous on $\Gamma_{X,\tau}$. If $S_{X,1}(\xi) - S_{X,2}(\xi) < 0$, then one can prove the same result. Thus, we show that $\mathcal{F}_1$ is continuous on $\Gamma_{X,\tau}$.

Step 2. The continuity of $\mathcal{F}_2$.

From (2.18), we have

$$
\begin{aligned}
&c(I'_{X,1}(\xi) - I'_{X,2}(\xi)) + (d_2 + \gamma)(I_{X,1}(\xi) - I_{X,2}(\xi)) + \varepsilon(I_{X,1}(\xi) + I_{X,2}(\xi))(I_{X,1}(\xi) - I_{X,2}(\xi)) \\
=\ & d_2\int_{\mathbb{R}} J(\xi - y)(\widehat{\varphi}_1(y) - \widehat{\varphi}_2(y))dy + [f(\phi_1(\xi))g(\varphi_1(\xi - c\tau)) - f(\phi_2(\xi))g(\varphi_2(\xi - c\tau))], \quad (2.25)
\end{aligned}
$$

where $\widehat{\varphi}_i(\xi)(i = 1, 2)$ is defined analogously as the $\widehat{\varphi}(\xi)$ in (2.16).

In view of $0 \le I_{X,1}(\xi) + I_{X,2}(\xi) \le 2K_\varepsilon$, we then deduce from (2.25) that there exists a nonnegative constant $\widetilde{c}$ such that

$$
\begin{aligned}
&c(I'_{X,1}(\xi) - I'_{X,2}(\xi)) + (d_2 + \gamma + \widetilde{c}\varepsilon)(I_{X,1}(\xi) - I_{X,2}(\xi)) \\
\le\ & d_2\int_{\mathbb{R}} J(\xi - y)(\widehat{\varphi}_1(y) - \widehat{\varphi}_2(y))dy + [f(\phi_1(\xi))g(\varphi_1(\xi - c\tau)) - f(\phi_2(\xi))g(\varphi_2(\xi - c\tau))]. \quad (2.26)
\end{aligned}
$$

Arguing as in the proof of (2.22) and (2.23), we obtain

$$
\left|d_2\int_{\mathbb{R}} J(\xi - y)(\widehat{\varphi}_1(y) - \widehat{\varphi}_2(y))dy\right| \le 2d_2\max_{\xi\in[-X,X]}|\varphi_1(\xi) - \varphi_2(\xi)| \qquad (2.27)
$$

and

$$
\begin{aligned}
&|f(\phi_1(\xi))g(\varphi_1(\xi - c\tau)) - f(\phi_2(\xi))g(\varphi_2(\xi - c\tau))| \\
\le\ & f(S_0)g'(0)\max_{\xi\in[-X,X]}|\varphi_1(\xi) - \varphi_2(\xi)| + Lg'(0)K_\varepsilon\max_{\xi\in[-X,X]}|\phi_1(\xi) - \phi_2(\xi)|. \qquad (2.28)
\end{aligned}
$$

Thus, it follows from (2.26)-(2.28) that

$$
\begin{aligned}
&c(I'_{X,1}(\xi) - I'_{X,2}(\xi)) + (d_2 + \gamma + \widetilde{c}\varepsilon)(I_{X,1}(\xi) - I_{X,2}(\xi)) \\
\le\ & (f(S_0)g'(0) + 2d_2)\max_{\xi\in[-X,X]}|\varphi_1(\xi) - \varphi_2(\xi)| + Lg'(0)K_\varepsilon\max_{\xi\in[-X,X]}|\phi_1(\xi) - \phi_2(\xi)|,
\end{aligned}
$$

which together with the Gronwall inequality implies that $\mathcal{F}_2$ is continuous on $\Gamma_{X,\tau}$. This completes the proof. □

From the definition of $\Gamma_{X,\tau}$, it is easy to see that $\Gamma_{X,\tau}$ is closed and convex. Thus, employing Propositions 2.1, 2.2 and Schauder's fixed point theorem, we can obtain the following result.

Proposition 2.3. *There exists $(S_X(\xi), I_X(\xi)) \in \Gamma_{X,\tau}$ such that*

$$(S_X(\xi), I_X(\xi)) = \mathcal{F}(S_X, I_X)(\xi)$$

and

$$\underline{S}(\xi) \le S_X(\xi) \le S_0,\ \underline{I}(\xi) \le I_X(\xi) \le \overline{I}(\xi),\ \xi \in (-X, X).$$

Next, we wish to obtain the existence of traveling wave solutions of (2.1) on $\mathbb{R}$. Before doing this, we need to give some estimates for $S_X(\xi)$ and $I_X(\xi)$ in the following space:

$$C^{1,1}([-X, X]) = \left\{u \in C^1[-X, X] | u \text{ and } u' \text{ are Lipschitz continuous}\right\}$$

with the norm

$$\|u(x)\|_{C^{1,1}([-X,X])} := \max_{x\in[-X,X]} |u(x)| + \max_{x\in[-X,X]} |u'(x)| + \sup_{x,y\in[-X,X],x\neq y} \frac{|u'(x) - u'(y)|}{|x - y|}.$$

Proposition 2.4. *Let $(S_X(\xi), I_X(\xi))$ be the fixed point of the operator $\mathcal{F}$ which is guaranteed by Proposition 2.3. Then there exists a positive constant C_1 independent of X such that*

$$\|S_X(\xi)\|_{C^{1,1}([-X,X])} < C_1, \quad \|I_X(\xi)\|_{C^{1,1}([-X,X])} < C_1$$

for all

$$X > \max\left\{\frac{1}{\eta}\ln M, \frac{1}{\alpha}\ln\frac{\rho}{S_0}\right\}.$$

Proof. First, we know that $(S_X(\xi), I_X(\xi))$ satisfies

$$\begin{cases} cS'_X(\xi) = d_1 \int_{\mathbb{R}} J(y)\widehat{S_X}(\xi - y)dy - d_1 S_X(\xi) - f(S_X(\xi))g(I_X(\xi - c\tau)),\ \xi \in [-X, X], \\ S_X(\xi) = \underline{S}(-X),\ \xi \in [-X - c\tau, -X] \end{cases} \tag{2.29}$$

and

$$\begin{cases} cI'_X(\xi) = d_2 \int_{\mathbb{R}} J(y)\widehat{I_X}(\xi - y)dy + f(S_X(\xi))g(I_X(\xi - c\tau)) - (d_2 + \gamma)I_X(\xi) - \varepsilon I_X^2(\xi),\ \xi \in [-X, X], \\ I_X(\xi) = \underline{I}(-X),\ \xi \in [-X - c\tau, -X], \end{cases} \tag{2.30}$$

where

$$\widehat{S_X}(\xi) = \begin{cases} S_X(X), & \xi > X, \\ S_X(\xi), & -X - c\tau \le \xi \le X, \\ \underline{S}(\xi + c\tau), & \xi < -X - c\tau \end{cases}$$

and

$$\widehat{I_X}(\xi) = \begin{cases} I_X(X), & \xi > X, \\ I_X(\xi), & -X - c\tau \le \xi \le X, \\ \underline{I}(\xi + c\tau), & \xi < -X - c\tau. \end{cases}$$

By the facts that $S_X(\xi) \le S_0,\ 0 \le \widehat{S_X}(\xi) \le S_0,\ 0 \le \widehat{I_X}(\xi) \le K_\varepsilon$ and $I_X(\xi - c\tau) \le K_\varepsilon$ for $\xi \in [-X, X]$, it follows from (A1), (A3), (2.5) and (2.29) that

$$|S'_X(\xi)| \quad \le \quad \frac{d_1}{c}\left|\int_{\mathbb{R}} J(y)\widehat{S_X}(\xi - y)dy\right| + \frac{d_1}{c}|S_X(\xi)| + \frac{1}{c}|f(S_X(\xi))g(I_X(\xi - c\tau))|$$

$$\leq \frac{1}{c}(2d_1S_0 + f(S_0)g'(0)K_\varepsilon).$$

Thus, there exists a positive constant C_2 independent of X such that

$$\|S_X(\xi)\|_{C^1([-X,X])} < C_2. \tag{2.31}$$

Similar arguments apply to the case $I'_X(\xi)$, we have

$$\|I_X(\xi)\|_{C^1([-X,X])} < C_2. \tag{2.32}$$

Next, we intend to show that $S_X(\xi)$, $I_X(\xi)$, $S'_X(\xi)$ and $I'_X(\xi)$ are Lipschitz continuous. For any $\xi, \eta \in [-X, X]$, it follows from (2.31) and (2.32) that

$$|S_X(\xi) - S_X(\eta)| < C_2|\xi - \eta|, \quad |I_X(\xi) - I_X(\eta)| < C_2|\xi - \eta|, \tag{2.33}$$

and so $S_X(\xi)$ and $I_X(\xi)$ are Lipschitz continuous.

In view of (2.29), we have

$$\begin{aligned}
& c|S'_X(\xi) - S'_X(\eta)| \\
\leq\ & d_1\left|\int_{\mathbb{R}} J(y)(\widehat{S_X}(\xi - y) - \widehat{S_X}(\eta - y))dy\right| + d_1|S_X(\xi) - S_X(\eta)| \\
& + |f(S_X(\xi))g(I_X(\xi - c\tau)) - f(S_X(\eta))g(I_X(\eta - c\tau))| \\
:=\ & B_1 + B_2 + B_3.
\end{aligned} \tag{2.34}$$

From (A3), we know that the kernel function J is Lipschitz continuous and compactly supported. Let L_J be the Lipschitz constant of J and R be the radius of supp J. Then,

$$\begin{aligned}
B_1 &= d_1\left|\int_{\mathbb{R}} J(y)\widehat{S_X}(\xi - y)dy - \int_{\mathbb{R}} J(y)\widehat{S_X}(\eta - y)dy\right| \\
&= d_1\left|\int_{-R}^{R} J(y)\widehat{S_X}(\xi - y)dy - \int_{-R}^{R} J(y)\widehat{S_X}(\eta - y)dy\right| \\
&= d_1\left|\int_{\xi-R}^{\xi+R} J(\xi - y)\widehat{S_X}(y)dy - \int_{\eta-R}^{\eta+R} J(\eta - y)\widehat{S_X}(y)dy\right| \\
&= d_1\left|\left(\int_{\eta+R}^{\xi+R} + \int_{\eta-R}^{\eta+R} + \int_{\xi-R}^{\eta-R}\right) J(\xi - y)\widehat{S_X}(y)dy - \int_{\eta-R}^{\eta+R} J(\eta - y)\widehat{S_X}(y)dy\right| \\
&= d_1\Bigg(\left|\int_{\eta+R}^{\xi+R} J(\xi - y)\widehat{S_X}(y)dy\right| + \left|\int_{\xi-R}^{\eta-R} J(\xi - y)\widehat{S_X}(y)dy\right| \\
&\quad + \left|\int_{\eta-R}^{\eta+R} [J(\xi - y) - J(\eta - y)]\widehat{S_X}(y)dy\right|\Bigg) \\
&\leq d_1(2S_0\|J\|_{L^\infty} + 2RL_JS_0)|\xi - \eta|
\end{aligned}$$

and

$$B_3 = |f(S_X(\xi))g(I_X(\xi - c\tau)) - f(S_X(\eta))g(I_X(\eta - c\tau))|$$

$$
\begin{aligned}
&\leq \ |f(S_X(\xi))||g(I_X(\xi - c\tau)) - g(I_X(\eta - c\tau))| + |g(I_X(\eta - c\tau))||f(S_X(\xi)) - f(S_X(\eta))| \\
&\leq \ f(S_0)g'(0)|I_X(\xi) - I_X(\eta)| + Lg'(0)K_\varepsilon|S_X(\xi) - S_X(\eta)|, \qquad (2.35)
\end{aligned}
$$

in which we have used the mean-value theorem, the assumptions (A1), (A2) and inequality (2.5). Combining (2.33), (2.34) and (2.35), there exists some positive constant L_1 independent of X such that

$$|S_X'(\xi) - S_X'(\eta)| \leq L_1|\xi - \eta|$$

and so S_X' is Lipschitz continuous. It follows from (2.30) that

$$
\begin{aligned}
& c|I_X'(\xi) - I_X'(\eta)| \\
\leq \ & d_2\left|\int_{\mathbb{R}} J(y)[\widehat{I_X}(\xi - y) - \widehat{I_X}(\eta - y)]dy\right| + (d_2 + \gamma)|I_X(\xi) - I_X(\eta)| \\
& + \varepsilon|I_X^2(\xi) - I_X^2(\eta)| + |f(S_X(\xi))g(I_X(\xi - c\tau)) - f(S_X(\eta))g(I_X(\eta - c\tau))|.
\end{aligned}
$$

Analogously, we have

$$|I_X'(\xi) - I_X'(\eta)| \leq L_1|\xi - \eta|$$

and so I_X' is Lipschitz continuous. Thus, there is a constant C_1 independent of X such that

$$\|S_X(\xi)\|_{C^{1,1}([-X,X])} < C_1, \quad \|I_X(\xi)\|_{C^{1,1}([-X,X])} < C_1.$$

This ends the proof. □

Now, we are in a position to derive the existence of solutions for (2.1) on $\mathbb{R}$ by a limiting argument.

Theorem 2.1. *Let $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma} > 1$. Then, for any $c > c^*$, (2.1) admits a solution $(S(\xi), I(\xi))$ such that*

$$\underline{S}(\xi) \leq S(\xi) \leq S_0, \ \underline{I}(\xi) \leq I(\xi) \leq \overline{I}(\xi). \qquad (2.36)$$

Proof. Choose a sequence $\{X_n\}_{n=1}^{\infty}$ satisfying

$$X_n > \max\left\{\frac{1}{\eta}\ln M, \frac{1}{\alpha}\ln\frac{\rho}{S_0}\right\}$$

and $\lim_{n\to+\infty} X_n = +\infty$. Then, for each $n \in \mathbb{N}$, the solution $(S_{X_n}(\xi), I_{X_n}(\xi)) \in \Gamma_{X_n,\tau}$ satisfies Propositions 2.3 and 2.4, Eqs.(2.29) and (2.30) in $\xi \in [-X_n - c\tau, X_n]$ for every $c > c^*$.

According to the estimates in Proposition 2.4, for the sequence $\{(S_{X_n}(\xi), I_{X_n}(\xi))\}$, we can extract a subsequence by a standard diagonal argument, denoted by $\{(S_{X_{n_k}}(\xi), I_{X_{n_k}}(\xi))\}_{k\in\mathbb{N}}$, such that

$$S_{X_{n_k}}(\xi) \to S(\xi), \ I_{X_{n_k}}(\xi) \to I(\xi) \text{ in } C^1_{loc}(\mathbb{R}) \text{ as } k \to \infty \qquad (2.37)$$

and

$$\begin{cases} cS_{X_{n_k}}'(\xi) = d_1\int_{\mathbb{R}} J(y)\widehat{S_{X_{n_k}}}(\xi - y)dy - d_1S_{X_{n_k}}(\xi) - f(S_{X_{n_k}}(\xi))g(I_{X_{n_k}}(\xi - c\tau)), \xi \in [-X_{n_k}, X_{n_k}], \\ S_{X_{n_k}}(\xi) = \underline{S}(-X_{n_k}), \ \xi \in [-X_{n_k} - c\tau, -X_{n_k}] \end{cases} \qquad (2.38)$$

with

$$\begin{cases} cI'_{X_{n_k}}(\xi) = d_2\int_{\mathbb{R}} J(y)\widehat{I_{X_{n_k}}}(\xi-y)dy + f(S_{X_{n_k}}(\xi))g(I_{X_{n_k}}(\xi-c\tau)) \\ \qquad\qquad -(d_2+\gamma)I_{X_{n_k}}(\xi) - \varepsilon I^2_{X_{n_k}}(\xi),\ \xi\in[-X_{n_k}, X_{n_k}], \\ I_{X_{n_k}}(\xi) = \underline{I}(-X_{n_k}),\ \xi\in[-X_{n_k}-c\tau, -X_{n_k}] \end{cases} \tag{2.39}$$

and

$$\underline{S}(\xi) \le S_{X_{n_k}}(\xi) \le S_0,\ \underline{I}(\xi) \le I_{X_{n_k}}(\xi) \le \overline{I}(\xi),\ \xi\in(-X_{n_k}, X_{n_k}), \tag{2.40}$$

where $\widehat{S_{X_{n_k}}}(\xi)$ and $\widehat{I_{X_{n_k}}}(\xi)$ are defined analogously as the $\widehat{\phi}(\xi)$ and $\widehat{\varphi}(\xi)$ in (2.15) and (2.16), respectively. Sine J is compactly supported (see(A3)), by the Lebesgue dominated convergence theorem, one has

$$\begin{aligned} \lim_{k\to+\infty}\int_{\mathbb{R}} J(y)\widehat{S_{X_{n_k}}}(\xi-y)dy &= \lim_{k\to+\infty}\int_{-R}^{R} J(y)\widehat{S_{X_{n_k}}}(\xi-y)dy \\ &= \lim_{k\to+\infty}\int_{\xi-R}^{\xi+R} J(\xi-y)\widehat{S_{X_{n_k}}}(y)dy \\ &= \int_{\xi-R}^{\xi+R} J(\xi-y)S(y)dy \\ &= \int_{\mathbb{R}} J(y)S(\xi-y)dy \\ &= J*S(\xi), \quad \forall\xi\in\mathbb{R}. \end{aligned} \tag{2.41}$$

Similarly, we can show that

$$\lim_{k\to+\infty}\int_{\mathbb{R}} J(y)\widehat{I_{X_{n_k}}}(\xi-y)dy = \int_{\mathbb{R}} J(y)I(\xi-y)dy = J*I(\xi), \quad \forall\xi\in\mathbb{R}. \tag{2.42}$$

Furthermore, in light of the continuity of f and g, we obtain

$$\lim_{k\to+\infty} f(S_{X_{n_k}}(\xi))g(I_{X_{n_k}}(\xi-c\tau)) = f(S(\xi))g(I(\xi-c\tau)), \quad \forall\xi\in\mathbb{R}. \tag{2.43}$$

Thus, passing to limits in (2.38), (2.39) and (2.40) as $k\to+\infty$, we derive from (2.37), (2.41)-(2.43) that $(S(\xi), I(\xi))$ satisfies (2.1) and (2.36). The proof of this theorem is finished. □

3. Existence of traveling wave solutions with critical speed

In this section, we will prove the existence of traveling wave solutions of (1.3) satisfying (1.5).

Theorem 3.1. *Let $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma} > 1$. Then for any $c \ge c^*$, (1.3) admits a pair of functions $(S(\xi), I(\xi))$ such that*

$$\underline{S}(\xi) \le S(\xi) \le S_0,\ \underline{I}(\xi) \le I(\xi) \le \overline{I}(\xi). \tag{3.1}$$

Proof. For $c > c^*$, let $\{\varepsilon_n\}$ be a sequence such that $0 < \varepsilon_{n+1} < \varepsilon_n < 1 (n = 1, 2, 3, \cdots)$ and $\lim_{n\to+\infty} \varepsilon_n = 0$. By Theorem 2.1 and Proposition 2.4, for each $n \in \mathbb{N}$, there exists a solution $\Phi_n(\xi) = (S_n(\xi), I_n(\xi))$ for $\varepsilon = \varepsilon_n$, such that

$$\begin{cases} cS_n'(\xi) = d_1(J * S_n(\xi) - S_n(\xi)) - f(S_n(\xi))g(I_n(\xi - c\tau)), \\ cI_n'(\xi) = d_2(J * I_n(\xi) - I_n(\xi)) + f(S_n(\xi))g(I_n(\xi - c\tau)) - \gamma I_n(\xi) - \varepsilon_n I_n^2(\xi) \end{cases} \tag{3.2}$$

and

$$\underline{S}(\xi) \le S_n(\xi) \le S_0,\ \underline{I}(\xi) \le I_n(\xi) \le \overline{I}(\xi) \tag{3.3}$$

for all $\xi \in \mathbb{R}$.

Furthermore, we know that

$$\|S_n(\xi)\|_{C^{1,1}(\mathbb{R})} + \|I_n(\xi)\|_{C^{1,1}(\mathbb{R})} < C_3, \tag{3.4}$$

where C_3 is a positive constant independent of ξ. Then we can assert that $\{\Phi_n(\xi)\}$ and $\{\Phi_n'(\xi)\}$ are equicontinuous and uniformly bounded on $\mathbb{R}$. By the Arzelà-Ascoli theorem, there exists a subsequence of $\{\varepsilon_n\}$, still denoted by $\{\varepsilon_n\}$, such that $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\Phi_n(\xi) \to \Phi(\xi),\ \Phi_n'(\xi) \to \Phi'(\xi)$$

uniformly on every closed bounded interval as $n \to \infty$, and hence pointwise on $\mathbb{R}$, where $\Phi(\xi) = (S(\xi), I(\xi))$ and $\Phi'(\xi) = (S'(\xi), I'(\xi))$ are bounded. Passing to the limits in (3.2) and (3.3) as $n \to \infty$ and employing the dominated convergence theorem and the continuity of f and g (see (A1) and (A2)), we obtain that $(S(\xi), I(\xi))$ satisfies (1.3) and (3.1).

For $c = c^*$, one can choose a decreasing sequence $\{c_n\} \in (c^*, c^* + 1)$ such that $\lim_{n\to\infty} c_n = c^*$ and the same reasoning applies to the above case $c > c^*$ and $\varepsilon_n \to 0$. For simplicity, we omit the details. This ends the proof.

Next we aim at the asymptotic behavior of solution $(S(\xi), I(\xi))$ of (1.3), whose existence is guaranteed by Theorem 3.1. For $\xi \in \mathbb{R}$, invoking the Squeeze theorem to (3.1), we deduce the asymptotic behavior of solution $(S(\xi), I(\xi))$ at $-\infty$.

Proposition 3.1. *Suppose that $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma} > 1$ and $c \ge c^*$. Then the solution $(S(\xi), I(\xi))$ of (1.3) satisfies*

$$S(-\infty) = S_0,\ I(-\infty) = 0 \tag{3.5}$$

and

$$\lim_{\xi\to-\infty} e^{-\lambda_1 \xi} I(\xi) = 1.$$

The following proposition shows the asymptotic behavior of $I(\xi)$ at ∞.

Proposition 3.2. *Assume that $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma} > 1$ and $c \ge c^*$. Then the solution $(S(\xi), I(\xi))$ of (1.3) satisfies*

$$0 < \int_{\mathbb{R}} f(S(\xi))g(I(\xi - c\tau))d\xi < \infty \tag{3.6}$$

with $\int_{\mathbb{R}} I(\xi)d\xi < \infty$ and $I(\infty) = 0$.

Proof. Using (3.1), (A1), (A2) and the definitions of $\underline{S}(\xi)$ and $\underline{I}(\xi)$, one has

$$\int_{\mathbb{R}} f(S(\xi))g(I(\xi - c\tau))d\xi \geq \int_{\mathbb{R}} f(\underline{S}(\xi))g(\underline{I}(\xi - c\tau))d\xi > 0.$$

Note that

$$\begin{aligned}
& \int_z^x (J * S(\xi) - S(\xi))d\xi \\
= & \int_z^x \int_{\mathbb{R}} J(y)(S(\xi - y) - S(\xi))dyd\xi \\
= & -\int_z^x \int_{\mathbb{R}} J(y)y \int_0^1 S'(\xi - ty)dtdyd\xi \\
= & \int_{\mathbb{R}} J(y)y \int_0^1 (S(z - ty) - S(x - ty))dtdy.
\end{aligned}$$

Then, by(3.5) and (A3), we get

$$\begin{aligned}
& \lim_{z\to-\infty} \int_z^x (J * S(\xi) - S(\xi))d\xi \\
= & \int_{\mathbb{R}} J(y)y \int_0^1 (S_0 - S(x - ty))dtdy \\
= & -\int_{\mathbb{R}} J(y)y \int_0^1 S(x - ty)dtdy,
\end{aligned}$$

which implies that, for $x \in \mathbb{R}$,

$$\left| \int_{-\infty}^x (J * S(\xi) - S(\xi))d\xi \right| \leq S_0 \int_{\mathbb{R}} J(y)|y|dy := \sigma_0, \tag{3.7}$$

where we used the fact that J is compactly supported (see(A3)). Taking an integration of the first equation in (1.3) over $(-\infty, x)$ and using (3.5) and (3.7), we get

$$\begin{aligned}
& \int_{-\infty}^x f(S(\xi))g(I(\xi - c\tau))d\xi \\
= & \; d_1 \int_{-\infty}^x (J * S(\xi) - S(\xi))d\xi + cS_0 - cS(x) \\
\leq & \; d_1\sigma_0 + cS_0,
\end{aligned}$$

which implies

$$\int_{\mathbb{R}} f(S(\xi))g(I(\xi - c\tau))d\xi < \infty. \tag{3.8}$$

Similar to the proof of (3.7), we have

$$\left| \int_{\mathbb{R}} (J * I(\xi) - I(\xi))d\xi \right| \leq K_\varepsilon \int_{\mathbb{R}} J(y)|y|dy := \sigma_1. \tag{3.9}$$

Taking an integration of the second equation in (1.3) over $\mathbb{R}$ gives

$$\begin{aligned} & cI(+\infty) + \gamma \int_{\mathbb{R}} I(\xi) d\xi \\ = \ & d_2 \int_{\mathbb{R}} (J * I(\xi) - I(\xi)) d\xi + \int_{\mathbb{R}} f(S(\xi)) g(I(\xi - c\tau)) d\xi \\ < \ & \infty, \end{aligned} \tag{3.10}$$

where we have used (3.8) and (3.9).

Consequently, it follows from (3.10) that

$$\int_{\mathbb{R}} I(\xi) d\xi < \infty.$$

Upon combining with the fact that $I'(\xi)$ is bounded on $\mathbb{R}$ (see (3.4)), we have

$$I(+\infty) = 0. \tag{3.11}$$

This completes the proof. □

The following proposition deals with the asymptotic behavior of $S(\xi)$ at ∞.

Proposition 3.3. *Assume that* $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma} > 1$ *and* $c \geq c^*$. *Then (1.3) has a solution* $(S(\xi), I(\xi))$ *such that* $\lim_{\xi \to +\infty} S(\xi)$ *exists and*

$$\lim_{\xi \to +\infty} S(\xi) := S_\infty < S_0.$$

Moreover, there holds

$$\int_{\mathbb{R}} f(S(\xi)) g(I(\xi - c\tau)) d\xi = \gamma \int_{\mathbb{R}} I(\xi) d\xi = c(S_0 - S_\infty).$$

Proof. We prove the existence of $\lim_{\xi \to +\infty} S(\xi)$ by a contradiction argument. Suppose

$$\lim_{\xi \to +\infty} \sup S(\xi) > \lim_{\xi \to +\infty} \inf S(\xi)$$

for a contrary. Then from Fluctuation Lemma (see Lemma 2.2 in [1]), we infer that there exists a sequence $\{\xi_n\}$ satisfying $\xi_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} S(\xi_n) = \lim_{\xi \to +\infty} \sup S(\xi) := \sigma_2 \text{ and } S'(\xi_n) = 0. \tag{3.12}$$

Meanwhile, there exists another sequence $\{\eta_n\}$ satisfying $\eta_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} S(\eta_n) = \lim_{\xi \to +\infty} \inf S(\xi) := \sigma_3 < \sigma_2 \text{ and } S'(\eta_n) = 0. \tag{3.13}$$

Following from the first equation in (1.3), we have

$$cS'(\xi_n) = d_1(J * S(\xi_n) - S(\xi_n)) - f(S(\xi_n)) g(I(\xi_n - c\tau)). \tag{3.14}$$

Passing to the limits in (3.14) as $n \to \infty$, and using (3.11), (3.12) and (A2), we obtain

$$\lim_{n\to\infty} J * S(\xi_n) = \lim_{n\to\infty} S(\xi_n) = \sigma_2. \tag{3.15}$$

Set

$$S_n(y) = S(\xi_n - y). \tag{3.16}$$

We will show that $\lim_{n\to\infty} S_n(y) \to \sigma_2$ for $y \in suppJ := \Omega$. Take sufficiently small $\varepsilon_1 > 0$ and let

$$\Omega_{\varepsilon_1} = \Omega \bigcap \Big\{ y \in \Omega | \lim_{n\to\infty} S_n(y) < \sigma_2 - \varepsilon_1 \Big\}. \tag{3.17}$$

Then from (3.12), (3.15)-(3.17) and (A3) we get

$$\begin{aligned}
\sigma_2 &= \lim_{n\to\infty} J * S(\xi_n) \\
&= \lim_{n\to\infty} \int_\Omega J(y) S(\xi_n - y) dy \\
&= \lim_{n\to\infty} \int_\Omega J(y) S_n(y) dy \\
&\le \limsup_{n\to\infty} \int_{\Omega\setminus\Omega_{\varepsilon_1}} J(y) S_n(y) dy + \limsup_{n\to\infty} \int_{\Omega_{\varepsilon_1}} J(y) S_n(y) dy \\
&\le \sigma_2 \int_{\Omega\setminus\Omega_{\varepsilon_1}} J(y) dy + (\sigma_2 - \varepsilon_1) \int_{\Omega_{\varepsilon_1}} J(y) dy \\
&= \sigma_2 - \varepsilon_1 \int_{\Omega_{\varepsilon_1}} J(y) dy,
\end{aligned}$$

which shows that $m(\Omega_{\varepsilon_1}) = 0$, where $m(\cdot)$ denotes the measure. Therefore, we have $\lim_{n\to\infty} S_n(y) = \sigma_2$ almost everywhere in Ω.

However, since $\{S_n\}$ is an equi-continuous family, the convergence is everywhere in Ω, that is,

$$\lim_{n\to\infty} S_n(y) = \lim_{n\to\infty} S(\xi_n - y) = \sigma_2,\ y \in \Omega. \tag{3.18}$$

Using the similar arguments, we can prove that

$$\lim_{n\to\infty} S(\eta_n - y) = \sigma_3 < \sigma_2,\ y \in \Omega. \tag{3.19}$$

Integrating two sides of the first equation in (1.3) from η_n to ξ_n, using (3.12), (3.13), (3.18), (3.19) and the fact that

$$\lim_{n\to\infty} \int_{\eta_n}^{\xi_n} f(S(\xi)) g(I(\xi - c\tau)) d\xi = 0,$$

we get

$$\begin{aligned}
0 < c(\sigma_2 - \sigma_3) &= c \lim_{n\to\infty} (S(\xi_n) - S(\eta_n)) \\
&= d_1 \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} (J * S(\xi) - S(\xi)) d\xi - \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} f(S(\xi)) g(I(\xi - c\tau)) d\xi
\end{aligned}$$

$$\begin{aligned}
&= d_1 \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} \int_{\mathbb{R}} J(y)(S(\xi-y)-S(\xi))dyd\xi \\
&= d_1 \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} \int_{\mathbb{R}} J(y)(-y) \int_0^1 S'(\xi-ty)dtdyd\xi \\
&= d_1 \lim_{n\to\infty} \int_{\mathbb{R}} J(y)y \int_0^1 (S(\eta_n-ty)-S(\xi_n-ty))dtdy \\
&= 0,
\end{aligned}$$

which leads to a contradiction. Thus, $\lim_{\xi\to\infty}\sup S(\xi) = \lim_{\xi\to\infty}\inf S(\xi)$ and so $\lim_{\xi\to\infty} S(\xi) := S_\infty$ exists.

Next, we will prove that $S_\infty < S_0$. Since $S(\xi) \le S_0$, we have $S_\infty \le S_0$. Assume that $S_\infty = S_0$. Then it follows from (3.5) that

$$S(-\infty) = S_\infty = S_0. \tag{3.20}$$

Taking an integration of the first equation in (1.3) over $\mathbb{R}$ yields

$$\begin{aligned}
&c(S_\infty - S(-\infty)) \\
&= d_1 \int_{\mathbb{R}} (J*S(\xi)-S(\xi))d\xi - \int_{\mathbb{R}} f(S(\xi))g(I(\xi-c\tau))d\xi \\
&= d_1 \left(\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)S(\xi-y)dyd\xi - \int_{\mathbb{R}} S(\xi)d\xi\right) - \int_{\mathbb{R}} f(S(\xi))g(I(\xi-c\tau))d\xi.
\end{aligned} \tag{3.21}$$

By Fubini's theorem and (A3), one has

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)S(\xi-y)dyd\xi - \int_{\mathbb{R}} S(\xi)d\xi \\
&= \int_{\mathbb{R}} J(y)\left(\int_{\mathbb{R}} S(\xi-y)d\xi\right)dy - \int_{\mathbb{R}} S(\xi)d\xi \\
&= \int_{\mathbb{R}} J(y)\left(\int_{\mathbb{R}} S(\xi)d\xi\right)dy - \int_{\mathbb{R}} S(\xi)d\xi \\
&= \int_{\mathbb{R}} S(\xi)d\xi - \int_{\mathbb{R}} S(\xi)d\xi \\
&= 0.
\end{aligned} \tag{3.22}$$

From (3.20)-(3.22), we obtain $\int_{\mathbb{R}} f(S(\xi))g(I(\xi-c\tau))d\xi = 0$, which contradicts (3.6). Thus, we have

$$S_\infty < S_0$$

and

$$\int_{\mathbb{R}} f(S(\xi))g(I(\xi-c\tau))d\xi = c(S_0 - S_\infty). \tag{3.23}$$

Moreover, integrating two sides of the second equation in (1.3) on $\mathbb{R}$ and recalling that $I(\pm\infty) = 0$, one has

$$0 = d_2 \int_{\mathbb{R}} [J*I(\xi)-I(\xi)]d\xi + \int_{\mathbb{R}} f(S(\xi))g(I(\xi-c\tau))d\xi - \int_{\mathbb{R}} \gamma I(\xi)d\xi. \tag{3.24}$$

Using the Fubini theorem again and repeating the above procedures, we can obtain $\int_{\mathbb{R}}[J*I(\xi)-I(\xi)]d\xi = 0$. It follows from (3.23) and (3.24) that

$$\gamma \int_{\mathbb{R}} I(\xi)d\xi = \int_{\mathbb{R}} f(S(\xi))g(I(\xi - c\tau))d\xi = c(S_0 - S_\infty).$$

The proof of this proposition is completed. □

Finally, combining Theorem 3.1 and Propositions 3.1-3.3, we obtain the existence of traveling wave solutions for system (1.2) satisfying (1.5).

Theorem 3.2. *Assume that $\mathcal{R}_0 = \frac{f(S_0)g'(0)}{\gamma} > 1$ and $c \geq c^*$. Then system (1.2) admits a nontrivial and nonnegative traveling wave solution $(S(x+ct), I(x+ct))$ satisfying (1.5).*

Remark 3.1. *(i) It is worth to point out that, in this paper, we have derived the existence of traveling wave solutions in the absence of assumption (H) and further obtained Theorem 3.2 to confirm that c^* is the minimal wave speed of the existence of traveling wave solutions for (1.2), which improves Theorem 1.1 ([18, Theorem 2.7]).*

(ii) Theorem 3.2 states that Theorem 1.2 ([19, Theorem 2.3]) still holds if we take the influences of delays into consideration.

(iii) In (1.1)*, the choice of $f(S) = \beta S$ and $g(I) = I$ leads to the model investigated by Cheng and Yuan [17]. Thus, Theorem 3.2 includes Theorem 3.2 of [17] as a special case.*

4. Application and conclusion

In this section, we will give a typical example to demonstrate the abstract results presented in Section 3. The choice of $f(S) = S$ and $g(I) = \frac{\beta I}{1+\alpha I}$ ($\alpha,\beta > 0$ are two coefficients) in (1.2) leads to

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1(J * S(x,t) - S(x,t)) - \beta S(x,t)\dfrac{I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2(J * I(x,t) - I(x,t)) + \beta S(x,t)\dfrac{I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - \gamma I(x,t). \end{cases} \tag{4.1}$$

Obviously, it is easy to verify that $f(S)$ and $g(I)$ satisfy assumptions (A1)-(A2). Applying Theorem 3.2, we obtain the following result.

Theorem 4.1. *There exists a positive constant c^* such that if $\mathcal{R}_0 = \frac{\beta S_0}{\gamma} > 1$ and $c \geq c^*$. Then system (4.1) admits a nontrivial and nonnegative traveling wave solution $(S(x+ct), I(x+ct))$ satisfying*

$$S(-\infty) = S_0,\ S(+\infty) = S_\infty < S_0,\ I(\pm\infty) = 0. \tag{4.2}$$

We further show that the minimal wave speed c^* is determined by the following system:

$$\Delta(\lambda,c) = 0 \quad \text{and} \quad \frac{\partial \Delta(\lambda,c)}{\partial \lambda} = 0, \quad \text{for} \lambda > 0, c > 0,$$

where

$$\Delta(\lambda,c) := d_2 \int_{\mathbb{R}} J(y)(e^{-\lambda y} - 1)dy - c\lambda + \beta S_0 e^{-\lambda c\tau} - \gamma.$$

It is noticed that the minimal wave speed c^* is relevant to the dispersal rate d_2 and the delay τ. Due to $\Delta(\lambda^*, c^*) = 0$, by the implicit function theorem, a direct calculation gives

$$\frac{dc^*}{dd_2} = \frac{\int_{\mathbb{R}} J(y)e^{-\lambda^* y}dy - 1}{\lambda^* + \lambda^*\tau\beta S_0 e^{-\lambda^* c^* \tau}} > 0,$$

which implies that the geographical movement of infected individuals can increase the speed of the spread of disease. Similarly, we have

$$\frac{dc^*}{d\tau} = \frac{-c^*\beta S_0 e^{-\lambda^* c^* \tau}}{1 + \beta\tau S_0 e^{-\lambda^* c^* \tau}} < 0.$$

That is, the longer the delay τ, the slower the spreading speed.

It is known that the existence and non-existence of the traveling wave solution to nonlinear partial equations have been investigated extensively since they can predict whether or not the disease spread in the individuals and how fast a disease invades geographically. In the present paper, we have studied the traveling wave solutions for a delayed nonlocal dispersal SIR epidemic model with the critical wave speed. It has been found that the existence of traveling wave solutions are totally determined by the basic reproduction number and the minimal wave speed c^*. More precisely, if $\mathcal{R}_0 > 1$ and $c \geq c^*$, then system (1.2) admits a nontrivial and nonnegative traveling wave solution $(S(x + ct), I(x + ct))$ satisfying (1.5). Results on this topic may help one predict how fast a disease invades geographically, and accordingly, take measures in advance to prevent the disease, or at least decrease possible negative consequences. The approaches applied in this paper have prospects for the study of the existence and non-existence of traveling wave solutions for nonlocal dispersal epidemic models with more general nonlinear incidences. Finally, we remark that there are quite a few spaces to deserve further investigations. For example, we can study the asymptotic speed of propagation, the uniqueness and stability of traveling wave solutions. Moreover, the exact boundary behavior of susceptible $S(\xi)$ at $+\infty$ is not obtained although the existence of $S(+\infty)$ is established. We leave these problems for future work.

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Conflict of interest

The authors declare there is no conflict of interest.

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