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Research article

Optimal control of stochastic system with Fractional Brownian Motion

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Abstract: In this paper, we introduce a class of stochastic harvesting population system with Fractional Brownian Motion (FBM), which is still unclear when the stochastic noise has the character of memorability. Stochastic optimal control problems with FBM can not be studied using classical methods, because FBM is neither a Markov pocess nor a semi-martingale. When the external environment impact on the system of FBM, the necessary and sufficient conditions for the optimization are offered through the stochastic maximum principle, Hamilton function and *Itô* formula in our work. To illustrate our study, we provide an example to demonstrate the obtained theoretical results, which is the expansion of certainty population system.

Keywords: optimal harvesting control; *Itô* Formula; Fractional Brownian Motion (FBM); maximum principle

1. Introduction

In the biological population system, it is affected by a variety of external factors, which they are likely to change the population's amount. In order to control the development of biological population reasonably, it is necessary to select appropriate control variables and establish reasonable performance indicators to study the optimal control of stochastic population systems. In the paper, a nonlinear population system equation (the harvesting equation) is discussed. The typical harvesting system can be described in the form:

$$\begin{cases} \frac{\partial p(r,t)}{\partial r} + \frac{\partial p(r,t)}{\partial t} = -\lambda(r,t,P(t))p(r,t) - u(r,t)p(r,t), \\ p(r,0) = p_0, \\ p(0,t) = \int_0^A \beta(r,t,P(t))p(r,t)dr, \\ P(t) = \int_0^A p(r,t)dt. \end{cases}$$

where $(r,t) \in Q$, $t \in (0,T)$, $r \in (0,A)$, $0 < A < \infty$. Since the system with the external factors, we are going to introduce the stochastic harvesting equations with Fractional Brownian Motion (FBM) as follows:

$$\begin{cases} \frac{\partial p(r,t)}{\partial r} + \frac{\partial p(r,t)}{\partial t} \\ = -\lambda_1(r,t,P(t))p(r,t) - u_1(r,t)p(r,t) + f_1(r,t,P(t)) + g_1(r,t,P(t))\frac{dB_H}{dt}, \\ p(r,0) = p_0, p(0,t) = \int_0^A \beta_1(r,t,P(t))p(r,t)dr, \\ P(t) = \int_0^A p(r,t)dt. \end{cases} \tag{1.1}$$

where $p(r,t)$ is the density of the population of age r at time t, A is the life expectancy, and $p(A,t) = 0$. λ_1 is the average mortality ratio of the population of age r at time t, β_1 is the average fertility ratio of the population of age r at time t, $u_1(r,t)$ is harvesting effort function, which is the control variable in the model and satisfies: $0 \leq u(t) \leq u_{max}.f_1(r,t,P(t)) + g_1(r,t,P(t))\frac{dB_H}{dt}$ is the stochastic perturbation, effecting of external environment on the population system, such as earthquakes, emigration, impacts of extra terrestrial objects, and so on.

The stochastic model has aroused concern in the recent years. Abel Cadenikls [1] used a stochastic maximum principle for systems with jumps, with applications to finance systems. Zhang [2–6] investigated the stability of numerical solutions for the stochastic age-dependent system. Zhang [7,8] provided the different methods for the numerical solutions in the stochastic system with age-dependent. Pei [10] focuses on asymptotic mean-square boundedness of several numerical methods applied to a class of stochastic age-dependent population equations with Poisson jumps. Emel Savku et al. [11,12] illustrated that they contributed to modern OR by hybrid (continuous-discrete) dynamics of stochastic differential equations with jumps and the optimal control. However, compared with stochastic system driven by the classical Brownian motion, FBM is a family of centered Gaussian random process indexed by the Hurst parameter $H \in (0,1)$ with continuous sample paths. Some special kinds of dynamical systems require both Wiener process and FBM to model their dynamics. Meanwhile, few has been done because classical methods to solve stochastic problems can not be used directly, since Fractional Brownian Motion (FBM) is not a semi-martingale and not a Markov process. Ma [13] developed a numerical scheme and show the convergence of the numerical approximation solution to the analytic solution for stochastic age-dependent population equations with FBM. Kloeden [14] used the multilevel Monte Carlo method introduced by Giles [15] to stochastic differential equations with Fractional Brownian Motion of Hurst parameter $H > 1/2$ and achieved a prescribed root mean square error of order ε with a computational effort of order ε^{-2}. Duncan [16] discussed the solutions semi-linear stochastic systems with FBM. Zhou [17] investigated the stability for the delayed neural networks with FBM.

On the other hand, optimal control problems have also attracted wide attention, due to their several applications in population system, economic system, finance system [11,12,18–20]. Luo [18] studied optimal harvesting control problem for an age-dependent competing system of n-dimension competing species. Zhao [19] and Chen [20] talked about optimal control of different stochastic system. He [21] investigated optimal harvesting problem for age-structured species. However, an optimal control problem requires the minimization of a criterion function of the states and control inputs of the system over a set of admissible control functions [22], which creates huge troublesome. Stochastic optimal control problem driven by FBM is the bottleneck problem. In this paper, all the previous fields are combined to consider the optimal control problem of stochastic harvesting population system with FBM, the necessary and sufficient conditions for the optimization are obtained, and the example for the obtained theoretical results is illustrated. We provide below a brief summary of our results.

• We introduce the fractional Brownian noise into a class of stochastic harvesting population system and establish necessary as well as sufficient conditions of optimal control, which has not been studied before;

• Using the stochastic maximum principle, Hamilton function and *Itô* formula to stochastic harvesting equations with Fractional Brownian Motion and study the optimal control of the system;

• The example is presented, and it supports our theoretical results.

The paper is divided into five sections. The assumption, notations and some basic definition are given in section 2. In section 3, we establish necessary as well as sufficient conditions of optimal control. In Section 4, an example is provided to illustrate the theoretical results. The conclusions are given in section 5.

2. Preliminaries of the problem

Definition 2.1. (Fractional Brownian Motion)

For $0 < H < 1$ Fractional Brownian Motion (FBM) $B^H = B^H(t), t \in \mathbf{R}$, for Hurst parameter $H \in (0,1)$ is the Gaussian process with mean 0 for all t : $E[B^H] = 0$ and covariance

$$E[B^{H(t)}B^H(s)] = \tfrac{1}{2}|t|^{2H} + |s|^{2H} - |t-s|^{2H} \qquad t, s \in \mathbf{R}$$

We take $B^H(0) = 0$. For $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is standard Brownian Motion.

Definition 2.2.

Let $f(x)$ is continuous functions, its H order fractional derivative is defined as:

$$f^H(x) = \tfrac{1}{\Gamma(-H)} \int_0^x (x-\xi)^{-H-1} f(\xi) d\xi, \qquad H < 0,$$

let $H > 0$,

$$f^{H-n}(x) = \tfrac{1}{\Gamma(-H+n)} \int_0^x (x-\xi)^{-H+n-1} f(\xi) d\xi,$$

such that

$$B(t, H) = D^{-(H+\frac{1}{2})}.$$

Where D is differential operator, H is Hurst parameter.

Lemma 2.1

Let $f(t)$ is continuous functions, such that

$$\int_0^t f^s (ds)^H = H \int_0^t (t-s)^{H-1} f(s) ds, \qquad 0 < H < 1.$$

In this paper, we discuss stochastic optimal control problems driven by fractional Brownian motion (fBm), and consider the following stochastic control harvesting population system with FBM interval [0,A]:

$$\begin{cases} \frac{dy}{dt} + \lambda(t)y + u(t)y - \beta(t)y = f(t) + g(t)\frac{dB_H}{dt}, t \in [0,T] \\ y(0) = y_0 \geq 0.y(T) = y_T \geq 0. \end{cases} \tag{2.1}$$

Getting motivation from the above facts, we discuss the optimal control problems in the system (2.1) is

$$J(u) = maxmin\tilde{J}(u) \equiv E\int_0^T u(t)p^u(t)dt. \tag{2.2}$$

Were $E(\cdot)$ is expectation operator.

As the standing hypotheses, we always assume that the following conditions are satisfied:

(A_1) $\lambda \in C(Q \times R^+)$ is nonnegative measurable function, where $\int_0^r \lambda(\tau)d\tau < +\infty$, $r < A$, $\int_0^A \lambda(\xi)d\xi = +\infty$.

(A_2) $\beta \in C(Q \times R^+)$ is nonnegative measurable function, where $\sup_{\xi\in(0,A)} \int_0^A \beta(\xi)d\xi \leq 1$.

(A_3) $u \in U_{ad} = U$ which is non-empty convex subset, where $U = L^2(Q)$.

(A_4) All $x_k, y_k \in R^n$, where $\|x_k\| \vee \|y_k\| \leq d(k = 1, 2)$, and there exists a constant $c_d > 0$, such that

$$\|f_1(x_1,y_1,t) - f_1(x_2,y_2,t)\|^2 \vee \|g_1(x_1,y_1,t) - g_1(x_2,y_2,t)\|^2 \vee \|h_1(x_1,y_1,t) - h_1(x_2,y_2,t)\|^2$$

$$\leq c_d(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2).$$

(A_5) All $x, y \in R^n$, and there exists a constant $L > 0$, such that all $t \in [0,T]$ satisfied

$$\|f_1(x,y,t)\|^2 \vee \|g_1(x,y,t)\|^2 \vee \|h_1(x,y,t)\|^2 \leq L(1 + \|x\|^2 \vee \|y\|^2).$$

(A_6) Let $f(t,y,u), g(t,y,u), h(t,y,u)$ are linear functions, we introduce

$$f(t,y,u) = D_t y + E_t u + F_t,$$

$$g(t,y,u) = G_t y + H_t u + I_t.$$

The Hamiltonian function is given by

$$H(t,q,\gamma,y,u) = -uy^u + \langle q, f - \lambda y + \beta y - uy\rangle + \langle\gamma, g(t,y,u)\rangle.$$

Now, we introduce the adjoint equation for our problem. The adjoint equation can be written as:

$$\begin{cases} -\frac{dq}{dt} + \lambda(t)q + uq - \beta(t)q = -L_y + qD_t + \gamma G_t - \gamma\frac{dB_t}{dt}, \\ q(A,t) = 0, q(r,T) = 0. \end{cases} \tag{2.3}$$

Note that the couple(q,γ) is the adjoint process corresponding to the stochastic system $p(r,t)$. The adjoint equation admits one and only one $\mathscr{F}_t-$ adapted solution (q,γ),where $L(t,y,u) = -uy^u$.

Moreover, to ensure that the above stochastic differential equation make sense, we shall consider only those $\mathscr{F}_t-$ predictable control processes $u : u \in U_{ad}$ that satisfy

$$p\{\int_0^T |F_t u_t|dt < \infty, \int_0^T |I_t u_t|dt < \infty\} = 1.$$

3. The main results

This is the main result of this paper, in this section, we derive necessary conditions for a control to be optimal.

Lemma 3.1. J is Gâteaux-differentiable with differential given by

$$\langle J'(u), u\rangle = E[\int_0^T \{\langle y_t^u, L_y\rangle + \langle u_t, L_u\rangle\}dt + \langle q_t, y_t^u\rangle].$$

To obtain Equation (3.1), we use Itô formula [24], Gronwall's inequality [25], and equivalently the formula of integration by parts.

$$\begin{aligned}
& \langle q_t, y_t^u\rangle - \langle q_0, y_0^u\rangle \\
= & \int_0^t \{\langle y_s^u, (L_y + \lambda(s)q_s - \beta(s)q_s - D_s q_s - \gamma_s A_s)\rangle \\
+ & \langle q_s, (-\lambda(s)y_s^u - u(s)y_s^u + \beta(s)y_s^u + f(s))\rangle \\
+ & \langle g, \gamma_s\rangle\}ds + \int_0^t \{\langle q_s, g\rangle + \langle y_s^u, \gamma_s\rangle\}dB_H + 2H\int_0^t s^{2H-1}\|g(s, P_s - P_0)\|_2^2 ds \\
= & \int_0^t \{\langle y_s^u, L_y\rangle + \langle q_s, E_s u + F_s\rangle + \langle \gamma_s, H_s u + I_s\rangle\}ds \\
+ & \int_0^t \{\langle q_s, g\rangle + \langle y_s^u, \gamma_s\rangle\}dB_s^H + 2H\int_0^t s^{2H-1}\|g(s, P_s - P_0)\|_2^2 ds.
\end{aligned} \tag{3.1}$$

The above equation may be rewritten as

$$R_t^u = \langle q_0, y_0^u\rangle + \int_0^t \{\langle y_s^u, L_y\rangle + \langle q_s, E_s u + F_s\rangle + \langle \gamma_s, H_s u + I_s\rangle\}ds + S_t^u.$$

Where we denote for every $u \in U$; $t \in [0, T]$:

$$R_t^u := \langle q_0, y_0^u\rangle - 2H\int_0^t s^{2H-1}\|g(s, P_t) - g(s, P_0\|_2^2 ds.$$

$$S_t^u := \int_0^t \{\langle q_s, g\rangle + \langle y_s^u, \gamma_s\rangle\}dB_s^H.$$

We have to consider the following tow cases:

Case 1: For every $u \in U$: $E[\bar{R}_t^u] \leq [R_t^u]$.

Case 2: For every $u \in U$: $E[\bar{R}_t^u] \geq [R_t^u]$.

Let us consider the function $\tilde{H}$: $[0, T] \times \Omega \times U \to R$ defined by

$$\tilde{H}(t, u) = L(t, y_t, u) - \langle q_t, E_t u\rangle - \langle \gamma_t, H_t u\rangle - 2H\int_0^t s^{2H-1}\|g(s, P_s - P_0)\|_2^2 ds.$$

We note that $H(t, u)$ is convex.

Theorem 3.1. If case 1 hold, then a necessary condition for a control u^* to be optimal for Problem (2.2) is that for every $u^* \in U$:

$$E[\int_0^t \{\langle \tilde{H}_u(t, u^*), u - u^*\rangle\}dt] \geq 0. \tag{3.2}$$

On the other hand, if case 2 holds, then inequality (3.2) is a sufficient condition of optimality for a control u^*.

Proof: Here, we apply previous knowledge and methods to obatain the results, such as Young inequality [2], Itô integral (Lemma 2.1) and the Hölder, Burkholder-Davis-Gundy (BDG) inequalities [22]. According to (2.3), u^* is an optimal control if and only if $u^* \in U$:

$$\langle -J'(u), u-u^*\rangle = E[\int_0^T \{\langle y_t^u - y_t^{u^*}, L_y\rangle + \langle u-u^*, L_u\rangle\}dt + \langle q_T, y_T^{u^*} - y_T^u\rangle] \geq 0. \tag{3.3}$$

In case 1, we see that for every $u \in U$:

$$\begin{aligned} & E[\int_0^T \{\langle \tilde{H}_u(t,u^*), u-u^*\rangle\}dt] \\ = \ & E[\int_0^T \{\langle L_u, u-u^*\rangle + \langle q_t, E_t(u^*-u)\rangle + \langle \gamma_t, H_t(u^*-u)\rangle \\ + \ & 2H\int_0^t s^{2H-1}\|g(s,P_s-P_0)\|_2^2(u^*-u)ds \\ = \ & E[\int_0^T \{\langle L_u, u-u^*\rangle + \langle y_t^u - y_t^{u^*}, L_y\rangle\}dt \\ - \ & \int_0^T \{\langle L_y, y_t^u\rangle + \langle q_t, E_t u\rangle + \langle \gamma_t, H_t u\rangle + 2H\int_0^t s^{2H-1}\|g(s,P_s-P_0)\|_2^2 u ds \\ + \ & \int_0^T \{\langle L_y, y_t^{u^*}\rangle + \langle q_t(\omega), E_t u^*\rangle + \langle \gamma_t(\omega), H_t u^*\rangle + 2H\int_0^t s^{2H-1}\|g(s,P_s-P_0)\|_2^2 u^* ds] \\ \geq \ & \int_0^T \{\langle y_t^u - y_t^{u^*}, L_y\rangle + \langle u-u^*), L_u\rangle\}dt + \langle q_T, y_T^{u^*} - y_T^u\rangle. \end{aligned} \tag{3.4}$$

Thus, in case 1 and in conjunction with (3.2), a necessary condition for a control u^* to be optimal is that $\forall u \in U$

$$\begin{aligned} & E[\int_0^T \{\langle L_u, u-u^*\rangle + \langle q_t, E_t(u-u^*)\rangle + \langle \gamma_t, H_t(u-u^*)\rangle \\ + \ & 2H\int_0^t s^{2H-1}\|g(s,P_s-P_0)\|_2^2(u^*-u)\}ds] \geq 0. \end{aligned} \tag{3.5}$$

Which is equivalent to (3.1).

On the other hand, in case 2, for every $\forall u \in U$:

$$\begin{aligned} & E[\int_0^T \{\langle \tilde{H}_u(t,u^*), u-u^*\rangle\}dt] \\ = \ & E[\int_0^T \{\langle L_u, u-u^*\rangle + \langle q_t, E_t(u^*-u)\rangle + \langle \gamma_t, H_t(u^*-u)\rangle \\ + \ & 2H\int_0^t s^{2H-1}\|g(s,P_s-P_0)\|_2^2(u^*-u)ds \\ = \ & E[\int_0^T \{\langle L_u, u-u^*\rangle + \langle y_t^u - y_t^{u^*}, L_y\rangle\}dt \\ - \ & \int_0^T \{\langle L_y, y_t^u\rangle + \langle q_t, E_t u\rangle + \langle \gamma_t, H_t u\rangle + 2H\int_0^t s^{2H-1}\|g(s,P_s-P_0)\|_2^2 u ds \\ + \ & \int_0^T \{\langle L_y, y_t^{u^*}\rangle + \langle q_t(\omega), E_t u^*\rangle + \langle \gamma_t(\omega), H_t u^*\rangle + 2H\int_0^t s^{2H-1}\|g(s,P_s-P_0)\|_2^2 u^* ds] \\ \leq \ & \int_0^T \{\langle y_t^u - y_t^{u^*}, L_y\rangle + \langle u-u^*), L_u\rangle\}dt + \langle q_T, y_T^{u^*} - y_T^u\rangle. \end{aligned} \tag{3.6}$$

Thus, in case 2, a su1cient condition for a control u^* to be optimal is that (3.6), or equivalently (3.2), holds for every $u^* \in U$.

4. An example

For convenience, we still adopt the notation introduced in Section 2.

Example 4.1. Let the admissible control domain be [0, 1], consider the following optimal control problem

$$J(u) = E[\int_0^T u * pdr],$$

subject to

$$\begin{cases} \frac{\partial p}{\partial r} + \frac{\partial p}{\partial t} = -\frac{1}{(1-r)^2} p - up + 2pt - pt\frac{dB_H}{dt}, \\ p(r,0) = exp(-\frac{1}{(1-r)^2}), \\ p(0,t) = 0, p(r,t) = t^2 \int_0^1 p(r,t)dr. \end{cases} \tag{4.1}$$

Here, B_H stands for Fractional Brownian Motion (FBM). Take $T = 1$, $A = 1$ in Eq. (4.1). We can set this problem in our formulation by taking $H = L^2([0,1] \times [0,1])$, $V = W_0^1([0,1])$(a Sobolev space with elements satisfying the boundary condition above), $f(r,t,P) = 2pt$, $\lambda(r,t) = \frac{1}{1-r^2}$, $\beta(r,t) = t^2$, $g(r,t,P) = p$, and $p(r,0) = exp(-\frac{1}{1-r})$.

Clearly, the operators f ,g and λ satisfy the assumption.

To solve this problem, we must write down Hamiltonian function [26]:

$$H(t,q,\gamma,y,u) = -uq + q(-\frac{1}{(1-r)^2}y + 2yt - uy + t^2 y) + \gamma pt.$$

And adjoint equation [22] is

$$\begin{cases} \frac{dq}{dt} = q(-\frac{1}{(1-r)^2}y + 2yt - uy + t^2 y) - \gamma t + \gamma\frac{dB_t}{dt}, \\ q(A,t) = 0, q(r,T) = 0. \end{cases} \tag{4.2}$$

Solving equation (4.1) and (4.2), for any admissible control $u(t,a) \in \mathcal{U}_{ad}$ and $t \in [0,1]$, if we have the results that the equation (3.2) is valid, which is the necessary and sufficient conditions for optimality in this control problem.

The corresponding Hamiltonian

$$H(t,q,\gamma,y,u) = -uq + q(-\frac{1}{(1-r)^2}y + 2yt - uy + t^2 y) + \gamma pt,$$

and

$$H(t,q,\gamma,y,u^*) = -u^* q + q(-\frac{1}{(1-r)^2}y + 2yt - u^* y + t^2 y) + \gamma pt.$$

If $u^*(r,t)$ is optimal,the necessary condition for (4.1) is

$$\begin{aligned} & E\int_0^1(-uq + q(-\tfrac{1}{(1-r)^2}y + 2yt - uy + t^2 y) + \gamma pt)dt \\ \leq\ & E\int_0^1(-u^* q + q(-\tfrac{1}{(1-r)^2}y + 2yt - u^* y + t^2 y) + \gamma pt)dt. \end{aligned} \tag{4.3}$$

Moreover, because of the solution for state equation $p(r,t)$ is the function of $u(r,t)$, $\lambda(r,t)$, $f(r,t,p)$ and $g(r,t,p)$ satisfying the assumption, we can conclude that

$$E[\int_0^1 \{\langle \tilde{H}_u(t,u^*), u - u^* \rangle\} dt] \geq 0.$$

So the Eq. (4.3) is also the sufficient condition for Eq. (4.1).

5. Conclusions

Existence and optimal control results of the stochastic model with Fractional Brownian Motion (FBM) is studied in this paper. Firstly, we introduce the fractional Brownian noise into a class of stochastic harvesting population system and establish necessary as well as sufficient conditions of optimal control, which has not been studied before. Secondly, Using the stochastic maximum principle, Hamilton function and *Itô* formula to stochastic harvesting equations with Fractional Brownian Motion and study the optimal control of the system. finally, the obtained theoretical results are verified by an illustrative example. As further direction, researchers are invited to investigate the optimal control problem for stochastic model by including Gâteaux-differentiable with differential.

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Conflict of interest

The authors declare that they have no conflict of interest.

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