Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 17(5): 5341–5368
DOI: 10.3934/mbe.2020289
Received: 18 May 2020
Accepted: 03 August 2020
Published: 11 August 2020

Research article

Asymptotic behavior of a stochastic delayed avian influenza model with saturated incidence rate

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Abstract: In this paper, we establish a stochastic delayed avian influenza model with saturated incidence rate. Firstly, we prove the existence and uniqueness of the global positive solution with any positive initial value. Then, we study the asymptotic behaviors of the disease-free equilibrium and the endemic equilibrium by constructing some suitable Lyapunov functions and applying the Young's inequality and Hölder's inequality. If $\mathscr{R}_0 < 1$, then the solution of stochastic system is going around disease-free equilibrium while the solution of stochastic system is going around endemic equilibrium as $\mathscr{R}_0 > 1$. Finally, some numerical examples are carried out to illustrate the accuracy of the theoretical results.

Keywords: stochastic avian influenza; time delays; saturated incidence rate; Lyapunov function; asymptotic behavior

1. Introduction

Avian influenza is an animal infectious disease caused by the transmission of influenza A viruses. Influenza A viruses are divided into subtypes according to two proteins on the surface of the virus: Hemagglutinin (HA) and neuraminidase (NA) [1]. Most avian influenza viruses infect only certain species and do not infect humans. However, a few avian influenza viruses have crossed the species barrier to infect humans and even kill them, such as H5N1, H7N1, H7N2, H7N3, H7N7, H9N2 and H7N9. Among them, H5N1 is a highly pathogenic avian influenza virus, which was first detected in human in Hong Kong in 1997. After that, humans infection with avian influenza have occurred from time to time. As of December 2019, the global cumulative number of cases of human infection with H5N1 avian influenza arrives 861, with 455 deaths. Unlike H5N1, H7N9 is classified as a low pathogenicity avian influenza virus [2]. In March 2013, there was the first case of human infection with the H7N9 avian influenza virus in Shanghai, China. In the following weeks, this virus spread to several provinces and municipalities in mainland China. As of May 2017, H7N9 has resulted in 1263 human cases in China, of whom 459 died, with a mortality rate of nearly 37%. The frequent outbreak of avian influenza in the world not only brings a serious threat to human health, but also causes psychological panic and huge social impact, and brings a huge blow to the national economy. Therefore, it has been important to understand the dynamical behavior of avian influenza and to predict what may occur. Mathematical modeling has been a useful tool to describe the dynamical behavior of avian influenza and to obtain a better understanding of transmission mechanisms. Recently, many avian influenza models have been built from different perspectives (see [2–12] and references therein).

As we all know, there exist time delays during the spread of avian influenza, which can be used to describe not only the infection period of avian influenza virus in poultry (human) population, but also the incubation period of avian influenza in poultry (human) population and the immune period of recovered human to avian influenza. Therefore, the time delays should be considered such that the avian influenza models are more realistic. Generally speaking, delayed differential equations exhibit more complex dynamical behavior than differential equations without delay because time delay can make a stable equilibrium position to be unstable [13–16]. Consequently, it is of great interest to describe the transmission mechanism of avian influenza by introducing time delay into the models. For example, Liu et al. [7] and Kang et al. [12] established avian influenza models with different time delays in the poultry and human populations by considering the incubation periods of avian influenza virus and the survival probabilities of infected poultry and humans. By considering the existence of intracellular delay between initial infection of a cell and the release of new virus particles, Samanta [17] established a non-autonomous ordinary differential equation with distributed delay to characterize the spread of avian influenza between poultry and humans. These surveys imply that the research of time delay on avian influenza is a meaningful issue and is still open for study.

On the other hand, many existing literatures only focus on the deterministic avian influenza models that do not consider the impact of environmental noise. However, in the real world, the spread of avian influenza is often affected by the variations of environmental factors, such as humidity, temperature and so on [18, 19]. Due to the fluctuations in the environment, an actual avian influenza system would not remain in a stable state, which would interfere with this stable state by acting directly on the density or indirectly affecting the parameter values. Therefore, it is of great significance to reveal the impact of environmental noise on avian influenza model by using stochastic model, so as to obtain more real benefits and accurately predict the future dynamics of avian influenza. To better understand the transmission dynamics of avian influenza, some authors have introduced stochastic perturbations into the deterministic models [20–22]. Zhang et al. [20] constructed a stochastic avian-human influenza model with logistic growth for avian population, and discussed the dynamical behavior of this model. Further, Zhang et al. [21] investigated a stochastic avian-human influenza epidemic model with psychological effect in human population and saturation effect within avian population. On the basis of the deterministic model established by Iwami et al. [3], Zhang et al. [22] established the corresponding stochastic model by introducing density disturbance. All the papers mentioned above only focused on the extinction and persistence of stochastic avian influenza models. However, to the best of our knowledge, there is no results related to the asymptotic behavior of stochastic avian influenza model around the equilibria of the corresponding deterministic model.

Motivated by the above discussions, in this paper, we investigate the asymptotic behavior of a stochastic delayed avian influenza model with saturated incidence rate. This work differs from existing results [7, 12, 17, 20–22] in that (a) time delays and white noise are taken into account to describe the latency period of avian influenza virus in both poultry and human population and the environmental fluctuations; (b) asymptotic behavior of a stochastic delayed avian influenza model is studied. Overview of the rest of the article is as follows: In section 3, we show that there exists a unique global positive solution of system (2.3) with the given initial value (2.4). In section 4, we prove that the solution of system (2.3) is going around E^0 under certain conditions. Further, we derive that the solution of system (2.3) is going around E^* under certain conditions in section 5. In section 6, some numerical examples are introduced to illustrate the effectiveness of theoretic results. Finally, some conclusions are given in section 7.

2. Model description and formulation

Although the avian influenza virus spreads between wild birds and poultry, and between poultry and humans, we will only consider the transmission dynamics of avian influenza between poultry and humans because poultry is the main source of infection. Moreover, we assume that the virus is not spread between humans and mutate. We denote the total population of poultry and humans at time t by $N_a(t)$ and $N_h(t)$, respectively. When the susceptible poultry contact with the infected poultry closely, there is usually no quick way to detect whether they are infected or the detection cost is too high, which makes it impossible to distinguish whether the close contacts of poultry are infected with the avian influenza virus. Therefore, the poultry population is divided into three sub-populations depending on the state of the disease: susceptible poultry $S_a(t)$, exposed poultry $E_a(t)$ and infected poultry $I_a(t)$. The total poultry population at time t is denoted by $N_a(t) = S_a(t) + E_a(t) + I_a(t)$. The human population is divided into three sub-populations, which are susceptible human $S_h(t)$, infected human with avian influenza $I_a(t)$ and recovered human from avian influenza $R_h(t)$. The total population of human at time t is given by $N_h(t) = S_h(t) + I_h(t) + R_h(t)$.

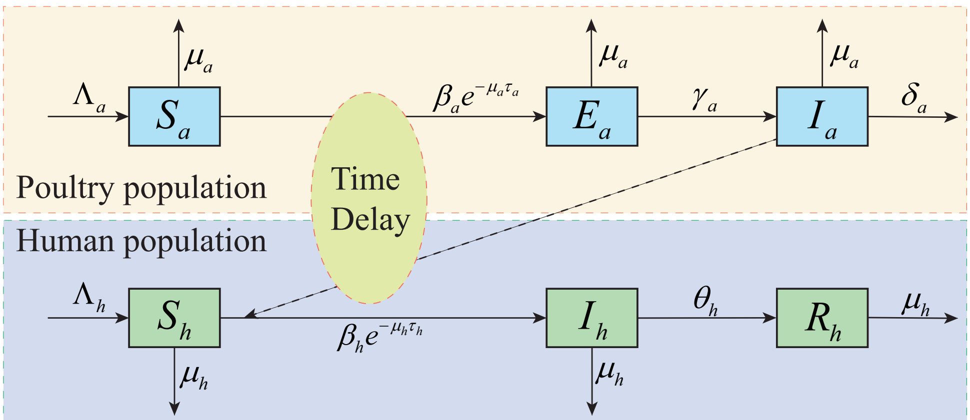

Figure 1. Schematic diagram of the model (2.1).

The reason why we do not consider the exposed class for human population is that the close contacts of human beings are usually isolated and tested to determine whether they are infected with the avian influenza virus. The poultry in E_a either shows symptoms after incubation period and move to I_a, or always stays in E_a until natural death. The number of susceptible poultry (human) is increased by new recruitment, but decreases by natural death and infection (moving to class I_a (I_h)). The number of infected poultry (human) is increased by the infection of susceptible poultry (human) and reduced through natural and disease-related death. In addition, the number of infected humans is also reduced by recovery from the disease (moving to class R_h). Based on the above discussions, we obtain the schematic diagram of our model (see Figure 1).

The corresponding avian influenza model can be represented by the following equations:

$$\begin{cases} \dfrac{dS_a(t)}{dt} = \Lambda_a - \mu_a S_a(t) - \dfrac{\beta_a S_a(t) I_a(t)}{1+\alpha_1 I_a(t)}, \\ \dfrac{dE_a(t)}{dt} = \dfrac{\beta_a e^{-\mu_a \tau_a} S_a(t-\tau_a) I_a(t-\tau_a)}{1+\alpha_1 I_a(t-\tau_a)} - (\mu_a+\gamma_a)E_a(t), \\ \dfrac{dI_a(t)}{dt} = \gamma_a E_a(t) - (\mu_a+\delta_a)I_a(t), \\ \dfrac{dS_h(t)}{dt} = \Lambda_h - \mu_h S_h(t) - \dfrac{\beta_h S_h(t) I_a(t)}{1+\alpha_2 I_a(t)}, \\ \dfrac{dI_h(t)}{dt} = \dfrac{\beta_h e^{-\mu_h \tau_h} S_h(t-\tau_h) I_a(t-\tau_h)}{1+\alpha_2 I_a(t-\tau_h)} - (\mu_h+\delta_h+\theta_h)I_h(t), \\ \dfrac{dR_h(t)}{dt} = \theta_h I_h(t) - \mu_h R_h(t). \end{cases} \tag{2.1}$$

All parameters in model (2.1) are assumed non-negative and described in Table 1.

Table 1. Parameters description in the model (2.1).

Parameter	Description
Λ_a	new recruitment of the poultry populations
Λ_h	new recruitment of the human population
β_a	the transmission rate from infective poultry to susceptible poultry
β_h	the transmission rate from infective poultry to susceptible human
μ_a	the natural death rate of poultry populations
μ_h	the natural death rate of human populations
δ_a	the disease-related death rate of poultry populations
δ_h	the disease-related death rate of humans populations
γ_a	the transfer rate of exposed poultry to infected poultry
θ_h	the recovery rate of the infective human
$\alpha_i (i = 1, 2)$	parameters that measure the inhibitory effect

Because the removed human populations $R_h(t)$ has no effect on the dynamics of the first five equations, system (2.1) can be decoupled to the following system:

$$\begin{cases} \dfrac{dS_a(t)}{dt} = \Lambda_a - \mu_a S_a(t) - \dfrac{\beta_a S_a(t) I_a(t)}{1 + \alpha_1 I_a(t)}, \\ \dfrac{dE_a(t)}{dt} = \dfrac{\beta_a e^{-\mu_a \tau_a} S_a(t-\tau_a) I_a(t-\tau_a)}{1 + \alpha_1 I_a(t-\tau_a)} - (\mu_a + \gamma_a) E_a(t), \\ \dfrac{dI_a(t)}{dt} = \gamma_a E_a(t) - (\mu_a + \delta_a) I_a(t), \\ \dfrac{dS_h(t)}{dt} = \Lambda_h - \mu_h S_h(t) - \dfrac{\beta_h S_h(t) I_a(t)}{1 + \alpha_2 I_a(t)}, \\ \dfrac{dI_h(t)}{dt} = \dfrac{\beta_h e^{-\mu_h \tau_h} S_h(t-\tau_h) I_a(t-\tau_h)}{1 + \alpha_2 I_a(t-\tau_h)} - (\mu_h + \delta_h + \theta_h) I_h(t). \end{cases} \tag{2.2}$$

A realistic avian influenza system would not remain in this stable state due to environmental fluctuations. In this paper, we will reveal how the environmental white noise affects the spread of avian influenza through investigating the dynamics of a stochastic delayed avian influenza model with saturated incidence rate. Taking the same approach as the literatures [23, 24], we assume that the environmental white noise is directly proportional to the variables $S_a(t)$, $E_a(t)$, $I_a(t)$, $S_h(t)$ and $I_h(t)$, respectively. Then, corresponding to system (2.2), the stochastic avian influenza model with time delay is of the following form

$$\begin{cases} dS_a(t) = \left(\Lambda_a - \mu_a S_a(t) - \dfrac{\beta_a S_a(t) I_a(t)}{1 + \alpha_1 I_a(t)}\right) dt + \sigma_1 S_a(t) dB_1(t), \\ dE_a(t) = \left(\dfrac{\beta_a e^{-\mu_a \tau_a} S_a(t-\tau_a) I_a(t-\tau_a)}{1 + \alpha_1 I_a(t-\tau_a)} - (\mu_a + \gamma_a) E_a(t)\right) dt + \sigma_2 E_a(t) dB_2(t), \\ dI_a(t) = \left(\gamma_a E_a(t) - (\mu_a + \delta_a) I_a(t)\right) dt + \sigma_3 I_a(t) dB_3(t), \\ dS_h(t) = \left(\Lambda_h - \mu_h S_h(t) - \dfrac{\beta_h S_h(t) I_a(t)}{1 + \alpha_2 I_a(t)}\right) dt + \sigma_4 S_h(t) dB_4(t), \\ dI_h(t) = \left(\dfrac{\beta_h e^{-\mu_h \tau_h} S_h(t-\tau_h) I_a(t-\tau_h)}{1 + \alpha_2 I_a(t-\tau_h)} - (\mu_h + \delta_h + \theta_h) I_h(t)\right) dt + \sigma_5 I_h(t) dB_5(t), \end{cases} \tag{2.3}$$

in which $B_i(t)$ $(i = 1, 2, \cdots, 5)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration$\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets), σ_i $(i = 1, 2, \cdots, 5)$ denote the intensities of the white noises. The initial value of system (2.3) are

$$\begin{cases} S_a(\theta) = \varphi_1(\theta), E_a(\theta) = \varphi_2(\theta), I_a(\theta) = \varphi_3(\theta), S_h(\theta) = \varphi_4(\theta), I_h(\theta) = \varphi_5(\theta), \\ \varphi_i(\theta) \in C([-\tau, 0], \mathbb{R}_+^5), i = 1, 2, 3, 4, 5, \tau = \max\{\tau_a, \tau_h\}, \end{cases} \tag{2.4}$$

where C is the Banach space $C([-\tau, 0]; \mathbb{R}_+^5)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+^5$, and $\mathbb{R}_+^5 = \{x = (x_1, x_2, x_3, x_4, x_5) : x_i > 0, i = 1, 2, 3, 4, 5\}$. By a biological meaning, we assume that $\varphi_i(0) > 0$ $(i = 1, 2, 3, 4, 5)$.

3. Existence and uniqueness of the global positive solution

In this section, we prove that the solution of system (2.3) is global and positive for any initial value (2.4).

Theorem 1. *For any initial value* (2.4), *system* (2.3) *has a unique positive solution* $(S_a(t), E_a(t), I_a(t), S_h(t), I_h(t))$ *on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}^5_+$ *with probability one, in other words,* $(S_a(t), E_a(t), I_a(t), S_h(t), I_h(t)) \in \mathbb{R}^5_+$ *for all* $t \geq 0$ *almost surely.*

Proof. Since the coefficients of system (2.3) satisfy the local Lipschitz conditions, then for any initial value (2.4), there exists a unique local solution $(S_a(t), E_a(t), I_a(t), S_h(t), I_h(t))$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosive time. To show this solution is global, we only need to show that $\tau_e = \infty$ a.s. To this end, let $k_0 \geq 1$ be sufficiently large such that $(S_a(\theta), E_a(\theta), I_a(\theta), S_h(\theta), I_h(\theta))$ $(\theta \in [-\tau, 0])$ all lie within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time as

$$\begin{aligned}\tau_k =& \inf\{t \in [0, \tau_e) : S_a(t) \notin (\frac{1}{k}, k) \text{ or } E_a(t) \notin (\frac{1}{k}, k)\\ &\text{or } I_a(t) \notin (\frac{1}{k}, k) \text{ or } S_h(t) \notin (\frac{1}{k}, k) \text{ or } I_h(t) \notin (\frac{1}{k}, k)\}.\end{aligned}$$

We set $\inf \varnothing = \infty$. Obviously, τ_k increasing when $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, where $\tau_\infty \leq \tau_e$ a.s. If we can verify $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(S_a(t), E_a(t), I_a(t), S_h(t), I_h(t)) \in \mathbb{R}^5_+$ a.s. for all $t \geq 0$. That is to say, to complete the proof we only need to show that $\tau_\infty = \infty$ a.s. If this assertion is not true, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

There exists an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \varepsilon \text{ for all } k \geq k_1. \tag{3.1}$$

Define a C^2-function V: $\mathbb{R}^5_+ \to \mathbb{R}_+$ by

$$\begin{aligned}V(S_a, E_a, I_a, S_h, I_h) =& e^{-\mu_a\tau_a}(S_a - a - a\ln\frac{S_a}{a}) + (E_a - 1 - \ln E_a) + (I_a - 1 - \ln I_a)\\ &+ \beta_a e^{-\mu_a\tau_a}\int_{t-\tau_a}^{t}\frac{S_a(s)I_a(s)}{1+\alpha_1 I_a(s)}ds + e^{-\mu_h\tau_h}(S_h - b - b\ln\frac{S_h}{b})\\ &+ (I_h - 1 - \ln I_h) + \beta_h e^{-\mu_h\tau_h}\int_{t-\tau_h}^{t}\frac{S_h(s)I_a(s)}{1+\alpha_2 I_a(s)}ds,\end{aligned}$$

in which a and b are positive constants to be determined later. The nonnegativity of this function can be derived from $x - 1 - \ln x \geq 0$ for any $x > 0$. Applying the Itô's formula to V, we get

$$\begin{aligned} dV =& e^{-\mu_a\tau_a}\left(1-\frac{a}{S_a}\right)dS_a + e^{-\mu_a\tau_a}\frac{a}{2S_a^2}(dS_a)^2 + \left(1-\frac{1}{E_a}\right)dE_a + \frac{a}{2E_a^2}(dE_a)^2 \\ &+ \left(1-\frac{1}{I_a}\right)dI_a + \frac{a}{2I_a^2}(dI_a)^2 + \frac{\beta_a e^{-\mu_a\tau_a}S_aI_a}{1+\alpha_1 I_a} - \frac{\beta_a e^{-\mu_a\tau_a}S_a(t-\tau_a)I_a(t-\tau_a)}{1+\alpha_1 I_a(t-\tau_a)} \\ &+ e^{-\mu_h\tau_h}\left(1-\frac{b}{S_h}\right)dS_h + e^{-\mu_h\tau_h}\frac{b}{2S_h^2}(dS_h)^2 + (1-\frac{1}{I_h})dI_h + \frac{1}{2I_h^2}(dI_h)^2 \\ &+ \frac{\beta_h e^{-\mu_h\tau_h}S_hI_h}{1+\alpha_2 I_a} - \frac{\beta_h e^{-\mu_h\tau_h}S_h(t-\tau_h)I_a(t-\tau_h)}{1+\alpha_2 I_a(t-\tau_h)} \\ =& LVdt + e^{-\mu_a\tau_a}\sigma_1(S_a - a)dB_1(t) + \sigma_2(E_a-1)dB_2(t) + \sigma_3(I_a-1)dB_3(t) \\ &+ e^{-\mu_h\tau_h}\sigma_4(S_h-b)dB_4(t) + \sigma_5(I_h-1)dB_5(t), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} LV =& e^{-\mu_a\tau_a}\left(1-\frac{a}{S_a}\right)(\Lambda_a - \mu_a S_a) - \left(1-\frac{1}{E_a}\right)(\mu_a+\gamma_a)E_a \\ &+ \left(1-\frac{1}{I_a}\right)(\gamma_a E_a - (\mu_a+\delta_a)I_a) + e^{-\mu_h\tau_h}\left(1-\frac{b}{S_h}\right)(\Lambda_h - \mu_h S_h) \\ &- \left(1-\frac{1}{I_h}\right)(\mu_h+\delta_h+\theta_h)I_h + e^{-\mu_a\tau_a}\frac{a\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + e^{-\mu_h\tau_h}\frac{b\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\ &\leq e^{-\mu_a\tau_a}\Lambda_a + a\mu_a e^{-\mu_a\tau_a} + \frac{a\sigma_1^2}{2}e^{-\mu_a\tau_a} + 2\mu_a + \delta_a + \gamma_a + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 \\ &+ e^{-\mu_h\tau_h}\Lambda_h + b\mu_h e^{-\mu_h\tau_h} + \mu_h + \delta_h + \theta_h + \frac{b\sigma_4^2}{2}e^{-\mu_h\tau_h} + \frac{1}{2}\sigma_5^2 \\ &+ \left(a\beta_a e^{-\mu_a\tau_a} + b\beta_h e^{-\mu_h\tau_h} - (\mu_a+\delta_a)\right)I_a. \end{aligned}$$

Choose $a = \frac{\mu_a e^{\mu_a\tau_a}}{\beta_a}$ and $b = \frac{\delta_a e^{\mu_h\tau_h}}{\beta_h}$ or $a = \frac{\delta_a e^{\mu_a\tau_a}}{\beta_a}$ and $b = \frac{\mu_a e^{\mu_h\tau_h}}{\beta_h}$ such that

$$a\beta_a e^{-\mu_a\tau_a} + b\beta_h e^{-\mu_h\tau_h} - (\mu_a+\delta_a) = 0.$$

Then, we can get

$$\begin{aligned} LV(S_a,E_a,I_a,S_h,I_h) \leq& e^{-\mu_a\tau_a}\Lambda_a + a\mu_a e^{-\mu_a\tau_a} + e^{-\mu_h\tau_h}\Lambda_h + b\mu_h e^{-\mu_h\tau_h} + 2\mu_a + \gamma_a \\ &+ \delta_a + \mu_h + \delta_h + \theta_h + \frac{a\sigma_1^2}{2}e^{-\mu_a\tau_a} + \frac{b\sigma_4^2}{2}e^{-\mu_h\tau_h} + \frac{1}{2}(\sigma_2^2+\sigma_3^2+\sigma_5^2) \\ &=: K, \end{aligned}$$

where K is a positive constant. It thus follows from (3.2) that

$$\begin{aligned} dV(S_a,E_a,I_a,S_h,I_h) \leq& Kdt + e^{-\mu_a\tau_a}\sigma_1(S_a-a)dB_1(t) + \sigma_2(E_a-1)dB_2(t) + \sigma_3(I_a-1)dB_3(t) \\ &+ e^{-\mu_h\tau_h}\sigma_4(S_h-b)dB_4(t) + \sigma_5(I_h-1)dB_5(t). \end{aligned} \tag{3.3}$$

Integrating both sides of (3.3) from 0 to $\tau_k \wedge T = \min\{\tau_k, T\}$ and then taking the expectation results in

$$\begin{aligned}&EV(S_a(\tau_k \wedge T), E_a(\tau_k \wedge T), I_a(\tau_k \wedge T), S_h(\tau_k \wedge T), I_h(\tau_k \wedge T))\\ \leq &V(S_a(0), E_a(0), I_a(0), S_h(0), I_h(0)) + KE(\tau_k \wedge T)\\ \leq &V(S_a(0), E_a(0), I_a(0), S_h(0), I_h(0)) + KT.\end{aligned} \tag{3.4}$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$, and according to (3.1), we have $P(\Omega_k) \geq \varepsilon$. For every $\omega \in \Omega_k$, there exists $S_a(\tau_k, \omega)$ or $E_a(\tau_k, \omega)$ or $I_a(\tau_k, \omega)$ or $S_h(\tau_k, \omega)$ or $I_h(\tau_k, \omega)$ equals either k or $\frac{1}{k}$. Therefore, $V(S_a(\tau_k, \omega), E_a(\tau_k, \omega), I_a(\tau_k, \omega), S_h(\tau_k, \omega), I_h(\tau_k, \omega))$ is no less either $k - 1 - \ln k$ or $\frac{1}{k} - 1 - \ln\frac{1}{k}$ or $k - a - a\ln\frac{k}{a}$ or $\frac{1}{k} - a + a\ln ak$ or $k - b - b\ln\frac{k}{b}$ or $\frac{1}{k} - b + b\ln bk$.
Therefore, we have

$$\begin{aligned}&V(S_a(\tau_k, \omega), E_a(\tau_k, \omega), I_a(\tau_k, \omega), S_h(\tau_k, \omega), I_h(\tau_k, \omega))\\ \geq &(k - 1 - \ln k) \wedge (\frac{1}{k} - 1 + \ln k) \wedge (k - a - a\ln\frac{k}{a})\\ &\wedge (\frac{1}{k} - a + a\ln ak) \wedge (k - b - b\ln\frac{k}{b}) \wedge (\frac{1}{k} - b + b\ln bk).\end{aligned}$$

It follows from (3.4) that

$$\begin{aligned}&V(S_a(0), E_a(0), I_a(0), S_h(0), I_h(0)) + KT\\ \geq &E[1_{\Omega_k} V(S_a(\tau_k, \omega), E_a(\tau_k, \omega), I_a(\tau_k, \omega), S_h(\tau_k, \omega), I_h(\tau_k, \omega))]\\ \geq &\varepsilon[(k - 1 - \ln k) \wedge (\frac{1}{k} - 1 + \ln k) \wedge (k - a - a\ln\frac{k}{a})\\ &\wedge (\frac{1}{k} - a + a\ln ak) \wedge (k - b - b\ln\frac{k}{b}) \wedge (\frac{1}{k} - b + b\ln bk)],\end{aligned}$$

where 1_{Ω_k} denotes the indicator function of Ω_k. Letting $k \to \infty$, then

$$\infty > V(S_a(0), E_a(0), I_a(0), S_h(0), I_h(0)) + KT = \infty,$$

which leads to the contradiction. This completes the proof. □

4. Asymptotic behavior of system (2.3) around the disease-free equilibrium E^0

In this section, we will investigate the solution of system (2.3) around disease-free equilibrium E^0 under certain conditions. It is worthwhile to mention that, if $\mathscr{R}_0 = \frac{\beta_a\gamma_a\Lambda_a e^{-\mu_a\tau_a}}{\mu_a(\mu_a+\delta_a)(\mu_a+\gamma_a)} < 1$, the deterministic system (2.2) is globally asymptotically stable around the unique disease-free equilibrium $E^0 = (S_a^0, 0, 0, S_h^0, 0) = (\frac{\Lambda_a}{\mu_a}, 0, 0, \frac{\Lambda_h}{\mu_h}, 0)$, but E^0 is not the equilibrium of the stochastic system (2.3). Thus, the result concerning the solution of stochastic system (2.3) around E^0 is presented by the following theorem.

Theorem 2. *Let $(S_a(t), E_a(t), I_a(t), S_h(t), I_h(t))$ be the solution of system* (2.3) *with the initial value* (2.4). *If $\mathscr{R}_0 < 1$ and the following conditions hold*

$$\sigma_1^2 < \mu_a, \sigma_2^2 < \mu_a + \gamma_a, \sigma_3^2 < \mu_a + \delta_a, \sigma_4^2 < \mu_h, \sigma_5^2 < \mu_h + \delta_h + \theta_h,$$

then,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left(S_a-\frac{\Lambda_a}{\mu_a}\right)^2ds\le\frac{\sigma_1^2\Lambda_a^2}{\mu_a^2(\mu_a-\sigma_1^2)},$$
$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t(E_a^2+I_a^2)ds\le\frac{P_1}{M_1},$$
$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left(S_h-\frac{\Lambda_h}{\mu_h}\right)^2ds\le\frac{\Lambda_h^2}{\mu_h^2(\mu_h-\sigma_4^2)}\left(\sigma_4^2+\frac{\beta_h}{\alpha_2}\right),$$
$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t I_h^2ds\le P_2,$$

where

$$M_1=\min\left\{\frac{\mu_a+\gamma_a-\sigma_2^2}{4},\frac{(\mu_a+\gamma_a-\sigma_2^2)(\mu_a+\delta_a-\sigma_3^2)(\mu_a+\delta_a)}{4\gamma_a^2}\right\},$$
$$P_1=\frac{e^{-2\mu_a\tau_a}\sigma_1^2\Lambda_a^2}{\mu_a^2}\left[\frac{1}{\mu_a-\sigma_1^2}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)+1\right],$$
$$P_2=\frac{2e^{-2\mu_h\tau_h}\Lambda_h}{\mu_h^2(\mu_h+\delta_h+\theta_h-\sigma_5^2)}\left[\frac{\alpha_2\sigma_4^2+\beta_h}{\alpha_2(\mu_h-\sigma_4^2)}\left(\frac{2\mu_h^2+2\mu_h\delta_h+2\mu_h\theta_h+(\delta_h+\theta_h)^2}{2(\mu_h+\delta_h+\theta_h)}+\sigma_4^2\right)+\sigma_4^2\right].$$

Proof. Since $(S_a^0,0,0,S_h^0,0)$ is the disease-free equilibrium of system (2.2), then

$$\Lambda_a=\mu_aS_a^0,\quad \Lambda_h=\mu_hS_h^0.$$

According to system (2.3), we can obtain that

$$\begin{aligned}dS_a(t)&=\left[-\mu_a\left(S_a-\frac{\Lambda_a}{\mu_a}\right)-\frac{\beta_aS_aI_a}{1+\alpha_1I_a}\right]dt+\sigma_1S_adB_1(t)\\&=\left[-\mu_a\left(S_a-\frac{\Lambda_a}{\mu_a}\right)-\beta_a\left(S_a-\frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1I_a}-\beta_a\frac{\Lambda_a}{\mu_a}\frac{I_a}{1+\alpha_1I_a}\right]dt+\sigma_1S_adB_1(t),\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}&d\left[E_a(t+\tau_a)+\frac{\mu_a+\gamma_a}{\gamma_a}I_a(t+\tau_a)\right]=dE_a(t+\tau_a)+\frac{\mu_a+\gamma_a}{\gamma_a}dI_a(t+\tau_a)\\&\le\left[\beta_ae^{-\mu_a\tau_a}\left(S_a-\frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1I_a}-\frac{(\mu_a+\gamma_a)(\mu_a+\delta_a)}{\gamma_a}I_a(t+\tau_a)\right.\\&\quad\left.+\beta_ae^{-\mu_a\tau_a}\frac{\Lambda_a}{\mu_a}I_a\right]dt+\sigma_2E_a(t+\tau_a)dB_2(t)+\frac{\sigma_3(\mu_a+\gamma_a)}{\gamma_a}I_a(t+\tau_a)dB_3(t)\\&\le\left[\beta_ae^{-\mu_a\tau_a}\left(S_a-\frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1I_a}+\frac{(\mu_a+\gamma_a)(\mu_a+\delta_a)}{\gamma_a}(I_a(t)-I_a(t+\tau_a))\right]dt\\&\quad+\sigma_2E_a(t+\tau_a)dB_2(t)+\frac{\sigma_3(\mu_a+\gamma_a)}{\gamma_a}I_a(t+\tau_a)dB_3(t).\end{aligned}\tag{4.2}$$

Let $V_1 = \frac{1}{2}\left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2$, then applying the Itô's formula to V_1, together with (4.1), we have

$$\begin{aligned} dV_1 =& \left[\left(S_a - \frac{\Lambda_a}{\mu_a}\right)\left(-\mu_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right) - \beta_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1 I_a} - \beta_a\frac{\Lambda_a}{\mu_a}\frac{I_a}{1+\alpha_1 I_a}\right) + \frac{1}{2}\sigma_1^2 S_a^2\right]dt \\ &+ \sigma_1 S_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)dB_1(t) \\ =& \left[-\mu_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2 - \beta_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2\frac{I_a}{1+\alpha_1 I_a} - \beta_a\frac{\Lambda_a}{\mu_a}\left(S_a - \frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1 I_a} + \frac{1}{2}\sigma_1^2 S_a^2\right]dt \\ &+ \sigma_1 S_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)dB_1(t) \\ =&: LV_1 dt + \sigma_1 S_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)dB_1(t), \end{aligned}$$

where

$$\begin{aligned} LV_1 &\le -\mu_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2 - \beta_a\frac{\Lambda_a}{\mu_a}\left(S_a - \frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1 I_a} + \sigma_1^2\left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2 + \frac{\sigma_1^2\Lambda_a^2}{\mu_a^2} \\ &= -(\mu_a - \sigma_1^2)\left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2 - \beta_a\frac{\Lambda_a}{\mu_a}\left(S_a - \frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1 I_a} + \frac{\sigma_1^2\Lambda_a^2}{\mu_a^2}. \end{aligned} \tag{4.3}$$

Similarly, let $V_2 = E_a(t+\tau_a) + \frac{\mu_a+\gamma_a}{\gamma_a}I_a(t+\tau_a) + \frac{(\mu_a+\gamma_a)(\mu_a+\delta_a)}{\gamma_a}\int_t^{t+\tau_a} I_a(s)ds$, it follows from (4.2) that

$$dV_2 \le \beta_a e^{-\mu_a\tau_a}\left(S_a - \frac{\Lambda_a}{\mu_a}\right)\frac{I_a}{1+\alpha_1 I_a} + \sigma_2 E_a(t+\tau_a)dB_2(t) + \frac{\sigma_3(\mu_a+\gamma_a)}{\gamma_a}I_a(t+\tau_a)dB_3(t).$$

Define $\bar{V} = e^{-\mu_a\tau_a}V_1 + \frac{\Lambda_a}{\mu_a}V_2$, then

$$\begin{aligned} d\bar{V} \le& \left[-e^{-\mu_a\tau_a}(\mu_a - \sigma_1^2)\left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2 + e^{-\mu_a\tau_a}\sigma_1^2\frac{\Lambda_a^2}{\mu_a^2}\right]dt \\ &+ \sigma_1 S_a\left(S_a - \frac{\Lambda_a}{\mu_a}\right)dB_1(t) + \sigma_2 E_a(t+\tau_a)dB_2(t) + \frac{\sigma_3(\mu_a+\gamma_a)}{\gamma_a}I_a(t+\tau_a)dB_3(t). \end{aligned} \tag{4.4}$$

Integrating both sides of (4.4) from 0 to t and taking expectation, we get

$$E\bar{V}(t) - E\bar{V}(0) \le -e^{-\mu_a\tau_a}(\mu_a - \sigma_1^2)E\int_0^t \left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2 ds + e^{-\mu_a\tau_a}\sigma_1^2\frac{\Lambda_a^2}{\mu_a^2}t.$$

Therefore, we can obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \left(S_a - \frac{\Lambda_a}{\mu_a}\right)^2 ds \le \frac{\sigma_1^2\Lambda_a^2}{\mu_a^2(\mu_a - \sigma_1^2)}.$$

Similarly, we define

$$V_3 = \frac{1}{2}\left[e^{-\mu_a\tau_a}\left(S_a - \frac{\Lambda_a}{\mu_a}\right) + E_a(t+\tau_a)\right]^2,$$

then,

$$
\begin{aligned}
LV_3 =& -e^{-2\mu_a\tau_a}\mu_a\left(S_a-\frac{\Lambda_a}{\mu_a}\right)^2-e^{-\mu_a\tau_a}(2\mu_a+\gamma_a)\left(S_a-\frac{\Lambda_a}{\mu_a}\right)E_a(t+\tau_a)\\
&-(\mu_a+\gamma_a)E_a^2(t+\tau_a)+\frac{1}{2}e^{-2\mu_a\tau_a}\sigma_1^2S_a^2+\frac{1}{2}\sigma_2^2E_a^2(t+\tau_a)\\
\le& -e^{-2\mu_a\tau_a}\mu_a\left(S_a-\frac{\Lambda_a}{\mu_a}\right)^2+\frac{\mu_a+\gamma_a}{2}E_a^2(t+\tau_a)+\frac{(2\mu_a+\gamma_a)^2e^{-2\mu_a\tau_a}}{2(\mu_a+\gamma_a)}\left(S_a-\frac{\Lambda_a}{\mu_a}\right)^2\\
&-(\mu_a+\gamma_a)E_a^2(t+\tau_a)+e^{-2\mu_a\tau_a}\sigma_1^2\left(S_a-\frac{\Lambda_a}{\mu_a}\right)^2+e^{-2\mu_a\tau_a}\frac{\sigma_1^2\Lambda_a^2}{\mu_a^2}+\frac{1}{2}\sigma_2^2E_a^2(t+\tau_a)\\
=&e^{-2\mu_a\tau_a}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)\left(S_a-\frac{\Lambda_a}{\mu_a}\right)^2\\
&-\frac{1}{2}(\mu_a+\gamma_a-\sigma_2^2)E_a^2(t+\tau_a)+e^{-2\mu_a\tau_a}\frac{\sigma_1^2\Lambda_a^2}{\mu_a^2}.
\end{aligned}
$$

Let $V_4=V_3+\frac{1}{2}(\mu_a+\gamma_a-\sigma_2^2)\int_t^{t+\tau_a}E_a^2(s)ds$, we get

$$
LV_4\le e^{-2\mu_a\tau_a}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)\left(S_a-\frac{\Lambda_a}{\mu_a}\right)^2-\frac{1}{2}(\mu_a+\gamma_a-\sigma_2^2)E_a^2+e^{-2\mu_a\tau_a}\frac{\sigma_1^2\Lambda_a^2}{\mu_a^2}.
$$

Let $V_5=\frac{1}{2}I_a^2$, the derivative of V_5 can be calculated as

$$
\begin{aligned}
LV_5 =&\gamma_aE_aI_a-(\mu_a+\delta_a)I_a^2+\frac{1}{2}\sigma_3^2I_a^2\\
\le&\frac{\mu_a+\delta_a}{2}I_a^2+\frac{\gamma_a^2}{2(\mu_a+\delta_a)}E_a^2-(\mu_a+\delta_a)I_a^2+\frac{1}{2}\sigma_3^2I_a^2\\
=&\frac{\gamma_a^2}{2(\mu_a+\delta_a)}E_a^2-\frac{1}{2}(\mu_a+\delta_a-\sigma_3^2)I_a^2.
\end{aligned}
$$

The Young's inequality is used above. Let

$$
\tilde{V}=V_4+\frac{e^{-\mu_a\tau_a}}{\mu_a-\sigma_1^2}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)\bar{V}+\frac{(\mu_a+\gamma_a-\sigma_2^2)(\mu_a+\delta_a)}{2\gamma_a^2}V_5,
$$

which implies that

$$
\begin{aligned}
L\tilde{V}\le& -\frac{1}{2}(\mu_a+\gamma_a-\sigma_2^2)E_a^2+e^{-2\mu_a\tau_a}\frac{\sigma_1^2\Lambda_a^2}{\mu_a^2}+\frac{e^{-2\mu_a\tau_a}\sigma_1^2\Lambda_a^2}{\mu_a^2(\mu_a-\sigma_1^2)}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)\\
&+\frac{1}{4}(\mu_a+\gamma_a-\sigma_2^2)E_a^2-\frac{(\mu_a+\gamma_a-\sigma_2^2)(\mu_a+\delta_a-\sigma_3^2)(\mu_a+\delta_a)}{4\gamma_a^2}I_a^2\\
=& -\frac{1}{4}(\mu_a+\gamma_a-\sigma_2^2)E_a^2-\frac{(\mu_a+\gamma_a-\sigma_2^2)(\mu_a+\delta_a-\sigma_3^2)(\mu_a+\delta_a)}{4\gamma_a^2}I_a^2\\
&+e^{-2\mu_a\tau_a}\frac{\sigma_1^2\Lambda_a^2}{\mu_a^2}\left[\frac{1}{\mu_a-\sigma_1^2}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)+1\right].
\end{aligned}
\tag{4.5}
$$

Integrating both sides of (4.5) from 0 to t and then taking expectation yields

$$\begin{aligned}E\tilde{V}(t) - E\tilde{V}(0) \le &- \frac{1}{4}(\mu_a + \gamma_a - \sigma_2^2)E\int_0^t E_a^2(s)ds \\ &- \frac{(\mu_a + \gamma_a - \sigma_2^2)(\mu_a + \delta_a - \sigma_3^2)(\mu_a + \delta_a)}{4\gamma_a^2}E\int_0^t I_a^2(s)ds \\ &+ e^{-2\mu_a\tau_a}\frac{\sigma_1^2\Lambda_a^2}{\mu_a^2}\left[\frac{1}{\mu_a - \sigma_1^2}\left(\frac{2\mu_a^2 + 2\mu_a\gamma_a + \gamma_a^2}{2(\mu_a + \gamma_a)} + \sigma_1^2\right) + 1\right]t.\end{aligned}$$

Consequently, we can obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t (E_a^2(s) + I_a^2(s))ds \le \frac{P_1}{M_1},$$

where M_1 and P_1 are defined in Theorem 2. Further, according to system (2.3), we have

$$\begin{aligned}dS_h(t) &= \left[-\mu_h\left(S_h - \frac{\Lambda_h}{\mu_h}\right) - \frac{\beta_h S_h I_a}{1 + \alpha_2 I_a}\right]dt + \sigma_4 S_h dB_4(t) \\ &= \left[-\mu_h\left(S_h - \frac{\Lambda_h}{\mu_h}\right) - \left(S_h - \frac{\Lambda_h}{\mu_h}\right)\frac{\beta_h I_a}{1 + \alpha_2 I_a} - \frac{\beta_h\Lambda_h I_a}{\mu_h(1 + \alpha_2 I_a)}\right]dt + \sigma_4 S_h dB_4(t),\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}dI_h(t + \tau_h) &= \left[\frac{\beta_h e^{-\mu_h\tau_h} S_h I_a}{1 + \alpha_2 I_a} - (\mu_h + \delta_h + \theta_h)I_h(t + \tau_h)\right]dt + \sigma_5 I_h(t + \tau_h)dB_5(t) \\ &\le \left[\frac{\beta_h e^{-\mu_h\tau_h} I_a}{1 + \alpha_2 I_a}\left(S_h - \frac{\Lambda_h}{\mu_h}\right) + \frac{\beta_h\Lambda_h e^{-\mu_h\tau_h}}{\alpha_2\mu_h} - (\mu_h + \delta_h + \theta_h)I_h(t + \tau_h)\right]dt \\ &\quad + \sigma_5 I_h(t + \tau_h)dB_5(t).\end{aligned} \tag{4.7}$$

Let $V_6 = \frac{1}{2}\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2$. Noting (4.6), we have

$$\begin{aligned}LV_6 &= -\mu_h\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 - \beta_h\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2\frac{I_a}{1 + \alpha_2 I_a} - \beta_h\frac{\Lambda_h}{\mu_h}\left(S_h - \frac{\Lambda_h}{\mu_h}\right)\frac{I_a}{1 + \alpha_2 I_a} + \frac{1}{2}\sigma_4^2 S_h^2 \\ &\le -\mu_h\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 - \beta_h\frac{\Lambda_h}{\mu_h}\left(S_h - \frac{\Lambda_h}{\mu_h}\right)\frac{I_a}{1 + \alpha_2 I_a} + \sigma_4^2\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 + \frac{\sigma_4^2\Lambda_h^2}{\mu_h^2} \\ &= -(\mu_h - \sigma_4^2)\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 - \beta_h\frac{\Lambda_h}{\mu_h}\left(S_h - \frac{\Lambda_h}{\mu_h}\right)\frac{I_a}{1 + \alpha_2 I_a} + \frac{\sigma_4^2\Lambda_h^2}{\mu_h^2}.\end{aligned}$$

Let $V_7 = e^{-\mu_h\tau_h}V_6 + \frac{\Lambda_h}{\mu_h}I_h(t + \tau_h)$, it follows from (4.7) that

$$\begin{aligned}LV_7 \le &- e^{-\mu_h\tau_h}(\mu_h - \sigma_4^2)\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 + e^{-\mu_h\tau_h}\frac{\sigma_4^2\Lambda_h^2}{\mu_h^2} + \frac{\beta_h\Lambda_h^2 e^{-\mu_h\tau_h}}{\alpha_2\mu_h^2} - \frac{\Lambda_h}{\mu_h}(\mu_h + \delta_h + \theta_h)I_h(t + \tau_h) \\ \le &- e^{-\mu_h\tau_h}(\mu_h - \sigma_4^2)\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 + e^{-\mu_h\tau_h}\frac{\Lambda_h^2}{\mu_h^2}(\sigma_4^2 + \frac{\beta_h}{\alpha_2}).\end{aligned} \tag{4.8}$$

Integrating both sides of (4.8) from 0 to t and then taking the expectation yields

$$EV_7(t) - EV_7(0) \leq -e^{-\mu_h\tau_h}(\mu_h - \sigma_4^2)E\int_0^t \left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 ds + e^{-\mu_h\tau_h}\frac{\Lambda_h^2}{\mu_h^2}(\sigma_4^2 + \frac{\beta_h}{\alpha_2})t,$$

therefore, we can get

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 ds \leq \frac{\Lambda_h^2}{\mu_h^2(\mu_h - \sigma_4^2)}(\sigma_4^2 + \frac{\beta_h}{\alpha_2}).$$

Let $V_8 = \frac{1}{2}\left[e^{-\mu_h\tau_h}\left(S_h - \frac{\Lambda_h}{\mu_h}\right) + I_h(t+\tau_h)\right]^2$, then

$$\begin{aligned}
LV_8 =&\left(e^{-\mu_h\tau_h}\left(S_h - \frac{\Lambda_h}{\mu_h}\right) + I_h(t+\tau_h)\right)[e^{-\mu_h\tau_h}(\Lambda_h - \mu_h S_h) - (\mu_h + \delta_h + \theta_h)I_h(t+\tau_h)]\\
&+ \frac{1}{2}e^{-2\mu_h\tau_h}\sigma_4^2 S_h^2 + \frac{1}{2}\sigma_5^2 I_h^2(t+\tau_h)\\
\leq& - e^{-2\mu_h\tau_h}\mu_h\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2 + \frac{(2\mu_h + \delta_h + \theta_h)^2 e^{-2\mu_h\tau_h}}{2(\mu_h + \delta_h + \theta_h)}\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2\\
&+ \frac{\mu_h + \delta_h + \theta_h}{2}I_h^2(t+\tau_h) - (\mu_h + \delta_h + \theta_h)I_h^2(t+\tau_h) + e^{-2\mu_h\tau_h}\sigma_4^2\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2\\
&+ \frac{e^{-2\mu_h\tau_h}\sigma_4^2\Lambda_h^2}{\mu_h^2} + \frac{1}{2}\sigma_5^2 I_h^2(t+\tau_h)\\
=&e^{-2\mu_h\tau_h}\left(\frac{2\mu_h^2 + 2\mu_h\delta_h + 2\mu_h\theta_h + (\delta_h + \theta_h)^2}{2(\mu_h + \delta_h + \theta_h)} + \sigma_4^2\right)\left(S_h - \frac{\Lambda_h}{\mu_h}\right)^2\\
&- \frac{1}{2}(\mu_h + \delta_h + \theta_h - \sigma_5^2)I_h^2(t+\tau_h) + \frac{e^{-2\mu_h\tau_h}\sigma_4^2\Lambda_h^2}{\mu_h^2}.
\end{aligned}$$

Defining

$$V_9 = V_8 + \frac{e^{-\mu_h\tau_h}}{\mu_h - \sigma_4^2}\left(\frac{2\mu_h^2 + 2\mu_h\delta_h + 2\mu_h\theta_h + (\delta_h + \theta_h)^2}{2(\mu_h + \delta_h + \theta_h)} + \sigma_4^2\right)V_7 + \frac{1}{2}(\mu_h + \delta_h + \theta_h - \sigma_5^2)\int_t^{t+\tau_h} I_h^2(s)ds,$$

we get

$$\begin{aligned}
LV_9 \leq& - \frac{1}{2}(\mu_h + \delta_h + \theta_h - \sigma_5^2)I_h^2\\
&+ \frac{e^{-2\mu_h\tau_h}\Lambda_h^2}{\mu_h^2}\left[\frac{1}{\mu_h - \sigma_4^2}(\sigma_4^2 + \frac{\beta_h}{\alpha_2})\left(\frac{2\mu_h^2 + 2\mu_h\delta_h + 2\mu_h\theta_h + (\delta_h + \theta_h)^2}{2(\mu_h + \delta_h + \theta_h)} + \sigma_4^2\right) + \sigma_4^2\right].
\end{aligned} \tag{4.9}$$

Integrating both sides of (4.9) from 0 to t and taking expectation, we obtain

$$\begin{aligned}
EV_9(t) - EV_9(0) \leq& - \frac{1}{2}(\mu_h + \delta_h + \theta_h - \sigma_5^2)E\int_0^t I_h^2(s)ds + \frac{e^{-2\mu_h\tau_h}\Lambda_h^2}{\mu_h^2}\Big[\frac{1}{\mu_h - \sigma_4^2}(\sigma_4^2 + \frac{\beta_h}{\alpha_2})\\
&\left(\frac{2\mu_h^2 + 2\mu_h\delta_h + 2\mu_h\theta_h + (\delta_h + \theta_h)^2}{2(\mu_h + \delta_h + \theta_h)} + \sigma_4^2\right) + \sigma_4^2\Big]t.
\end{aligned}$$

Consequently, we can obtain

$$\limsup_{t\to\infty} E\int_0^t I_h^2(s)ds \le P_2,$$

where P_2 is defined in Theorem 2. This completes the proof. □

5. Asymptotic behavior of system (2.3) around the endemic equilibrium E^*

If $\mathscr{R}_0 > 1$, there exists an endemic equilibrium $E^* = (S_a^*, E_a^*, I_a^*, S_h^*, I_h^*)$ of system (2.2), but it is not the equilibrium of system (2.3), where $S_a^* = \frac{\Lambda_a(1+\alpha_1 I_a^*)}{\mu_a(1+\alpha_1 I_a^*)+\beta_a I_a^*}$, $E_a^* = \frac{\beta_a\Lambda_a e^{-\mu_a\tau_a} I_a^*}{(\mu_a+\gamma_a)[\mu_a(1+\alpha_1 I_a^*)+\beta_a I_a^*]}$, $I_a^* = \frac{\mu_a(\mathscr{R}_0-1)}{\alpha_1\mu_a+\beta_a}$, $S_h^* = \frac{\Lambda_h(1+\alpha_2 I_a^*)}{\mu_h(1+\alpha_2 I_a^*)+\beta_h I_a^*}$, $E_a^* = \frac{\beta_h e^{-\mu_h\tau_h} S_h^* I_a^*}{(\mu_h+\delta_h+\theta_h)(1+\alpha_2 I_a^*)}$. In this section, we show that the solution of system (2.3) is going around E^* under certain conditions.

Theorem 3. *Let* $(S_a(t), E_a(t), I_a(t), S_h(t), I_h(t))$ *be the solution of system* (2.3) *with initial value* (2.4). *If* $\mathscr{R}_0 > 1$ *and the following conditions hold*

(i) $\sigma_1^2 < \mu_a, \sigma_2^2 < \frac{1}{2}(\mu_a+\gamma_a), \sigma_3^2 < \frac{1}{2}(\mu_a+\delta_a), \sigma_4^2 < \mu_h, \sigma_5^2 < \mu_h+\delta_h+\theta_h$;
(ii) $\max(\sqrt{P_3}, \sqrt{P_4}, \sqrt{P_5}, \sqrt{P_6}) < d(E^*, E^0)$,

then

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}E\int_0^t (S_a - S_a^*)^2 ds \le P_3,\\
&\limsup_{t\to\infty} E\int_0^t [(E_a(s)-E_a^*)^2 + (I_a(s)-I_a^*)^2]ds \le \frac{L_1}{L_2} =: P_4,\\
&\limsup_{t\to\infty} E\int_0^t (S_h - S_h^*)^2 ds \le P_5,\\
&\limsup_{t\to\infty} E\int_0^t (I_h - I_h^*)^2 ds \le P_6,
\end{aligned}$$

where

$$\begin{aligned}
d(E^*, E^0) &= \sqrt{\left(S_a^* - \frac{\Lambda_a}{\mu_a}\right)^2 + (E_a^*)^2 + (I_a^*)^2 + \left(S_h^* - \frac{\Lambda_h}{\mu_h}\right)^2 + (I_h^*)^2}\\
P_3 &= \frac{1}{\mu_a-\sigma_1^2}\left[\sigma_1^2(S_a^*)^2 + \frac{\sigma_1^2 S_a^* L_3}{2\mu_a} + \left(e^{\mu_a\tau_a}S_a^* + \frac{L_3}{\mu_a e^{-\mu_a\tau_a}}\right)\left(\frac{1}{2}\sigma_2^2 E_a^* + \frac{\mu_a+\gamma_a}{2\gamma_a}\sigma_3^2 I_a^*\right)\right],\\
P_4 &= \frac{L_1}{L_2}, \qquad P_5 = \frac{\sigma_4^2(S_h^*)^2}{\mu_h-\sigma_4^2},\\
P_6 &= \frac{\sigma_4^2 L_4^2}{(\mu_h-\sigma_4^2)(\mu_h+\delta_h+\theta_h-\sigma_5^2)^2} + \frac{2\sigma_5^2(I_h^*)^2}{\mu_h+\delta_h+\theta_h-\sigma_5^2},\\
L_1 &= \frac{e^{-\mu_a\tau_a}}{\mu_a-\sigma_1^2}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)} + \sigma_1^2\right)\left[\sigma_1^2(S_a^*)^2 + \frac{\sigma_1^2 S_a^* L_3}{2\mu_a}\right.\\
&\quad \left. + \left(e^{\mu_a\tau_a}S_a^* + \frac{L_3}{\mu_a e^{-\mu_a\tau_a}}\right)\left(\frac{1}{2}\sigma_2^2 E_a^* + \frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2 I_a^*\right)\right] + e^{-2\mu_a\tau_a}\sigma_1^2(S_a^*)^2
\end{aligned}$$

$$+\sigma_2^2(E_a^*)^2+\frac{\sigma_3^2(\mu_a+\delta_a)(\mu_a+\gamma_a-2\sigma_2^2)}{2\gamma_a^2}(I_a^*)^2,$$

$$L_2=\min\left\{\frac{1}{4}(\mu_a+\gamma_a-2\sigma_2^2),\ \frac{(\mu_a+\delta_a)(\mu_a+\gamma_a-2\sigma_2^2)(\mu_a+\delta_a-2\sigma_3^2)}{4\gamma_a^2}\right\},$$

$$L_3=\frac{\beta_a S_a^* I_a^*}{1+\alpha_1 I_a^*},\qquad L_4=\frac{\beta_h^* S_h^* I_a^*}{1+\alpha_2 I_a^*}.$$

Proof. Since $(S_a^*, E_a^*, I_a^*, S_h^*, I_h^*)$ is the interior equilibrium of system (2.2), then

$$\begin{aligned}&\Lambda_a=\mu_a S_a^*+\frac{\beta_a S_a^* I_a^*}{1+\alpha_1 I_a^*},\quad (\mu_a+\gamma_a)E_a^*=\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*},\qquad \frac{I_a^*}{E_a^*}=\frac{\gamma_a}{\mu_a+\delta_a},\\ &\Lambda_h=\mu_h S_h^*+\frac{\beta_h S_h^* I_a^*}{1+\alpha_2 I_a^*},\quad (\mu_h+\delta_h+\theta_h)I_h^*=\frac{\beta_h e^{-\mu_h\tau_h}S_h^* I_a^*}{1+\alpha_2 I_a^*}.\end{aligned}\tag{5.1}$$

Define the Lyapunov function W_1 as $W_1=S_a-S_a^*-S_a^*\ln\frac{S_a}{S_a^*}$, from which we have

$$\begin{aligned}dW_1=&\left(\Lambda_a-\mu_a S_a-\frac{\beta_a S_a I_a}{1+\alpha_1 I_a}-\frac{\Lambda_a S_a^*}{S_a}+\mu_a S_a^*+\frac{\beta_a S_a^* I_a}{1+\alpha_1 I_a}+\frac{1}{2}S_a^*\sigma_1^2\right)dt+\sigma_1(S_a-S_a^*)dB_1(t)\\ =&\Bigg[\left(\mu_a S_a^*+\frac{\beta_a S_a^* I_a^*}{1+\alpha_1 I_a^*}\right)(2-\frac{S_a^*}{S_a}-\frac{S_a}{S_a^*})+\frac{\beta_a S_a^* I_a^*}{1+\alpha_1 I_a^*}\Bigg(-\frac{S_a I_a(1+\alpha_1 I_a^*)}{S_a^* I_a^*(1+\alpha_1 I_a)}\\ &+\frac{S_a}{S_a^*}+\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-1\Bigg)+\frac{1}{2}S_a^*\sigma_1^2\Bigg]dt+\sigma_1(S_a-S_a^*)dB_1(t)\\ =&LW_1dt+\sigma_1(S_a-S_a^*)dB_1(t),\end{aligned}$$

where

$$LW_1=-\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\frac{(S_a-S_a^*)^2}{S_a}-\beta_a(S_a-S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a}-\frac{I_a^*}{1+\alpha_1 I_a^*}\right)+\frac{1}{2}S_a^*\sigma_1^2.\tag{5.2}$$

Similarly, we can define W_2 as

$$W_2=E_a(t+\tau_a)-E_a^*-E_a^*\ln\frac{E_a(t+\tau_a)}{E_a^*}+\frac{\mu_a+\gamma_a}{\gamma_a}\left(I_a(t+\tau_a)-I_a^*-I_a^*\ln\frac{I_a(t+\tau_a)}{I_a^*}\right).$$

By using the Itô's formula, the derivative of W_2 is calculated as follows

$$\begin{aligned}LW_2=&\left(1-\frac{E_a^*}{E_a(t+\tau_a)}\right)\left(\frac{\beta_a e^{-\mu_a\tau_a}S_a I_a}{1+\alpha_1 I_a}-(\mu_a+\gamma_a)E_a(t+\tau_a)\right)+\frac{\mu_a+\gamma_a}{\gamma_a}\\ &\left(1-\frac{I_a^*}{I_a(t+\tau_a)}\right)(\gamma_a E_a(t+\tau_a)-(\mu_a+\delta_a)I_a(t+\tau_a))+\frac{1}{2}\sigma_2^2E_a^*+\frac{\mu_a+\gamma_a}{2\gamma_a}\sigma_3^2 I_a^*\\ =&\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}(S_a-S_a^*)\left(\frac{1+\alpha_1 I_a^*}{I_a^*}\frac{I_a}{1+\alpha_1 I_a}-1\right)+\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\Bigg(\frac{S_a}{S_a^*}\\ &-\frac{1+\alpha_1 I_a^*}{S_a^* I_a^*}\frac{S_a I_a}{1+\alpha_1 I_a}\frac{E_a^*}{E_a(t+\tau_a)}+\frac{1+\alpha_1 I_a^*}{I_a^*}\frac{I_a}{1+\alpha_1 I_a}-\frac{I_a(t+\tau_a)}{I_a^*}\\ &-\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}\Bigg)+\frac{1}{2}\sigma_2^2E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2 I_a^*.\end{aligned}\tag{5.3}$$

Since $x-1-\ln x\geq 0$ for $x>0$, the following estimate can be obtained

$$\begin{aligned}&\frac{1+\alpha_1 I_a^*}{S_a^* I_a^*}\frac{S_a I_a}{1+\alpha_1 I_a}\frac{E_a^*}{E_a(t+\tau_a)}\\ \geq& 1+\ln\left(\frac{1+\alpha_1 I_a^*}{S_a^* I_a^*}\frac{S_a I_a}{1+\alpha_1 I_a}\frac{E_a^*}{E_a(t+\tau_a)}\right)\\ =&1+\ln\frac{S_a}{S_a^*}-\ln\frac{I_a(t+\tau_a)}{I_a^*}+\ln\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-\ln\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}.\end{aligned}\tag{5.4}$$

Substituting (5.4) into (5.3), we can get

$$\begin{aligned}LW_2\leq&\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}(S_a-S_a^*)\left(\frac{1+\alpha_1 I_a^*}{I_a^*}\frac{I_a}{1+\alpha_1 I_a}-1\right)+\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\Big(\frac{S_a}{S_a^*}-1-\ln\frac{S_a}{S_a^*}\\ &+\ln\frac{I_a(t+\tau_a)}{I_a^*}-\ln\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}+\ln\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}+\frac{1+\alpha_1 I_a^*}{I_a^*}\frac{I_a}{1+\alpha_1 I_a}\\ &-\frac{I_a(t+\tau_a)}{I_a^*}-\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}\Big)+\frac{1}{2}\sigma_2^2 E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2 I_a^*\\ =&\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}(S_a-S_a^*)\left(\frac{1+\alpha_1 I_a^*}{I_a^*}\frac{I_a}{1+\alpha_1 I_a}-1\right)+\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\Big[(\frac{S_a}{S_a^*}-\ln\frac{S_a}{S_a^*})\\ &-\left(\frac{I_a(t+\tau_a)}{I_a^*}-\ln\frac{I_a(t+\tau_a)}{I_a^*}\right)+\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-\ln\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}\right)\\ &-\left(\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}-\ln\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}\right)-1\Big]+\frac{1}{2}\sigma_2^2 E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2 I_a^*.\end{aligned}\tag{5.5}$$

Choose $W_3=W_2+\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\int_t^{t+\tau_a}\left(\frac{I_a(s)}{I_a^*}-\ln\frac{I_a(s)}{I_a^*}-1\right)ds$. Therefore, LW_3 can be obtained as follows by using (5.5):

$$\begin{aligned}LW_3\leq&\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}(S_a-S_a^*)\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-1\right)+\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\Big[\left(\frac{S_a}{S_a^*}-\ln\frac{S_a}{S_a^*}\right)\\ &-\left(\frac{I_a(t+\tau_a)}{I_a^*}-\ln\frac{I_a(t+\tau_a)}{I_a^*}\right)+\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-\ln\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}\right)\\ &-\left(\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}-\ln\frac{E_a(t+\tau_a)}{E_a^*}\frac{I_a^*}{I_a(t+\tau_a)}\right)-1\Big]+\frac{1}{2}\sigma_2^2 E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2 I_a^*\\ &+\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\left(\frac{I_a(t+\tau_a)}{I_a^*}-\ln\frac{I_a(t+\tau_a)}{I_a^*}-1\right)-\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\left(\frac{I_a}{I_a^*}-\ln\frac{I_a}{I_a^*}-1\right)\\ \leq&\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}(S_a-S_a^*)\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-1\right)+\frac{\beta_a e^{-\mu_a\tau_a}S_a^* I_a^*}{1+\alpha_1 I_a^*}\Big[\frac{S_a}{S_a^*}+\frac{S_a^*}{S_a}-1-\frac{I_a}{I_a^*}\\ &+\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}+\ln\frac{I_a^*(1+\alpha_1 I_a)}{I_a(1+\alpha_1 I_a^*)}\frac{I_a}{I_a^*}-1\Big]+\frac{1}{2}\sigma_2^2 E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2 I_a^*.\end{aligned}\tag{5.6}$$

Noting that $x - 1 - \ln x \geq 0$ holds for $x > 0$, we also have

$$\begin{aligned}
&-\frac{I_a}{I_a^*}+\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}+\ln\frac{I_a^*(1+\alpha_1 I_a)}{I_a(1+\alpha_1 I_a^*)}\frac{I_a}{I_a^*}\\
\leq&-\frac{I_a}{I_a^*}+\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}+\frac{I_a^*(1+\alpha_1 I_a)}{I_a(1+\alpha_1 I_a^*)}\frac{I_a}{I_a^*}-1\\
\leq&\frac{I_a^*(1+\alpha_1 I_a)}{I_a(1+\alpha_1 I_a^*)}\frac{I_a}{I_a^*}\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}\frac{I_a^*}{I_a}-1\right)\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-1\right)\\
=&\frac{(1+\alpha_1 I_a)(1+\alpha_1 I_a^*)}{I_a^*}\left(\frac{1}{1+\alpha_1 I_a}-\frac{1}{1+\alpha_1 I_a^*}\right)\left(\frac{I_a}{1+\alpha_1 I_a}-\frac{I_a^*}{1+\alpha_1 I_a^*}\right)<0,
\end{aligned} \tag{5.7}$$

substituting (5.7) into (5.6) and using $\frac{S_a}{S_a^*}+\frac{S_a^*}{S_a}-2=\frac{(S_a-S_a^*)^2}{S_aS_a^*}$, we know that

$$LW_3 \leq \frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}(S_a-S_a^*)\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-1\right)+\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}\frac{(S_a-S_a^*)^2}{S_a}+\frac{1}{2}\sigma_2^2E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2I_a^*. \tag{5.8}$$

Let $W_4 = W_1 + \frac{1+\alpha_1 I_a^*}{\beta_a e^{-\mu_a\tau_a}I_a^*}\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)W_3$. Applying the Itô's formula, together with (5.2) and (5.8), derives that

$$\begin{aligned}
LW_4 =&LW_1+\frac{1+\alpha_1 I_a^*}{\beta_a e^{-\mu_a\tau_a}I_a^*}\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)LW_3\\
\leq&-\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\frac{(S_a-S_a^*)^2}{S_a}-\beta_a(S_a-S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a}-\frac{I_a^*}{1+\alpha_1 I_a^*}\right)+\frac{1}{2}\sigma_1^2S_a^*\\
&+\frac{1+\alpha_1 I_a^*}{\beta_a e^{-\mu_a\tau_a}I_a^*}\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\left[\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}(S_a-S_a^*)\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-1\right)\right.\\
&\left.+\frac{\beta_a e^{-\mu_a\tau_a}I_a^*}{1+\alpha_1 I_a^*}\frac{(S_a-S_a^*)^2}{S_a}+\frac{1}{2}\sigma_2^2E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2I_a^*\right]\\
=&\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)(S_a-S_a^*)\left(\frac{I_a(1+\alpha_1 I_a^*)}{I_a^*(1+\alpha_1 I_a)}-1\right)-\beta_a(S_a-S_a^*)\Bigg(\frac{I_a}{1+\alpha_1 I_a}\\
&-\frac{I_a^*}{1+\alpha_1 I_a^*}\Bigg)+\frac{1}{2}\sigma_1^2S_a^*+\frac{1+\alpha_1 I_a^*}{\beta_a e^{-\mu_a\tau_a}I_a^*}\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\left(\frac{1}{2}\sigma_2^2E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2I_a^*\right)\\
=&(S_a-S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a}-\frac{I_a^*}{1+\alpha_1 I_a^*}\right)\left[\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\frac{1+\alpha_1 I_a^*}{I_a^*}-\beta_a\right]+\frac{1}{2}\sigma_1^2S_a^*\\
&+\frac{1+\alpha_1 I_a^*}{\beta_a e^{-\mu_a\tau_a}I_a^*}\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\left(\frac{1}{2}\sigma_2^2E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2I_a^*\right)\\
=&\frac{\mu_a(1+\alpha_1 I_a^*)}{I_a^*}(S_a-S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a}-\frac{I_a^*}{1+\alpha_1 I_a^*}\right)+\frac{1}{2}\sigma_1^2S_a^*\\
&+\frac{1+\alpha_1 I_a^*}{\beta_a e^{-\mu_a\tau_a}I_a^*}\left(\mu_a+\frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\left(\frac{1}{2}\sigma_2^2E_a^*+\frac{1}{2}\frac{\mu_a+\gamma_a}{\gamma_a}\sigma_3^2I_a^*\right).
\end{aligned} \tag{5.9}$$

Choose Lyapunov function W_5 as $W_5 = \frac{(S_a-S_a^*)^2}{2}$, then its derivative is

$$LW_5 =(S_a-S_a^*)\left[\Lambda_a-\mu_aS_a-\frac{\beta_aS_aI_a}{1+\alpha_1 I_a}\right]+\frac{1}{2}\sigma_1^2S_a^2$$

$$
\begin{aligned}
&=(S_a - S_a^*)\left[\mu_a S_a^* - \mu_a S_a + \frac{\beta_a S_a^* I_a^*}{1+\alpha_1 I_a^*} - \frac{\beta_a S_a I_a}{1+\alpha_1 I_a}\right] + \frac{1}{2}\sigma_1^2 S_a^2\\
&= -\mu_a (S_a - S_a^*)^2 - \beta_a S_a^*(S_a - S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a} - \frac{I_a^*}{1+\alpha_1 I_a^*}\right) - \beta_a (S_a - S_a^*)^2 \frac{I_a}{1+\alpha_1 I_a} + \frac{1}{2}\sigma_1^2 S_a^2\\
&\leq -\mu_a (S_a - S_a^*)^2 - \beta_a S_a^*(S_a - S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a} - \frac{I_a^*}{1+\alpha_1 I_a^*}\right) + \sigma_1^2 (S_a - S_a^*)^2 + \sigma_1^2 (S_a^*)^2\\
&= -(\mu_a - \sigma_1^2)(S_a - S_a^*)^2 - \beta_a S_a^*(S_a - S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a} - \frac{I_a^*}{1+\alpha_1 I_a^*}\right) + \sigma_1^2 (S_a^*)^2.
\end{aligned}
$$

Let $\bar{W} = W_5 + \frac{\beta_a S_a^* I_a^*}{\mu_a(1+\alpha_1 I_a^*)} W_4$, one can derive that

$$
\begin{aligned}
L\bar{W} \leq& -(\mu_a - \sigma_1^2)(S_a - S_a^*)^2 - \beta_a S_a^*(S_a - S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a} - \frac{I_a^*}{1+\alpha_1 I_a^*}\right) + \sigma_1^2 (S_a^*)^2\\
&+ \frac{\beta_a S_a^* I_a^*}{\mu_a(1+\alpha_1 I_a^*)}\Bigg[\frac{\mu_a(1+\alpha_1 I_a^*)}{I_a^*}(S_a - S_a^*)\left(\frac{I_a}{1+\alpha_1 I_a} - \frac{I_a^*}{1+\alpha_1 I_a^*}\right) + \frac{1}{2}\sigma_1^2 S_a^*\\
&+ \frac{1+\alpha_1 I_a^*}{\beta_a e^{-\mu_a \tau_a} I_a^*}\left(\mu_a + \frac{\beta_a I_a^*}{1+\alpha_1 I_a^*}\right)\left(\frac{1}{2}\sigma_2^2 E_a^* + \frac{1}{2}\frac{\mu_a + \gamma_a}{\gamma_a}\sigma_3^2 I_a^*\right)\Bigg] \qquad (5.10)\\
=& -(\mu_a - \sigma_1^2)(S_a - S_a^*)^2 + \sigma_1^2 (S_a^*)^2 + \frac{\beta_a S_a^* I_a^*}{2\mu_a(1+\alpha_1 I_a^*)}\sigma_1^2 S_a^*\\
&+ \left(e^{\mu_a \tau_a} S_a^* + \frac{\beta_a S_a^* I_a^*}{\mu_a e^{-\mu_a \tau_a}(1+\alpha_1 I_a^*)}\right)\left(\frac{1}{2}\sigma_2^2 E_a^* + \frac{1}{2}\frac{\mu_a + \gamma_a}{\gamma_a}\sigma_3^2 I_a^*\right).
\end{aligned}
$$

Integrating both sides of (5.10) from 0 to t and then taking expectation yields

$$
\begin{aligned}
E\bar{W}(t) - E\bar{W}(0) \leq& -(\mu_a - \sigma_1^2) E\int_0^t (S_a(s) - S_a^*)^2 ds + \Bigg[\sigma_1^2 (S_a^*)^2 + \frac{\beta_a S_a^* I_a^*}{2\mu_a(1+\alpha_1 I_a^*)}\sigma_1^2 S_a^*\\
&+ \left(e^{\mu_a \tau_a} S_a^* + \frac{\beta_a S_a^* I_a^*}{\mu_a e^{-\mu_a \tau_a}(1+\alpha_1 I_a^*)}\right)\left(\frac{1}{2}\sigma_2^2 E_a^* + \frac{1}{2}\frac{\mu_a + \gamma_a}{\gamma_a}\sigma_3^2 I_a^*\right)\Bigg] t.
\end{aligned}
$$

Then, we can get

$$
\limsup_{t\to\infty} \frac{1}{t} E\int_0^t (S_a(s) - S_a^*)^2 ds \leq P_3,
$$

where P_3 is defined in Theorem 3. Defining $W_6 = \frac{1}{2}\left[e^{-\mu_a \tau_a}(S_a - S_a^*) + E_a(t+\tau_a) - E_a^*\right]^2$, the use of Itô's formula yields that

$$
\begin{aligned}
LW_6 =& -\mu_a e^{-2\mu_a \tau_a}(S_a - S_a^*)^2 - (\mu_a + \gamma_a)(E_a(t+\tau_a) - E_a^*)^2 - (2\mu_a + \gamma_a) e^{-\mu_a \tau_a}(S_a - S_a^*)\\
&(E_a(t+\tau_a) - E_a^*) + \frac{1}{2} e^{-2\mu_a \tau_a}\sigma_1^2 S_a^2 + \frac{1}{2}\sigma_2^2 E_a^2(t+\tau_a)\\
\leq& -\mu_a e^{-2\mu_a \tau_a}(S_a - S_a^*)^2 - (\mu_a + \gamma_a)(E_a(t+\tau_a) - E_a^*)^2 + \frac{\mu_a + \gamma_a}{2}(E_a(t+\tau_a) - E_a^*)^2\\
&+ \frac{(2\mu_a + \gamma_a)^2 e^{-2\mu_a \tau_a}}{2(\mu_a + \gamma_a)}(S_a - S_a^*)^2 + e^{-2\mu_a \tau_a}\sigma_1^2 (S_a - S_a^*)^2 + e^{-2\mu_a \tau_a}\sigma_1^2 (S_a^*)^2\\
&+ \sigma_2^2 (E_a(t+\tau_a) - E_a^*)^2 + \sigma_2^2 (E_a^*)^2
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-2\mu_a\tau_a}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)(S_a-S_a^*)^2-\left(\frac{\mu_a+\gamma_a}{2}-\sigma_2^2\right)(E_a(t+\tau_a)-E_a^*)^2\\
&\quad + e^{-2\mu_a\tau_a}\sigma_1^2(S_a^*)^2+\sigma_2^2(E_a^*)^2.
\end{aligned}
$$

Let $W_7 = W_6 + \left(\frac{\mu_a+\gamma_a}{2}-\sigma_2^2\right)\int_t^{t+\tau_a}(E_a(s)-E_a^*)^2ds$ and $W_8 = \frac{1}{2}(I_a-I_a^*)^2$. We have

$$
\begin{aligned}
LW_7 \le & e^{-2\mu_a\tau_a}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)(S_a-S_a^*)^2-\left(\frac{\mu_a+\gamma_a}{2}-\sigma_2^2\right)(E_a-E_a^*)^2\\
&+ e^{-2\mu_a\tau_a}\sigma_1^2(S_a^*)^2+\sigma_2^2(E_a^*)^2,
\end{aligned}
\tag{5.11}
$$

and

$$
\begin{aligned}
LW_8 =& (I_a-I_a^*)(\gamma_aE_a-(\mu_a+\delta_a)I_a)+\frac{1}{2}\sigma_3^2I_a^2\\
=& \gamma_a(E_a-E_a^*)(I_a-I_a^*)-(\mu_a+\delta_a)(I_a-I_a^*)^2+\frac{1}{2}\sigma_3^2I_a^2\\
\le& \frac{\mu_a+\delta_a}{2}(I_a-I_a^*)^2+\frac{\gamma_a^2}{2(\mu_a+\delta_a)}(E_a-E_a^*)^2\\
&-(\mu_a+\delta_a)(I_a-I_a^*)^2+\sigma_3^2(I_a-I_a^*)^2+\sigma_3^2(I_a^*)^2\\
=& \frac{\gamma_a^2}{2(\mu_a+\delta_a)}(E_a-E_a^*)^2-\left(\frac{\mu_a+\delta_a}{2}-\sigma_3^2\right)(I_a-I_a^*)^2+\sigma_3^2(I_a^*)^2.
\end{aligned}
\tag{5.12}
$$

Let $\tilde{W} = W_7 + \frac{e^{-\mu_a\tau_a}}{\mu_a-\sigma_1^2}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)\bar{W}+\frac{(\mu_a+\delta_a)(\mu_a+\gamma_a-2\sigma_2^2)}{2\gamma_a^2}W_8$. Making use of (5.10), (5.11) and (5.12) yields that

$$
\begin{aligned}
L\tilde{W} =& LW_7+\frac{e^{-\mu_a\tau_a}}{\mu_a-\sigma_1^2}\left(\frac{2\mu_a^2+2\mu_a\gamma_a+\gamma_a^2}{2(\mu_a+\gamma_a)}+\sigma_1^2\right)L\bar{W}+\frac{(\mu_a+\delta_a)(\mu_a+\gamma_a-2\sigma_2^2)}{2\gamma_a^2}LW_8\\
\le& -\frac{1}{4}(\mu_a+\gamma_a-2\sigma_2^2)(E_a-E_a^*)^2-\frac{(\mu_a+\delta_a)(\mu_a+\gamma_a-2\sigma_2^2)(\mu_a+\delta_a-2\sigma_3^2)}{4\gamma_a^2}(I_a-I_a^*)^2+L_1.
\end{aligned}
\tag{5.13}
$$

Integrating both sides of (5.13) from 0 to t and then taking expectation yields

$$
\begin{aligned}
E\tilde{W}(t)-E\tilde{W}(0) \le& -\frac{1}{4}(\mu_a+\gamma_a-2\sigma_2^2)E\int_0^t(E_a(s)-E_a^*)^2ds\\
&-\frac{(\mu_a+\delta_a)(\mu_a+\gamma_a-2\sigma_2^2)(\mu_a+\delta_a-2\sigma_3^2)}{4\gamma_a^2}E\int_0^t(I_a(s)-I_a^*)^2ds+L_1t.
\end{aligned}
$$

Therefore, we can obtain

$$
\limsup_{t\to\infty} E\int_0^t[(E_a(s)-E_a^*)^2+(I_a(s)-I_a^*)^2]ds \le \frac{L_1}{L_2} =: P_4,
$$

where L_1, L_2 have been defined in Theorem 3. Taking $U_1 = \frac{1}{2}(S_h - S_h^*)^2$, we have

$$\begin{aligned}
LU_1 =&(S_h - S_h^*)(\Lambda_h - \mu_h S_h - \frac{\beta_h S_h I_a^*}{1+\alpha_2 I_a^*}) + \frac{1}{2}\sigma_4^2 S_h^* \\
=&(S_h - S_h^*)\left[\mu_h S_h^* - \mu_h S_h + \frac{\beta_h S_h^* I_a^*}{1+\alpha_2 I_a^*} - \frac{\beta_h S_h I_a^*}{1+\alpha_2 I_a^*}\right] + \frac{1}{2}\sigma_4^2 S_h^* \\
=&-(\mu_h + \frac{\beta_h I_a^*}{1+\alpha_2 I_a^*})(S_h - S_h^*)^2 + \sigma_4^2(S_h - S_h^*)^2 + \sigma_4^2(S_h^*)^2 \\
\le&-(\mu_h - \sigma_4^2)(S_h - S_h^*)^2 + \sigma_4^2(S_h^*)^2.
\end{aligned} \tag{5.14}$$

Integrating both sides of (5.14) from 0 to t and then taking expectation, we get

$$EU_1(t) - EU_1(0) \le -(\mu_h - \sigma_4^2)E\int_0^t (S_h - S_h^*)^2 ds + \sigma_4^2(S_h^*)^2 t.$$

Therefore, we can obtain

$$\limsup_{t\to\infty} E\int_0^t (S_h - S_h^*)^2 ds \le \frac{\sigma_4^2(S_h^*)^2}{\mu_h - \sigma_4^2}.$$

Let $U_2 = \frac{1}{2}[I_h(t+\tau_h) - I_h^*]^2$, we have

$$\begin{aligned}
LU_2 =&(I_h(t+\tau_h) - I_h^*)\left[\frac{\beta_h S_h I_a^*}{1+\alpha_2 I_a^*} - (\mu_h + \delta_h + \theta_h)I_h(t+\tau_h)\right] + \frac{1}{2}\sigma_5^2 I_h^2(t+\tau_h) \\
=&\frac{\beta_h I_a^*}{1+\alpha_2 I_a^*}(I_h(t+\tau_h) - I_h^*)(S_h - S_h^*) - (\mu_h + \delta_h + \theta_h)(I_h(t+\tau_h) - I_h^*)^2 + \frac{1}{2}\sigma_5^2 I_h^2(t+\tau_h) \\
\le&\frac{\beta_h^2 (I_a^*)^2}{2(1+\alpha_2 I_a^*)^2(\mu_h + \delta_h + \theta_h - \sigma_5^2)}(S_h - S_h^*)^2 - \frac{\mu_h + \delta_h + \theta_h - \sigma_5^2}{2}(I_h(t+\tau_h) - I_h^*)^2 \\
&-(\mu_h + \delta_h + \theta_h)(I_h(t+\tau_h) - I_h^*)^2 + \sigma_5^2(I_h(t+\tau_h) - I_h^*)^2 + \sigma_5^2(I_h^*)^2 \\
=&\frac{\beta_h^2 (I_a^*)^2 (S_h - S_h^*)^2}{2(1+\alpha_2 I_a^*)^2(\mu_h + \delta_h + \theta_h - \sigma_5^2)} - \frac{\mu_h + \delta_h + \theta_h - \sigma_5^2}{2}(I_h(t+\tau_h) - I_h^*)^2 + \sigma_5^2(I_h^*)^2.
\end{aligned}$$

Let $\overline{U} = \frac{\beta_h^2 (I_a^*)^2}{2(\mu_h - \sigma_4^2)(1+\alpha_2 I_a^*)^2(\mu_h + \delta_h + \theta_h - \sigma_5^2)} U_1 + U_2$, then

$$L\overline{U} = -\frac{\mu_h + \delta_h + \theta_h - \sigma_5^2}{2}(I_h(t+\tau_h) - I_h^*)^2 + \frac{\beta_h^2 (I_a^*)^2 \sigma_4^2 (S_h^*)^2}{2(\mu_h - \sigma_4^2)(1+\alpha_2 I_a^*)^2(\mu_h + \delta_h + \theta_h - \sigma_5^2)} + \sigma_5^2(I_h^*)^2.$$

Let $U_3 = \frac{\mu_h + \delta_h + \theta_h - \sigma_5^2}{2}\int_t^{t+\tau_h}(I_h(s) - I_h^*)^2 ds$, we obtain

$$LU_3 = \frac{\mu_h + \delta_h + \theta_h - \sigma_5^2}{2}\left[(I_h(t+\tau_h) - I_h^*)^2 + (I_h - I_h^*)^2\right].$$

Let $\widetilde{U} = \overline{U} + U_3$, then,

$$\begin{aligned} L\widetilde{U} =& -\frac{\mu_h+\delta_h+\theta_h-\sigma_5^2}{2}(I_h(t+\tau_h)-I_h^*)^2 + \frac{\beta_h^2(I_a^*)^2\sigma_4^2(S_h^*)^2}{2(\mu_h-\sigma_4^2)(1+\alpha_2 I_a^*)^2(\mu_h+\delta_h+\theta_h-\sigma_5^2)} \\ &+ \sigma_5^2(I_h^*)^2 + \frac{\mu_h+\delta_h+\theta_h-\sigma_5^2}{2}\left[(I_h(t+\tau_h)-I_h^*)^2 + (I_h-I_h^*)^2\right] \\ =& -\frac{\mu_h+\delta_h+\theta_h-\sigma_5^2}{2}(I_h-I_h^*)^2 + \frac{\beta_h^2(I_a^*)^2\sigma_4^2(S_h^*)^2}{2(\mu_h-\sigma_4^2)(1+\alpha_2 I_a^*)^2(\mu_h+\delta_h+\theta_h-\sigma_5^2)} + \sigma_5^2(I_h^*)^2. \end{aligned} \tag{5.15}$$

Integrating both sides of (5.15) from 0 to t and then taking expectation, we have

$$\begin{aligned} E\widetilde{U}(t) - E\widetilde{U}(0) \le & -\frac{\mu_h+\delta_h+\theta_h-\sigma_5^2}{2} E\int_0^t (I_h-I_h^*)^2 ds \\ &+ \frac{\beta_h^2(I_a^*)^2\sigma_4^2(S_h^*)^2}{2(\mu_h-\sigma_4^2)(1+\alpha_2 I_a^*)^2(\mu_h+\delta_h+\theta_h-\sigma_5^2)} t + \sigma_5^2(I_h^*)^2 t. \end{aligned}$$

Therefore, we can obtain

$$\limsup_{t\to\infty} E\int_0^t (I_h-I_h^*)^2 ds \le P_6,$$

where P_6 has been defined in Theorem 3. The proof is completed. □

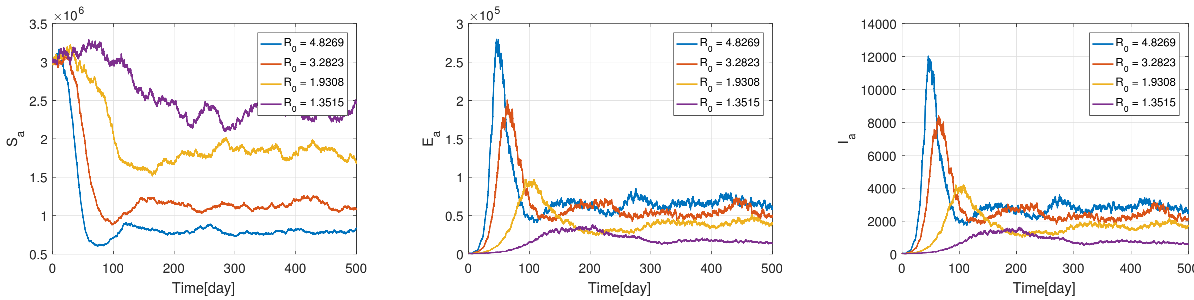

Figure 2. The behavior of avian population under different $\mathscr{R}_0 > 1$.

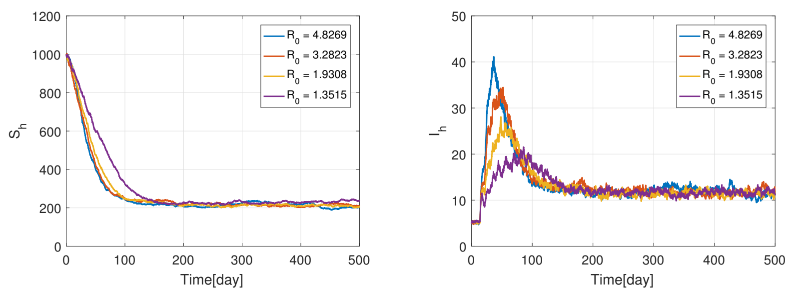

Figure 3. The behavior of human population under different $\mathscr{R}_0 > 1$.

6. Numerical simulation

This section is devoted to illustrating the theoretical results by numerical examples. The parameters of system (2.3) are selected as in Table 2, α_1 and α_2 are varying parameters that is taken value from 0.001 to 0.1, and $\sigma_1 = 0.01$, $\sigma_2 = \sigma_3 = \sigma_5 = 0.04$, $\sigma_4 = 0.008$. The initial conditions of system (2.3) are $S_a(\theta) = 3,000,000, E_a(\theta) = 1,000, I_a(\theta) = 10, S_h(\theta) = 1,000, I_h(\theta) = 5,\ \theta \in [-\tau, 0]$. The Milstein method [25] is used to obtain the discrete form of system (2.3) as follows:

$$\begin{cases} S_a(k+1) = S_a(k) + \left(\Lambda_a - \mu_a S_a(k) - \dfrac{\beta_a S_a(k) I_a(k)}{1+\alpha_1 I_a(k)}\right)\Delta t + \sigma_1 S_a(k)\sqrt{\Delta t}\xi_1(k) \\ \qquad +\dfrac{1}{2}\sigma_1^2 S_a(k)(\xi_1^2(k)-1)\Delta t, \\ E_a(k+1) = E_a(k) + \left(\dfrac{\beta_a e^{-\mu_a\tau_a} S_a(k-\frac{\tau_a}{\Delta t}) I_a(k-\frac{\tau_a}{\Delta t})}{1+\alpha_1 I_a(k-\frac{\tau_a}{\Delta t})} - (\mu_a+\gamma_a)E_a(k)\right)\Delta t \\ \qquad +\sigma_2 E_a(k)\sqrt{\Delta t}\xi_2(k) + \dfrac{1}{2}\sigma_2^2 E_a(k)(\xi_2^2(k)-1)\Delta t, \\ I_a(k+1) = I_a(k) + (\gamma_a E_a(k) - (\mu_a+\delta_a)I_a(k))\,\Delta t + \sigma_3 I_a(k)\sqrt{\Delta t}\xi_3(k) + \dfrac{1}{2}\sigma_3^2 I_a(k)(\xi_3^2(k)-1)\Delta t, \\ S_h(k+1) = S_h(k) + \left(\Lambda_h - \mu_h S_h(k) - \dfrac{\beta_h S_h(k) I_a(k)}{1+\alpha_2 I_a(k)}\right)\Delta t + \sigma_4 S_h(k)\sqrt{\Delta t}\xi_4(k) \\ \qquad +\dfrac{1}{2}\sigma_4^2 S_h(k)(\xi_4^2(k)-1)\Delta t, \\ I_h(k+1) = I_h(k) + \left(\dfrac{\beta_h e^{-\mu_h\tau_h} S_h(k-\frac{\tau_h}{\Delta t}) I_a(k-\frac{\tau_h}{\Delta t})}{1+\alpha_2 I_a(k-\frac{\tau_h}{\Delta t})} - (\mu_h+\delta_h+\theta_h)I_h(k)\right)\Delta t \\ \qquad +\sigma_5 I_h(k)\sqrt{\Delta t}\xi_5(k) + \dfrac{1}{2}\sigma_5^2 I_h(k)(\xi_5^2(k)-1)\Delta t, \end{cases} \tag{6.1}$$

where $\xi_i(k) \sim N(0,1)(i = 1, \cdots, 5; k = 1, 2, \cdots)$ are independent Gaussian random variables. Initially, we study the effect of $\mathscr{R}_0$, which, by Theorems 2 and 3, can govern the asymptotic behavior.

Table 2. Parameter values used in numerical simulations for model (2.3).

Parameter	Value	Source of data
Λ_a	30000	Assumed
Λ_h	$\mu_h \times 1000$	Assumed
β_a	$(0.5—12.5) \times 10^{-6}\text{day}^{-1}$	[10]
β_h	3×10^{-4}	[10]
μ_a	$1/100\text{day}^{-1}$	[10]
μ_h	$200/(70 \times 365)\text{day}^{-1}$	Assumed
δ_a	5day^{-1}	[10]
δ_h	0.03day^{-1}	[10, 11]
γ_a	0.3day^{-1}	[11]
θ_h	0.16day^{-1}	[11]
τ_a	7 day	Assumed
τ_h	14 day	Assumed

Example 1. Effect of basic reproduction number $\mathscr{R}_0$.

Choose different β_a such that $\mathscr{R}_0$ take different values, which are shown in Table 3. Since $\sigma_1^2 = 10^{-4} < \mu_a = 10^{-2}, \sigma_2^2 = 0.0016 < \frac{1}{2}(\mu_a + \gamma_a) = 0.155, \sigma_3^2 = 0.0016 < \frac{1}{2}(\mu_a + \delta_a) = 5.01, \sigma_4^2 = 0.000064 < \mu_h = 0.0078, \sigma_5^2 = 0.0016 < \mu_h + \delta_h + \theta_h = 0.1978$, the condition (i) of Theorem 3 is satisfied. From Table 3, we see that for each $\mathscr{R}_0$, the inequality $P_m < d_E$ holds, which means the condition (ii) of Theorem 3 is also satisfied. Thus, all the conclusions of Theorem 3 hold. It follows from Table 3 that the change of $\mathscr{R}_0$ can result in different values of E^*, which also illustrate the value of E^* is related to $\mathscr{R}_0$. By the discrete form of system (2.3), the numerical results under different $\mathscr{R}_0$ are presented by Figures 2 and 3 when $\mathscr{R}_0 > 1$, which show that the solution of system (2.3) goes around the endemic equilibrium E^*. The effectiveness of Theorem 3 is also indicated by these two figures. In addition, we can see from Figures 2 and 3 and Table 3 that the number of infected poultry and humans will reduce with the decrease of $\mathscr{R}_0$. On the other hand, in order to explore if the results of Theorem 3 hold, we enhance the intensity of perturbation as $\sigma = (\sigma_1, \cdots, \sigma_5) = (0.02, 0.08, 0.08, 0.016, 0.08)$ (Case I: condition (i) of Theorem 3 is satisfied but condition (ii) is not satisfied), $\sigma = (0.06, 0.24, 0.24, 0.048, 0.24)$ (Case II: Both conditions (i) and (ii) are not satisfied) and $\sigma = (0.10, 0.40, 0.40, 0.080, 0.40)$ (Case III: Both conditions (i) and (ii) are not satisfied). The simulation results are presented in Figure 4, which are obtained by computing the average of 800 simulations. The equilibrium of corresponding deterministic model is $E^* = (2.3931 \times 10^6, 1.8254 \times 10^4, 0.7812 \times 10^3, 227.3975, 11.5855)$. From Figure 4, we see that the curves will move away from the equilibrium point E^* with the increasing of intensity of perturbation, which violate the conclusions of Theorem 3.

Table 3. Value of $E^*(S_a^*, E_a^*, I_a^*, S_h^*, I_h^*)$ under different $\mathscr{R}_0$.

$\mathscr{R}_0$	$S_a^*(\times 10^6)$	$E_a^*(\times 10^4)$	$I_a^*(\times 10^3)$	S_h^*	I_h^*	$P_m\ (\times 10^5)^\sharp$	$d_E\ (\times 10^5)^\sharp$
4.8269	0.7977	6.6239	2.8348	212.6795	11.8062	2.5226	22.033
3.2823	1.1336	5.6137	2.4024	213.7049	11.7908	2.7378	18.673
1.9308	1.7948	3.6249	1.5513	217.3720	11.7359	3.2294	12.057
1.3515	2.3931	1.8254	0.7812	227.3975	11.5855	3.7266	6.072

$^\sharp$ $P_m = \max(\sqrt{P_3}, \sqrt{P_4}, \sqrt{P_5}, \sqrt{P_6})$, $d_E = d(E^*, E^0)$.

According to the values of $\sigma_1, \cdots, \sigma_5$ and the parameters values in Table 2, we easily verify the conditions of Theorem 2 are satisfied. Therefore, from Theorem 2 we know that the solution of system (2.3) will go around the disease-free equilibrium E^0 when $\mathscr{R}_0 < 1$. The numerical simulation results of $\mathscr{R}_0$ are presented in Figures 5 and 6. These figures show E_a, I_a and I_h all go to zero when $\mathscr{R}_0 < 1$, which illustrate the effectiveness of the theoretical results in Theorem 2. Meanwhile, Figures 5 and 6 also show that the rate of E_a, I_a and I_h converges to zero is increasing with the decrease of $\mathscr{R}_0$. The conditions of Theorem 2 are only a sufficient ones, so we want to know whether the conclusions of Theorem 2 hold when the intensity of perturbation increase such that these conditions are not satisfied. Thus, we choose $\sigma = (\sigma_1, \cdots, \sigma_5) = (0.02, 0.08, 0.08, 0.016, 0.08)$ (Case I), $\sigma = (0.06, 0.24, 0.24, 0.048, 0.24)$ (Case II) and $\sigma = (0.12, 0.48, 0.48, 0.096, 0.48)$ (Case III), and the simulation results are presented in Figure 7. Figure 7 shows E_a, I_a and I_h converge to zero for each cases, so the results of Theorem 2 also hold.

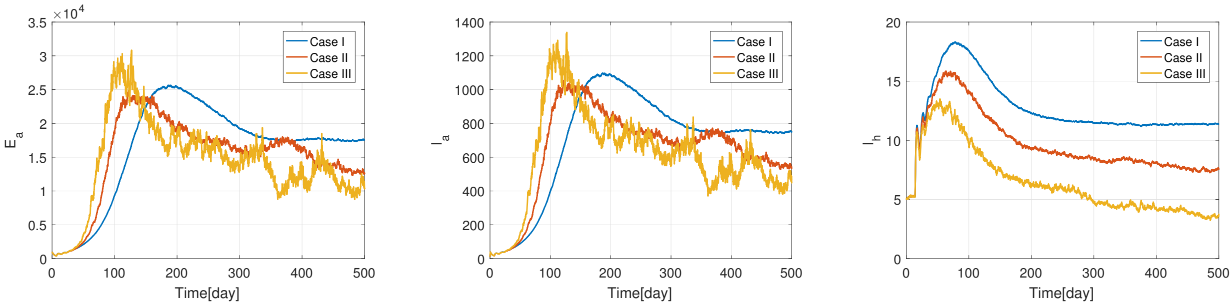

Figure 4. The trajectories of E_a, I_a and I_h for large perturbation when $\mathscr{R}_0 = 1.3515 > 1$.

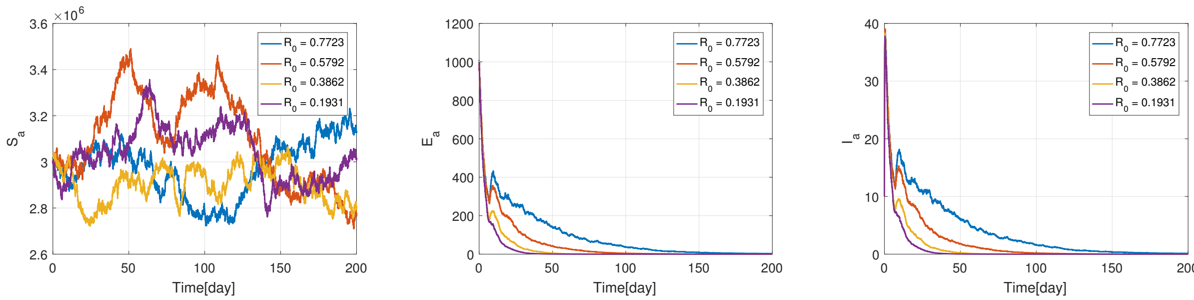

Figure 5. The behavior of infected avian population under different $\mathscr{R}_0 < 1$.

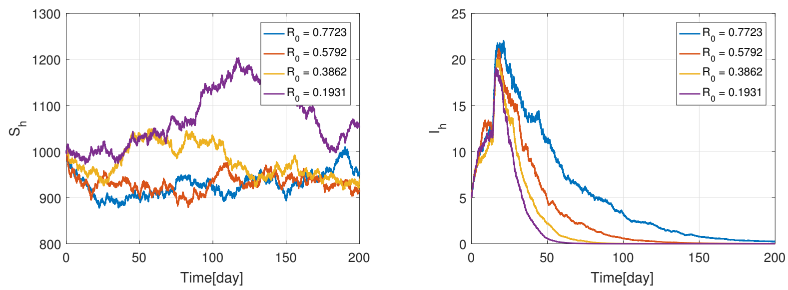

Figure 6. The behavior of infected human population under different $\mathscr{R}_0 < 1$.

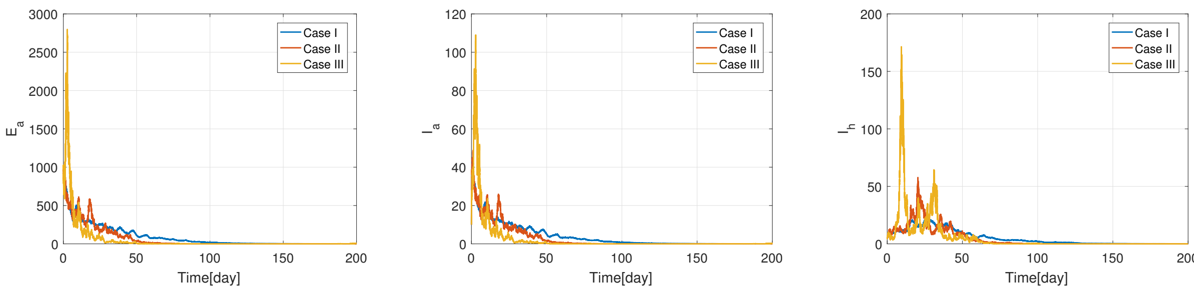

Figure 7. The trajectories of E_a, I_a and I_h for large perturbation when $\mathscr{R}_0 = 0.7723 < 1$.

Example 2. Effect of time delays τ_a and τ_h.

In order to study the effect of time delays, we consider the average peak values of E_a, I_a and I_h, and the time of reaching average peak values by 300 simulation runs. The simulation results are shown in Figures 8 and 9. It follows from Figure 8 that the increase of time delay τ_a or τ_h can reduce the peak value of both infected poultry and human population. Meanwhile, from Figure 9, we know that the large time delay also lead to the delay of reaching peak value. Thus, we may conclude that time delays have significate influence for the spread of avian influenza. According to the practical meaning of τ_a and τ_h, related department can adopt some measures to increase the spread delay to suppress the outbreak of influenza, such as isolation. In addition, the adopting of those control measures will win time for taking drug control.

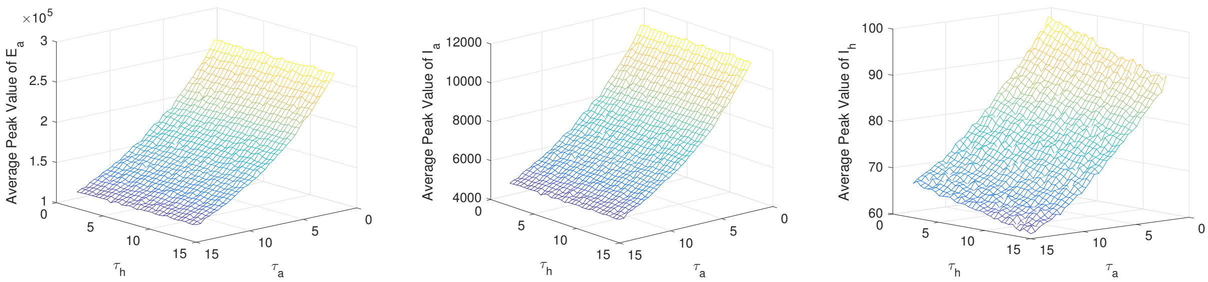

Figure 8. The average peak values of E_a, I_a and I_h under different τ_a and τ_h by 300 simulation runs.

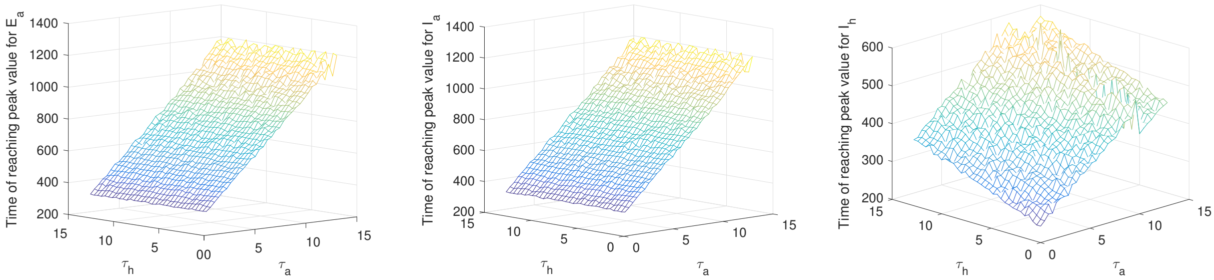

Figure 9. The time of reaching average peak values of E_a, I_a and I_h by 300 simulation runs.

Example 3. Effect of saturation constants α_1 and α_2.

According to the analysis of Introduction, the saturation constants α_1 and α_2 are important parameters for avian influenza. We thus explore the effects of α_1 and α_2 in this example. In order to explore the effect of α_1 under fixed α_2, we run 1000 simulations and take their average values. The results are shown in Figure 10. It follows from Figure 10 that α_1 can influence the rate of convergence to the equilibria of the poultry population, while it can not significantly influence the rate of convergence to the equilibria of the human population. In addition, we study the influence of α_2 under fixed α_1. The simulation results are presented in Figure 11, which implies that α_2 can not change the rate of convergence to the equilibria of the poultry population. Figure 11 also means that α_2 can not increase the rate of convergence to the equilibria of the human population, but it can evidently reduce the peak value of $I_h(t)$. In summary, α_1 and α_2 have evidently influence to the spreading of avian influenza among both avian and human population.

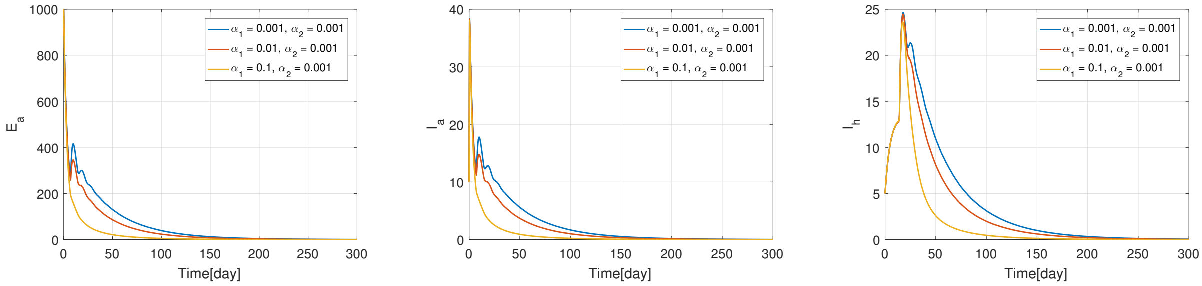

Figure 10. The effect of α_1 and α_2 by 1000 simulation runs (for fixed α_2).

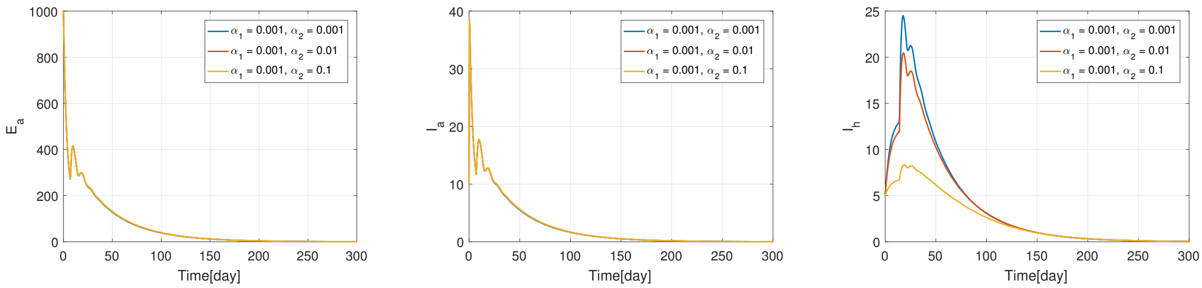

Figure 11. The effect of α_1 and α_2 by 1000 simulation runs (for fixed α_1).

7. Concluding remarks

In this paper, we establish a stochastic delayed avian influenza model with saturated incidence rate. To begin with, we investigate the existence and uniqueness of the global positive solution to the system (2.3) with any positive initial value (2.4). Since there is no equilibrium point in the system (2.3) at this time, thus, the asymptotic behaviors of the disease-free equilibrium and the endemic equilibrium are given by constructing some suitable Lyapunov functions and applying the Young's inequality and Hölder's inequality. Theorem 2 shows that if $\mathscr{R}_0 < 1$, then the solution of system (2.3) is going around E^0 while from Theorem 3, we obtain that if $\mathscr{R}_0 > 1$, then the solution of system (2.3) is going around E^*. Finally, some numerical examples are given to illustrate the accuracy of the theoretical results.

There are some interesting issues deserve further investigations. On the one hand, we can formulate some more realistic but complex avian influenza models, such as considering the effects of Lévy jumps or impulsive perturbations on system (2.3). On the other hand, the coefficients in our model studied in this paper are all constants. If the coefficients are with Markov switching, how will the properties change? We leave these investigations as our future work.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (11661064), Ningxia Natural Science Foundation Project (2019AAC03069) and the Funds for Improving the International Education Capacity of Ningxia University (030900001921).

Conflict of interest

The authors declare that they have no conflict of interest.

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