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Research article

Dynamical analysis of a stochastic SIRS epidemic model with saturating contact rate

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Abstract: In this paper, a stochastic SIRS epidemic model with saturating contact rate is constructed. First, for the deterministic system, the stability of the equilibria is discussed by using eigenvalue theory. Second, for the stochastic system, the threshold conditions of disease extinction and persistence are established. Our results indicate that a large environmental noise intensity can suppress the spread of disease. Conversely, if the intensity of environmental noise is small, the system has a stationary solution which indicates the disease is persistent. Eventually, we introduce some computer simulations to validate the theoretical results.

Keywords: stochastic SIRS epidemic model; saturating contact rate; extinction; persistence; stationary solution

1. Introduction

Recently, a new type of pneumonia caused by the coronavirus, named COVID-19, is spreading around the world. The issue of infectious diseases has once again aroused people's great concern. How to prevent and control infectious diseases has been an important subject facing human beings [1–8]. The SIR model assumes that the infected person can obtain permanent immunity after recovery. However, for smallpox, cholera, malaria and other diseases, individuals recovered from treatment can return to the susceptible category after temporary immunization, which can be described by SIRS model. Moreover, for some bacterial infectious diseases, such as meningitis and sexually transmitted diseases, some individuals can not produce effective antibodies after treatment and may be infected again. Others may gain temporary immunity, but then lose immunity and become susceptible [9–12]. Literature [10] established an SIRS model with a general population-size dependent contact rate $\lambda(N)$ and proportional transfer rate from the infective class to susceptible class. The authors studied the threshold conditions of disease extinction and discussed the stability of disease-free equilibrium and endemic equilibrium.

Infection rate is an important index to measure the intensity of disease transmission. Employing an appropriate infection rate based on a specific disease for the mathematical model plays a vital role in the disease prevention and control. In the literature [13], Thieme and Castillo-Chavez proposed the incidence $\frac{\beta\varrho(N(t))S(t)I(t)}{N(t)}$, where $N(t)$ represents the total population. On that basis, Heesterbeek et al. [14] gave the following saturating contact rate

$$\varrho(N(t)) = \frac{bN(t)}{1 + bN(t) + \sqrt{1 + 2bN(t)}}.$$

Obviously, $\varrho(N(t))$ is a non-decreasing function of $N(t)$. $\frac{\varrho(N(t))}{N(t)}$ is a non-increasing function of $N(t)$. If $N(t)$ is sufficiently small, $\varrho(N(t)) \sim bN$. Conversly, if $N(t)$ is fully large, $\varrho(N(t)) \sim 1$. Compared with the bilinear incidence $\beta S(t) I(t)$ and the standard incidence $\frac{\beta S(t)I(t)}{N(t)}$, the saturating contact rate is more closer to the transmission of many diseases. The saturated contact rate is widely used in the study of infectious disease modeling. For example, Zhang et al. [15] constructed an SEIS model with general saturated incidence rate, and proved the global asymptotic stability of the endemic equilibrium by using the autonomous convergence theorem. Lan et al. [16] considered an SIS epidemic model with saturating contact rate, by using Itô's formula, the conditions for disease extinction and the existence of stationary solutions were obtained. In reference [11], Li et al. established an SIRS epidemic model with a general incidence, which considered both the transfer from the infected to the susceptible and the transfer from the recovered to the susceptible. Motivated by the above literature, we formulate a deterministic SIRS epidemic model with saturating contact rate and transfer from infectious to susceptible:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - uS(t) - \frac{\beta bS(t)I(t)}{g(N(t))} + \gamma_1 I(t) + \delta R(t), \\ \frac{dI(t)}{dt} = \frac{\beta bS(t)I(t)}{g(N(t))} - (u + \gamma_1 + \gamma_2 + \alpha) I(t), \\ \frac{dR(t)}{dt} = \gamma_2 I(t) - (u + \delta) R(t), \end{cases} \tag{1.1}$$

where $g(N(t)) = 1 + bN(t) + \sqrt{1 + 2bN(t)}$, $N(t) = S(t) + I(t) + R(t)$ is the total population. $S(t)$, $I(t)$, $R(t)$ represent the number of susceptible individuals, infected individuals and recovered individuals, respectively. Λ is the recruitment rate of susceptible individuals. u denotes the natural mortality rate. α represents the mortality rate caused by diseases. γ_1 is the transfer rate from the infected individuals to the susceptible individuals. γ_2 is the transfer rate from the infected individuals to the recovered individuals, and δ denotes the immunity loss rate.

Due to the influence of environmental noise, the prevalence and transmission of diseases is often random. For example, the change of temperature and the influence of climate will lead to the fluctuation of mortality, morbidity and so on. In recent years, mathematical models of infectious diseases described by stochastic differential equations have been widely concerned [17–22]. There are many ways to construct a stochastic differential equation model, such as adding random perturbations to the parameters of deterministic system [23–26], or introducing proportional perturbations to state variables [27–31]. Recently, considering the effect of two different white noises on the model parameters, reference [32] established a stochastic SIS model with two correlated Brownian motions, in which the threshold of disease extinction as well as the variance and mean of the stationary distribution were investigated. In this paper, we consider that the incidence coefficient βb is disturbed by white noise, that is $\beta b \to \beta b + \sigma dB(t)$, where σ^2 is the intensity of white noise, $B(t)$ is defined as the standard Brownian motion in the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$. Thus, the above model (1.1) is transformed into the following SIRS stochastic epidemic model:

$$\begin{cases} dS(t) = \left[\Lambda - uS(t) - \frac{\beta bS(t)I(t)}{g(N(t))} + \gamma_1 I(t) + \delta R(t)\right]dt - \frac{\sigma S(t)I(t)}{g(N(t))}dB(t), \\ dI(t) = \left[\frac{\beta bS(t)I(t)}{g(N(t))} - (u + \gamma_1 + \gamma_2 + \alpha) I(t)\right]dt + \frac{\sigma S(t)I(t)}{g(N(t))}dB(t), \\ dR(t) = \left[\gamma_2 I(t) - (u + \delta) R(t)\right]dt. \end{cases} \tag{1.2}$$

As far as we know, there have been a lot of studies on the epidemic model disturbed by environmental noise, but there are few stochastic models considering saturating contact rate and transfer from infectious to susceptible, especially the existence of stationary solution. The paper is organized as follows: In section 2, we give some notations and related lemmas. The thresholds of deterministic system and stochastic system are established in sections 3 and 4, respectively. Sufficient condition for the existence of stationary solution in the stochastic system (1.2) is provided in section 5. In section 6, we verify the results of theoretical derivation by numerical simulations.

2. Preliminaries

Throughout this paper, we let $R^3_+ = \{x_i > 0, i = 1, 2, 3\}$. For an integrable function h on $[0, +\infty)$, we define $\langle h(t)\rangle = \frac{1}{t}\int_0^t h(\pi)d\pi$. By using the methods from Liu et al. [33], we can prove that the region

$$\Gamma = \left\{(S(t), I(t), R(t)) \in R^3_+, \frac{\Lambda}{u+\alpha} \leq N(t) \leq \frac{\Lambda}{u}\right\}$$

is a positively invariant set of system (1.2).

Lemma 2.1. *For any given initial value $(S(0), I(0), R(0)) \in R^3_+$, then the model (1.2) has a unique positive solution $(S(t), I(t), R(t))$ on $t \geq 0$, and the solution will remain in R^3_+ with probability 1.*

Next, we will introduce some contents of stationary Markov process. The n-dimensional stochastic differential equation can be expressed by the following formula

$$dx(t) = f(x(t), t)dt + \sum_{i=1}^{l} g_i(x(t), t)dB_i(t), \qquad \forall t \geq t_0, \tag{2.1}$$

with the initial value $x(t_0) = x_0 \in \mathbb{R}^n$. Integrating from 0 to t for both sides of the Eq (2.1), one can obtain that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(\theta), \theta)d\theta + \sum_{i=1}^{l}\int_{t_0}^{t} g_i(x(\theta), \theta)dB_i(\theta), \qquad \forall t \geq t_0. \tag{2.2}$$

Assume that the vectors $f(x,t)$, $g_1(x,t)$,..., $g_l(x,t)(t \geq t_0, x \in \mathbb{R}^n)$ are continuous functions of (x,t), satisfying the following conditions for some constant D,

$$\begin{aligned} &(A_1) \quad |f(x,t) - f(y,t)| + \sum_{i=1}^{l} |g_i(x,t) - g_i(y,t)| \leq D|x-y|, \\ &(A_2) \quad |f(x,t)| + \sum_{i=1}^{l} |g_i(x,t)| \leq D(1+|x|). \end{aligned} \tag{2.3}$$

According to the literature [34] Theorem 3.7, we have the following Lemma.

Lemma 2.2. *Suppose that the coefficients of* (2.2) *are independent of t, and the conditions* (2.3) *hold in* U_G $(\forall G > 0)$. *There exists a function* $V(x) \in C^2$ *with the following properties in* $\mathbb{R}^n$

$$V(x) \geq 0 \quad \textit{and} \quad \sup_{|x|>G} LV(x) = -M_G \to -\infty \,(G \to \infty), \tag{2.4}$$

where C^2 *represents a class of functions that are twice continuously differentiable relative to* x *in* $\mathbb{R}^n$. *Further, we assume that there is at least one* $x \in \mathbb{R}^n$, *such that the process* $X^x(t)$ *is regular. Then there exists a solution of system* (2.2) *which is a stationary Markov process.*

For Lemma 2.2, we need to note the following two points.

Remark 2.1 (i) Condition (2.3) can be replaced by the global existence of solution of system (2.2) (see [35] Remark 5);

(ii) Condition (2.4) can be replaced by the weaker condition $LV(x) \leq -1$ (see [34] Chapter 4).

3. Dynamics of system (1.1)

In this section, we concentrate on the stability of the equilibria. Firstly, using $N(t)$ as a variable instead of the variable $S(t)$, we convert system (1.1) into the following form

$$\begin{cases} \frac{dN(t)}{dt} = \Lambda - uN(t) - \alpha I(t), \\ \frac{dI(t)}{dt} = \frac{\beta b I(t)}{g(N(t))}(N(t) - I(t) - R(t)) - (u + \gamma_1 + \gamma_2 + \alpha) I(t), \\ \frac{dR(t)}{dt} = \gamma_2 I(t) - (u + \delta) R(t). \end{cases} \tag{3.1}$$

Obviously, the system (3.1) exists a boundary equilibrium $P_0 = \left(\frac{\Lambda}{u}, 0, 0\right)$. Define

$$R_0 = \frac{\Lambda \beta b}{u(u + \gamma_1 + \gamma_2 + \alpha) g\left(\frac{\Lambda}{u}\right)}.$$

By direct calculation, if $R_0 > 1$, we get system (3.1) has a positive equilibrium $P^* = (N^*, I^*, R^*)$ with

$$I^* = \frac{\Lambda - uN^*}{\alpha}, \qquad R^* = \frac{\gamma_2(\Lambda - uN^*)}{\alpha(u + \delta)},$$

where N^* is the unique positive root of the following function

$$\phi(N) = \beta b\left[N - \frac{\Lambda - uN}{\alpha} - \frac{\gamma_2(\Lambda - uN)}{\alpha(u + \delta)}\right] - (u + \gamma_1 + \gamma_2 + \alpha) g(N).$$

In fact, we have

$$\phi\left(\frac{\Lambda}{u}\right) = (u + \gamma_1 + \gamma_2 + \alpha) g\left(\frac{\Lambda}{u}\right)(R_0 - 1) > 0$$

and

$$\phi\left(\frac{\Lambda}{u + \alpha}\right) = -\frac{\beta b \gamma_2 \Lambda}{(u + \alpha)(u + \delta)} - (u + \gamma_1 + \gamma_2 + \alpha) g\left(\frac{\Lambda}{u + \alpha}\right) < 0.$$

Let $\frac{d\phi(N)}{dN} = 0$, it can be got

$$N_{**} = \frac{\left\{\frac{\alpha(u+\delta)(u+\gamma_1+\gamma_2+\alpha)}{\beta\left[(\alpha+u)(u+\delta)+u\gamma_2\right]-\alpha(u+\delta)(u+\gamma_1+\gamma_2+\alpha)}\right\}^2 - 1}{2b},$$

which implies that $\phi(N)$ is increasing if $N \geq N_{**}$, and $\phi(N)$ is decreasing if $N < N_{**}$. Therefore, function $\phi(N)$ has a unique positive root, then the system (3.1) exists a unique positive equilibrium $P^* = (N^*, I^*, R^*)$.

Theorem 3.1. *For system* (3.1)*, we have*

(i) If $R_0 < 1$*, then* $P_0 = \left(\frac{\Lambda}{u}, 0, 0\right)$ *is a unique stable equilibrium, which implies the disease of system* (3.1) *goes extinct.*

(ii) If $R_0 > 1$*, then* $P^* = (N^*, I^*, R^*)$ *is a stable positive equilibrium, which implies the disease of system* (3.1) *is permanent.*

Proof. (i) The Jacobian matrix of system (3.1) evaluated at $P_0 = \left(\frac{\Lambda}{u}, 0, 0\right)$ is

$$J_0 = \begin{bmatrix} -u & -\alpha & 0 \\ 0 & \frac{\Lambda\beta b}{ug\left(\frac{\Lambda}{u}\right)} - (u + \gamma_1 + \gamma_2 + \alpha) & 0 \\ 0 & \gamma_2 & -(u+\delta) \end{bmatrix},$$

which has three eigenvalues:

$$\lambda_1 = -u < 0, \qquad \lambda_2 = -(u+\delta) < 0, \qquad \lambda_3 = (u + \gamma_1 + \gamma_2 + \alpha)(R_0 - 1) < 0.$$

Therefore, according to stability theory, P_0 is stable if $R_0 < 1$.

(ii) The Jacobian matrix at $P^* = (N^*, I^*, R^*)$ is

$$J^* = \begin{bmatrix} -u & -\alpha & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & \gamma_2 & -(u+\delta) \end{bmatrix},$$

where

$$\begin{aligned} a_{21} &= \frac{\beta b I^* \left[g(N^*) - (N^* - I^* - R^*)\left(b + \frac{b}{\sqrt{1+2bN^*}}\right)\right]}{g^2(N^*)} \\ &= \frac{\beta b I^* \left(1 + \sqrt{1+2bN^*} - \frac{bN^*}{\sqrt{1+2bN^*}}\right)}{g^2(N^*)} + \frac{\beta b I^* (I^* + R^*)}{g^2(N^*)}\left(b + \frac{b}{\sqrt{1+2bN^*}}\right) \\ &= \frac{\beta b I^*}{g(N^*)\sqrt{1+2bN^*}} + \frac{\beta b I^* (I^* + R^*)}{g^2(N^*)}\left(b + \frac{b}{\sqrt{1+2bN^*}}\right) > 0, \\ a_{22} &= \frac{\beta b (N^* - R^* - 2I^*)}{g(N^*)} - (u + \gamma_1 + \gamma_2 + \alpha) = -\frac{\beta b I^*}{g(N^*)} < 0, \\ a_{23} &= -\frac{\beta b I^*}{g(N^*)} = a_{22} < 0. \end{aligned}$$

Hence, the characteristic equation of J^* is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$\begin{aligned} a_1 &= u + \frac{\beta bI^*}{g(N^*)} + u + \delta > 0, \\ a_2 &= \alpha\left[\frac{\beta bI^*}{g(N^*)\sqrt{1+2bN^*}} + \frac{\beta bI^*(I^*+R^*)}{g^2(N^*)}\left(b + \frac{b}{\sqrt{1+2bN^*}}\right)\right] \\ &\quad + (2u + \delta + \gamma_2)\frac{\beta bI^*}{g(N^*)} + u(u+\delta) > 0 \end{aligned}$$

and

$$\begin{aligned} a_3 &= \alpha(u+\delta)\left[\frac{\beta bI^*}{g(N^*)\sqrt{1+2bN^*}} + \frac{\beta bI^*(I^*+R^*)}{g^2(N^*)}\left(b + \frac{b}{\sqrt{1+2bN^*}}\right)\right] \\ &\quad + u(u + \delta + \gamma_2)\frac{\beta bI^*}{g(N^*)} > 0. \end{aligned}$$

Then,

$$\begin{aligned} a_1a_2 - a_3 &= \alpha u a_{21} - \left[(2u+\delta)(2u+\delta+\gamma_2) + u\gamma_2\right]a_{22} - \alpha a_{21}a_{22} \\ &\quad + (2u+\delta+\gamma_2)a_{22}^2 + u(u+\delta)(2u+\delta) > 0. \end{aligned}$$

Therefore, the equilibrium P^* is stable when it exists. This completes the proof of Theorem 3.1. □

4. The threshold of the system (1.2)

In the previous section, we have obtained the threshold for ordinary differential equation (ODE) system (1.1). Similarly, the threshold of stochastic differential equation (SDE) system (1.2) is also crucial, which determines the extinction and persistence of the disease. Define the following parameter

$$R_0^s = \frac{\Lambda\beta b}{u\left(u + \gamma_1 + \gamma_2 + \alpha + \frac{\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}\right)g\left(\frac{\Lambda}{u}\right)},$$

where $g(x) = 1 + bx + \sqrt{1+2bx}$. In this section, we will prove that R_0^s is the threshold of system (1.2). Now, we give the following definition.

Definition 4.1.

(i) The disease $I(t)$ is said to be extinct if $\lim_{t\to+\infty} I(t) = 0$;

(ii) The disease $I(t)$ is said to be permanent in mean if there is a positive constant ϕ such that $\liminf_{t\to\infty}\langle I(t)\rangle \geq \phi$.

4.1. Extinction

Theorem 4.1. *Set* $(S(t), I(t), R(t))$ *be a solution of system* (1.2) *with any given initial value* $(S(0), I(0), R(0)) \in \Gamma$.

(i) If $\sigma^2 > \max\left\{\frac{\beta^2 b^2}{2(u+\gamma_1+\gamma_2+\alpha)}, \frac{\beta b u g\left(\frac{\Lambda}{u}\right)}{\Lambda}\right\}$ *holds, then*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \frac{\beta^2 b^2}{2\sigma^2} - (u+\gamma_1+\gamma_2+\alpha) < 0 \qquad a.s.,$$

(ii) If $R_0^s < 1$ *and* $\sigma^2 < \frac{\beta b u g\left(\frac{\Lambda}{u}\right)}{\Lambda}$ *holds, then*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \left(u+\gamma_1+\gamma_2+\alpha+\frac{\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}\right)(R_0^s - 1) < 0 \qquad a.s.,$$

which implies that the disease dies out with probability 1. In addition, we have

$$\lim_{t\to\infty} \langle S(t)\rangle = \frac{\Lambda}{u} \qquad a.s.,$$

$$\lim_{t\to\infty} \langle R(t)\rangle = 0 \qquad a.s..$$

Proof. Set $\tilde{V} = \ln I(t)$, by the Itô's formula, one can get

$$\begin{aligned} d(\ln I(t)) &= \left[\frac{\beta b S(t)}{g(N(t))} - (u+\gamma_1+\gamma_2+\alpha) - \frac{\sigma^2 S^2(t)}{2g^2(N(t))}\right]dt + \frac{\sigma S(t)}{g(N(t))}dB(t) \\ &= \Phi\left(\frac{S(t)}{g(N(t))}\right)dt + \frac{\sigma S(t)}{g(N(t))}dB(t), \end{aligned} \tag{4.1}$$

where $\Phi(x) = -\frac{\sigma^2}{2}x^2 + \beta b x - (u+\gamma_1+\gamma_2+\alpha)$.

Case (i): If condition (i) is satisfied, then

$$\begin{aligned} \Phi\left(\frac{S(t)}{g(N(t))}\right) &= -\frac{\sigma^2}{2}\left(\frac{S(t)}{g(N(t))} - \frac{\beta b}{\sigma^2}\right)^2 + \frac{\beta^2 b^2}{2\sigma^2} - (u+\gamma_1+\gamma_2+\alpha) \\ &\le \frac{\beta^2 b^2}{2\sigma^2} - (u+\gamma_1+\gamma_2+\alpha). \end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1), one can obtain that

$$d(\ln I(t)) \le \left[\frac{\beta^2 b^2}{2\sigma^2} - (u+\gamma_1+\gamma_2+\alpha)\right]dt + \frac{\sigma S(t)}{g(N(t))}dB(t). \tag{4.3}$$

By using the strong law of large numbers for martingales, we obtain

$$\lim_{t\to\infty} \frac{\int_0^t \frac{\sigma S(\xi)}{g(N(\xi))} d\xi}{t} = 0.$$

Integrating from 0 to t , dividing by t , and taking superior limit on both sides of Eq (4.3) yields

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \frac{\beta^2 b^2}{2\sigma^2} - (u + \gamma_1 + \gamma_2 + \alpha) < 0. \qquad a.s. \tag{4.4}$$

Case (ii): If condition (ii) $R_0^s < 1$ and $\sigma^2 < \frac{\beta b u g\left(\frac{\Lambda}{u}\right)}{\Lambda}$ are satisfied, then

$$\begin{aligned} \Phi\left(\frac{S(t)}{g(N(t))}\right) &= -\frac{\sigma^2}{2}\left(\frac{S(t)}{g(N(t))} - \frac{\beta b}{\sigma^2}\right)^2 + \frac{\beta^2 b^2}{2\sigma^2} - (u + \gamma_1 + \gamma_2 + \alpha) \\ &\le -\frac{\sigma^2}{2}\left(\frac{\Lambda}{ug\left(\frac{\Lambda}{u}\right)} - \frac{\beta b}{\sigma^2}\right)^2 + \frac{\beta^2 b^2}{2\sigma^2} - (u + \gamma_1 + \gamma_2 + \alpha) \\ &= \left(u + \gamma_1 + \gamma_2 + \alpha + \frac{\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}\right)(R_0^s - 1). \end{aligned} \tag{4.5}$$

From Eq (4.1), one can get

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \left(u + \gamma_1 + \gamma_2 + \alpha + \frac{\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}\right)(R_0^s - 1) < 0 \qquad a.s.. \tag{4.6}$$

The inequalities (4.4) and (4.6) imply $\lim_{t\to\infty} I(t) = 0$ and the disease goes to extinction.

Next, adding up the three equations of system (1.2), integrating both sides from 0 to t and dividing by t, one can see that

$$\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{\delta}{u+\delta}\frac{R(t) - R(0)}{t} = \Lambda - u\langle S(t)\rangle - \left(u + \alpha + \frac{u\gamma_2}{u+\delta}\right)\langle I(t)\rangle. \tag{4.7}$$

So, from Eq (4.7) we have

$$\langle S(t)\rangle = \frac{\Lambda}{u} - \left(\frac{u+\alpha}{u} + \frac{\gamma_2}{u+\delta}\right)\langle I(t)\rangle - \Theta(t), \tag{4.8}$$

where $\Theta(t) = \frac{1}{u}\left[\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{\delta}{u+\delta}\frac{R(t)-R(0)}{t}\right]$. Obviously, $\lim_{t\to\infty} \Theta(t) = 0$ and we have

$$\lim_{t\to\infty} \langle S(t)\rangle = \frac{\Lambda}{u} \qquad a.s..$$

Similarly, from the third equation of the system (1.2) yields

$$\langle R(t)\rangle = \frac{\gamma_2}{u+\delta}\langle I(t)\rangle - \frac{R(t) - R(0)}{(u+\delta)t}. \tag{4.9}$$

Let $t \to \infty$, we get that

$$\lim_{t\to\infty} \langle R(t)\rangle = 0 \qquad a.s.$$

This finishes the proof. □

4.2. Permanence in mean

Theorem 4.2. *If $R_0^s > 1$, the disease is persistence in mean. Moreover, we have*

$$\liminf_{t\to\infty} \langle I(t)\rangle \geq I^{**} > 0,$$
$$\liminf_{t\to\infty} \left\langle \frac{\Lambda}{u} - S(t)\right\rangle \geq \left(\frac{u+\alpha}{u} + \frac{\gamma_2}{u+\delta}\right) I^{**} > 0,$$
$$\liminf_{t\to\infty} \langle R(t)\rangle \geq \frac{\gamma_2}{u+\delta} I^{**} > 0,$$

where

$$I^{**} = \frac{u(u+\delta)\left[u+\gamma_1+\gamma_2+\alpha+\frac{\Lambda^2\sigma^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}\right] g\left(\frac{\Lambda}{u}\right)\left(R_0^s-1\right)}{\beta b\left[(u+\alpha)(u+\delta)+\gamma_2 u\right]}.$$

Proof. In view of Eq (4.1), we have

$$d(\ln I(t)) \geq \left[\frac{\beta b S(t)}{g\left(\frac{\Lambda}{u}\right)} - (u+\gamma_1+\gamma_2+\alpha) - \frac{\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}\right]dt + \frac{\sigma S(t)}{g(N(t))}dB(t). \tag{4.10}$$

Integrating from 0 to t and dividing by t on both sides of Eq (4.10), one can get that

$$\frac{\ln I(t)-\ln I(0)}{t} \geq \frac{\beta b}{g\left(\frac{\Lambda}{u}\right)}\langle S(t)\rangle - (u+\gamma_1+\gamma_2+\alpha) - \frac{\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)} + \frac{\int_0^t \frac{\sigma S(\xi)}{g(N(\xi))}d\xi}{t}. \tag{4.11}$$

Substituting Eq (4.8) into Eq (4.11), we can get

$$\begin{aligned}\frac{\ln I(t)-\ln I(0)}{t} \geq& \left[u+\gamma_1+\gamma_2+\alpha+\frac{\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}\right](R_0^s-1) - \frac{\beta b\Theta(t)}{ug\left(\frac{\Lambda}{u}\right)}\\ &+\frac{\int_0^t \frac{\sigma S(\xi)}{g(N(\xi))}d\xi}{t} - \frac{\beta b\left[(u+\alpha)(u+\delta)+\gamma_2 u\right]}{u(u+\delta)g\left(\frac{\Lambda}{u}\right)}\langle I(t)\rangle.\end{aligned} \tag{4.12}$$

Hence,

$$\begin{aligned}\ln I(t) \geq& \left[u+\gamma_1+\gamma_2+\alpha+\frac{\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}\right](R_0^s-1)t + F(t)\\ &-\frac{\beta b\left[(u+\alpha)(u+\delta)+\gamma_2 u\right]}{u(u+\delta)g\left(\frac{\Lambda}{u}\right)}\int_0^t I(\xi)d\xi.\end{aligned} \tag{4.13}$$

where $F(t) = \ln I(0) + \int_0^t \frac{\sigma S(\xi)}{g(N(\xi))}d\xi - \frac{\beta b t\Theta(t)}{ug\left(\frac{\Lambda}{u}\right)}$. Obviously, $\lim_{t\to\infty}\frac{F(t)}{t} = 0$ *a.s.*. From Lemma 1 in [12], we obtain

$$\liminf_{t\to\infty}\langle I(t)\rangle \geq \frac{u(u+\delta)\left[u+\gamma_1+\gamma_2+\alpha+\frac{\Lambda^2\sigma^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}\right] g\left(\frac{\Lambda}{u}\right)\left(R_0^s-1\right)}{\beta b\left[(u+\alpha)(u+\delta)+\gamma_2 u\right]} = I^{**}.$$

According to Eqs (4.8) and (4.9), we can see that

$$\liminf_{t\to\infty}\left\langle \frac{\Lambda}{u}-S(t)\right\rangle=\left(\frac{u+\alpha}{u}+\frac{\gamma_2}{u+\delta}\right)\liminf_{t\to\infty}\langle I(t)\rangle\geq\left(\frac{u+\alpha}{u}+\frac{\gamma_2}{u+\delta}\right)I^{**}$$

and

$$\liminf_{t\to\infty}\langle R(t)\rangle=\frac{\gamma_2}{u+\delta}\liminf_{t\to\infty}\langle I(t)\rangle\geq\frac{\gamma_2}{u+\delta}I^{**}.$$

The proof of Theorem 4.2 is completed. □

Remark 4.1. *According to Theorem 4.1 (i), if the intensity of white noise is large enough that the condition* $\sigma^2 > \max\left\{\frac{\beta^2 b^2}{2(u+\gamma_1+\gamma_2+\alpha)}, \frac{\beta bug\left(\frac{\Lambda}{u}\right)}{\Lambda}\right\}$ *holds, then the disease goes to extinction. Therefore, a large environmental nose intensity can suppress the spread of disease. In addition, by comparing the thresholds of systems* (1.2) *and* (1.1)*, it can be found that if the intensity of environmental noise* σ^2=0*, then* $R_0^s = R_0$*; if* $\sigma^2 \neq 0$*, then* $R_0^s < R_0$*. When* $R_0^s < 1 < R_0$*, the deterministic system* (1.1) *has a stable positive equilibrium, while the disease of the stochastic system* (1.2) *dies out with probability 1. This means that the presence of environmental noise is conducive to disease control.*

5. The existence of stationary solution

Theorem 5.1. *If* $R_0^s > 1$*, there exists a solution of system* (1.2) *which is a stationary Markov process.*

Proof. Let $(S(t), I(t), R(t))$ be a solution of system (1.2) with any given initial value $(S(0), I(0), R(0)) \in \Gamma$. Then we construct a C^2-function G as following:

$$\begin{aligned}G(S(t), I(t), R(t)) &= M\left(-\ln I(t)-c_1\ln S(t)-c_2 g^2(N(t))+\frac{4bc_2 g\left(\frac{\Lambda}{u}\right)}{u+\delta}R(t)\right)\\&\quad-\ln S(t)-\ln R(t)-\ln\left(N(t)-\frac{\Lambda}{u+\alpha}\right)-\ln\left(\frac{\Lambda}{u}-N(t)\right)\\&= MV_1+V_2+V_3+V_4+V_5,\end{aligned}$$

where $V_1 = -\ln I(t) - c_1\ln S(t) - c_2 g^2(N(t)) + \frac{4bc_2 g\left(\frac{\Lambda}{u}\right)}{u+\delta}R(t)$, $V_2 = -\ln S(t)$, $V_3 = -\ln R(t)$, $V_4 = -\ln\left(N(t)-\frac{\Lambda}{u+\alpha}\right)$, $V_5 = -\ln\left(\frac{\Lambda}{u}-N(t)\right)$, $c_1 = \frac{u+\gamma_1+\gamma_2+\alpha+\frac{\Lambda^2\sigma^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}}{u}$ and $c_2 = \frac{u+\gamma_1+\gamma_2+\alpha+\frac{\Lambda^2\sigma^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}}{2b\Lambda g\left(\frac{\Lambda}{u}\right)}$. The M is a positive constant and satisfies the following condition

$$-3M\left[u+\gamma_1+\gamma_2+\alpha+\frac{\Lambda^2\sigma^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}\right]\left(\sqrt[3]{R_0^s}-1\right)+\frac{\beta b\Lambda}{ug\left(\frac{\Lambda}{u}\right)}+\frac{\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u}\right)}+4u+\delta+\alpha\leq-2. \quad (5.1)$$

Furthermore, $G(S(t), I(t), R(t))$ is a continuous function, which exists a minimum point $\left(\bar{S}_0, \bar{I}_0, \bar{R}_0\right)$. Next, we define a nonnegative C^2-function V

$$V(S(t), I(t), R(t)) = G(S(t), I(t), R(t)) - G\left(\bar{S}_0, \bar{I}_0, \bar{R}_0\right).$$

Applying Itô's formula for V_1, we can see

$$\begin{aligned}
LV_1 &= -\frac{1}{I(t)}\left[\frac{\beta bS(t)I(t)}{g(N(t))} - (u+\gamma_1+\gamma_2+\alpha)I(t)\right] + \frac{\sigma^2 S^2(t)}{g^2(N(t))} \\
&\quad -\frac{c_1}{S(t)}\left[\Lambda - uS(t) - \frac{\beta bS(t)I(t)}{g(N(t))} + \gamma_1 I(t) + \delta R(t)\right] + \frac{c_1\sigma^2 I^2(t)}{2g^2(N(t))} \\
&\quad -2bc_2 g(N(t))\left(1+\frac{1}{\sqrt{1+2bN(t)}}\right)(\Lambda - uN(t) - \alpha I(t)) \\
&\quad +\frac{4bc_2 g\left(\frac{\Lambda}{u}\right)}{u+\delta}\left[\gamma_2 I(t) - (u+\delta)R(t)\right] \\
&\le -\frac{\beta bS(t)}{g(N(t))} - \frac{c_1\Lambda}{S(t)} - 2b\Lambda c_2 g(N(t)) + (u+\gamma_1+\gamma_2+\alpha) + \frac{\Lambda^2\sigma^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)} \\
&\quad +c_1 u + \frac{c_1\beta bI(t)}{g\left(\frac{\Lambda}{u+\alpha}\right)} + \frac{c_1\sigma^2 I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} + 2b\Lambda c_2 g\left(\frac{\Lambda}{u}\right) - 4bc_2 g\left(\frac{\Lambda}{u}\right)R(t) \\
&\quad +2b\alpha c_2 g(N(t))\left(1+\frac{1}{\sqrt{1+2bN(t)}}\right)I(t) + \frac{4bc_2\gamma_2 g\left(\frac{\Lambda}{u}\right)}{u+\delta}I(t) \\
&\le -3\sqrt[3]{2\Lambda^2\beta b^2 c_1 c_2} + 3\left[u+\gamma_1+\gamma_2+\alpha+\frac{\Lambda^2\sigma^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}\right] + \frac{c_1\sigma^2 I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} \\
&\quad +\left[\frac{c_1\beta b}{g\left(\frac{\Lambda}{u+\alpha}\right)} + 4b\alpha c_2 g\left(\frac{\Lambda}{u}\right) + \frac{4bc_2\gamma_2 g\left(\frac{\Lambda}{u}\right)}{u+\delta}\right]I(t) \\
&= -3\left[u+\gamma_1+\gamma_2+\alpha+\frac{\Lambda^2\sigma^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}\right]\left(\sqrt[3]{R_0^s}-1\right) + \frac{c_1\sigma^2 I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} \\
&\quad +\left[\frac{c_1\beta b}{g\left(\frac{\Lambda}{u+\alpha}\right)} + 4bc_2 g\left(\frac{\Lambda}{u}\right)\left(\alpha+\frac{\gamma_2}{u+\delta}\right)\right]I(t). \qquad (5.2)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
LV_2 &= -\frac{\Lambda}{S(t)} + u + \frac{\beta bI(t)}{g(N(t))} - \frac{\gamma_1 I(t)}{S(t)} - \frac{\delta R(t)}{S(t)} + \frac{\sigma^2 I^2(t)}{2g^2(N(t))} \\
&\le -\frac{\Lambda}{S(t)} + u + \frac{\beta b\Lambda}{ug\left(\frac{\Lambda}{u}\right)} + \frac{\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}, \qquad (5.3)
\end{aligned}$$

$$LV_3 = -\frac{\gamma_2 I(t)}{R(t)} + u + \delta, \qquad (5.4)$$

$$LV_4 = -\frac{\Lambda - uN(t) - \alpha I(t)}{N(t) - \frac{\Lambda}{u+\alpha}} = u + \alpha - \frac{\alpha(S(t)+R(t))}{N(t) - \frac{\Lambda}{u+\alpha}}, \qquad (5.5)$$

$$LV_5 = \frac{\Lambda - uN(t) - \alpha I(t)}{\frac{\Lambda}{u} - N(t)} = u - \frac{\alpha I(t)}{\frac{\Lambda}{u} - N(t)}. \tag{5.6}$$

Therefore, in view of Eqs (5.2)–(5.6), we get that

$$\begin{aligned}
LV \leq\ & -3M\left[u + \gamma_1 + \gamma_2 + \alpha + \frac{\Lambda^2\sigma^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)}\right]\left(\sqrt[3]{R_0^s} - 1\right) + \frac{Mc_1\sigma^2 I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} - \frac{\Lambda}{S(t)} \\
& + M\left[\frac{c_1\beta b}{g\left(\frac{\Lambda}{u+\alpha}\right)} + 4bc_2 g\left(\frac{\Lambda}{u}\right)\left(\alpha + \frac{\gamma_2}{u+\delta}\right)\right] I(t) + u + \frac{\beta b\Lambda}{ug\left(\frac{\Lambda}{u}\right)} + \frac{\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u}\right)} \\
& - \frac{\gamma_2 I(t)}{R(t)} + u + \delta + u + \alpha - \frac{\alpha(S(t) + R(t))}{N(t) - \frac{\Lambda}{u+\alpha}} + u - \frac{\alpha I(t)}{\frac{\Lambda}{u} - N(t)} \\
\leq\ & -2 + M\left[\frac{c_1\beta b}{g\left(\frac{\Lambda}{u+\alpha}\right)} + 4bc_2 g\left(\frac{\Lambda}{u}\right)\left(\alpha + \frac{\gamma_2}{u+\delta}\right)\right] I(t) + \frac{Mc_1\sigma^2 I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} \\
& - \frac{\Lambda}{S(t)} - \frac{\gamma_2 I(t)}{R(t)} - \frac{\alpha(S(t) + R(t))}{N(t) - \frac{\Lambda}{u+\alpha}} - \frac{\alpha I(t)}{\frac{\Lambda}{u} - N(t)} \\
:=\ & -2 + M\lambda I(t) + \frac{Mc_1\sigma^2 I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} - \frac{\Lambda}{S(t)} - \frac{\gamma_2 I(t)}{R(t)} - \frac{\alpha(S(t) + R(t))}{N(t) - \frac{\Lambda}{u+\alpha}} - \frac{\alpha I(t)}{\frac{\Lambda}{u} - N(t)},
\end{aligned}$$

where $\lambda = \frac{c_1\beta b}{g\left(\frac{\Lambda}{u+\alpha}\right)} + 4bc_2 g\left(\frac{\Lambda}{u}\right)\left(\alpha + \frac{\gamma_2}{u+\delta}\right)$.

Define the following bounded closed set

$$D_\varepsilon = \left\{(S(t), I(t), R(t)) \in \Gamma : \varepsilon \leq S(t) \leq \frac{\Lambda}{u}, \varepsilon \leq I(t) \leq \frac{\Lambda}{u}, \varepsilon^2 \leq R(t) \leq \frac{\Lambda}{u}, \right.$$

$$\left. \frac{\Lambda}{u+\alpha} + \varepsilon^3 \leq N(t) \leq \frac{\Lambda}{u} - \varepsilon^3 \right\},$$

where ε is a sufficiently small constant satisfying the following inequalities (5.7)–(5.11)

$$-2 + M\lambda\frac{\Lambda}{u} + \frac{Mc_1\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u+\alpha}\right)} - \frac{\Lambda}{\varepsilon} \leq -1, \tag{5.7}$$

$$-2 + M\lambda\varepsilon + \frac{Mc_1\sigma^2\varepsilon^2}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} \leq -1, \tag{5.8}$$

$$-2 + M\lambda\frac{\Lambda}{u} + \frac{Mc_1\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u+\alpha}\right)} - \frac{\gamma_2}{\varepsilon} \leq -1, \tag{5.9}$$

$$-2 + M\lambda\frac{\Lambda}{u} + \frac{Mc_1\sigma^2\Lambda^2}{2u^2 g^2\left(\frac{\Lambda}{u+\alpha}\right)} - \frac{\alpha(1+\varepsilon)}{\varepsilon^2} \leq -1, \tag{5.10}$$

$$-2+M\lambda\frac{\Lambda}{u}+\frac{Mc_1\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u+\alpha}\right)}-\frac{\alpha}{\varepsilon^2}\leq -1. \tag{5.11}$$

For convenience, we divide $\Gamma\backslash D_\varepsilon$ into five domains

$$\begin{aligned}
D_1 &= \{(S(t),I(t),R(t))\in\Gamma, 0<S(t)<\varepsilon\},\\
D_2 &= \{(S(t),I(t),R(t))\in\Gamma, 0<I(t)<\varepsilon\},\\
D_3 &= \left\{(S(t),I(t),R(t))\in\Gamma, I(t)\geq\varepsilon, 0<R(t)<\varepsilon^2\right\},\\
D_4 &= \left\{(S(t),I(t),R(t))\in\Gamma, S(t)\geq\varepsilon, I(t)\geq\varepsilon, R(t)\geq\varepsilon^2, \frac{\Lambda}{u+\alpha}<N(t)<\frac{\Lambda}{u+\alpha}+\varepsilon^2\right\},\\
D_5 &= \left\{(S(t),I(t),R(t))\in\Gamma, S(t)\geq\varepsilon, I(t)\geq\varepsilon, R(t)\geq\varepsilon^2, \frac{\Lambda}{u}-\varepsilon^2<N(t)<\frac{\Lambda}{u}\right\}.
\end{aligned}$$

So, we only need to prove $LV\leq -1$ on the above five domains.

Case i: If $(S(t),I(t),R(t))\in D_1$, from Eq (5.7), one can obtain that

$$\begin{aligned}
LV &\leq -2+M\lambda I(t)+\frac{Mc_1\sigma^2I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)}-\frac{\Lambda}{S(t)}\\
&\leq -2+M\lambda\frac{\Lambda}{u}+\frac{Mc_1\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u+\alpha}\right)}-\frac{\Lambda}{\varepsilon}\leq -1.
\end{aligned} \tag{5.12}$$

Case ii: If $(S(t),I(t),R(t))\in D_2$, in view of Eq (5.8), we have

$$\begin{aligned}
LV &\leq -2+M\lambda I(t)+\frac{Mc_1\sigma^2I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)}\\
&\leq -2+M\lambda\varepsilon+\frac{Mc_1\sigma^2\varepsilon^2}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)}\leq -1.
\end{aligned} \tag{5.13}$$

Case iii: If $(S(t),I(t),R(t))\in D_3$, according to Eq (5.9), we obtain

$$\begin{aligned}
LV &\leq -2+M\lambda I(t)+\frac{Mc_1\sigma^2I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)}-\frac{\gamma_2I(t)}{R(t)}\\
&\leq -2+M\lambda\frac{\Lambda}{u}+\frac{Mc_1\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u+\alpha}\right)}-\frac{\gamma_2}{\varepsilon}\leq -1.
\end{aligned} \tag{5.14}$$

Case iv: If $(S(t),I(t),R(t))\in D_4$, by Eq (5.10), one can see that

$$\begin{aligned}
LV &\leq -2+M\lambda I(t)+\frac{Mc_1\sigma^2I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)}-\frac{\alpha(S(t)+R(t))}{N(t)-\frac{\Lambda}{u+\alpha}}\\
&\leq -2+M\lambda\frac{\Lambda}{u}+\frac{Mc_1\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u+\alpha}\right)}-\frac{\alpha(1+\varepsilon)}{\varepsilon^2}\leq -1.
\end{aligned} \tag{5.15}$$

Case v: If $(S(t), I(t), R(t)) \in D_5$, in view Eq (5.11), we derive

$$\begin{aligned} LV &\leq -2 + M\lambda I(t) + \frac{Mc_1\sigma^2 I^2(t)}{2g^2\left(\frac{\Lambda}{u+\alpha}\right)} - \frac{\alpha I(t)}{\frac{\Lambda}{u} - N(t)} \\ &\leq -2 + M\lambda\frac{\Lambda}{u} + \frac{Mc_1\sigma^2\Lambda^2}{2u^2g^2\left(\frac{\Lambda}{u+\alpha}\right)} - \frac{\alpha}{\varepsilon^2} \leq -1. \end{aligned} \tag{5.16}$$

Consequently, for a sufficiently small ε, we have

$$LV(S(t), I(t), R(t)) \leq -1, \quad for \quad \forall (S(t), I(t), R(t)) \in \Gamma \backslash D_\varepsilon.$$

In view of Lemma 2.2, there exists a solution of system (1.2) which is a stationary Markov process. This completes the proof. □

Remark 5.1. *Theorem 5.1 shows that if the intensity of white noise is small enough to make* $R_0^s > 1$*, then the system* (1.2) *has a stationary solution. That is to say, disease will exist for a long time and form endemic disease.*

6. Conclusions and numerical simulations

In this paper, we study the dynamics of a stochastic SIRS epidemic model with saturating contact rate. Firstly, the threshold of disease extinction for deterministic system (1.1) is obtained by using the Jacobian matrix. If $R_0 < 1$, system (1.1) has a unique stable equilibrium and the disease goes to extinct. If $R_0 > 1$, system (1.1) forms endemic disease after a sufficiently long time. Secondly, the threshold parameter R_0^s of stochastic system (1.2) is established. If the intensity of the white noise is small enough to satisfy the condition $R_0^s > 1$, the disease is persistent. Otherwise, the disease will die out. Finally, we prove that there exists a stationary solution under condition $R_0^s > 1$ in system (1.2).

In order to demonstrate the above theoretical derivation, we use MATLAB software to carry out some numerical simulations. Next, we choose the relevant parameters of system (1.1) as follows:

$$\Lambda = 1, \ u = 0.1, \ b = 0.6, \ \delta = 0.13, \ \gamma_1 = 0.1, \ \gamma_2 = 0.08, \ \alpha = 0.12.$$

Firstly, we simulate the deterministic system and set $\beta = 0.66$. By an ordinary computation, $R_0 = 0.9335 < 1$. From the first condition of the Theorem 3.1, one can obtain that the system (1.1) has a unique stable equilibrium point $E_0 = (10, 0, 0)$ (Figure 1). Furthermore, if we increase β to 0.8, in this case, we have $R_0 = 1.1315 > 1$. The condition (ii) in Theorem 3.1 is established, then by Theorem 3.1, the disease of system (1.1) is persistent (Figure 2).

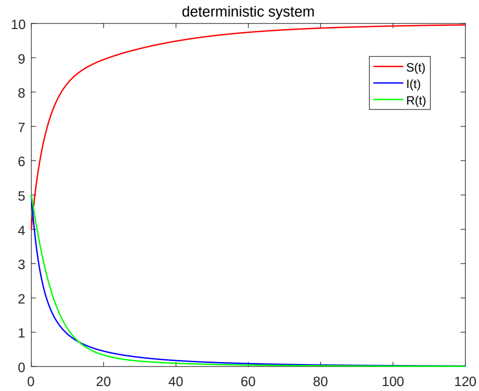

Figure 1. Numerical simulation of the deterministic system (1.1), where $\Lambda = 1,\ u = 0.1,\ b = 0.6,\ \delta = 0.13,\ \gamma_1 = 0.1,\ \gamma_2 = 0.08,\ \alpha = 0.12, \beta = 0.66, R_0 = 0.9335 < 1$.

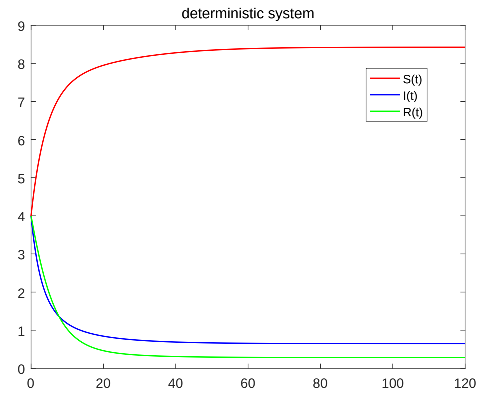

Figure 2. Computer simulation of the deterministic system (1.1), where $\Lambda = 1,\ u = 0.1,\ b = 0.6,\ \delta = 0.13,\ \gamma_1 = 0.1,\ \gamma_2 = 0.08,\ \alpha = 0.12, \beta = 0.8, R_0 = 1.1315 > 1$.

Next, we perform numerical simulations on stochastic system. We keep the parameters of deterministic system (1.1) unchanged, and only select different intensities of white noise σ in system (1.2).

Case i: In order to verify the conclusion (i) of the Theorem 4.1, we let $\sigma = 0.8$. By computing, we can obtain $\sigma^2 = 0.64$, $g\left(\frac{\Lambda}{u}\right) = 10.6056$ and $\max\left\{\frac{\beta^2 b^2}{2(u+\gamma_1+\gamma_2+\alpha)}, \frac{\beta bug\left(\frac{\Lambda}{u}\right)}{\Lambda}\right\} = 0.5091$, which implies the parameters satisfy the condition (i) $\sigma^2 > \max\left\{\frac{\beta^2 b^2}{2(u+\gamma_1+\gamma_2+\alpha)}, \frac{\beta bug\left(\frac{\Lambda}{u}\right)}{\Lambda}\right\}$. This shows that the disease in system (1.2) dies out with probability 1 (Figure 3).

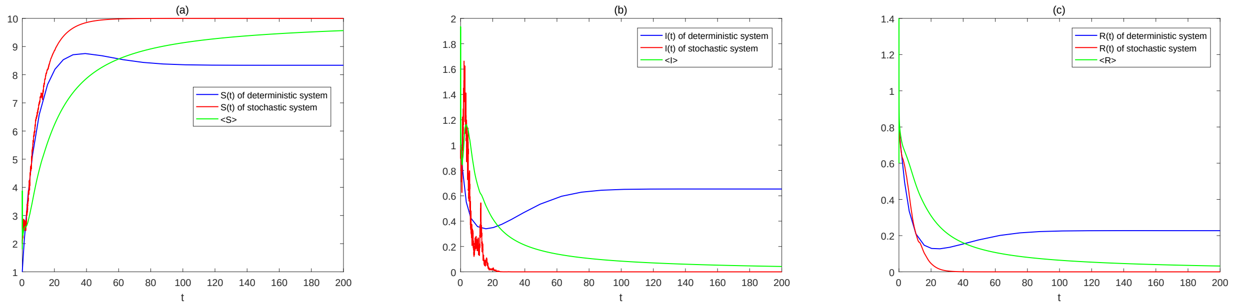

Figure 3. Time series diagram of $(S(t), I(t), R(t))$, where $\Lambda = 1$, $u = 0.1$, $b = 0.6$, $\delta = 0.13$, $\gamma_1 = 0.1$, $\gamma_2 = 0.08$, $\alpha = 0.12, \beta = 0.8, \sigma = 0.8, R_0 = 1.1315 > 1$.

Case ii: In Figure 4, we assume that $\sigma = 0.45$. It is not difficult to obtain that σ satisfies the second condition of Theorem 4.1. Then, we have $R_0^s = 0.9236 < 1$ and $0.2025 = \sigma^2 < \frac{\beta bug(\frac{\Lambda}{u})}{\Lambda} = 0.5091$. As can be seen from Figure 4, the disease is going extinct.

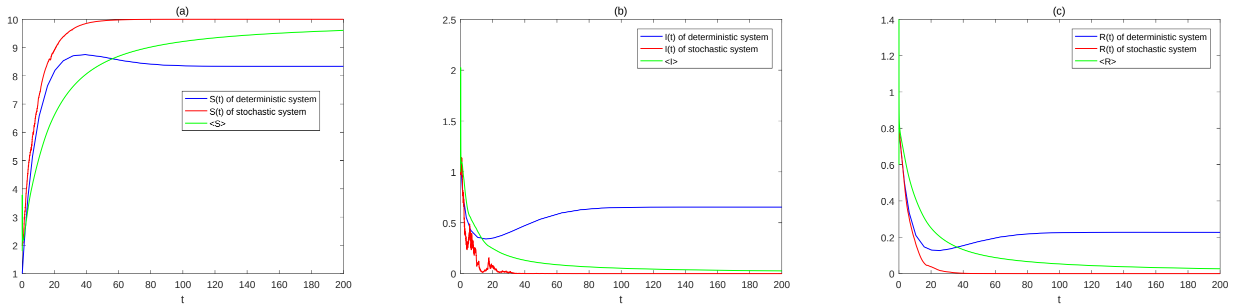

Figure 4. Time series diagram of $(S(t), I(t), R(t))$, where $\Lambda = 1$, $u = 0.1$, $b = 0.6$, $\delta = 0.13$, $\gamma_1 = 0.1$, $\gamma_2 = 0.08$, $\alpha = 0.12, \beta = 0.8, \sigma = 0.45, R_0 = 1.1315 > 1, R_0^s = 0.9236$.

Case iii: If $\sigma = 0.1$, by calculation, we get $R_0^s = 1.1190 > 1$. According to Theorem 4.2, the disease in system (1.2) is permanent in the time mean (Figure 5). Furthermore, by Theorem 5.1, the system (1.2) has a stationary solution (Figure 6).

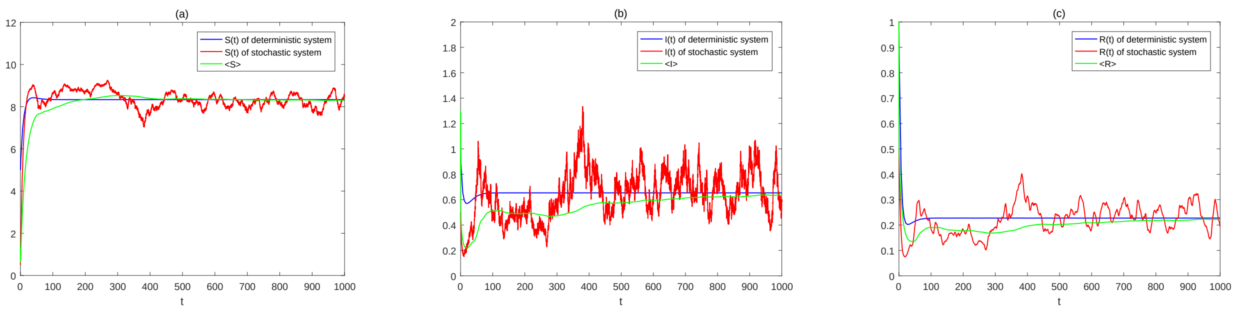

Figure 5. Time series diagram of $(S(t), I(t), R(t))$, where $\Lambda = 1$, $u = 0.1$, $b = 0.6$, $\delta = 0.13$, $\gamma_1 = 0.1$, $\gamma_2 = 0.08$, $\alpha = 0.12, \beta = 0.8, \sigma = 0.1, R_0 = 1.1315 > 1, R_0^s = 1.1190$.

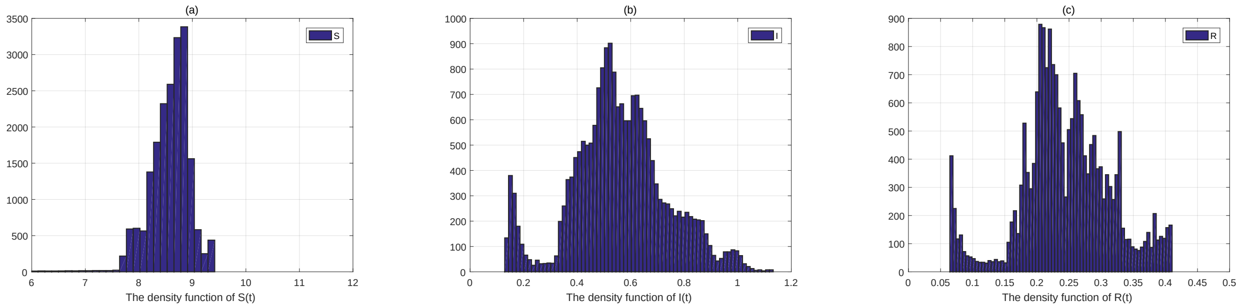

Figure 6. The density function distribution of $(S(t), I(t), R(t))$ with $\sigma = 0.1, R_0^s = 1.1190$.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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