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Research article

Global analysis and optimal harvesting for a hybrid stochastic phytoplankton-zooplankton-fish model with distributed delays

Yuanpei Xia[1], **Weisong Zhou**[2] **and Zhichun Yang**[1,*]

[1] School of Mathematical Sciences, Chongqing Normal University, Chongqing 401131, China
[2] School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, China

* **Correspondence:** Email: yangzhch@126.com.

Abstract: In this paper, we formulate a phytoplankton-zooplankton-fish model with distributed delays and hybrid stochastic noises involving Brownian motion and Markov chain, and propose an optimal harvesting problem pursuing the maximum of total economic income. By global analysis in terms of some system parameters, we investigate the dynamical behaviors on the well-posedness, bounded- ness, persistence, extinction, stability and attractiveness of the solutions for the stochastic delayed system. Moreover, we provide sufficient and necessary condition ensuring the existence of the optimization solution for the optimization problem and obtain the optimal harvesting effect and the maximum of sustainable yield. Lastly, two numerical examples and their simulations are given to illustrate the effectiveness of our results.

Keywords: phytoplankton-zooplankton-fish model; white noise; Markov switching; distributed delays; optimal harvesting

1. Introduction

In aquatic ecosystems, phytoplankton are taken as basic food source and the first trophic level while zooplankton are primary consumers of phytoplankton in food chains [1]. Some plankton models composed of phytoplankton and zooplankton were formulated, and dynamical behaviors of those models were investigated in the past two decades (e.g., [2–8]). Besides plankton, fish is an essential part in aquatic environments like fishponds, lakes, rivers, oceans, etc.. According to an experiment given in [9], it was clearly showed that the addition of fish to the chain of phytoplankton-zooplankton caused the reduction in the algae intake of zooplankton and the rapid growth of phytoplankton. Thus, it is meaningful to incorporate plankton-feeding fish into the plankton model to form the food chain relationship with three species involving fish, phytoplankton and zooplankton. Actually, the study of asymptotical behavior for the plankton-fish model is closely related to the sustainable development of aquatic ecosystems. Scheffe [10] originally accounted for the effects of planktivorous fish in the phytoplankton-zooplankton interaction model, and Malchow et al. [11] have extended the model to a spatial one. Recently, Prabir Panja et al. [12] formulated a toxin-producing phytoplankton-zooplankton-fish model and obtained some sufficient conditions on stability, the existence of equilibrium and bifurcation of the model. Amit Sharma et al. [13] proposed a delayed plankton-fish model with harvesting, and the bifurcation analysis of the system was carried out by taking the rate of harvesting as the bifurcation parameter. Meng and Wu [14] proposed a delayed phytoplankton-zooplankton-fish model with taxation and nonlinear fish harvesting, and gave the Hopf-bifurcation analysis for the model. Wei et al. [15] considered a time-varying phytoplankton-zooplankton-fish system, and gave some sufficient conditions ensuring global asymptotical stability. Since many omnivorous fishes (like crucian carp, grass carp, carp, engraulis japonicus, cockerel, etc.) feed on both phytoplankton and zooplankton, there are more enriching and complex behaviors for the phytoplankton-zooplankton-fish system with the food chain relationship given in Figure 1 (e.g., [12, 13]).

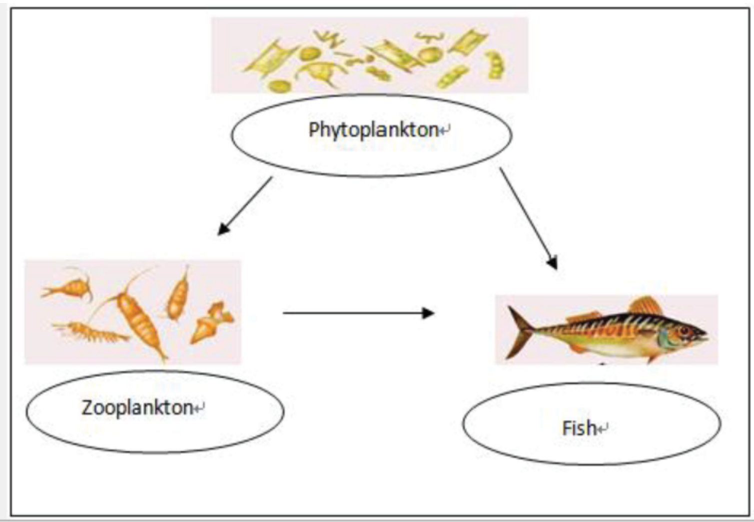

Figure 1. The food chain of the phytoplankton, zooplankton and fish.

On the other hand, delay effects and environmental disturbances are unavoidable in the real world. Time delays occur frequently in many predator-prey models because predators can increase their quantity through digestion, absorption, reproduction and other processes after ingesting the food [16, 17]. Meanwhile, environmental noises often affect dynamic behaviors of the population system [18–20] since the birth rate, death rate, environmental carrying rate and other system parameters of species are easily disturbed by noises. These noises usually involve white noise and telegraph noise. Telegraph noise can be regarded as switching without memory between two or more states, and the time spent in switching between two states is exponentially distributed [21, 22]. Therefore, it is more realistic to formulate a phytoplankton-zooplankton-fish model with time delays and hybrid stochastic noises involving Brownian motion and Markov chain, which essentially belongs to the stochastic predator-prey system. In past decades, some interesting results on stochastic predator-prey models have been investigated. In [23], the authors studied the stationary distribution and global asymptotic stability of a three-species stochastic food-chain system without time-delay. In [24], the authors considered a three-species food chain stochastic system with a hidden Markov chain and proposed two kinds of special dissipative control strategies. One can refer to recent publications on dynamical behaviors of stochastic predator-prey systems [25–28]. To the best of our knowledge, however, there are few works to discuss the dynamics for the stochastic delayed phytoplankton-zooplankton-fish model with the food chain relationship given in Figure 1.

What's more, the optimal harvesting strategy is of great significance to the development of the ecosystem. In the phytoplankton-zooplankton-fish system, the eutrophication of the water body will be controlled by harvesting plankton [29]. Ones can directly benefit from fish, but overfishing may break the balance of the ecosystem. In response to the issue of resource sustainability, Clark and Mesterton-Gibbons et al. have established several types of predator-prey ecological models with optimal harvesting strategies, and discussed how to implement harvesting strategies to maintain fisheries sustainable development [30–33]. The optimal harvesting problem of the stochastic predator-prey model with time delays was investigated by the ergodic method in [34–36]. In the obtained optimization strategies, authors focused on the maximum of total species that has been harvested rather than the maximum of total economic income. By using Pontryagin's maximum principle, the optimal harvesting policy with the maximized present value of revenues was given for a deterministic phytoplankton-zooplankton model [3]. Nevertheless, there are few publications to investigate the optimal harvesting problem in economic income for the stochastic phytoplankton-zooplankton-fish model with time delays.

Motivated by the above discussion, we propose the following hybrid stochastic phytoplankton-zooplankton-fish system with distributed delays and harvesting

$$
\begin{cases}
\mathrm{d}x_1(t) = x_1(t)\Big[a_1(\gamma(t)) - h_1 - c_{11}x_1(t) - c_{12}\int_{-\tau_{12}}^{0} x_2(t+\theta)\mathrm{d}\mu_{12}(\theta) - c_{13}\int_{-\tau_{13}}^{0} x_3(t+\theta)\mathrm{d}\mu_{13}(\theta)\Big]\mathrm{d}t \\
\qquad + \sigma_1(\gamma(t))\, x_1(t)\mathrm{d}\beta_1(t), \\
\mathrm{d}x_2(t) = x_2(t)\Big[-a_2(\gamma(t)) - h_2 + c_{21}\int_{-\tau_{21}}^{0} x_1(t+\theta)\mathrm{d}\mu_{21}(\theta) - c_{22}x_2(t) - c_{23}\int_{-\tau_{23}}^{0} x_3(t+\theta)\mathrm{d}\mu_{23}(\theta)\Big]\mathrm{d}t \\
\qquad + \sigma_2(\gamma(t))\, x_2(t)\mathrm{d}\beta_2(t), \\
\mathrm{d}x_3(t) = x_3(t)\Big[-a_3(\gamma(t)) - h_3 + c_{31}\int_{-\tau_{31}}^{0} x_1(t+\theta)\mathrm{d}\mu_{31}(\theta) + c_{32}\int_{-\tau_{32}}^{0} x_2(t+\theta)\mathrm{d}\mu_{32}(\theta) - c_{33}x_3(t)\Big]\mathrm{d}t \\
\qquad + \sigma_3(\gamma(t))\, x_3(t)\mathrm{d}\beta_3(t),
\end{cases}
\tag{1}
$$

where $x_i(t), i = 1, 2, 3$, is the population size of phytoplankton, zooplankton and fish at time t, respectively, $a_1(\cdot) > 0$ means the growth rate of species x_1, $a_j(\cdot) > 0 (j = 2, 3)$ stands for the death rate of species x_j, $c_{ii} > 0$ shows the intra-specific competition of the ith species, c_{12}, c_{13} and $c_{23} > 0$ denote the capture rates, c_{21}, c_{31} and $c_{32} > 0$ represent the conversation rate of the food, $h_i \geq 0$ is harvesting effort of phytoplankton, zooplankton and fish, respectively, τ_{ij} is time delay and $\mu_{ij}(\theta)$ is a deterministic and nondecreasing function defined on $[-\tau_{ij}, 0]$ satisfying $\int_{-\tau_{ij}}^{0} \mathrm{d}\mu_{ij}(\theta) = 1$, $\beta_i(t)_{t\geq 0}$ is standard independent Brownian motion defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $\sigma_i^2(\cdot)$ represents the intensity of the stochastic noise, $\{\gamma(t), t \geq 0\}$ is a continuous-time Markov chain in a finite state space $\mathbb{S} = \{1, \cdots n\}$ with the generator $Q = (q_{ij})_{n\times n}$, which follows

$$P\{\gamma(t+\delta) = j \mid \gamma(t) = i\} = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } i \neq j; \\ 1 + q_{ij}\delta + o(\delta), & \text{if } i = j, \end{cases}$$

and $\delta>0$, q_{ij} is the transition rate from i to j satisfying $q_{ij} > 0, q_{ii} = -\sum_{j\neq i} q_{ij}$.

In this paper, we mainly study dynamical behaviors and optimal harvesting policy for the above phytoplankton-zooplankton-fish model with distributed delays, hybrid stochastic noises and harvesting. The main purpose and contribution of this paper are listed as follows. Firstly, we formulate a stochastic phytoplankton-zooplankton-fish model with distributed delays and harvesting, in which hybrid stochastic noises involve Brownian motion and Markov chain. Secondly, we give a global analysis of dynamics on persistence, extinction, stability and attractivity in terms of some system parameters for the stochastic delayed system. Lastly, we provide the optimal harvesting policy by solving the following optimization problem with the maximum of total economic income

$$\max \Phi(H) = \lim_{t\to\infty} \sum_{j=1}^{3} r_j h_j E(x_j(t)) - W,$$

$$s.t. \ \lim_{t\to\infty} \frac{\int_0^t x_i(s)ds}{t} > 0, \ \ H = (h_1, h_2, h_3)^T \geq 0, \tag{2}$$

where the harvesting effort H is the decision variable, and the total profit Φ is the objective function, the unit profit $r_j = p_j - q_j > 0$, p_j represents the unit market price of the species x_j, and q_j is the cost unit price of harvesting the species x_j, $j = 1, 2, 3$, W stands for the total fixed cost for harvesting three species.

The remaining part of this paper is organized as follows. In section 2, we give some definitions, assumptions and basic lemmas. In section 3, we give the global analysis of dynamic behavior on stability, persistence and extinction for the system (1). We obtain the sufficient and necessary condition for the optimal harvesting strategy and the maximum of harvesting yield in section 4. Lastly, we illustrate our main results in some examples and their simulations in section 5.

2. Preliminary

In the model (1), we always suppose that Brownian motions $\beta_i(t)$ and the Markov chain $\gamma(t)$ are independent, and the Markov chain is irreducible. According to [37], $\gamma(t)$ is ergodic and has a unique stationary distribution $\xi = (\xi_1, \cdots \xi_n)$ satisfying

$$\xi Q = 0, \sum_{i=1}^{n} \xi_i = 1, \xi_i>0, i = 1, \cdots, n.$$

For simplicity, we define the following notions.

$$\bar{a}_j = \sum_{k=1}^{n} \xi_k a_j(k), \ \ \frac{\bar{\sigma}_j^2}{2} = \sum_{k=1}^{n} \xi_k \frac{\sigma_j^2(k)}{2}, j = 1, 2, 3,$$

$$d_1(\gamma(t)) = a_1(\gamma(t)) - \frac{\sigma_1^2(\gamma(t))}{2}, d_i(\gamma(t)) = a_i(\gamma(t)) + \frac{\sigma_i^2(\gamma(t))}{2}, i = 2, 3,$$

$$\bar{d}_j = \sum_{k=1}^{n} \xi_k d_j(k), j = 1, 2, 3, d = (\bar{d}_1, \bar{d}_2, \bar{d}_3)^T, u_1 = a_1(\gamma(t)) - h_1 - \frac{\sigma_1^2(\gamma(t))}{2},$$

$$u_i = a_i(\gamma(t)) + h_i + \frac{\sigma_i{}^2(\gamma(t))}{2}, i = 2, 3, \bar{u}_j = \sum_{k=1}^{n} \xi_k u_j(k), j = 1, 2, 3,$$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ -c_{21} & c_{22} & c_{23} \\ -c_{31} & -c_{32} & c_{33} \end{bmatrix}, R = \begin{vmatrix} c_{11} & \bar{a}_1 & \bar{\sigma}_1^2/2 + h_1 \\ -c_{21} & -\bar{a}_2 & \bar{\sigma}_2^2/2 + h_2 \\ -c_{31} & -\bar{a}_3 & \bar{\sigma}_3^2/2 + h_3 \end{vmatrix},$$

$$C_1 = \begin{vmatrix} \bar{a}_1 & c_{12} & c_{13} \\ -\bar{a}_2 & c_{22} & c_{23} \\ -\bar{a}_3 & -c_{32} & c_{33} \end{vmatrix}, \tilde{C_1} = \begin{vmatrix} \bar{\sigma}_1^2/2 + h_1 & c_{12} & c_{13} \\ \bar{\sigma}_2^2/2 + h_2 & c_{22} & c_{23} \\ \bar{\sigma}_3^2/2 + h_3 & -c_{32} & c_{33} \end{vmatrix},$$

$$C_2 = \begin{vmatrix} c_{11} & \bar{a}_1 & c_{13} \\ -c_{21} & -\bar{a}_2 & c_{23} \\ -c_{31} & -\bar{a}_3 & c_{33} \end{vmatrix}, \tilde{C_2} = \begin{vmatrix} c_{11} & \bar{\sigma}_1^2/2 + h_1 & c_{13} \\ -c_{21} & \bar{\sigma}_2^2/2 + h_2 & c_{23} \\ -c_{31} & \bar{\sigma}_3^2/2 + h_3 & c_{33} \end{vmatrix},$$

$$C_3 = \begin{vmatrix} c_{11} & c_{12} & \bar{a}_1 \\ -c_{21} & c_{22} & -\bar{a}_2 \\ -c_{31} & -c_{32} & -\bar{a}_3 \end{vmatrix}, \tilde{C_3} = \begin{vmatrix} c_{11} & c_{12} & \bar{\sigma}_1^2/2 + h_1 \\ -c_{21} & c_{22} & \bar{\sigma}_2^2/2 + h_2 \\ -c_{31} & -c_{32} & \bar{\sigma}_3^2/2 + h_3 \end{vmatrix},$$

$$\Delta_1 = c_{22}\bar{a}_1 + c_{12}\bar{a}_2, \ \ \Delta_2 = c_{21}\bar{a}_1 - c_{11}\bar{a}_2, \ \ \Delta_3 = c_{31}\bar{a}_1 - c_{11}\bar{a}_3,$$

$$\tilde{\Delta}_1 = c_{22}(\frac{\bar{\sigma}_1^2}{2} + h_1) - c_{12}(\frac{\bar{\sigma}_2^2}{2} + h_1), \tilde{\Delta}_2 = c_{21}(\frac{\bar{\sigma}_1^2}{2} + h_1) + c_{11}(\frac{\bar{\sigma}_2^2}{2} + h_2), \tilde{\Delta}_3 = c_{31}(\frac{\bar{\sigma}_1^2}{2} + h_1) + c_{11}(\frac{\bar{\sigma}_3^2}{2} + h_3),$$

$$\omega_1 = \frac{\bar{a}_1}{\bar{\sigma}_1^2/2 + h_1}, \ \ \omega_2 = \frac{\Delta_2}{\tilde{\Delta}_2}, \ \ \omega_3 = \frac{C_3}{\tilde{C}_3}, \ \ \Theta_2 = \frac{\Delta_3}{\tilde{\Delta}_3}, \ \ \Theta_3 = \frac{C_2}{\tilde{C}_2},$$

$$\hat{\sigma}_i = \max_{k \in \mathbb{S}}\{|\sigma_i(k)|\}, \hat{a}_i = \max_{k \in \mathbb{S}}\{a_i(k)\}, \check{a}_i = \min_{k \in \mathbb{S}}\{a_i(k)\}, \langle Y_j(t) \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t Y_j(s) \mathrm{d}s,$$

$|C|$ represents the determinant of C, C_{ij} stands for the complement minor of $|C|, i, j = 1, 2, 3$, for $m, n \in R$, $m \wedge n = \min\{m, n\}$, and let I be the unit matrix.

Next, we will give the following assumptions.

(**H₁**) $|C| > 0, C_i > 0, \Delta_j > 0, j = 2, 3$;

(**H₂**) $\tilde{C_i} > 0, i = 1, 2, 3, C_{23} > 0, C_{12} > 0, C_{31} < 0$.

According to the above assumptions, we can see that the system (1) has a positive equilibrium state if there is no stochastic noise and harvesting and that species 1 and species $j, j = 2, 3$ can coexist without stochastic noise, harvesting and other predators [38].

For delayed stochastic system (1), the initial conditions are given in the following form

$$X(t_0 + s) = \phi(s), s \in [-\tau, 0], \phi \in \mathbb{C}\left([-\tau, 0], \mathbb{R}_+^3\right), \gamma(t_0) = k \in \mathbb{S}.$$

Here, we denote $\mathbb{C}\left([-\tau, 0]; \mathbb{R}_+^3\right)$ represents the family of all continuous functions from $[-\tau, 0] \rightarrow \left\{x \in \mathbb{R}^3 | x_1 > 0, x_2 > 0, x_3 > 0\right\}$ and $\tau = \max_{i \neq j}\{\tau_{ij}\}$.

In the following, we give some definitions and lemmas to obtain our main results.

Definition 1. (see [39]) Let $X(t) = (x(t), \gamma(t)) \in \mathbb{R}_+^3 \times \mathbb{S}$ be the solution of the system (1). Then for $i = 1, 2, 3$,
(a) the population $x_i(t)$ is said to be extinct if $\lim_{t\to\infty} x_i(t) = 0$, a.s.;
(b) the population $x_i(t)$ is said to be stable in the mean if $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\mathrm{d}s = c > 0$, a.s..

Definition 2. (see [40]) The system (1) is said to be asymptotically stable in distribution if there exists a probability measure $\pi(\cdot \times \cdot)$ on $\mathbb{R}_+^3 \times \mathbb{S}$ such that transition probability $p(t, \phi, i, \mathrm{d}y \times j)$ of the stochastic process $y(t)$ converges weakly to $\pi(dy \times j)$ as $t \to \infty$ for every initial value $\phi \in C([-\tau, 0], \mathbb{R}_+^3)$ with $\gamma(0) = i \in \mathbb{S}$.

For the biological significance, we first give the well-posedness and boundedness of the system (1).

Lemma 1. For any given initial value $\phi \in \mathbb{C}\left([-\tau, 0], \mathbb{R}_+^3\right)$ with $\gamma(0) = k \in \mathbb{S}$, the system (1) has a unique global positive solution $X(t) = (x(t), \gamma(t)) \in \mathbb{R}_+^3 \times \mathbb{S}$, a.s.. Furthermore, for any $p > 0$, there exist constants $K_i(p) > 0$ such that

$$\limsup_{t\to\infty} E[x_i^p(t)] \leq K_i(p), i = 1, 2, 3.$$

Proof. It is easy to see the function defined in right side of the system (1) obeys the local Lipschitz condition. Then, the system has a unique local solution $X(t)$ on $[0, \tau_e)$, where τ_e stands for the explosion time. We may prove $\tau_e = \infty$ a.s.. Fix a $k_0 > 0$ sufficiently large for $x_1(\phi), x_2(\phi), x_3(\phi) \in (1/k_0, k_0)$. For each integer $k > k_0$, define stopping times as follows

$$\tau_k = \inf\left\{t \in [0, \tau_e) : x_1(t) \notin (1/k, k), x_2(t) \notin (1/k, k), x_3(t) \notin (1/k, k)\right\}.$$

It is clear that τ_k is increasing with k. Setting $\tau_\infty = \lim_{t\to\infty} \tau_k$, we have $\tau_\infty \leq \tau_e$ a.s.. Thus, we only need to prove $\tau_\infty = \infty$, a.s.. For if this statement is false, there exists a $T > 0$ and an $\epsilon \in (0, 1)$ such that $P(\tau_\infty \leq T) > \epsilon$. We can find an integer $k_1 > k_0$ such that $P(\tau_k \leq T) > \epsilon$ for any $k > k_1$. Then take three positive constants χ, κ and $n > 0$ for

$$-\chi c_{11} + \frac{nc_{21} + n\kappa c_{31}}{2} < 0, -c_{22} + \frac{\chi c_{12} + nc_{21} + n^2 c_{32}\kappa}{2n^2} < 0, -c_{33}\kappa + \frac{\chi c_{13} + c_{23} + n\kappa(c_{31} + c_{32})}{2n^2} < 0.$$

We choose n sufficiently large and two positive constants $\kappa > \frac{2c_{22}}{c_{32}}$ and $\chi > \frac{nc_{21}+n\kappa c_{31}}{2c_{11}}$. Define $v_i(x_i) = x_i - 1 - \ln x_i, i = 1, 2, 3,$

$$\begin{aligned} v_4(x_1, x_2, x_3) \quad = \quad & \frac{\chi c_{12}}{2}\int_{-\tau_{12}}^0 \int_{t+\theta}^t x_2^2(s)\mathrm{d}s\mathrm{d}\mu_{12}(\theta) + \frac{\chi c_{13}}{2}\int_{-\tau_{13}}^0 \int_{t+\theta}^t x_3^2(s)\mathrm{d}s\mathrm{d}\mu_{13}(\theta) \\ & + \frac{c_{23}}{2}\int_{-\tau_{23}}^0 \int_{t+\theta}^t x_3^2(s)\mathrm{d}s\mathrm{d}\mu_{23} + \frac{c_{21}}{2}\int_{-\tau_{21}}^0 \int_{t+\theta}^t x_1^2(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta) \\ & + \frac{\kappa c_{31}}{2}\int_{-\tau_{31}}^0 \int_{t+\theta}^t x_1^2(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta) + \frac{\kappa c_{32}}{2}\int_{-\tau_{32}}^0 \int_{t+\theta}^t x_2^2(s)\mathrm{d}s\mathrm{d}\mu_{32}(\theta). \end{aligned}$$

In particular, $v_i(x_i), i = 1, 2, 3$ and $v_4(x_1, x_2, x_3)$ are independent of γ. By the generalized Itô's formula, we can obtain

$$\begin{aligned} \mathrm{d}v_1(x_1) &= \mathcal{L}[v_1(x_1)]\,\mathrm{d}t + \sigma_1(\gamma(t))[x_1 - 1]\,\mathrm{d}\beta_1, \\ \mathrm{d}v_2(x_2) &= \mathcal{L}[v_2(x_2)]\,\mathrm{d}t + \sigma_2(\gamma(t))[x_2 - 1]\,\mathrm{d}\beta_2, \\ \mathrm{d}v_3(x_3) &= \mathcal{L}[v_3(x_3)]\,\mathrm{d}t + \sigma_3(\gamma(t))[x_3 - 1]\,\mathrm{d}\beta_3, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}[v_1(x_1)] &= (x_1 - 1)\Big[a_1(\gamma(t)) - h_1 - c_{11}x_1 - c_{12}\int_{-\tau_{12}}^{0} x_2(t+\theta)\mathrm{d}\mu_{12}(\theta) \\ &\quad -c_{13}\int_{-\tau_{13}}^{0} x_3(t+\theta)\mathrm{d}\mu_{13}(\theta)\Big] + \frac{\sigma_1^2(\gamma(t))}{2}, \\ \mathcal{L}[v_2(x_2)] &= (x_2 - 1)\Big[-a_2(\gamma(t)) - h_2 + c_{21}\int_{-\tau_{21}}^{0} x_1(t+\theta)\mathrm{d}\mu_{21}(\theta) - c_{22}x_2 \\ &\quad -c_{23}\int_{-\tau_{23}}^{0} x_3(t+\theta)\mathrm{d}\mu_{23}(\theta)\Big] + \frac{\sigma_2^2(\gamma(t))}{2}, \\ \mathcal{L}[v_3(x_3)] &= (x_3 - 1)\Big[-a_3(\gamma(t)) - h_3 + c_{31}\int_{-\tau_{31}}^{0} x_1(t+\theta)\mathrm{d}\mu_{31}(\theta) + c_{32}\int_{-\tau_{32}}^{0} x_2(t+\theta)\mathrm{d}\mu_{32}(\theta) \\ &\quad -c_{33}x_3\Big] + \frac{\sigma_3^2(\gamma(t))}{2}. \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}[v_1(x_1)] &\le \frac{\hat{\sigma}_1^2}{2} - (\check{a}_1 - h_1) + \frac{(c_{12}+c_{13})n^2}{2} + (c_{11} + \hat{a}_1 - h_1)x_1 + \frac{c_{12}}{2n^2}\int_{-\tau_{12}}^{0} x_2^2(t+\theta)\mathrm{d}\mu_{12}(\theta) \\ &\quad + \frac{c_{13}}{2n^2}\int_{-\tau_{13}}^{0} x_3^2(t+\theta)\mathrm{d}\mu_{13}(\theta) - c_{11}x_1^2, \\ \mathcal{L}[v_2(x_2)] &\le \frac{\hat{\sigma}_2^2}{2} + \hat{a}_2 + h_2 + \frac{n^2}{2}c_{23} + (c_{22} - \check{a}_2 - h_2)x_2 + (\frac{c_{21}}{2n} - c_{22})x_2^2 + \frac{c_{23}}{2n^2}\int_{-\tau_{23}}^{0} x_3^2(t+\theta)\mathrm{d}\mu_{23}(\theta) \\ &\quad + \frac{nc_{21}}{2}\int_{-\tau_{21}}^{0} x_1^2(t+\theta)\mathrm{d}\mu_{21}(\theta), \\ \mathcal{L}[v_3(x_3)] &\le \frac{\hat{\sigma}_3^2}{2} + \hat{a}_3 + h_3 + (\frac{c_{31}+c_{32}}{2n} - c_{33})x_3^2 + (c_{33} - \check{a}_3 - h_3)x_3 + \frac{nc_{31}}{2}\int_{-\tau_{31}}^{0} x_1^2(t+\theta)\mathrm{d}\mu_{31}(\theta) \\ &\quad + \frac{nc_{32}}{2}\int_{-\tau_{32}}^{0} x_2^2(t+\theta)\mathrm{d}\mu_{32}(\theta). \end{aligned}$$

Define $V(x_1, x_2, x_3) = \chi v_1(x_1) + v_2(x_2) + \kappa v_3(x_3) + v_4(x_1, x_2, x_3)$

$$\begin{aligned} \mathrm{d}V(x_1, x_2, x_3) &= \Big[\chi\mathcal{L}v_1(x_1) + \mathcal{L}v_2(x_2) + \kappa\mathcal{L}v_3(x_3) + \frac{\mathrm{d}}{\mathrm{d}t}v_4(x_1, x_2, x_3)\Big]\mathrm{d}t + \chi(x_1 - 1)\sigma_1(\gamma(t))\,\mathrm{d}\beta_1(t) \\ &\quad + (x_2 - 1)\sigma_2(\gamma(t))\,\mathrm{d}\beta_2(t) + \kappa(x_3 - 1)\sigma_3(\gamma(t))\,\mathrm{d}\beta_3(t), \end{aligned}$$

where

$$
\begin{aligned}
&\chi\mathcal{L}v_1(x_1)+\mathcal{L}v_2(x_2)+\kappa\mathcal{L}v_3(x_3)+\frac{\mathrm{d}}{\mathrm{d}t}v_4(x_1,x_2,x_3)\\
\leq\ & \frac{\hat{\sigma}_1^2}{2}\chi-\chi(\breve{a}_1-h_1)+\chi(c_{11}+\hat{a}_1-h_1)x_1+\frac{(c_{12}+c_{13})n^2}{2}\chi+\frac{c_{12}x_2^2+c_{13}x_3^2}{2n^2}\chi\\
&-\chi c_{11}x_1^2+\frac{\hat{\sigma}_2^2}{2}+\hat{a}_2+h_2+\frac{n^2c_{23}}{2}+(c_{22}-\breve{a}_2-h_2)x_2+(\frac{c_{21}}{2n}-c_{22})x_2^2\\
&+\frac{c_{23}}{2n^2}x_3^2+\frac{nc_{21}}{2}x_1^2+\frac{\hat{\sigma}_3^2}{2}\kappa+\kappa(\hat{a}_3+h_3)+\kappa(\frac{c_{31}+c_{32}}{2n}-c_{33})x_3^2\\
&+\kappa(c_{33}-\breve{a}_3-h_3)x_3+\frac{\kappa nc_{31}}{2}x_1^2+\frac{n\kappa c_{32}}{2}x_2^2\\
=\ &\left(-\chi c_{11}+\frac{nc_{21}}{2}+\frac{\kappa nc_{31}}{2}\right)x_1^2+\chi(c_{11}+\hat{a}_1-h_1)x_1+\left(\chi\frac{c_{12}}{2n^2}+\frac{c_{21}}{2n}-c_{22}\right.\\
&\left.+\frac{\kappa nc_{32}}{2}\right)x_2^2+(c_{22}-\breve{a}_2-h_2)x_2+\left(\frac{\chi c_{13}+c_{23}}{2n^2}+\frac{\kappa(c_{31}+c_{32})}{2n}-\kappa c_{33}\right)x_3^2\\
&+\kappa(c_{33}-\breve{a}_3-h_3)x_3+\frac{\chi\hat{\sigma}_1^2+\hat{\sigma}_2^2+\kappa\hat{\sigma}_3^2}{2}-\chi(\breve{a}_1-h_1)+\frac{\chi n^2}{2}(c_{12}+c_{13})\\
&+\hat{a}_2+h_2+\frac{n^2}{2}c_{23}+\kappa(\breve{a}_3+h_3).
\end{aligned}
$$

Thus there is a $L>0$

$$
\mathrm{d}V(x_1,x_2,x_3)\leq L\mathrm{d}t+\chi(x_1-1)\sigma_1(\gamma(t))\,\mathrm{d}\beta_1(t)+(x_2-1)\sigma_2(\gamma(t))\,\mathrm{d}\beta_2(t)+\kappa(x_3-1)\sigma_3(\gamma(t))\,\mathrm{d}\beta_3(t). \quad (3)
$$

Integral on both sides of Eq (3) from 0 to $\tau_k\wedge T$ and then take expectation, for $k\to\infty$, we can get the following contradiction

$$
\infty=EV(x(\tau_k\wedge T))\leq V(x(0))+\int_0^{\tau_k\wedge T}L\mathrm{d}r\leq V(x(0))+LT<\infty.
$$

Thus, $\tau_e=\infty$, a.s.. We next to prove $\limsup_{t\to\infty}E[x_i^p(t)]\leq K_i(p), i=1,2,3$. Define a function $U_1(x_1)=e^tx_1^p$. According to the generalised Itô's formula

$$
\mathrm{d}U_1(x_1)=\mathcal{L}U_1(x_1)\mathrm{d}t+pe^tx_1^p\sigma_1(\gamma(t))\,\mathrm{d}\beta_1(t), \quad (4)
$$

where

$$
\begin{aligned}
\mathcal{L}U_1(x_1)\ =\ & e^t{x_1}^p\left\{1+\frac{p(p-1)\sigma_1^2(\gamma(t))}{2}+p\left[a_1(\gamma(t))-h_1-c_{11}x_1(t)-c_{12}\int_{-\tau_{12}}^0x_2(t+\theta)\mathrm{d}\mu_{12}(\theta)\right.\right.\\
&\left.\left.-c_{13}\int_{-\tau_{13}}^0x_3(t+\theta)\mathrm{d}\mu_{13}(\theta)\right]\right\}.
\end{aligned}
$$

For $p\geq1$

$$
\mathcal{L}U_1(x_1)\leq e^t\left\{\left[1+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\hat{a}_1\right]x_1^p-pc_{11}x_1^{p+1}\right\}\leq K_1^*(p)e^t,
$$

where $K_1^*(p) = \sup_{x_1>0}\left\{\left[1+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\left(\hat{a}_1-h_1\right)\right]x_1^p - pc_{11}x_1^{p+1}\right\}$ is a constant. For $0<p<1$

$$\mathcal{L}U_1(x_1) \le e^t\left\{\left(1+p\hat{a}_1\right)x_1^p - pc_{11}x_1^{p+1}\right\} \le K_1^{**}(p)e^t,$$

where $K_1^{**}(p) = \sup_{x_1>0}\left\{\left(1+p\hat{a}_1\right)x_1^p - pc_{11}x_1^{p+1}\right\}$. Integral on both sides of the Eq (4) from 0 to t and then take the expectation, We can then show that $E[e^t x_1^p(t)] \le x_1^p(0) + K_1(p)(e^t-1)$, where $K_1(p) = \max\{K_1^*(p), K_1^{**}(p)\}$. Thus, $\limsup_{t\to\infty} E[x_1^p(t)] \le K_1(p)$. Continuing this approach we define

$$U_2(x_1,x_2) = c_1^*U_1(x_1) + e^t {x_2}^p(t) + e^{\tau_{21}}\frac{pn^{1+p}}{1+p}c_{21}\int_{-\tau_{21}}^0\int_{t+\theta}^t e^s {x_1}^{1+p}(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta).$$

Letting an appropriate n>0 such that $c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}}$>0, $p(c_{33}-\frac{pc_{31}}{1+p}n^{-\frac{1+p}{p}}-\frac{pc_{32}}{1+p}n^{-\frac{1+p}{p}})$>0. The constants $c_1^*, {c_2}^*$ and ${c_3}^*$ satisfying

$$c_1^* = \frac{c_{21}}{c_{11}}e^{\tau_{21}}n^{1+p},\quad c_2^* = \frac{e^{\tau_{31}}c_{31}n^{1+p}}{c_{11}(1+p)},\quad c_3^* = \frac{e^{\tau_{32}}c_{32}n^{1+p}}{c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}}}.$$

By the generalized Itô's formula, we obtain

$$\mathrm{d}U_2(x_1,x_2) = \mathcal{L}U_2(x_1,x_2)\mathrm{d}t + c_1^*pe^tx_1^p\sigma_1\left(\gamma(t)\right)\mathrm{d}\beta_1(t) + pe^t{x_2}^p\sigma_2\left(\gamma(t)\right)\mathrm{d}\beta_2(t),$$

where

$$\begin{aligned}\mathcal{L}U_2(x_1,x_2) &= c_1^*\mathcal{L}U_1(x_1) + \mathcal{L}[e^t{x_2}^p] + \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\tau_{21}}\frac{p}{1+p}n^{1+p}c_{21}\int_{-\tau_{21}}^0\int_{t+\theta}^t e^s x_1^{1+p}(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta)\right]\\ &= c_1^*\mathcal{L}U_1(x_1) + e^t{x_2}^p\left\{1+\frac{p(p-1)\sigma_2^2(\gamma(t))}{2} + p\left[-a_2(\gamma(t)) - h_2 + c_{21}\int_{-\tau_{21}}^0 x_1(t+\theta)\mathrm{d}\mu_{21}(\theta)\right.\right.\\ &\quad \left.\left. -c_{22}x_2 - c_{23}\int_{-\tau_{23}}^0 x_3(t+\theta)\mathrm{d}\mu_{23}(\theta)\right]\right\} + e^{\tau_{21}}\frac{pn^{1+p}}{1+p}c_{21}\left[e^tx_1^{1+p} - \int_{-\tau_{21}}^0 e^{t+\theta}x_1^{1+p}(t+\theta)\mathrm{d}\mu_{21}(\theta)\right].\end{aligned}$$

For $p>1$

$$\begin{aligned}\mathcal{L}U_2(x_1,x_2) &\le c_1^*e^t\left\{\left[1+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\hat{a}_1\right]x_1^p - pc_{11}x_1^{1+p}\right\} + e^t\left\{\left[1+\frac{p(p-1)\hat{\sigma}_2^2}{2}\right]{x_2}^p\right.\\ &\quad \left. -p(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}}){x_2}^{1+p} + e^{\tau_{21}}\frac{pn^{1+p}}{1+p}c_{21}x_1^{1+p}\right\}\\ &= e^t\left\{c_1^*\left[1+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\hat{a}_1\right]x_1^p + \left[1+\frac{p(p-1)\hat{\sigma}_2^2}{2}\right]{x_2}^p - \frac{p^2e^{\tau_{21}}}{1+p}n^{1+p}c_{21}x_1^{1+p}\right.\\ &\quad \left. -p(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}}){x_2}^{1+p}\right\}\\ &\le e^tK_2^*(p),\end{aligned}$$

where $K_2^*(p) = \sup_{x_1,x_2>0}\left\{c_1^*\left[1+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\hat{a}_1\right]x_1^p+\left[1+\frac{p(p-1)\hat{\sigma}_2^2}{2}\right]x_2^p-\frac{p^2e^{\tau_{21}}}{1+p}n^{1+p}c_{21}x_1^{1+p}-p(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}})x_2^{1+p}\right\}$, and for $0 < p < 1$

$$\begin{aligned}\mathcal{L}U_2(x_1,x_2) &\leq e^t\left\{c_1^*\left(1+p\hat{a}_1\right)x_1^p + {x_2}^p - \frac{p^2e^{\tau_{21}}}{1+p}n^{1+p}c_{21}x_1^{1+p} - p(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}}){x_2}^{1+p}\right\}\\ &= e^tK_2^{**}(p),\end{aligned}$$

where $K_2^{**}(p) = \sup_{x_1,x_2>0}\left\{c_1^*\left(1+p\hat{a}_1\right)x_1^p + x_2^p - \frac{p^2e^{\tau_{21}}}{1+p}n^{1+p}c_{21}x_1^{1+p} - p(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}})x_2^{1+p}\right\}$. Take the same method as above, we obtain there's a positive constant $K_2(p) = \max\{K_2^*(p), K_2^{**}(p)\}$ such that $\limsup_{t\to\infty} E[x_2^p(t)] \leq K_2(p)$. Next, we define

$$\begin{aligned}U_3(x_1,x_2,x_3) &= c_2^*U_1(x_1) + c_3^*U_2(x_1,x_2) + e^t{x_3}^p(t) + e^{\tau_{32}}\frac{p}{1+p}n^{1+p}c_{32}\int_{-\tau_{32}}^0\int_{t+\theta}^t e^s x_2^{1+p}(s)\mathrm{d}s\mathrm{d}\mu_{32}(\theta)\\ &\quad +e^{\tau_{31}}\frac{p}{1+p}n^{1+p}c_{31}\int_{-\tau_{31}}^0\int_{t+\theta}^t e^s x_1^{1+p}(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta).\end{aligned}$$

Then

$$\mathrm{d}U_3(t) = \mathcal{L}U_3(t)\mathrm{d}t + c_2^*pe^tx_1^p\sigma_1(\gamma(t))\,\mathrm{d}\beta_1(t) + c_3^*pe^tx_2^p\sigma_2(\gamma(t))\,\mathrm{d}\beta_2(t) + pe^tx_3^p\sigma_3(\gamma(t))\,\mathrm{d}\beta_3(t),$$

where

$$\begin{aligned}\mathcal{L}U_3(t) &= c_2^*\mathcal{L}U_1(t) + c_3^*\mathcal{L}U_2(t) + \mathcal{L}[e^t{x_3}^p(t)]\\ &\quad +\frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{\tau_{32}}\frac{p}{1+p}n^{1+p}c_{32}\int_{-\tau_{32}}^0\int_{t+\theta}^t e^s x_2^{1+p}(s)\mathrm{d}s\mathrm{d}\mu_{32}(\theta)\\ &\quad +e^{\tau_{31}}\frac{p}{1+p}n^{1+p}c_{31}\int_{-\tau_{31}}^0\int_{t+\theta}^t e^s x_1^{1+p}(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta)\Big].\end{aligned}$$

For $p > 1$

$$\begin{aligned}\mathcal{L}U_3(t) &\leq c_2^*e^t\left\{\left[1+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\hat{a}_1\right]x_1^p - pc_{11}x_1^{1+p}\right\} + c_3^*e^t\left\{c_1^*\left[1\right.\right.\\ &\quad \left.+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\hat{a}_1\right]x_1^p - \frac{p^2e^{\tau_{21}}}{1+p}n^{1+p}c_{21}x_1^{1+p} + \left[1+\frac{p(p-1)\hat{\sigma}_2^2}{2}\right.\\ &\quad \left.\left.\right]x_2^p - p(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}})x_2^{1+p}\right\} + e^t\left\{\left[1+\frac{p(p-1)\hat{\sigma}_3^2}{2}\right]{x_3}^p\right.\\ &\quad -p\left[c_{33}-\frac{pc_{31}}{1+p}n^{-\frac{1+p}{p}}-\frac{pc_{32}}{1+p}n^{-\frac{1+p}{p}}\right]{x_3}^{1+p} + \frac{pc_{31}}{1+p}n^{1+p}\int_{-\tau_{31}}^0 x_1^{1+p}(t+\theta)\mathrm{d}\mu_{31}(\theta)\\ &\quad \left.+\frac{pc_{32}}{1+p}n^{1+p}\int_{-\tau_{32}}^0 x_2^{1+p}(t+\theta)\mathrm{d}\mu_{32}(\theta)\right\} + e^{\tau_{32}}\frac{p}{1+p}n^{1+p}c_{32}\Big[e^tx_2^{1+p} -\\ &\quad \int_{-\tau_{32}}^0 e^{t+\theta}x_2^{1+p}(t+\theta)\mathrm{d}\mu_{32}(\theta)\Big] + e^{\tau_{31}}\frac{p}{1+p}n^{1+p}c_{31}\Big[e^tx_1^{1+p} - \int_{-\tau_{31}}^0 e^{t+\theta}x_1^{1+p}(t+\theta)\mathrm{d}\mu_{31}(\theta)\Big]\end{aligned}$$

$$\leq \ e^t\Bigg\{\left(1+\frac{p(p-1)\hat{\sigma}_1^2}{2}+p\hat{a}_1\right)(c_2^*+c_3^*c_1^*)x_1^p-\frac{p^2e^{\tau_{21}+\tau_{32}}n^{2+2p}c_{32}c_{21}}{(1+p)(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}})}x_1^{1+p}$$
$$+c_3^*\left(1+\frac{p(p-1)\hat{\sigma}_2^2}{2}\right)x_2^p-\frac{p^2}{1+p}e^{\tau_{32}}c_{32}n^{1+p}x_2^{1+p}+\left(1+\frac{p(p-1)\hat{\sigma}_3^2}{2}\right){x_3}^p$$
$$-p\left(c_{33}-\frac{pc_{31}}{1+p}n^{-\frac{1+p}{p}}-\frac{pc_{32}}{1+p}n^{-\frac{1+p}{p}}\right){x_3}^{1+p}\Bigg\}.$$

For $0 < p < 1$

$$\mathcal{L}U_3(t) \ \leq \ e^t\Bigg\{(1+p\hat{a}_1)(c_2^*+c_3^*c_1^*)x_1^p-\frac{p^2e^{\tau_{21}+\tau_{32}}n^{2+2p}c_{32}c_{21}}{(1+p)(c_{22}-\frac{pc_{21}}{1+p}n^{-\frac{1+p}{p}})}x_1^{1+p}+c_3^*x_2^p$$
$$-\frac{p^2}{1+p}e^{\tau_{32}}c_{32}n^{1+p}x_2^{1+p}+{x_3}^p-p\left(c_{33}-\frac{pc_{31}}{1+p}n^{-\frac{1+p}{p}}-\frac{pc_{32}}{1+p}n^{-\frac{1+p}{p}}\right){x_3}^{1+p}\Bigg\}.$$

Similarly, there's a positive constant $K_3(p)$ satisfying $\mathcal{L}U_3(t) \leq K_3(p)e^t$ and $\limsup_{t\to\infty} E[{x_3}^p(t)] \leq K_3(p)$. This completes the proof.

In the following, we give some basic lemmas.

Lemma 2. [41] Let $Z(t) \in \mathbb{R}_+$ and $g(t)$ be two stochastic processes satisfying $\lim_{t\to\infty} \frac{g(t)}{t} = 0$, a.s.. We have the following conclusion.

(i) If there are three constants $T > 0, l_2 > 0$ and l_1 such that for all $t \geq T$

$$\ln Z(t) \leq l_1 t - l_2 \int_0^t Z(s)\mathrm{d}s + g(t),$$

then

$$\begin{cases} \limsup_{t\to\infty} t^{-1}\int_0^t Z(s)\mathrm{d}s \leq l_1/l_2, \quad a.s., \ \text{if} \quad l_1 \geq 0; \\ \lim_{t\to\infty} Z(t) = 0 \quad a.s., \ \text{if} \quad l_1 < 0. \end{cases}$$

(ii) If there exist three positive constants T, l_1 and l_2 such that

$$\ln Z(t) \geq l_1 t - l_2 \int_0^t Z(s)\mathrm{d}s + g(t), \quad a.s..$$

for all $t \geq T$, then

$$\liminf_{t\to\infty} t^{-1}\int_0^t Z(s)\mathrm{d}s \geq l_1/l_2, \quad a.s..$$

Consider the following auxiliary system:

$$\begin{cases} \mathrm{d}Y_1(t) = Y_1(t)\Big[a_1(\gamma(t)) - h_1 - c_{11}Y_1(t)\Big]\mathrm{d}t + \sigma_1(\gamma(t))Y_1(t)\mathrm{d}\beta_1(t), \\ \mathrm{d}Y_2(t) = Y_2(t)\Big[-a_2(\gamma(t)) - h_2 + c_{21}\int_{-\tau_{21}}^0 Y_1(t+\theta)\mathrm{d}\mu_{21}(\theta) - c_{22}Y_2(t)\Big]\mathrm{d}t + \sigma_2(\gamma(t))Y_2(t)\mathrm{d}\beta_2(t), \\ \mathrm{d}Y_3(t) = Y_3(t)\Big[-a_3(\gamma(t)) - h_3 + c_{31}\int_{-\tau_{31}}^0 Y_1(t+\theta)\mathrm{d}\mu_{31}(\theta) + c_{32}\int_{-\tau_{32}}^0 Y_2(t+\theta)\mathrm{d}\mu_{32}(\theta) - c_{33}Y_3(t)\Big]\mathrm{d}t \\ \qquad + \sigma_3(\gamma(t))Y_3(t)\mathrm{d}\beta_3(t). \end{cases} \tag{5}$$

with initial date $\varphi \in \mathbb{C}\left([-\tau, 0], \mathbb{R}_+^3\right)$ with $\gamma(0) = i \in \mathbb{S}$. Obviously, the system (5) has a unique global positive solution [23].

Lemma 3. For the system (5), we have the following conclusion.
(a) If $\bar{u}_1 < 0$, then $\lim_{t\to\infty} Y_j(t) = 0, a.s., j = 1, 2, 3$;
(b) If $\bar{u}_1 = 0$, then $\langle Y_1(t)\rangle = 0, \lim_{t\to\infty} Y_i(t) = 0, a.s., i = 2, 3$;
(c) If $\bar{u}_1 > 0, \Delta_2 - \tilde{\Delta}_2 < 0$, and $\Delta_3 - \tilde{\Delta}_3 < 0$, then

$$\langle Y_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \lim_{t\to\infty} Y_i(t) = 0, a.s., i = 2, 3.$$

(d) If $\bar{u}_1 > 0, \Delta_2 - \tilde{\Delta}_2 < 0$ and $\Delta_3 - \tilde{\Delta}_3 \geq 0$, then

$$\langle Y_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \lim_{t\to\infty} Y_2(t) = 0, \quad \langle Y_3(t)\rangle = \frac{\Delta_3 - \tilde{\Delta}_3}{c_{11}c_{33}}, a.s..$$

(e) If $\bar{u}_1 > 0, \Delta_2 - \tilde{\Delta}_2 \geq 0$ and $c_{22}(\Delta_3 - \tilde{\Delta}_3) + c_{32}(\Delta_2 - \tilde{\Delta}_2) < 0$, then

$$\langle Y_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \langle Y_2(t)\rangle = \frac{\Delta_2 - \tilde{\Delta}_2}{c_{11}c_{22}}, \quad \lim_{t\to\infty} Y_3(t) = 0, a.s..$$

(f) If $\bar{u}_1 > 0, \Delta_2 - \tilde{\Delta}_2 \geq 0$ and $c_{22}(\Delta_3 - \tilde{\Delta}_3) + c_{32}(\Delta_2 - \tilde{\Delta}_2) \geq 0$, then

$$\langle Y_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \langle Y_2(t)\rangle = \frac{\Delta_2 - \tilde{\Delta}_2}{c_{11}c_{22}}, \quad \langle Y_3(t)\rangle = \frac{c_{22}(\Delta_3 - \tilde{\Delta}_3) + c_{32}(\Delta_2 - \tilde{\Delta}_2)}{c_{11}c_{22}c_{33}}, a.s..$$

Proof. Firstly, let us prove the conclusion (a). By the generalized Itô's formula, we have

$$\ln Y_1(t) = \int_0^t u_1(\gamma(s))\,\mathrm{d}s - c_{11}\int_0^t Y_1(s)\mathrm{d}s + \int_0^t \sigma_1(\gamma(s))\,\mathrm{d}\beta_1(s) + \ln Y_1(0), \tag{6}$$

$$\begin{aligned}\ln Y_2(t) = & -\int_0^t u_2(\gamma(s))\,\mathrm{d}s - c_{22}\int_0^t Y_2(s)\mathrm{d}s + c_{21}\int_0^t\int_{-\tau_{21}}^0 Y_1(s+\theta)\mathrm{d}\mu_{21}(\theta)\mathrm{d}s \\ & + \int_0^t \sigma_2(\gamma(s))\,\mathrm{d}\beta_2(s) + \ln Y_2(0),\end{aligned} \tag{7}$$

$$\begin{aligned}\ln Y_3(t) = & -\int_0^t u_3(\gamma(s))\,\mathrm{d}s + c_{31}\int_0^t\int_{-\tau_{31}}^0 Y_1(s+\theta)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s + c_{32}\int_0^t\int_{-\tau_{32}}^0 Y_2(s+\theta)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s \\ & - c_{33}\int_0^t Y_3(s)\mathrm{d}s + \int_0^t \sigma_3(\gamma(s))\,\mathrm{d}\beta_3(s) + \ln Y_3(0).\end{aligned} \tag{8}$$

Because of the ergodicity of $\gamma(t)$, one gets

$$\lim_{t\to\infty} t^{-1}\int_0^t u_j(\gamma(s))\,\mathrm{d}s = \bar{u}_j, a.s., j = 1, 2, 3. \tag{9}$$

Obviously, for any $\epsilon \in (0, \bar{u}_1)$, there exist a set $\Omega_\epsilon \subset \Omega$ satisfying $P(\Omega_\epsilon) \geq 1 - \epsilon$ and a positive constant $T = T(\epsilon)$ such that for $t \geq T$

$$\bar{u}_1 - \epsilon \leq t^{-1}\ln Y_1(0) + t^{-1}\int_0^t u_1(\gamma(s))\,\mathrm{d}s \leq \bar{u}_1 + \epsilon. \tag{10}$$

Substituting Eq (10) into Eq (6), then

$$
\begin{aligned}
&(\bar{u}_1-\epsilon)t-c_{11}\int_0^t Y_1(s)\mathrm{d}s+\int_0^t \sigma_1(\gamma(s))\,\mathrm{d}\beta_1(s)\\
&\le \ln Y_1(t)\\
&\le (\bar{u}_1+\epsilon)t-c_{11}\int_0^t Y_1(s)\mathrm{d}s+\int_0^t \sigma_1(\gamma(s))\,\mathrm{d}\beta_1(s).
\end{aligned} \tag{11}
$$

Meanwhile, the quadratic variation of the stochastic integral $\int_0^t \sigma_j(\gamma(s))\,\mathrm{d}\beta_j(s)$ is $\int_0^t \sigma_j^2(\gamma(s))\,\mathrm{d}s \le \hat{\sigma}_j^2 t$. The strong law of large number theorem shows

$$
\lim_{t\to\infty}\frac{\int_0^t \sigma_j(\gamma(s))\,\mathrm{d}\beta_j(s)}{t}=0, a.s., j=1,2,3. \tag{12}
$$

When $\bar{u}_1<0$, for Eq (11) and Lemma 2, we can show that $\lim_{t\to\infty} Y_1(t)=0, \lim_{t\to\infty} t^{-1}\int_0^t Y_1(s)\mathrm{d}s=0$. From Eqs (7) and (8), for $t\to\infty$, then

$$
t^{-1}\ln Y_2(t) = -t^{-1}\int_0^t u_2(\gamma(s))\,\mathrm{d}s - c_{22}\langle Y_2(t)\rangle + t^{-1}\int_0^t \sigma_2(\gamma(s))\,\mathrm{d}\beta_2(s)+t^{-1}\ln Y_2(0), \tag{13}
$$

$$
\begin{aligned}
t^{-1}\ln Y_3(t) = {}& -t^{-1}\int_0^t u_3(\gamma(s))\,\mathrm{d}s + c_{32}t^{-1}\langle Y_2(t)\rangle + c_{32}t^{-1}\Big[\int_{-\tau_{32}}^0\int_\theta^0 Y_2(s)\mathrm{d}\mu_{32}(\theta)\\
&-\int_{-\tau_{32}}^0\int_{t+\theta}^t Y_2(s)\mathrm{d}\mu_{32}(\theta)\Big] - c_{33}\langle Y_3(t)\rangle + t^{-1}\int_0^t \sigma_3(\gamma(s))\,\mathrm{d}\beta_3(s)+t^{-1}\ln Y_3(0).
\end{aligned} \tag{14}
$$

For any $\epsilon\in(0,\bar{u}_2)$ satisfying $(-\bar{u}_2+\epsilon)<0$, by Eqs (9) and (13), we have

$$
\ln Y_2(t)\le(-\bar{u}_2+\epsilon)t-c_{22}\int_0^t Y_2(s)\mathrm{d}s+\int_0^t \sigma_2(\gamma(s))\,\mathrm{d}\beta_2(s)+\ln Y_2(0).
$$

It follows from Lemma 2 that $\lim_{t\to\infty} Y_2(t)=0$ and $\langle Y_2(t)\rangle=0$, for any $t\ge T$. From Eq (14),

$$
t^{-1}\ln Y_3(t)=-\bar{u}_3-c_{33}\langle Y_3(t)\rangle+t^{-1}\int_0^t \sigma_3(\gamma(s))\,\mathrm{d}\beta_3(s)+t^{-1}\ln Y_3(0).
$$

It's easy to see $\lim_{t\to\infty} Y_3(t)=0, a.s.$. This completes the proof of (a).

Now we are in the position to prove (b). By Eq (11) and Lemma 2, we have

$$
\frac{\bar{u}_1-\epsilon}{c_{11}}\le\inf\langle Y_1(t)\rangle\le\sup\langle Y_1(t)\rangle\le\frac{\bar{u}_1+\epsilon}{c_{11}}.
$$

According to the arbitraryiness of ϵ, we get $\langle Y_1(t)\rangle=0$ when $\bar{u}_1=0$. Similarly, we obtain the Eqs (13) and (14) from Eqs (7) and (8). This implies $\lim_{t\to\infty} Y_j(t)=0$, $j=2,3$. The proof of (b) is complete.

Next, we shall prove (c). It can be shown from Eq (11) that $\langle Y_1(t)\rangle=\frac{\bar{u}_1}{c_{11}}$ *a.s.* when $\bar{u}_1>0$. Fix a positive constant T, for any $t\ge T$, we may shift Eq (7) to obtain

$$
\ln Y_2(t)=\frac{\Delta_2-\tilde{\Delta}_2}{c_{11}}t-c_{22}\int_0^t Y_2(s)\mathrm{d}s+\ln Y_2(0)+\phi_2(t), \tag{15}
$$

where $\phi_2(t) = c_{21}\int_{-\tau_{21}}^{0}\int_{\theta}^{0} Y_1(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta) - c_{21}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} Y_1(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta)$ and then $\lim_{t\to\infty} t^{-1}\phi_2(t) = 0$. When $\Delta_2 - \tilde{\Delta}_2 < 0$, we further have $\lim_{t\to\infty} Y_2(t) = 0, a.s.$ by Lemma 2. For any $t \geq T$, Eq (8) follows

$$\ln Y_3(t) = \left(\frac{\Delta_3 - \tilde{\Delta}_3}{c_{11}}\right)t - c_{33}\int_0^t Y_3(s)\mathrm{d}s + \phi_3(t) + \ln Y_3(0), \tag{16}$$

where $\phi_3(t) = c_{31}\int_{-\tau_{31}}^{0}\int_{\theta}^{0} Y_1(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta) - c_{31}\int_{-\tau_{31}}^{0}\int_{t+\theta}^{t} Y_1(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta)$. Clearly, $\lim_{t\to\infty}\frac{1}{t}\phi_3(t) = 0$. From Lemma 2, when $\Delta_3 - \tilde{\Delta}_3 < 0, \lim_{t\to\infty} Y_3(t) = 0, a.s.$. Thus the required conclusion (c) follows.

Next, we shall give the proof of (d). Conclusion (c) shows $\langle Y_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}$ and $\lim_{t\to\infty} Y_2(t) = 0, a.s.$ for $\bar{u} > 0$ and $\Delta_2 - \tilde{\Delta}_2 < 0$. Assume $\Delta_3 - \tilde{\Delta}_3 \geq 0$, there is a conclusion that $\langle Y_3(t)\rangle = \frac{\Delta_3 - \tilde{\Delta}_3}{c_{11}c_{33}}$ from Eq (16). The proof of (d) is completed.

In the following, we shall prove (e). From Eqs (11) and (15), assume $\Delta_2 - \tilde{\Delta}_2 \geq 0$, we get $\langle Y_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}$ and $\langle Y_2(t)\rangle = \frac{\Delta_2 - \tilde{\Delta}_2}{c_{11}c_{22}}$. Then, for any $t \geq T$, we have

$$\ln Y_3(t) = \frac{c_{22}(\Delta_3 - \tilde{\Delta}_3) + c_{32}(\Delta_2 - \tilde{\Delta}_2)}{c_{11}c_{22}} - c_{33}\int_0^t Y_3(s)\mathrm{d}s + \phi_3^*(t), \tag{17}$$

where

$$\begin{aligned}\phi_3^*(t) =& c_{31}\int_{-\tau_{31}}^{0}\int_{\theta}^{0} Y_1(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta) - c_{31}\int_{-\tau_{31}}^{0}\int_{t+\theta}^{t} Y_1(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta)\\ &+ c_{32}\int_{-\tau_{32}}^{0}\int_{\theta}^{0} Y_2(s)\mathrm{d}s\mathrm{d}\mu_{32}(\theta) - c_{32}\int_{-\tau_{32}}^{0}\int_{t+\theta}^{t} Y_2(s)\mathrm{d}s\mathrm{d}\mu_{32}(\theta).\end{aligned}$$

Then, $\lim_{t\to\infty} t^{-1}\phi_3^*(t) = 0$. When $c_{22}(\Delta_3 - \tilde{\Delta}_3) + c_{32}(\Delta_2 - \tilde{\Delta}_2) < 0$, we get $\lim_{t\to\infty} Y_3(t) = 0$. This completes the proof of (e).

Lastly, we give the proof of (f). Let $\bar{u} > 0$ and $\Delta_2 - \tilde{\Delta}_2 > 0$, we can then show that $\langle Y_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}$ and $\langle Y_2(t)\rangle = \frac{\Delta_2 - \tilde{\Delta}_2}{c_{11}c_{22}}$ $a.s.$ by Eqs (11) and (15). From Eq (17), we directly obtain $\langle Y_3(t)\rangle = \frac{c_{22}\Gamma_3 + c_{32}\Gamma_2}{c_{11}c_{22}c_{33}}$ provided that $c_{22}(\Delta_3 - \tilde{\Delta}_3) + c_{32}(\Delta_2 - \tilde{\Delta}_2) \geq 0$.

This completes the proof of all cases.

Lemma 4. Let $X(t) = (x(t), \gamma(t)) \in \mathbb{R}_+^3 \times \mathbb{S}$ be a global positive solution of the system (1). Then the solution has the following properties

$$\limsup_{t\to\infty}\frac{\ln x_j(t)}{t} \leq 0, \quad \lim_{t\to\infty} t^{-1}\int_{t-\tau}^{t} x_j(s)ds = 0, a.s., j = 1, 2, 3.$$

Proof. It is easy to see

$$\begin{aligned}\mathrm{d}x_1(t) &\leq x_1(t)\Big[a_1(\gamma(t)) - h_1 - c_{11}x_1(t)\Big]\mathrm{d}t + \sigma_1(\gamma(t))x_1(t)\mathrm{d}\beta_1(t),\\ \mathrm{d}x_2(t) &\leq x_2(t)\Big[-a_2(\gamma(t)) - h_2 + c_{21}\int_{-\tau_{21}}^{0} x_1(t+\theta)\mathrm{d}\mu_{21}(\theta) - c_{22}x_2(t)\Big]\mathrm{d}t + \sigma_2(\gamma(t))x_2(t)\mathrm{d}\beta_2(t),\end{aligned}$$

$$\mathrm{d}x_3(t) \le x_3(t)\Big[-a_3(\gamma(t)) - h_3 + c_{31}\int_{-\tau_{31}}^{0} x_1(t+\theta)\mathrm{d}\mu_{31}(\theta) + c_{32}\int_{-\tau_{32}}^{0} x_2(t+\theta)\mathrm{d}\mu_{32}(\theta) - c_{33}x_3(t)\Big]\mathrm{d}t$$
$$+\sigma_3(\gamma(t))\,x_3(t)\mathrm{d}\beta_3(t).$$

By comparison theorem, we obtain $x_j(t) \le Y_j(t)$, $a.s.$, $j = 1, 2, 3$. It follows from Lemma 3 that $\lim_{t\to\infty} Y_j(t) = 0$ or $\langle Y_j(t)\rangle = a$, $j = 1, 2, 3$, where the constant $a \ge 0$. From Eqs (6)–(8), then

$$\limsup_{t\to\infty} \tfrac{\ln x_j(t)}{t} \le \limsup_{t\to\infty} \tfrac{\ln Y_j(t)}{t} \le 0, \quad j = 1, 2, 3,$$
$$\lim_{t\to\infty} t^{-1}\int_{t-\tau}^{t} x_j(s)ds \le \lim_{t\to\infty} t^{-1}\int_{t-\tau}^{t} Y_j(s)ds = \lim_{t\to\infty} t^{-1}\left(\int_0^t Y_j(s)ds - \int_0^{t-\tau} Y_j(s)ds\right) = 0, a.s..$$

This completes the proof.

Lemma 5. Let condition ($\mathbf{H_2}$) hold. For the defined parameters $R, \omega_i, \Theta_i, i = 1, 2, 3$, we have the following conclusions.
(a). If $R > 0$, then $\omega_1 > \omega_2 > \omega_3$.
(b). If $R < 0$, then $\omega_1 > \Theta_2 > \Theta_3$.
(c). If $R = 0$, then $\omega_1 > \Theta_2 = \Theta_3$.

Proof. From the definition of $R, \omega_i, \Theta_i, i = 1, 2, 3$, we have

$$\omega_1 - \omega_2 = \omega_1 - \tfrac{\Delta_2}{\tilde\Delta_2} = \tfrac{\bar a_1\tilde\Delta_2 - \Delta_2(\bar\sigma_1^2/2+h_1)}{\tilde\Delta_2(\bar\sigma_1^2/2+h_1)} = \tfrac{c_{11}}{\tilde\Delta_2(\bar\sigma_1^2/2+h_1)}[(\bar\sigma_2^2/2 + h_2)\bar a_1 + (\bar\sigma_1^2/2 + h_1)\bar a_2] > 0,$$
$$\omega_1 - \Theta_2 = \omega_1 - \tfrac{\Delta_3}{\tilde\Delta_3} = \tfrac{\bar a_1\tilde\Delta_3 - (\bar\sigma_1^2/2+h_1)\Delta_3}{(\bar\sigma_1^2/2+h_1)\tilde\Delta_3} = c_{11}\tfrac{\bar a_1(\bar\sigma_3^2/2+h_3) + \bar a_3(\bar\sigma_1^2/2+h_1)}{(\bar\sigma_1^2/2+h_1)\tilde\Delta_3} > 0,$$
$$\Theta_3 - \omega_2 = \tfrac{C_2}{\tilde C_2} - \tfrac{\Delta_2}{\tilde\Delta_2} = \tfrac{C_2\tilde\Delta_2 - \tilde C_2\Delta_2}{\tilde C_2\tilde\Delta_2} = \tfrac{C_{32}R}{\tilde C_2\tilde\Delta_2},$$
$$\omega_2 - \Theta_2 = \tfrac{\Delta_2}{\tilde\Delta_2} - \tfrac{\Delta_3}{\tilde\Delta_3} = \tfrac{\Delta_2\tilde\Delta_3 - \tilde\Delta_2\Delta_3}{\tilde\Delta_2\tilde\Delta_3} = \tfrac{c_{11}R}{\tilde\Delta_2\tilde\Delta_3},$$
$$\Theta_2 - \omega_3 = \tfrac{\Delta_3}{\tilde\Delta_3} - \tfrac{C_3}{\tilde C_3} = \tfrac{\Delta_3\tilde C_3 - \tilde\Delta_3 C_3}{\tilde\Delta_3\tilde C_3} = \tfrac{C_{23}R}{\tilde\Delta_3\tilde C_3}.$$

When $R > 0$, we can see $\omega_1 > \omega_2 > \Theta_2 > \omega_3$. When $R < 0$, then $\Theta_3 < \omega_2 < \Theta_2 < \omega_1$. When $R = 0$, then $\omega_3 = \Theta_3 = \omega_2 = \Theta_2 < \omega_1$. Thus the proof is complete.

3. Global analysis for dynamical behaviors

In this section, we shall investigate dynamical behaviors involving persistence, extinction, stability and attractiveness of the system (1). In terms of parameters R, ω_i and Θ_i, we first give a global analysis of dynamical behaviors according to Lemma 5.

Theorem 1. Let conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) be satisfied and $X(t) = (x(t), \gamma(t)) \in \mathbb{R}^3_+ \times \mathbb{S}$ be a global positive solution of the system (1). We have the following conclusions.
(i). When $R > 0$
(a). If $\omega_1 < 1$, then $\lim_{t\to\infty} x_i(t) = 0$, $a.s.$, $i = 1, 2, 3$;
(b). If $\omega_1 = 1$, then $\langle x_1(t)\rangle = 0, \lim_{t\to\infty} x_j(t) = 0, a.s., j = 2, 3$;
(c). If $\omega_1 > 1 > \omega_2$, then

$$\langle x_1(t)\rangle = \frac{\bar u_1}{c_{11}}, \quad \lim_{t\to\infty} x_j(t) = 0, a.s., j = 2, 3;$$

(d). If $\omega_2 = 1$, then

$$\langle x_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \langle x_2(t)\rangle = 0, \quad \lim_{t\to\infty} x_3(t) = 0, a.s.;$$

(e). If $\omega_2 > 1 > \omega_3$, then

$$\langle x_i(t)\rangle = \frac{\Delta_i - \tilde{\Delta}_i}{C_{33}}, i = 1, 2, \quad \lim_{t\to\infty} x_3(t) = 0, a.s.;$$

(f). If $\omega_3 = 1$, then

$$\langle x_i(t)\rangle = \frac{\Delta_i - \tilde{\Delta}_i}{C_{33}}, i = 1, 2, \quad \langle x_3(t)\rangle = 0, a.s.;$$

(g). If $\omega_3 > 1$, then

$$\langle x_i(t)\rangle = \frac{C_i - \tilde{C}_i}{|C|}, a.s., i = 1, 2, 3.$$

(ii). When $R < 0$

(h). If $\omega_1 < 1$, then $\lim_{t\to\infty} x_i(t) = 0$, $a.s.$, $i = 1, 2, 3$;

(i). If $\omega_1 = 1$, then $\langle x_1(t)\rangle = 0, \lim_{t\to\infty} x_j(t) = 0, a.s., j = 2, 3$;

(j). If $\omega_1 > 1 > \Theta_2$, then

$$\langle x_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \lim_{t\to\infty} x_j(t) = 0, a.s., j = 2, 3;$$

(k). If $\Theta_2 = 1$, then

$$\langle x_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \lim_{t\to\infty} x_2(t) = 0, \langle x_3(t)\rangle = 0, a.s.;$$

(l). If $\Theta_2 > 1 > \Theta_3$, then

$$\langle x_1(t)\rangle = \frac{c_{33}\bar{u}_1 + c_{13}\bar{u}_3}{C_{22}}, \lim_{t\to\infty} x_2(t) = 0, \langle x_3(t)\rangle = \frac{\Delta_3 - \tilde{\Delta}_3}{C_{22}}, a.s.;$$

(m). If $\Theta_3 = 1$, then

$$\langle x_1(t)\rangle = \frac{c_{33}\bar{u}_1 + c_{13}\bar{u}_3}{C_{22}}, \langle x_2(t)\rangle = 0, \langle x_3(t)\rangle = \frac{\Delta_3 - \tilde{\Delta}_3}{C_{22}}, a.s.;$$

(n). If $\Theta_3 > 1$, then

$$\langle x_i(t)\rangle = \frac{C_i - \tilde{C}_i}{|C|}, a.s., i = 1, 2, 3.$$

(iii). When $R = 0$

(o). If $\omega_1 < 1$, then $\lim_{t\to\infty} x_i(t) = 0$, $a.s.$, $i = 1, 2, 3$;

(p). If $\omega_1 = 1$, then $\langle x_1(t)\rangle = 0, \lim_{t\to\infty} x_j(t) = 0$, $a.s., j = 2, 3$;

(q). If $\Theta_2 < 1 < \omega_1$, then

$$\langle x_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \lim_{t\to\infty} x_j(t) = 0, a.s., j = 2, 3;$$

(r). If $\Theta_2 = 1$, then

$$\langle x_1(t)\rangle = \frac{\bar{u}_1}{c_{11}}, \quad \langle x_j(t)\rangle = 0, a.s., j = 2, 3;$$

(s). If $\Theta_2 > 1$, then

$$\langle x_i(t)\rangle = \frac{C_i - \tilde{C}_i}{|C|}, a.s., i = 1, 2, 3.$$

Proof. By the stochastic comparison theorem and the condition (a) in Lemma 3. when $\omega_1 < 1$, we have the conclusion

$$\lim_{t\to\infty} x_j(t) \le \lim_{t\to\infty} Y_j(t) = 0, a.s., j = 1, 2, 3.$$

Next, we prove the conclusion (b) holds. Thanks to the condition (b) in Lemma 3, we can immediately obtain that $\langle x_1(t)\rangle \le \langle Y_1(t)\rangle = 0$ and $\lim_{t\to\infty} x_i(t) \le \lim_{t\to\infty} Y_i(t) = 0, a.s., i = 2, 3$.

Then, we shall prove the condition (c). When $\omega_1 > 1 > \omega_2$, that is, $\bar{u}_1 > 0, \Delta_2 - \tilde{\Delta}_2 < 0$ and $\Delta_3 - \tilde{\Delta}_3 < 0$. we further get $\lim_{t\to\infty} x_j(t) \le \lim_{t\to\infty} Y_j(t) = 0, a.s., j = 2, 3$ by condition (c) in Lemma 3. Fix a constant $T > 0$ and any $t \ge T$, Applying generalized Itô's formula to the system (1) leads to

$$\begin{aligned}
\ln x_1(t) &= \int_0^t u_1(\gamma(s))\,\mathrm{d}s - c_{11}\int_0^t x_1(s)\mathrm{d}s - c_{12}\int_0^t x_2(s)\mathrm{d}s - c_{13}\int_0^t x_3(s)\mathrm{d}s \\
&\quad +\Phi_1(t), \qquad (18)\\
\ln x_2(t) &= -\int_0^t u_2(\gamma(s))\,\mathrm{d}s + c_{21}\int_0^t x_1(s)\mathrm{d}s - c_{22}\int_0^t x_2(s)\mathrm{d}s - c_{23}\int_0^t x_3(s)\mathrm{d}s \\
&\quad +\Phi_2(t), \qquad (19)\\
\ln x_3(t) &= -\int_0^t u_3(\gamma(s))\,\mathrm{d}s + c_{31}\int_0^t x_1(s)\mathrm{d}s + c_{32}\int_0^t x_2(s)\mathrm{d}s - c_{33}\int_0^t x_3(s)\mathrm{d}s \\
&\quad +\Phi_3(t), \qquad (20)
\end{aligned}$$

where

$$\begin{aligned}
\Phi_1(t) &= c_{12}\int_{-\tau_{12}}^0\int_{t+\theta}^t x_2(s)\mathrm{d}s\mathrm{d}\mu_{12}(\theta) - c_{12}\int_{-\tau_{12}}^0\int_{\theta}^0 x_2(s)\mathrm{d}s\mathrm{d}\mu_{12}(\theta) + c_{13}\int_{-\tau_{13}}^0\int_{t+\theta}^t x_3(s)\mathrm{d}s\mathrm{d}\mu_{13}(\theta) \\
&\quad -c_{13}\int_{-\tau_{13}}^0\int_{\theta}^0 x_3(s)\mathrm{d}s\mathrm{d}\mu_{13}(\theta) + \int_0^t \sigma_1(\gamma(t))\mathrm{d}\beta_1(t) + \ln x_1(0),\\
\Phi_2(t) &= -c_{21}\int_{-\tau_{21}}^0\int_{t+\theta}^t x_1(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta) + c_{21}\int_{-\tau_{21}}^0\int_{\theta}^0 x_1(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta) + c_{23}\int_{-\tau_{23}}^0\int_{t+\theta}^t x_3(s)\mathrm{d}s\mathrm{d}\mu_{23}(\theta) \\
&\quad -c_{23}\int_{-\tau_{23}}^0\int_{\theta}^0 x_3(s)\mathrm{d}s\mathrm{d}\mu_{23}(\theta) + \int_0^t \sigma_2(\gamma(t))\mathrm{d}\beta_2(t) + \ln x_2(0),\\
\Phi_3(t) &= -c_{31}\int_{-\tau_{31}}^0\int_{t+\theta}^t x_1(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta) + c_{31}\int_{-\tau_{31}}^0\int_{\theta}^0 x_1(s)\mathrm{d}s\mathrm{d}\mu_{31}(\theta) - c_{32}\int_{-\tau_{32}}^0\int_{t+\theta}^t x_2(s)\mathrm{d}s\mathrm{d}\mu_{32}(\theta)
\end{aligned}$$

$$+c_{32}\int_{-\tau_{32}}^{0}\int_{\theta}^{0}x_2(s)\mathrm{d}s\mathrm{d}\mu_{32}(\theta)+\int_0^t\sigma_3(\gamma(t))\mathrm{d}\beta_3(t)+\ln x_3(0).$$

When $\lim_{t\to\infty}x_j(t)$, $j=2,3$, for Eq (18), one can observe that

$$t^{-1}\ln x_1(t)=t^{-1}\int_0^t u_1(\gamma(s))\,\mathrm{d}s-c_{11}\langle x_1(t)\rangle\mathrm{d}s+t^{-1}\ln x_1(0). \tag{21}$$

From Lemma 2, we have $\langle x_1\rangle=\frac{\bar{u}_1}{c_{11}}$. This completes the proof of (c).

Then, we shall prove the conclusion (d). Similar to the above method, when $\omega_2=1$, that is, $\bar{u}_1>0,\Delta_2-\tilde{\Delta}_2=0$ and $\Delta_3-\tilde{\Delta}_3<0$. The condition (e) in Lemma 3 shows $\lim_{t\to\infty}Y_3(t)=0$ and $\langle Y_2(t)\rangle=0$. Thus we can check that $\langle x_2(t)\rangle\le\langle Y_2(t)\rangle=0$ and $\lim_{t\to\infty}x_3(t)\le\lim_{t\to\infty}Y_3(t)=0$. Eq (18) changes into Eq (21) then we get $\langle x_1(t)\rangle=\frac{\bar{u}_1}{c_{11}}$.

Next, we prove the conclusion (e). It is not difficult to show that $\lim_{t\to\infty}t^{-1}\Phi_i(t)=0, i=1,2,3$. Take two negative constants m and n satisfying

$$\begin{cases}-c_{11}m+c_{31}n=-c_{21},\\-c_{13}m-c_{33}n=c_{23}.\end{cases}$$

Then

$$\begin{cases}m=-\dfrac{C_{12}}{C_{22}},\\[2mm] n=-\dfrac{C_{32}}{C_{22}}.\end{cases}$$

Compute $m\times$ Eq (18) + Eq (19) + $n\times$ Eq (20)

$$\begin{aligned}m\ln x_1(t)+\ln x_2(t)+n\ln x_3(t)=&m\int_0^t u_1(\gamma(s))\,\mathrm{d}s-\int_0^t u_2(\gamma(s))\,\mathrm{d}s-n\int_0^t u_3(\gamma(s))\,\mathrm{d}s\\&-(mc_{12}+c_{22}-nc_{32})\int_0^t x_2(s)\mathrm{d}s+m\Phi_1(t)+\Phi_2(t)+n\Phi_3(t).\end{aligned} \tag{22}$$

From Lemma 4, there exists an $\epsilon>0$ and a $T=T(\epsilon)>0$. For $t\ge T$, one has that

$$\begin{aligned}t^{-1}\ln x_2(t)&\le m\bar{u}_1-\bar{u}_2-n\bar{u}_3-(mc_{12}+c_{22}-nc_{32})\langle x_2(t)\rangle+\frac{m\Phi_1(t)+\Phi_2(t)+n\Phi_3(t)}{t}\\&=(C_2-\tilde{C}_2)-|C|\langle x_2(t)\rangle+\frac{m\Phi_1(t)+\Phi_2(t)+n\Phi_3(t)}{t}.\end{aligned}$$

Because of $\frac{C_2}{\tilde{C}_2}>\omega_2>1$, it's easy to see $C_2-\tilde{C}_2>0$. we further have $\sup\langle x_2(t)\rangle\le\frac{C_2-\tilde{C}_2}{|C|}$. Then take two negative constants m^* and n^* satisfying

$$\begin{cases}-c_{22}m^*+c_{32}n^*=c_{12},\\-c_{23}m^*-c_{33}n^*=c_{13}.\end{cases}$$

That is

$$\begin{cases} m^* = -\dfrac{C_{21}}{C_{11}}, \\ n^* = \dfrac{C_{31}}{C_{11}}. \end{cases}$$

Compute Eq (18) + $m^*\times$ Eq (19) + $n^*\times$ Eq (20)

$$\begin{aligned} \ln x_1(t) + m^* \ln x_2(t) + n^* \ln x_3(t) = &\int_0^t u_1(\gamma(s))\,\mathrm{d}s - m^* \int_0^t u_2(\gamma(s))\,\mathrm{d}s - n^* \int_0^t u_3(\gamma(s))\,\mathrm{d}s \\ &- (c_{11} - m^* c_{21} - n^* c_{31}) \int_0^t x_1(s)\mathrm{d}s + \Phi_1(t) + m^*\Phi_2(t) + n^*\Phi_3(t). \end{aligned} \tag{23}$$

Thus

$$\begin{aligned} t^{-1} \ln x_1(t) &\le \bar{u}_1 - m^*\bar{u}_2 - n^*\bar{u}_3 - (c_{11} - m^*c_{21} - n^*c_{31})\langle x_1(t)\rangle + \frac{\Phi_1(t) + m^*\Phi_2(t) + n^*\Phi_3(t)}{t} \\ &= (C_1 - \tilde{C}_1) - |C|\langle x_1(t)\rangle + \frac{\Phi_1(t) + m^*\Phi_2(t) + n^*\Phi_3(t)}{t}. \end{aligned}$$

From ($\mathbf{H_2}$), we gain $C_1 - \tilde{C}_1 > 0$ and then $\sup\langle x_1(t)\rangle \le \frac{C_1 - \tilde{C}_1}{|C|}$. For Eq (20)

$$\begin{aligned} t^{-1} \ln x_3(t) &\le -t^{-1} \int_0^t u_3(\gamma(s))\,\mathrm{d}s + c_{31} \sup\langle x_1(t)\rangle + c_{32} \sup\langle x_2(t)\rangle - c_{33}\langle x_3(t)\rangle + t^{-1}\Phi_3(t) \\ &= c_{33}\frac{C_3 - \tilde{C}_3}{|C|} - c_{33}t^{-1} \int_0^t x_3(s)\mathrm{d}s + t^{-1}\Phi_3(t). \end{aligned} \tag{24}$$

Assume $\omega_3 = \frac{C_3}{\tilde{C}_3} < 1$, we obtain $\lim\limits_{t\to\infty} x_3(t) = 0$. Equations (18) and (19) follow that:

$$t^{-1} \ln x_1(t) = t^{-1} \int_0^t u_1(\gamma(s))\,\mathrm{d}s - c_{11}\langle x_1(t)\rangle - c_{12}\langle x_2(t)\rangle + t^{-1}\Phi_1^*(t), \tag{25}$$

$$t^{-1} \ln x_2(t) = -t^{-1} \int_0^t u_2(\gamma(s))\,\mathrm{d}s + c_{21}\langle x_1(t)\rangle - c_{22}\langle x_2(t)\rangle + t^{-1}\Phi_2^*(t), \tag{26}$$

where

$$\begin{aligned} \Phi_1^*(t) &= c_{12} \int_{-\tau_{12}}^0 \int_{t+\theta}^t x_2(s)\mathrm{d}s\mathrm{d}\mu_{12}(\theta) - c_{12} \int_{-\tau_{12}}^0 \int_\theta^0 x_2(s)\mathrm{d}s\mathrm{d}\mu_{12}(\theta) + \int_0^t \sigma_1(\gamma(s))\,\mathrm{d}\beta_1(s) \\ &\quad + \ln x_1(0), \\ \Phi_2^*(t) &= -c_{21} \int_{-\tau_{21}}^0 \int_{t+\theta}^t x_1(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta) + c_{21} \int_{-\tau_{21}}^0 \int_\theta^0 x_1(s)\mathrm{d}s\mathrm{d}\mu_{21}(\theta) + \int_0^t \sigma_2(\gamma(s))\,\mathrm{d}\beta_2(s) \\ &\quad + \ln x_2(0). \end{aligned}$$

Analogously, $\lim\limits_{t\to\infty} t^{-1}\Phi_i^*(t) = 0, a.s., i = 1, 2$. Compute $c_{21}\times$ Eq (25) + $c_{11}\times$ Eq (26)

$$c_{21}t^{-1} \ln x_1(t) + c_{11}t^{-1} \ln x_2(t) = c_{21}t^{-1} \int_0^t u_1(\gamma(s))\,\mathrm{d}s - c_{11}t^{-1} \int_0^t u_2(\gamma(s))\,\mathrm{d}s - C_{33}\langle x_2(t)\rangle$$

$$+c_{21}t^{-1}\Phi_1^*(t) + c_{11}t^{-1}\Phi_2^*(t). \tag{27}$$

For any $t \geq T$

$$c_{11}t^{-1}\ln x_2(t) \leq (\Delta_2 - \tilde{\Delta}_2 + \epsilon) - C_{33}\langle x_2(t)\rangle + t^{-1}c_{21}\Phi_1^*(t) + t^{-1}c_{11}\Phi_2^*(t).$$

It means $\sup\langle x_2(t)\rangle \leq \frac{\Delta_2-\tilde{\Delta}_2}{C_{33}}$ and then

$$\begin{aligned} t^{-1}\ln x_1(t) &\geq t^{-1}\int_0^t u_1(\gamma(s))\,\mathrm{d}s - c_{11}\langle x_1(t)\rangle - c_{12}\sup\langle x_2(t)\rangle + t^{-1}\Phi_1^*(t) \\ &= c_{11}\frac{\Delta_1 - \tilde{\Delta}_1}{C_{33}} - c_{11}\langle x_1(t)\rangle + t^{-1}\Phi_1^*(t). \end{aligned}$$

we can see if $\omega_1 > 1$,

$$\Delta_1 - \tilde{\Delta}_1 = c_{22}\left(\bar{a}_1 - \bar{\sigma}_1^2/2 - h_1\right) + c_{12}\left(\bar{a}_2 + \bar{\sigma}_2^2/2 + h_2\right) > 0.$$

Then $\inf\langle x_1(t)\rangle \geq \frac{\Delta_1-\tilde{\Delta}_1}{C_{33}}$ and

$$\begin{aligned} t^{-1}\ln x_2(t) &\geq -t^{-1}\int_0^t u_2(\gamma(s))\,\mathrm{d}s + c_{21}\inf\langle x_1(t)\rangle - c_{22}\langle x_2(t)\rangle + t^{-1}\Phi_2^*(t) \\ &= c_{22}\frac{\Delta_2 - \tilde{\Delta}_2}{C_{33}} - c_{22}\langle x_2(t)\rangle + t^{-1}\Phi_2^*(t). \end{aligned}$$

We can now easily establish $\langle x_2(t)\rangle = \frac{\Delta_2-\tilde{\Delta}_2}{C_{33}}$ and then

$$\begin{aligned} t^{-1}\ln x_1(t) &= t^{-1}\int_0^t u_1(\gamma(s))\,\mathrm{d}s - c_{11}\langle x_1(t)\rangle - c_{12}\frac{\Delta_2 - \tilde{\Delta}_2}{C_{33}} + t^{-1}\Phi_1^*(t) \\ &= c_{11}\frac{\Delta_1 - \tilde{\Delta}_1}{C_{33}} - c_{11}t^{-1}\int_0^t x_1(s)\mathrm{d}s + t^{-1}\Phi_1^*(t). \end{aligned}$$

Finally, we have $\langle x_1(t)\rangle = \frac{\Delta_1-\tilde{\Delta}_1}{C_{33}}, \langle x_2(t)\rangle = \frac{\Delta_2-\tilde{\Delta}_2}{C_{33}}$ and $\lim_{t\to\infty} x_3(t) = 0$. This completes the proof of (e).

Next, we prove the case (f). For Eq (24), If $\omega_3 = 1$, this is, $C_3 - \tilde{C}_3 = 0$, then we get $\sup_{t\to\infty}\langle x_3(t)\rangle \leq 0$. Because of $\langle x_3(t)\rangle \geq 0$, we finally derive $\langle x_3(t)\rangle = 0$. Equations (18) and (19) become Eqs (25) and (26) respectively. Finally we derive $\langle x_1(t)\rangle = \frac{\Delta_1-\tilde{\Delta}_1}{C_{33}}, \langle x_2(t)\rangle = \frac{\Delta_2-\tilde{\Delta}_2}{C_{33}}$ and $\langle x_3(t)\rangle = 0$.

Lastly, we shall prove the condition (g) holds. If $\omega_3 = \frac{C_3}{\tilde{C}_3} > 1$, for Eq (24), we have $\sup\langle x_3(t)\rangle \leq \frac{C_3-\tilde{C}_3}{|C|}$. and

$$\begin{aligned} t^{-1}\ln x_1(t) &\geq \int_0^t t^{-1}u_1(\gamma(s))\,\mathrm{d}s - c_{11}\langle x_1(t)\rangle - c_{12}\sup\langle x_2(t)\rangle - c_{13}\sup\langle x_3(t)\rangle + t^{-1}\Phi_1(t) \\ &= \bar{u}_1 - c_{11}\langle x_1(t)\rangle - c_{12}\frac{C_2 - \tilde{C}_2}{|C|} - c_{13}\frac{C_3 - \tilde{C}_3}{|C|} + t^{-1}\Phi_1(t) \\ &= c_{11}\frac{C_1 - \tilde{C}_1}{|C|} - c_{11}\langle x_1(t)\rangle + t^{-1}\Phi_1(t). \end{aligned}$$

Thus, $\frac{C_1-\tilde{C}_1}{|C|} \leq \inf\langle x_1(t)\rangle \leq \sup\langle x_1(t)\rangle \leq \frac{C_1-\tilde{C}_1}{|C|}$, $\langle x_1(t)\rangle = \frac{C_1-\tilde{C}_1}{|C|}$. By the same way, for Eq (19)

$$\begin{aligned} t^{-1}\ln x_2(t) &\geq -t^{-1}\int_0^t u_2(\gamma(s))\,\mathrm{d}s - c_{11}\langle x_1(t)\rangle - c_{12}\langle x_2(t)\rangle - c_{13}\sup\langle x_3(t)\rangle + t^{-1}\Phi_2(t) \\ &= -\bar{u}_2 - c_{11}\frac{C_1-\tilde{C}_1}{|C|} - c_{12}\langle x_2(t)\rangle - c_{13}\frac{C_3-\tilde{C}_3}{|C|} + t^{-1}\Phi_2(t) \\ &= c_{12}\frac{C_2-\tilde{C}_2}{|C|} - c_{12}\langle x_2(t)\rangle + t^{-1}\Phi_2(t). \end{aligned}$$

We have $\frac{C_2-\tilde{C}_2}{|C|} \leq \inf\langle x_2(t)\rangle \leq \sup\langle x_2(t)\rangle \leq \frac{C_2-\tilde{C}_2}{|C|}$ and then $\langle x_2(t)\rangle = \frac{C_2-\tilde{C}_2}{|C|}$. Similarly, $\langle x_3(t)\rangle = \frac{C_3-\tilde{C}_3}{|C|}$.

The proof of (ii) and (iii) are similar to one of (i). Here we omit the remaining part and all the proof is complete.

In the following, we further give the attractiveness of all positive solutions by M-matrix.

Theorem 2. If the following matrix is a non-singular M-matrix

$$C_M := \begin{pmatrix} c_{11} & -c_{21} & -c_{31} \\ -c_{12} & c_{22} & -c_{32} \\ -c_{13} & -c_{23} & c_{33} \end{pmatrix},$$

then

$$\lim_{t\to\infty} E|X^{\eta,k}(t) - X^{\eta^*,k}(t)| = 0,$$

where $X^{\eta,k}(t)$ and $X^{\eta^*,k}(t)$ are two positive solutions of the system (1) with initial conditions $\eta \in \mathbb{C}\left([-\tau,0],\mathbb{R}^3_+\right), \gamma(0) = k$ and $\eta^* \in \mathbb{C}\left([-\tau,0],\mathbb{R}^3_+\right), \gamma(0) = k$.

Proof. Notice that

$$\lim_{t\to\infty} E|X^{\eta,k}(t) - X^{\eta^*,k}(t)| = \lim_{t\to\infty} E\sqrt{(x_1(t;\eta) - x_1(t;\eta^*))^2 + (x_2(t;\eta) - x_2(t;\eta^*))^2 + (x_3(t;\eta) - x_3(t;\eta^*))^2}.$$

Since C_M is an non-singular M-matrix, there exists positive vector $\zeta = (\zeta_1,\zeta_2,\zeta_3)^T, \zeta_i > 0, i = 1,2,3$ such that $C_M\zeta > 0$. We just need to proof $\lim_{t\to\infty} E|(x_j(t;\eta) - x_j(t;\eta^*)| = 0$, $j = 1,2,3$, define

$$v_j(x_j) = |\ln x_j(t;\eta) - \ln x_j(t;\eta^*)|, j = 1,2,3. \tag{28}$$

$$\begin{aligned} v_4(x_1,x_2,x_3) = &\ \zeta_1 c_{12}\int_{-\tau_{12}}^0\int_{t+\theta}^t |x_2(s,\eta) - x_2(s,\eta^*)|\mathrm{d}s\mathrm{d}\mu_{12}(\theta) \\ &+\zeta_1 c_{13}\int_{-\tau_{13}}^0\int_{t+\theta}^t |x_3(s,\eta) - x_3(s,\eta^*)|\mathrm{d}s\mathrm{d}\mu_{13}(\theta) \\ &+\zeta_2 c_{21}\int_{-\tau_{21}}^0\int_{t+\theta}^t |x_1(s,\eta) - x_1(s,\eta^*)|\mathrm{d}s\mathrm{d}\mu_{21}(\theta) \\ &+\zeta_2 c_{23}\int_{-\tau_{23}}^0\int_{t+\theta}^t |x_3(s,\eta) - x_3(s,\eta^*)|\mathrm{d}s\mathrm{d}\mu_{23}(\theta) \\ &+\zeta_3 c_{31}\int_{-\tau_{31}}^0\int_{t+\theta}^t |x_1(s,\eta) - x_1(s,\eta^*)|\mathrm{d}s\mathrm{d}\mu_{31}(\theta) \end{aligned}$$

$$+\zeta_3 c_{32}\int_{-\tau_{32}}^{0}\int_{t+\theta}^{t}|x_2(s,\eta)-x_2(s,\eta^*)|\mathrm{d}s\mathrm{d}\mu_{32}(\theta).$$

Calculating the upper right derivative of $v_j(x_j)$ along the solution of Eq (28), it follows that

$$\begin{aligned}
\mathrm{d}^+v_1(x_1) =& sgn\,(x_1(t;\eta)-x_1(t;\eta^*))\,\mathrm{d}\left(\ln x_j(t;\eta)-\ln x_j(t;\eta^*)\right)\\
&\le -c_{11}|x_1(t;\eta)-x_1(t;\eta^*)|+c_{12}\int_{-\tau_{12}}^{0}|x_2(t+\theta;\eta)-x_2(t+\theta;\eta^*)|\mathrm{d}\mu_{12}(\theta)\\
&+c_{13}\int_{-\tau_{13}}^{0}|x_3(t+\theta;\eta)-x_3(t+\theta;\eta^*)|\mathrm{d}\mu_{13}(\theta),\\
\mathrm{d}^+v_2(x_2) \le& -c_{22}|x_2(t;\eta)-x_2(t;\eta^*)|+c_{21}\int_{-\tau_{21}}^{0}|x_1(t+\theta;\eta)-x_1(t+\theta;\eta^*)|\mathrm{d}\mu_{21}(\theta)\\
&+c_{23}\int_{-\tau_{23}}^{0}|x_3(t+\theta;\eta)-x_3(t+\theta;\eta^*)|\mathrm{d}\mu_{23}(\theta),\\
\mathrm{d}^+v_3(x_3) \le& -c_{33}|x_3(t;\eta)-x_3(t;\eta^*)|+c_{31}\int_{-\tau_{31}}^{0}|x_1(t+\theta;\eta)-x_1(t+\theta;\eta^*)|\mathrm{d}\mu_{31}(\theta)\\
&+c_{32}\int_{-\tau_{32}}^{0}|x_2(t+\theta;\eta)-x_2(t+\theta;\eta^*)|\mathrm{d}\mu_{32}(\theta).
\end{aligned}$$

Let

$$V(t)=\zeta_1 v_1(x_1)+\zeta_2 v_2(x_2)+\zeta_3 v_3(x_3)+v_4(x_1,x_2,x_3). \tag{29}$$

Then

$$\begin{aligned}
\mathrm{d}^+V(t) \le& -(c_{11}\zeta_1-c_{21}\zeta_2-c_{31}\zeta_3)\mid x_1(t,\eta)-x_1(t,\eta^*)\mid -(c_{22}\zeta_2-c_{12}\zeta_1-c_{32}\zeta_3)\mid x_2(t,\eta)-x_2(t,\eta^*)\mid\\
&-(c_{33}\zeta_3-c_{13}\zeta_1-c_{23}\zeta_2)\mid x_3(t,\eta)-x_3(t,\eta^*)\mid .
\end{aligned}$$

Integrate both sides of the Eq (29) from 0 to t and then take expectation

$$\begin{aligned}
E[V(t)] \le& E[V(0)]-(c_{11}\zeta_1-c_{21}\zeta_2-c_{31}\zeta_3)\int_0^t E|x_1(s,\eta)-x_1(s,\eta^*)|\mathrm{d}s\\
&-(c_{22}\zeta_2-c_{12}\zeta_1-c_{32}\zeta_3)\int_0^t E|x_2(s,\eta)-x_2(s,\eta^*)|\mathrm{d}s\\
&-(c_{33}\zeta_3-c_{13}\zeta_1-c_{23}\zeta_2)\int_0^t E|x_3(s,\eta)-x_3(s,\eta^*)|\mathrm{d}s.
\end{aligned}$$

From $C_M\zeta>0$, we see $\int_0^t E|x_j(s,\eta)-x_j(s,\eta^*)|\mathrm{d}s<\infty$, $j=1,2,3$. Next, we define function

$$G_j(t)=E|x_j(t,\eta)-x_j(t,\eta^*)|,\ j=1,2,3.$$

For arbitrary $t_1,t_2\in[-\tau,+\infty)$,

$$\begin{aligned}
\left|G_j(t_2)-G_j(t_1)\right| =& \left|E\left|x_j(t_2,\eta)-x_j(t_2,\eta^*)\right|-E\left|x_j(t_1,\eta)-x_j(t_1,\eta^*)\right|\right|\\
\le& E|x_j(t_2,\eta)-x_j(t_1,\eta)|+E|x_j(t_2,\eta^*)-x_j(t_1,\eta^*)|.
\end{aligned}$$

Integrating both sides of the system (1) from t_1 to t_2, we see

$$\begin{aligned}
x_1(t_2,\eta)-x_1(t_1,\eta) &= \int_{t_1}^{t_2}\Big[x_1(s,\eta)\Big(a_1(\gamma(s))-h_1-c_{11}x_1(s,\eta)-c_{12}\int_{-\tau_{12}}^{0}x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\\
&\quad -c_{13}\int_{-\tau_{13}}^{0}x_3(s+\theta,\eta)\mathrm{d}\mu_{13}(\theta)\Big)\Big]\mathrm{d}s+\int_{t_1}^{t_2}\sigma_1(\gamma(s))\,x_1(s,\eta)\mathrm{d}\beta_1(s),\\
x_2(t_2,\eta)-x_2(t_1,\eta) &= \int_{t_1}^{t_2}\Big[x_2(s,\eta)\Big(-a_2(\gamma(s))-h_2+c_{21}\int_{-\tau_{21}}^{0}x_1(s+\theta,\eta)\mathrm{d}\mu_{21}(\theta)-c_{22}x_2(s,\eta)\\
&\quad -c_{23}\int_{-\tau_{23}}^{0}x_3(s+\theta,\eta)\mathrm{d}\mu_{23}(\theta)\Big)\Big]\mathrm{d}s+\int_{t_1}^{t_2}\sigma_2(\gamma(s))\,x_2(s,\eta)\mathrm{d}\beta_2(s),\\
x_3(t_2,\eta)-x_3(t_1,\eta) &= \int_{t_1}^{t_2}\Big[x_3(s,\eta)\Big(-a_3(\gamma(s))-h_3+c_{31}\int_{-\tau_{31}}^{0}x_1(s+\theta,\eta)\mathrm{d}\mu_{31}(\theta)\\
&\quad +c_{32}\int_{-\tau_{32}}^{0}x_2(s+\theta,\eta)\mathrm{d}\mu_{32}(\theta)-c_{33}x_3(s,\eta)\Big)\Big]\mathrm{d}s+\int_{t_1}^{t_2}\sigma_3(\gamma(s))\,x_3(s,\eta)\mathrm{d}\beta_3(s).
\end{aligned}$$

For any $t_2>t_1$, $p>0$

$$\begin{aligned}
[E|x_1(t_2,\eta)-x_1(t_1,\eta)|]^p &\le E|x_1(t_2,\eta)-x_1(t_1,\eta)|^p\\
&\le E\Big\{\int_{t_1}^{t_2}\Big[x_1(s,\eta)\Big|a_1(\gamma(t))-h_1-c_{11}x_1(s,\eta)-c_{12}\int_{-\tau_{12}}^{0}x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\\
&\quad -c_{13}\int_{-\tau_{13}}^{0}x_3(s+\theta,\eta)\mathrm{d}\mu_{13}(\theta)\Big|\Big]\mathrm{d}s+\Big|\int_{t_1}^{t_2}\sigma_1(\gamma(s))\,x_1(s,\eta)\mathrm{d}\beta_1(s)\Big|\Big\}^p\\
&\le 2^pE\Big\{\int_{t_1}^{t_2}\Big[x_1(s,\eta)\Big|\hat{a}_1-h_1-c_{11}x_1(s,\eta)-c_{12}\int_{-\tau_{12}}^{0}x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\\
&\quad -c_{13}\int_{-\tau_{13}}^{0}x_3(s+\theta,\eta)\mathrm{d}\mu_{13}(\theta)\Big|\Big]\mathrm{d}s\Big\}^p+2^pE\Big|\int_{t_1}^{t_2}\sigma_1(\gamma(s))\,x_1(s)\mathrm{d}\beta_1(s)\Big|^p.
\end{aligned} \tag{30}$$

Moreover,

$$\begin{aligned}
&E\Big\{\int_{t_1}^{t_2}\Big[x_1(s,\eta)\Big|\hat{a}_1-h_1-c_{11}x_1(s,\eta)-c_{12}\int_{-\tau_{12}}^{0}x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)-c_{13}\int_{-\tau_{13}}^{0}x_3(s+\theta,\eta)\mathrm{d}\mu_{13}(\theta)\Big|\Big]\mathrm{d}s\Big\}^p\\
\le\ & E\Big\{\int_{t_1}^{t_2}\Big[x_1(s,\eta)|\hat{a}_1-h_1|+c_{11}x_1^2(s,\eta)+c_{12}\int_{-\tau_{12}}^{0}x_1(s,\eta)x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\\
&+c_{13}\int_{-\tau_{13}}^{0}x_1(s,\eta)x_3(s+\theta,\eta)\mathrm{d}\mu_{13}(\theta)\Big]\mathrm{d}s\Big\}^p\\
\le\ & (t_2-t_1)^{p-1}E\int_{t_1}^{t_2}4^p\Big[x_1^p(s,\eta)|\hat{a}_1-h_1|^p+c_{11}^px_1^{2p}(s,\eta)+c_{12}^p\Big(\int_{-\tau_{12}}^{0}x_1(s,\eta)x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\Big)^p\\
&+c_{13}^p\Big(\int_{-\tau_{13}}^{0}x_1(s,\eta)x_3(s+\theta,\eta)\mathrm{d}\mu_{13}(\theta)\Big)^p\Big]\mathrm{d}s\\
=\ & 4^p(t_2-t_1)^{p-1}|\hat{a}_1-h_1|^p\int_{t_1}^{t_2}E[x_1^p(s,\eta)]\mathrm{d}s+4^p(t_2-t_1)^{p-1}c_{11}^p\int_{t_1}^{t_2}E[x_1^{2p}(s,\eta)]\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+4^p(t_2-t_1)^{p-1}c_{12}^pE\int_{t_1}^{t_2}\left(\int_{-\tau_{12}}^{0}x_1(s,\eta)x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\right)^p\mathrm{d}s\\
&+4^p(t_2-t_1)^{p-1}c_{13}^pE\int_{t_1}^{t_2}\left(\int_{-\tau_{13}}^{0}x_1(s,\eta)x_3(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\right)^p\mathrm{d}s, \qquad (31)
\end{aligned}
$$

where

$$
\begin{aligned}
&E\int_{t_1}^{t_2}\left(\int_{-\tau_{12}}^{0}x_1(s,\eta)x_2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\right)^p\mathrm{d}s\\
\le\ & E\int_{t_1}^{t_2}\left[\frac{x_1^2(s,\eta)}{2}+\frac{1}{2}\int_{-\tau_{12}}^{0}x_2^2(s+\theta,\eta)\mathrm{d}\mu_{12}(\theta)\right]^p\mathrm{d}s\\
\le\ & \int_{t_1}^{t_2}E[x_1^{2p}(s,\eta)]+\int_{t_1}^{t_2}\int_{-\tau_{12}}^{0}E[x_2^{2p}(s+\theta,\eta)]\mathrm{d}\mu_{12}(\theta)\mathrm{d}s. \qquad (32)
\end{aligned}
$$

Similarly,

$$
E\int_{t_1}^{t_2}\left(\int_{-\tau_{13}}^{0}x_1(s,\eta)x_3(s+\theta,\eta)\mathrm{d}\mu_{13}(\theta)\right)^p\mathrm{d}s\le\int_{t_1}^{t_2}E[x_1^{2p}(s,\eta)]+\int_{t_1}^{t_2}\int_{-\tau_{13}}^{0}E[x_3^{2p}(s+\theta,\eta)]\mathrm{d}\mu_{13}(\theta)\mathrm{d}s. \qquad (33)
$$

In view of the Theorem 2.11 in [40], for any $t_2>t_1$ and $p\ge 2$, we have

$$
E\left|\int_{t_1}^{t_2}x(s;\eta)\mathrm{d}\beta(s)\right|^p\le(\frac{p(p-1)}{2})^{\frac{p}{2}}(t_1-t_2)^{\frac{p-2}{2}}\int_{t_1}^{t_2}E[x^p(s,\eta)]\mathrm{d}s. \qquad (34)
$$

From Lemma 1 and Eqs (30)–(34), for $p\ge 2$, we get

$$
\begin{aligned}
[E|x_1(t_2,\eta)-x_1(t_1,\eta)|]^p\ \le\ & 2^p|\hat{\sigma}_1|^p(\frac{p(p-1)}{2})^{\frac{p}{2}}|t_2-t_1|^{\frac{p}{2}}K_1(p)+8^p\Big\{|\hat{a}_1-h_1|^p|t_2-t_1|^pK_1(p)\\
&+|t_2-t_1|^pc_{11}^pK_1(2p)+|t_2-t_1|^pc_{12}^pK_1(2p)+|t_2-t_1|^pc_{12}^pK_2(2p)\\
&+|t_2-t_1|^pc_{13}^pK_1(2p)+|t_2-t_1|^pc_{13}^pK_3(2p)\Big\}\\
\le\ & L_1|t_2-t_1|^{\frac{p}{2}},
\end{aligned}
$$

where, for $|t_2-t_1|<\epsilon$, the constant

$$
\begin{aligned}
L_1=\ &2^p|\hat{\sigma}_1|^p(\frac{p(p-1)}{2})^{\frac{p}{2}}K_1(p)+(64\epsilon)^{\frac{p}{2}}[|\hat{a}_1-h_1|^pK_1(p)+(c_{11}^p+c_{12}^p+c_{13}^p)K_1(2p)\\
&+c_{12}^pK_2(2p)+c_{13}^pK_3(2p)]>0.
\end{aligned}
$$

By the same method, we get $[E|x_1(t_2,\eta^*)-x_1(t_1,\eta^*)|]^p\le L_1|t_2-t_1|^{\frac{p}{2}}$. Thus $|G_1(t_2)-G_1(t_1)|\le 2L_1^{\frac{1}{p}}\sqrt{|t_2-t_1|}$. Similarly, there exist two constants $L_2>0$ and $L_3>0$ such that

$$
|G_2(t_2)-G_2(t_1)|\le 2L_2^{\frac{1}{p}}\sqrt{|t_2-t_1|},|G_3(t_2)-G_3(t_1)|\le 2L_3^{\frac{1}{p}}\sqrt{|t_2-t_1|},\quad a.s..
$$

So the $G_j(t)$, $j=1,2,3$, is uniformly continuous function. According to Barbălat Lemma [42], we have the conclusion. The proof is complete.

4. The optimal harvesting policy

In this section, we shall provide the sufficient and necessary conditions ensuring the existence of optimal solution of the system (1) with harvesting.

Assume $y(t)$ is the stochastic process of $(X(t), \gamma(t))$ and $p(t, \phi, i, \mathrm{d}y \times \{j\})$ is the transition probability of the process $\gamma(t)$. $P(t, \phi, i, A \times D)$ stands for the probability of event $\{y(t) \in A \times D\}$ with the initial condition $\phi \in \mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right), \gamma(0) = i$ and

$$P(t, \phi, i, A \times D) = \sum_{j \in D} \int_A p(t, \phi, i, \mathrm{d}y \times \{j\}).$$

Now, let $\mathcal{P}$ stands for all probability measures on $\mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right) \times \mathbb{S}$. For any measures $P_1, P_2 \in \mathcal{P}$ define the metric

$$d_{\mathbb{L}}(P_1, P_2) = \sup_{f \in \mathbb{L}} \Big| \sum_{i=1}^{n} \int_{\mathbb{R}^3_+} f(\phi, i) P_1(\mathrm{d}\phi, i) - \sum_{i=1}^{n} \int_{\mathbb{R}^3_+} f(\phi, i) P_2(\mathrm{d}\phi, i) \Big|,$$

where

$$\mathbb{L} = \left\{ f : \mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right) \to \mathbb{R} :| f(\phi, i) - f(\psi, j) | \le |\phi - \psi| + | i - j |, | f(\cdot \times \cdot) | \le 1 \right\}.$$

Theorem 3. Let all positive solutions of (1) be globally attractive. Then the solution $(X(t), \gamma(t))$ has a unique ergodic invariant distribution $\bar{\nu}(\cdot \times \cdot)$ defined on $\mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right) \times \mathbb{S}$.

Proof. We observe from Lemma 1 that for any initial data $\phi \in \mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right)$ with $\gamma(0) = i \in \mathbb{S}$, the family of transition probability $\{p(t, \phi, i, \mathrm{d}y \times \{j\}) : t \ge 0\}$ is tight by the Chebyshev inequality. That is to say, for a compact subset $K = K(\epsilon, \phi, i)$ and any $\epsilon > 0$

$$P(t, \phi, i, K \times \mathbb{S}) \ge 1 - \epsilon$$

Next, based on the Theorem 2, for any $\epsilon > 0, t \ge T$ and any compact subset $K \in \mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right)$, we have

$$\lim_{t \to \infty} d_{\mathbb{L}} \left(p(t, \phi, i, \cdot \times \cdot), p(t, \tilde{\phi}, j, \cdot \times \cdot) \right) = 0$$

uniformly in $\phi, \tilde{\phi} \in K$ and $i, j \in \mathbb{S}$. The proof is similar as Lemma 5.6 in [40], here we omit it. And then for an arbitrary initial data $\eta \in \mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right)$ with $\gamma(0) = j \in \mathbb{S}$, the $\{p(t, \eta, j, \cdot \times \cdot) : t \ge 0\}$ is cauchy in the metric space $\mathcal{P}(\mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right) \times \mathbb{S})$ (see Lemma 5.7 in [40]). Hence, there exists a unique probability measure $\bar{\nu}(\cdot)$ such that

$$\lim_{t \to \infty} d_{\mathbb{L}} \left(p(t, \eta, j, \cdot \times \cdot), \bar{\nu}(\cdot) \right) = 0.$$

The last, for any initial value $\phi \in \mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right)$ with $\gamma(0) = i \in \mathbb{S}$, we obtain that

$$\lim_{t \to \infty} d_{\mathbb{L}} \left(p(t, \phi, i, \cdot \times \cdot), \bar{\nu}(\cdot) \right) \le \lim_{t \to \infty} d_{\mathbb{L}} \left(p(t, \phi, i, \cdot \times \cdot), p(t, \eta, j, \cdot \times \cdot) \right) + \lim_{t \to \infty} d_{\mathbb{L}} \left(p(t, \eta, j, \cdot \times \cdot), \bar{\nu}(\cdot) \right) = 0.$$

It means that the $p(t, \phi, i, \cdot \times \cdot)$ converges weakly to $\bar{\nu}(\cdot)$. [43] (Corollary 3.4.3 and Theorem 3.2.6) show $\bar{\nu}(\cdot)$ is strong mixing and ergodic.

This completes the proof.

When C_M is a non-singular matrix, it follows from Theorems 2 and 3 that any positive solution $(X(t), \gamma(t))$ of the system (1) has a unique ergodic invariant distribution $\bar{\nu}(\cdot \times \cdot)$ defined on $\mathbb{C}\left([-\tau, 0], \mathbb{R}^3_+\right) \times \mathbb{S}$. This guarantees that $\lim_{t\to\infty} \sum_{j=1}^{3} E(h_j x_j(t))$ exists for the positive solution $x(t) = (x_1(t), x_2(t), x_3(t))^T$. That is, the harvesting effort $H^* = (h_1^*, h_2^*, h_3^*)^T$ is the optimal solution of Eq (2) if

(i) $\Phi(H^*)$ is the maximum value of Φ;

(ii) All species in the system (1) are stable in mean (see also Definition 1).

The following theorem gives sufficient and necessary conditions ensuring the existence of optimal solution $H^* = (h_1^*, h_2^*, h_3^*)^T$ for Eq (2).

Theorem 4. Let C_M be a non-singular matrix. Then the system (1) has the optimal harvesting policy if and only if (a), (b) and (c) all hold.

(a) $\tilde{C}_i \,|_{h_j^*} > 0, h_j^* \geq 0, i, j = 1, 2, 3;$

(b) $R \,|_{h_j^*} > 0, \omega_3 \,|_{h_j^*} > 1,$ or $R \,|_{h_j^*} < 0, \Theta_3 \,|_{h_j^*} > 1,$ or $R \,|_{h_j^*} = 0, \omega_2 > 1, j = 1, 2, 3;$

(c) $C^{-1} + (C^{-1})^T$ is positive definite.

Furthermore, we obtain the optimal harvesting effort $H^* = \left[\left(C(C^{-1})^T + I\right)\tilde{R}\right]^{-1} d$ and the maximum of sustainable yield $\Phi(H^*) = (H^*)^T \tilde{R} C^{-1}(d - \tilde{R}H^*) - W$, where $\tilde{R} = diag\{r_1, r_2, r_3\}$.

Proof. Denote $\mu = \{H = (h_1, h_2, h_3)^T \in \mathbb{R}^3 \mid (a), (b) \text{ all hold}\}$. If we take $h_1 = h_2 = h_3 = 0, r_1 = r_2 = r_3 = 1$ and choose the appropriate parameters, we can see μ is not empty. Denote ρ means the stationary probability density of model (1) and $x(t) = (x_1(t), x_2(t), x_3(t))^T$. For notation convenience, we use x instead of $x(t)$, so we get

$$\Phi(H) = \lim_{t\to\infty} \sum_{j=1}^{3} r_j h_j E(x_j(t)) - W = \sum_{k\in\mathbb{S}} \int_{\mathbb{R}^3_+} H^T \tilde{R} x \rho(x, k) \mathrm{d}x - W.$$

Theorem 3 shows the system (1) has a unique stable distribution $\bar{\nu}$. Due to the one-to-one corresponded between ρ and $\bar{\nu}$, we have the following relations (see Theorem 3.3.1 in [43])

$$\lim_{t\to\infty} t^{-1} \int_0^t x(s) \mathrm{d}s = \sum_{k\in\mathbb{S}} \int_{\mathbb{R}^3_+} x\rho(x, k)\mathrm{d}x = \sum_{k\in\mathbb{S}} \int_{\mathbb{R}^3_+} x\bar{\nu}(\mathrm{d}x, k).$$

For arbitrary $(h_1, h_2, h_3)^T \in \mu$, we have

$$\Phi(H) = \sum_{j=1}^{3} r_j h_j \lim_{t\to\infty} t^{-1} \int_0^t x_j(s)\mathrm{d}s - W = \sum_{j=1}^{3} r_j h_j \frac{C_j - \tilde{C}_j}{|C|} = H^T \tilde{R} C^{-1}(d - \tilde{R}H) - W.$$

Thus

$$\Phi(H) = H^T \tilde{R} C^{-1}(d - \tilde{R}H) - W.$$

Next

$$\frac{\mathrm{d}\Phi(H)}{\mathrm{d}H} = \tilde{R}C^{-1}d - \left[\tilde{R}\left(C^{-1} + (C^{-1})^T\right)\tilde{R}\right] H.$$

Assume $H^* = (h_1^*, h_2^*, h_3^*)^T$ is the solution of

$$\frac{\mathrm{d}\Phi(H)}{\mathrm{d}H} = 0.$$

Thus one get

$$H^* = \left[\left(C(C^{-1})^T + I\right)\tilde{R}\right]^{-1} d.$$

The Hessian matrix

$$\frac{\mathrm{d}}{\mathrm{d}H^T}\left[\frac{\mathrm{d}\Phi(H)}{\mathrm{d}H}\right] = -\tilde{R}\left(C^{-1} + (C^{-1})^T\right)\tilde{R}.$$

Due to $C^{-1}+(C^{-1})^T$ is positive definite, then H^* is a unique extreme point of $\Phi(H)$. And then the optimal harvesting effort $H^* = \left[\left(C(C^{-1})^T + I\right)\tilde{R}\right]^{-1} d$ and the maximum value of expectation of sustainable yield $\Phi(H^*) = (H^*)^T\tilde{R}C^{-1}(d - \tilde{R}H^*) - W$.

If conditions (a) and (b) do not hold, the species of the system (1) will not stable in mean or to be extinct. Then the optimal harvesting policy does not exist. What's more, if (c) not hold. That is to say $C^{-1} + (C^{-1})^T$ is not positive definite. Set $Q = C^{-1} + (C^{-1})^T$, then we can see

$$Q_{11} = \frac{2(c_{22}c_{33})}{c_{11}c_{22}c_{33} + c_{11}c_{23}c_{32} + c_{12}c_{21}c_{33} - c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31}} = \frac{2(c_{22}c_{33})}{|C|} > 0.$$

So Q must not be negative definite, and Q is indefinite. This implies that $\Phi(H)$ has no extreme point. Thus, the optimal harvesting policy does not exist. The proof is complete.

5. Examples and numerical simulations

In this section, we shall give some numerical examples and their simulations to illustrate the effectiveness of the obtained results for the hybrid stochastic phytoplankton-zooplankton-fish model with distributed delays (1). For the practical biological significance, some parameter values of the system (1) are given such as the growth rate of phytoplankton, the death rate of zooplankton and fish [44–46]. According to these publications, we take the mean growth rate of phytoplankton ranged from $0.26d^{-1}$ to $1.04d^{-1}$, the death rate of diaphanosoma brachyurum (a kind of zooplankton) ranged from $0.01d^{-1}$ to $0.92d^{-1}$, and the death rate of fish ranged from $0.15d^{-1}$ to $0.95d^{-1}$. Due to technical reasons, the rate of intra-specific competition, capture and food conversation are not easy to be obtained. In the following examples, we will refer to the value range of system parameters.

Example 1. Consider the stochastic delayed phytoplankton-zooplankton-fish model (1) with $\mathbb{S} = \{1, 2\}, \xi = (0.3, 0.7)$, and parameters $a_1(1) = 0.56, a_2(1) = 0.27, a_3(1) = 0.45$, $a_1(2) = 0.80, a_2(2) = 0.33, a_3(2) = 0.64$, $h_1 = h_2 = h_3 = 0$, $c_{11} = 0.35, c_{12} = 0.2, c_{13} = 0.1, c_{21} = 0.15, c_{22} = 0.65, c_{23} = 0.3, c_{31} = 0.28, c_{32} = 0.1, c_{33} = 0.55$, $\sigma_1^2(1) = \sigma_1^2(2) = 0.02, \sigma_2^2(1) = \sigma_2^2(2) = 0.01, \sigma_3^2(1) = \sigma_3^2(2) = 0.012$.

We compute that $C_{23} = 0.021 > 0, C_{31} = -0.005 < 0, C_{12} = 0.0015 > 0, |C| = 0.155025 > 0$, $\bar{a}_1 = 0.73, \bar{a}_2 = 0.31, \bar{a}_3 = 0.58, C_1 = 0.323, C_2 = 0.00015, C_3 = 0.00097, \tilde{C}_1 = 0.0032, \tilde{C}_2 = 0.000367, \tilde{C}_3 = 0.0034, R = 0.000001 > 0, \omega_3 = 0.2845 < 1, \omega_2 = 0.3077 < 1, \omega_1 = 73 > 1$. It follows from Case (c) of Theorem 1 that

$$\lim_{t\to\infty} x_j(t) = 0, j = 2, 3, \quad \lim_{t\to\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{u_1}{C_{11}} = 2.05.$$

Figure 2 is shown to illustrate the species x_2, x_3 are extinct and x_1 is stable in mean. So, there is no optimal solution for the optimization problem (2).

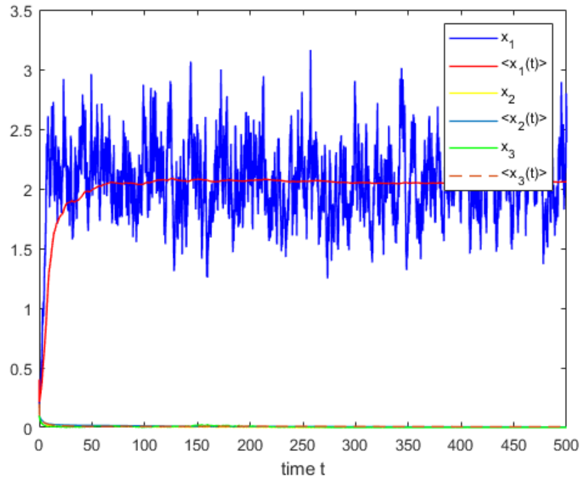

Figure 2. Species x_1 are stable in mean and species x_2 and x_3 die out.

The above example shows the effect of time delay and environmental noise on the survival and extinction of species without harvesting. According to Theorem 1, the time delay has no effect on the survival and extinction of the three species, but the harvesting and environmental noise have great influence on it. This is also verified by our numerical experiments. The results of example 1 showed that when Zooplankton and Fish were extinct, Phytoplankton could still stable in mean under certain environmental noise. But it is meaningless to discuss the optimal harvesting policy in the case of species extinction.

In order to demonstrate some results about optimal harvesting policy for the system (1), we give the following example.

Example 2. Consider the stochastic delayed phytoplankton-zooplankton-fish model (1) with $\mathbb{S} = \{1, 2\}, \xi = (0.9, 0.1)$, $a_1(1) = 0.9789, a_2(1) = 0.0211, a_3(1) = 0.1044$, $a_1(2) = 0.59, a_2(2) = 0.01, a_3(2) = 0.06$, we compute $\bar{a}_1 = 0.94, \bar{a}_2 = 0.02, \bar{a}_3 = 0.1$.

When there is no harvesting, i.e. $h_i = 0, i = 1, 2, 3$, we compute that $C_1 = 0.3672, C_2 = 0.0062, C_3 = 0.1599, \tilde{C}_1 = 0.0032, \tilde{C}_2 = 0.000367, \tilde{C}_3 = 0.0034, R = -0.000243 < 0, \Theta_3 = \frac{C_2}{\tilde{C}_2} = 16.8163 > 1$. It follows from Case (n) of Theorem 1 that

$$\lim_{t\to\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{C_1 - \tilde{C}_1}{|C|} = 2.3474,$$
$$\lim_{t\to\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{C_2 - \tilde{C}_2}{|C|} = 0.0375,$$
$$\lim_{t\to\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{C_3 - \tilde{C}_3}{|C|} = 1.0091.$$

Figure 3 is shown to illustrate the species $x_i, i = 1, 2, 3$ are stable in mean for the system (1) with no harvesting.

Furthermore, we select the appropriate values of the unit profit parameters $r_1 = 0.6, r_2 = 2.7, r_3 = 4$, and fixed cost $W = 0.2$. Then, all the conditions in Theorem 4 are checked as follows: $d = (0.93, 0.025, 0.106)^T, H^* = (h_1^*, h_2^*, h_3^*)^T = \left[\left(C(C^{-1})^T + I\right)\tilde{R}\right]^{-1} d = (0.6915, 0.0713, 0.0764)^T, \tilde{C}_1|_{H=H^*} = 0.2622, \tilde{C}_2|_{H=H^*} = 0.0059, \tilde{C}_3|_{H=H^*} = 0.1578, R|_{H=H^*} = 0.000225 > 0, \omega_3 = \frac{C_3}{\tilde{C}_3} = 1.0129 > 1$ and $C_M^{-1} \geq 0$. Thus, the optimal harvesting effort is $H^* = (0.6915, 0.0713, 0.0764)^T$ and the maximum of total economic income is 0.4792. Under the optimal harvesting condition, by Case (g) in Theorem 1, we see that

$$\lim_{t\to\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{C_1 - \tilde{C}_1}{|C|} = 0.6767,$$

$$\lim_{t\to\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{C_2 - \tilde{C}_2}{|C|} = 0.0019,$$

$$\lim_{t\to\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{C_3 - \tilde{C}_3}{|C|} = 0.0131.$$

Figure 4 is shown to illustrate the species $x_i, i = 1, 2, 3$ are stable in mean for the system (1) with the optimal harvesting H^*.

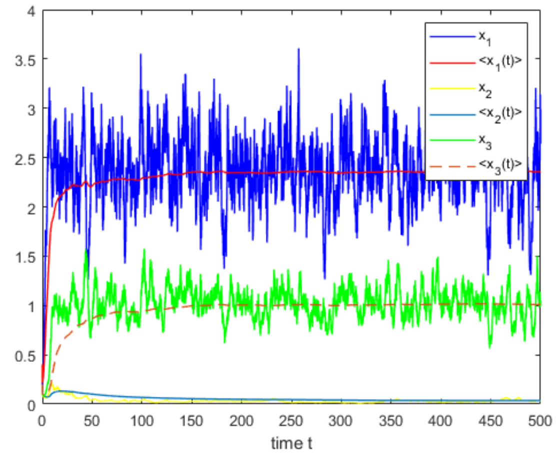

Figure 3. All species are stable in mean without harvesting.

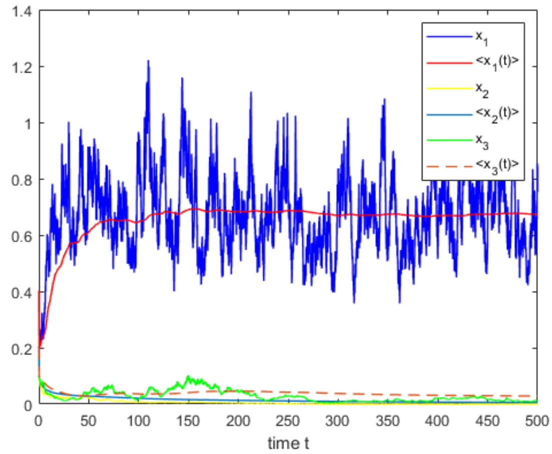

Figure 4. All species are stable in mean with harvesting.

In example 2, we see that all species are stable in mean without harvesting. According to Theorem 4, we get the maximum harvesting effort H^*, and verify that all species are still stable in mean. This means that the system (1) has an optimal harvesting strategy with the maximum of total economic income. However, excessive harvesting or strong enough environmental noise may cause the extinction of species, and the optimal harvesting strategy does not exist.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. J. Steele, *Stability of Plankton Ecosystem*, Chapman and Hall, London, 1974.
2. T. Saha, M. Bandyopadhyay, Dynamical analysis of toxin producing phytoplankton-zooplankton interactions, *Nonlinear Anal. Real World Appl.*, **10** (2009), 314–332.
3. Y. Lv, Y. Pei, S. Gao, C. Li, Harvesting of a phytoplankton-zooplankton model, *Nonlinear Anal. Real World Appl.*, **11** (2010), 3608-3619.
4. C. Liu, L. Wang, Q. Zhang, Y. Yun, Dynamical analysis in a bioeconomic phytoplankton zooplankton system with double time delays and environmental stochasticity, *Phys. A*, **482** (2017), 682–698.
5. X. Meng, J. Li, Stability and Hopf bifurcation analysis of a delayed phytoplankton-zooplankton model with Allee effect and linear harvesting, *Math. Biosci. Eng.*, **17** (2019), 1973–2002.
6. G. Denaro, D. Valenti, B. Spagnolo, G. Basilone, S. Mazzola, S. W. Zgoz, et al., Dynamics of two picophytoplankton groups in mediterranean sea: Analysis of the deep chlorophyll maximum by a stochastic advection-reaction-diffusion model, *PLoS One*, **8** (2013), e66765.
7. G. Denaro, D. Valenti, A. La Cognata, B. Spagnolo, A. Bonanno, G. Basilone, et al., Spatio-temporal behaviour of the deep chlorophyll maximum in Mediterranean Sea: Development of a stochastic model for picophytoplankton dynamics, *Ecol. Complex*, **13** (2013), 21–34.
8. D. Valenti, G. Denaro, A. L. Cognata, B. Spagnolo, Picophytoplankton dynamics in noisy marine environment, *Acta. Phys. Pol B*, **43** (2012), 1227–1240.
9. A. S. Heiskanen, T. Tamminen, K. Gundersen, Impact of planktonic food web structure on nutrient retention and loss from a late summer pelagic system in the coastal northern Baltic Sea, *Mar. Ecol. Prog. Ser.*, **145** (1996), 195–208.
10. M. Scheffe, Fish and nutrients interplay determines algal biomass: a minimal model, *Oikos*, **62** (1991), 271–282.
11. A. B. Medvinsky, S. V. Petrovskii, I. A. Tikhonova, H. Malchow, B. L. Li, Spatio-temporal complexity of plankton and fish dynamics, *SIAM Rev.*, **44** (2002), 311–370.
12. P. Panja, S. K. Mondal, Stability analysis of coexistence of three species prey-predator model, *Nonlinear Anal.*, **81** (2015), 373–382.

13. A. Sharma, A. K. Sharma, K. Agnihotri, Complex dynamic of plankton-fish interaction with quadratic harvesting and time delay, *Model Earth Syst. Environ.*, **2** (2016), 1–17.
14. X. Y. Meng, Y.Q. Wu, Bifurcation and control in a singular phytoplankton-zooplankton-fish model with nonlinear fish harvesting and taxation, *Int. J. Bifurcat. Chaos*, **28** (2018), 1850042.
15. Z. Wei, J. Sugie, Global asymptotic stability and equiasymptotic stability for a time-varying phytoplankton-zooplankton-fish system, *Nonlinear Anal. Real World Appl.*, **46** (2019), 116–136.
16. B. Buonomo, M. Cerasuolo, The effect of time delay in plant–pathogen interactions with host demography, *Math. Biosci. Eng.*, **12** (2015), 473–490.
17. B. Tian, Y. Qiu, N. Chen, Periodic and almost periodic solution for a non-autonomous epidemic predator-prey system with time-delay, *Appl. Math. Copmut.*, **215** (2009), 779–790.
18. O. A. Chichigina, A. A. Dubkov, D, Valenti, B. Spagnolo, Stability in a system subject to noise with regulated periodicity, *Phys. Rev. E*, **84** (2011), 021134.
19. D. Valenti, L. Tranchina, M. Brai, A. Caruso, C. Cosentino, B. Spagnolo, Environmental metal pollution considered as noise: Effects on the spatial distribution of benthic foraminifera in two coastal marine areas of Sicily (Southern Italy), *Ecol. Model*, **213** (2008), 449–462.
20. A. A. Dubkov, B. Spagnolo, Verhulst model with Lévy white noise excitation, *Eur. Phys. J. B*, **65** (2008), 361–367.
21. Q. Luo, X. Mao, Stochastic population dynamics under regime switching, *J. Math. Anal. Appl.*, **355** (2009), 577–593.
22. M. Slatkin, The dynamics of a Population in a Markovian environment, *Ecology*, **59** (1978), 249–256.
23. H. Qiu, W. Deng, Stationary distribution and global asymptotic stability of a three-species stochastic food-chain system, *Turk. J. Math.*, **41** (2017), 1292–1307.
24. Y. Ma, Q. Zhang, L. Wang, T. Kang, Dissipative control of a three-species food chain stochastic system with a hidden Markovchain, *Adv. Differ. Equ-Ny.*, **2017** (2017), 1–22.
25. Y. Lin, D. Jiang, Long-time behavior of a stochastic predator-prey model with modified Leslie-Gower and Holling-type II schemes, *Int. J. Biomath*, **9** (2016), 1650039.
26. J. Lv, K. Wang, Asymptotic properties of a stochastic predator-prey system with Holling II functional response, *Commun. Nonlinear. Sci. Numer Simulat.*, **16** (2011), 4037–4048.
27. Z. Liu, N. Shi, D. Jiang, C. Ji, The Asymptotic behavior of a stochastic predator-prey system with Holling II functional response, *Abstr. Appl. Anal.*, **2012** (2012), 1–14.
28. G. Gilioli, S. Pasquali, F. Ruggeri, Nonlinear functional response parameter estimation in a stochastic predator-prey model, *Math. Biosci. Eng.*, **9** (2012), 75–96.
29. C. S. Reynolds, *The Ecology of Freshwater Phytoplankton*, Cambridge University Press, Cambridge, 1984.
30. C. W. Clark, *Mathematical Bio-Economics: The Optimal Management of Renewable Resources*, Wiley, New York, 1976.
31. C. W. Clark, Bioeconomic Modeling and Resource Management, in *Applied Mathematical Ecology* (eds. S. A. Levin), Springer, 1989, 11–57.

32. M. Mesterton-Gibbons, On the optimal policy for combining harvesting of predator and prey, *Nat. Resour. Model*, **3** (1988), 63–90.

33. M. Mesterton-Gibbons, A technique for finding optimal two-species harvesting policies, *Ecol. Model*, **92** (1996), 235–244

34. S. Wang, L. Wang, T. Wei, Optimal harvesting for a stochastic predator-prey model with S-type distributed time delays, *Methodol Comput. Appl.*, **20** (2016), 37–68.

35. M. Liu, C. Bai, Analysis of a stochastic tri-trophic food-chain model with harvesting, *J. Math. Biol.*, **73** (2016), 597–625.

36. M. Liu, X. He, J. Yu, Dynamics of a stochastic regime-switching predator-prey model with harvesting and distributed delay, *Nonlinear Anal. Hybrid Syst.*, **28** (2018), 87–104.

37. W. J. Anderson, *Continuous-Time Markov Chains*, Springer, New York, 1991.

38. R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, 1975.

39. X. Zhang, W. Li, M. Liu, K. Wang, Dynamics of a stochastic Holling II one-predator two-prey system with jumps, *Phys. A*, **421** (2015), 571–582.

40. X. Mao, C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, 2006.

41. M. Liu, J. Yu, P. S. Mandal, Dynamics of a stochastic delay competitive model with harvesting and Markovian switching, *Appl. Math. Comput.*, **337** (2018), 335–349.

42. V. M. Popov, *Hyperstability of Control Systems*, Springer-Verlag, New York, 1973.

43. D. Prato, J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

44. L. Thomas, Estimating Phytoplankton Growth Rates from Compositional Data, in *Oceanography/Biological Oceanography Massachusetts Institute of Technology and Woods Hole Oceanographic Institution*, Massachusetts Institute Of Technology, 2008.

45. T. Nanazato, M. Yasuno, Population dynamics and production of cladoceran zooplankton in the highly eutrophic Lake Kasumigaura, *Hydrobiologia*, **124** (1981), 13–22.

46. Y. Wang, Q. Liu, Estimating natural mortality from stock size and catch data (in Chinese), *Period. Ocean Univ. China*, **35** (2005), 020–024.