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Research article

Global boundedness and stability for a chemotaxis model of Boló's concentric sclerosis

Xiaoli Hu, Shengmao Fu*

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: fusm@nwnu.edu.cn; Tel: +8613893617107.

Abstract: Baló's concentric sclerosis (BCS) is considered a variant of inflammatory demyelinating disease closely related to multiple sclerosis characterized by a discrete concentrically layered lesion in the cerebal white matter. Khonsari and Calvez (Plos ONE. 2(2007)) proposed a parabolic-elliptic-ODE chemotaxis model for BCS which describes the evolution of the densities of activated macrophages, cytokine and apoptotic oligodendrocytes. Because "classically activated" M1 microglia can produce cytotoxicity, we introduce a linear production term from the activated microglia in the ODE for pro-inflammatory cytotoxic. For the new BCS chemotaxis model, we first investigate the uniform boundedness and global existence of classical solutions, and then get a range of the chemosensitive rate χ where the unique positive equilibrium point is exponentially asymptotically stable.

Keywords: Baló's concentric sclerosis; chemotaxis model; uniform boundedness; asymptotic behavior

1. Introduction

Baló's concentric sclerosis (BCS) was first described by Marburg [1] in 1906, but became more widely known until 1928 when the Hungarian neuropathologist Josef Baló published a report of a 23-year-old student with right hemiparesis, aphasia, and papilledema, who at autopsy had several lesions of the cerebral white matter, with an unusual concentric pattern of demyelination [2]. Traditionally, BCS is often regarded as a rare variant of multiple sclerosis (MS). Clinically, BCS is most often characterized by an acute onset with steady progression to major disability and death with months, thus resembling Marburg's acute MS [3, 4]. Its pathological hallmarks are oligodendrocyte loss and large demyelinated lesions characterized by the annual ring-like alternating pattern of demyelinating and myelin-preserved regions. In [5], the authors found that tissue preconditioning might explain why Baló lesions develop a concentric pattern. According to the tissue preconditioning theory and the analogies between Baló's sclerosis and the Liesegang periodic precipitation

phenomenon, Khonsari and Calvez [6] established the following chemotaxis model

$$\begin{aligned}
\tilde{u}_\tau = & \underbrace{D\Delta_X\tilde{u}}_{\substack{\text{diffusion of}\\ \text{activated macrophages}}} - \underbrace{\nabla_X\cdot(\tilde{\chi}\tilde{u}(\bar{u}-\tilde{u})\nabla\tilde{v})}_{\substack{\text{chemoattractant attracts}\\ \text{surrounding activated macrophages}}} + \underbrace{\mu\tilde{u}(\bar{u}-\tilde{u})}_{\text{production of activated macrophages}},\\
\underbrace{-\tilde{\epsilon}\Delta_X\tilde{v}}_{\text{diffusion of chemoattractant}} = & \underbrace{-\tilde{\alpha}\tilde{v}+\tilde{\beta}\tilde{w}}_{\text{degradation\production of chemoattractant}},\\
\tilde{w}_\tau = & \underbrace{\kappa\frac{\tilde{u}}{\bar{u}+\tilde{u}}\tilde{u}(\bar{w}-\tilde{w})}_{\text{destruction of oligodendrocytes}},
\end{aligned} \tag{1.1}$$

where $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are, respectively, the density of activated macrophages, the concentration of chemoattractants and density of destroyed oligodendrocytes. $\bar{u}$ and $\bar{w}$ represent the characteristic densities of macrophages and oligodendrocytes respectively.

By numerical simulation, the authors in [6, 7] indicated that model (1.1) only produces heterogeneous concentric demyelination and homogeneous demyelinated plaques as χ value gradually increases. In addition to the chemoattractant produced by destroyed oligodendrocytes, "classically activated" M1 microglia also can release cytotoxicity [8]. Therefore we introduce a linear production term into the second equation of model (1.1), and establish the following BCS chemotaxis model with linear production term

$$\begin{cases}
\tilde{u}_\tau = D\Delta_X\tilde{u} - \nabla_X\cdot(\tilde{\chi}\tilde{u}(\bar{u}-\tilde{u})\nabla\tilde{v}) + \mu\tilde{u}(\bar{u}-\tilde{u}),\\
-\tilde{\epsilon}\Delta_X\tilde{v} + \tilde{\alpha}\tilde{v} = \tilde{\beta}\tilde{w} + \tilde{\gamma}\tilde{u},\\
\tilde{w}_\tau = \kappa\dfrac{\tilde{u}}{\bar{u}+\tilde{u}}\tilde{u}(\bar{w}-\tilde{w}).
\end{cases} \tag{1.2}$$

Before going to details, let us simplify model (1.2) with the following scaling

$$\begin{aligned}
&u = \tfrac{\tilde{u}}{\bar{u}}, \quad v = \tfrac{\mu\bar{u}\tilde{\epsilon}}{D}\tilde{v}, \quad w = \tfrac{\tilde{w}}{\bar{w}}, \quad t = \mu\bar{u}\tau, \quad x = \sqrt{\tfrac{\mu\bar{u}}{D}}X,\\
&\chi = \tfrac{\tilde{\chi}}{\tilde{\epsilon}\mu}, \quad \alpha = \tfrac{D\tilde{\alpha}}{\tilde{\epsilon}\mu\bar{u}}, \quad \beta = \tilde{\beta}\bar{w}, \quad \gamma = \tilde{\gamma}\bar{u}, \quad \delta = \tfrac{\kappa}{\mu},
\end{aligned}$$

then model (1.2) takes the form

$$\begin{cases}
u_t = \Delta u - \nabla\cdot(\chi u(1-u)\nabla v) + u(1-u), & x\in\Omega, t>0,\\
-\Delta v + \alpha v = \beta w + \gamma u, & x\in\Omega, t>0,\\
w_t = \delta\dfrac{u}{1+u}u(1-w), & x\in\Omega, t>0,\\
\partial_\eta u = \partial_\eta v = 0, & x\in\partial\Omega,\ t>0,\\
u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x\in\Omega,
\end{cases} \tag{1.3}$$

where $\Omega\subset\mathbb{R}^n$ ($n\geq 1$) is a smooth bounded domain, η is the outward normal vector to $\partial\Omega$, $\partial_\eta = \partial/\partial\eta$, δ balances the speed of the front and the intensity of the macrophages in damaging the myelin. The parameters χ, α and δ are positive constants as well as β, γ are nonnegative constants.

If $\delta = 0$, then model (1.3) is a parabolic-elliptic chemotaxis system with volume-filling effect and logistic source. In order to be more line with biologically realistic mechanisms, Hillen and Painter [9, 10] considered the finite size of individual cells-"volume-filling" and derived volume-filling models

$$\begin{cases}
u_t = \nabla\cdot(D_u(q(u)-q'(u)u)\nabla u - q(u)u\chi(v)\nabla v) + f(u,v),\\
v_t = D_v\Delta v + g(u,v).
\end{cases} \tag{1.4}$$

$q(u)$ is the probability of the cell finding space at its neighbouring location. It is also called the squeezing probability, which reflects the elastic properties of cells. For the linear choice of $q(u) = 1 - u$, global existence of solutions to model (1.4) in any space dimension are investigated in [9]. Wang and Thomas [11] established the global existence of classical solutions and given necessary and sufficient conditions for spatial pattern formation to a generalized volume-filling chemotaxis model. For a chemotaxis system with generalized volume-filling effect and logistic source, the global boundedness and finite time blow-up of solutions are obtained in [12]. Furthermore, the pattern formation of the volume-filling chemotaxis systems with logistic source and both linear diffusion and nonlinear diffusion are shown in [13–15] by the weakly nonlinear analysis. For parabolic-elliptic Keller-Segel volume-filling chemotaxis model with linear squeezing probability, asymptotic behavior of solutions is studied both in the whole space $\mathbb{R}^n$ [16] and on bounded domains [17]. Moreover, the boundedness and singularity formation in parabolic-elliptic Keller-Segel volume-filling chemotaxis model with nonlinear squeezing probability are discussed in [18, 19].

Very recently, we [20] investigated the uniform boundedness and global asymptotic stability for the following chemotaxis model of multiple sclerosis

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(u)\nabla v) + u(1-u), \ \ \chi(u) = \chi\dfrac{u}{1+u}, & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - \beta v + \alpha w + \gamma u, & x \in \Omega, t > 0, \\ w_t = \delta\dfrac{u}{1+u}u(1-w), & x \in \Omega, t > 0, \end{cases}$$

subject to the homogeneous Neumann boundary conditions.

In this paper, we are first devoted to studying the local existence and uniform boundedness of the unique classical solution to system (1.3) by using Neumann heat semigroup arguments, Banach fixed point theorem, parabolic Schauder estimate and elliptic regularity theory. Then we discuss that exponential asymptotic stability of the positive equilibrium point to system (1.3) by constructing Lyapunov function.

Although, in the pathological mechanism of BCS, the initial data in model (1.3) satisfy $0 < u_0(x) \leq 1, w_0(x) = 0$, we mathematically assume that

$$\begin{cases} u_0(x) \in C^0(\bar{\Omega}) \text{ with } 0 \leq, \not\equiv u_0(x) \leq 1 \text{ in } \Omega, \\ w_0(x) \in C^{2+\nu}(\bar{\Omega}) \text{ with } 0 < \nu < 1 \text{ and } 0 \leq w_0(x) \leq 1 \text{ in } \Omega. \end{cases} \tag{1.5}$$

It is because the condition (1.5) implies $u(x, t_0) > 0$ for any $t_0 > 0$ by the strong maximum principle.

The following theorems give the main results of this paper.

Theorem 1.1. *Assume that the initial data $(u_0(x), w_0(x))$ satisfy the condition (1.5). Then model (1.3) possesses a unique global solution $(u(x,t), v(x,t), w(x,t))$ satisfying*

$$\begin{aligned} &u(x,t) \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ &v(x,t) \in C^0((0,\infty), C^2(\bar{\Omega})), \\ &w(x,t) \in C^{2,1}(\bar{\Omega} \times [0,\infty)), \end{aligned} \tag{1.6}$$

and

$$0 < u(x,t) \leq 1, \ \ 0 \leq v(x,t) \leq \frac{\beta+\gamma}{\alpha}, \ \ w_0(x) \leq w(x,t) \leq 1, \ \ \text{in } \bar{\Omega} \times (0,\infty).$$

Moreover, there exist a $\nu \in (0,1)$ *and* $M > 0$ *such that*

$$\|u\|_{C^{2+\nu,1+\nu/2}(\bar{\Omega}\times[1,\infty))} + \|v\|_{C^0([1,\infty),C^{2+\nu}(\bar{\Omega}))} + \|w\|_{C^{\nu,1+\nu/2}(\bar{\Omega}\times[1,\infty))} \le M. \tag{1.7}$$

Theorem 1.2. *Assume that* $\beta \ge 0,\ \gamma \ge 0,\ \beta+\gamma > 0$ *and*

$$\chi < \begin{cases} \min\left\{\frac{2\sqrt{2\alpha}}{\beta}, \frac{2\sqrt{2\alpha}}{\gamma}\right\}, & \beta > 0, \gamma > 0, \\ \frac{2\sqrt{2\alpha}}{\beta}, & \beta > 0, \gamma = 0, \\ \frac{2\sqrt{2\alpha}}{\gamma}, & \beta = 0, \gamma > 0. \end{cases} \tag{1.8}$$

Let (u,v,w) *be a positive classical solution of the problem (1.3), (1.5). Then*

$$\|u(\cdot,t)-u^*\|_{L^\infty(\Omega)} + \|v(\cdot,t)-v^*\|_{L^\infty(\Omega)} + \|w(\cdot,t)-w^*\|_{L^\infty(\Omega)} \to 0, \quad as\ t \to \infty. \tag{1.9}$$

Furthermore, there exist positive constants $\lambda = \lambda(\chi,\alpha,\gamma,\delta,n)$ *and* $C = C(|\Omega|,\chi,\alpha,\beta,\gamma,\delta)$ *such that*

$$\|u-u^*\|_{L^\infty(\Omega)} \le Ce^{-\lambda t},\ \|v-v^*\|_{L^\infty(\Omega)} \le Ce^{-\lambda t},\ \|w-w^*\|_{L^\infty(\Omega)} \le Ce^{-\lambda t},\ \ t > 0, \tag{1.10}$$

where $(u^*,v^*,w^*) = (1, \frac{\beta+\gamma}{\alpha}, 1)$ *is the unique positive equilibrium point of the model (1.3).*

The paper is organized as follows. In section 2, we prove the local existence, the boundedness and global existence of a unique classical solution. In section 3, we firstly establish the uniform convergence of the positive global classical solution, then discuss the exponential asymptotic stability of positive equilibrium point in the case of weak chemotactic sensitivity. The paper ends with a brief concluding remarks.

2. Boundedness and global existence

The aim of this section is to develop the existence and boundedness of a global classical solution by employing Neumann heat semigroup arguments, Banach fixed point theorem, parabolic Schauder estimate and elliptic regularity theory.

Proof of Theorem 1.1 (i) **Existence.** For $p \in (1,\infty)$, let A denote the sectorial operator defined by

$$Au := -\Delta u \text{ for } u \in D(A) := \left\{\varphi \in W^{2,p}(\Omega)\Big|\frac{\partial}{\partial\eta}\varphi\Big|_{\partial\Omega} = 0\right\}.$$

$\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω with zero-flux boundary condition. Let $A_1 = -\Delta + \alpha$ and X^l be the domains of fractional powers operator $A^l,\ l \ge 0$. From the Theorem 1.6.1 in [21], we know that for any $p > n$ and $l \in (\frac{n}{2p}, \frac{1}{2})$,

$$\|z\|_{L^\infty(\Omega)} \le C\|A_1^l z\|_{L^p(\Omega)} \text{ for all } z \in X^l. \tag{2.1}$$

We introduce the closed subset

$$S := \left\{u \in X\Big|\|u\|_{L^\infty((0,T);L^\infty(\Omega))} \le R+1\right\}$$

in the space $X := C^0([0,T];C^0(\bar{\Omega}))$, where R is a any positive number satisfying

$$\|u_0(x)\|_{L^\infty(\Omega)} \le R$$

and $T > 0$ will be specified later. Note $F(u) = \frac{u}{1+u}$, we consider an auxiliary problem with $F(u)$ replaced by its extension $\tilde{F}(u)$ defined by

$$\tilde{F}(u) = \begin{cases} F(u)u & \text{if } u \geq 0, \\ -F(-u)(-u) & \text{if } u < 0. \end{cases}$$

Notice that $\tilde{F}(u)$ is a smooth globally Lipshitz function. Given $\hat{u} \in S$, we define $\Psi\hat{u} = u$ by first writing

$$w(x,t) = (w_0(x) - 1)e^{-\delta \int_0^t \tilde{F}(\hat{u})\hat{u}ds} + 1, \quad x \in \Omega, \ t > 0, \tag{2.2}$$

and

$$w_0 \leq w(x,t) \leq 1, \quad x \in \Omega, \ t > 0,$$

then letting v solve

$$\begin{cases} -\Delta v + \alpha v = \beta w + \gamma \hat{u}, & x \in \Omega, \ t \in (0,T), \\ \partial_\eta v = 0, & x \in \partial\Omega, \ t \in (0,T), \end{cases} \tag{2.3}$$

and finally taking u to be the solution of the linear parabolic problem

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\hat{u}(1-\hat{u})\nabla v) + \hat{u}(1-\hat{u}), & x \in \Omega, \ t \in (0,T), \\ \partial_\eta u = 0, & x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Applying Agmon-Douglas-Nirenberg Theorem [22, 23] for the problem (2.3), there exists a constant C such that

$$\begin{aligned} \|v\|_{W_p^2(\Omega)} &\leq C(\beta\|w\|_{L^p(\Omega)} + \gamma\|\hat{u}\|_{L^p(\Omega)}) \\ &\leq C(\beta|\Omega|^{\frac{1}{p}} + \gamma(R+1)) \end{aligned} \tag{2.4}$$

for all $t \in (0,T)$. From a variation-of-constants formula, we define

$$\Psi(\hat{u}) = e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla \cdot (\hat{u}(1-\hat{u})\nabla v(s))\, ds + \int_0^t e^{(t-s)\Delta}\hat{u}(s)(1-\hat{u}(s))ds.$$

First we shall show that for T small enough

$$\|\Psi(\hat{u})\|_{L^\infty((0,T);L^\infty(\Omega))} \leq R+1$$

for any $\hat{u} \in S$. From the maximum principle, we can give

$$\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}, \tag{2.5}$$

and

$$\begin{aligned} \int_0^t \|e^{t\Delta}\hat{u}(s)(1-\hat{u}(s))\|_{L^\infty(\Omega)}ds &\leq \int_0^t \|\hat{u}(s)(1-\hat{u}(s))\|_{L^\infty(\Omega)}ds \\ &\leq (R+1)(R+2)T \end{aligned} \tag{2.6}$$

for all $t \in (0, T)$. We use inequalities (2.1) and (2.4) to estimate

$$\begin{aligned}
&\chi \int_0^t \|e^{(t-s)\Delta}\nabla \cdot (\hat{u}(1-\hat{u})\nabla v(s))\|_{L^\infty(\Omega)} ds \\
\leq & C \int_0^t (t-s)^{-l} \|e^{\frac{t-s}{2}\Delta}\nabla \cdot (\hat{u}(1-\hat{u})\nabla v(s))\|_{L^p(\Omega)} ds \\
\leq & C \int_0^t (t-s)^{-l-\frac{1}{2}} \|(\hat{u}(1-\hat{u})\nabla v(s)\|_{L^p(\Omega)} ds \\
\leq & C T^{\frac{1}{2}-l}(R+1)(R+2)(\beta|\Omega|^{\frac{1}{p}} + \gamma(R+1))
\end{aligned} \tag{2.7}$$

for all $t \in (0, T)$. This estimate is attributed to $T < 1$ and the inequality in [24, Lemma 1.3 iv]

$$\|e^{t\Delta}\nabla z\|_{L^p(\Omega)} \leq C_1(1 + t^{-\frac{1}{2}})e^{-\lambda_1 t}\|z\|_{L^p(\Omega)} \text{ for all } z \in C_c^\infty(\Omega).$$

From inequalities (2.5), (2.6) and (2.7) we can deduce that Ψ maps S into itself for T small enough.

Next we prove that the map Ψ is a contractive on S. For $\hat{u}_1, \hat{u}_2 \in S$, we estimate

$$\begin{aligned}
&\|\Psi(\hat{u}_1) - \Psi(\hat{u}_2)\|_{L^\infty(\Omega)} \\
\leq & \chi \int_0^t (t-s)^{-l-\frac{1}{2}} \| \left[\hat{u}_2(s)(1-\hat{u}_2(s)) - \hat{u}_1(s)(1-\hat{u}_1(s))\right] \nabla v_2(s)\|_{L^p(\Omega)} ds \\
& + \chi \int_0^t \|\hat{u}_1(s)(1-\hat{u}_1(s))(\nabla v_1(s) - \nabla v_2(s))\|_{L^p(\Omega)} ds \\
& + \int_0^t \|e^{(t-s)\Delta}[\hat{u}_1(s)(1-\hat{u}_1(s)) - \hat{u}_2(s)(1-\hat{u}_2(s))]\|_{L^\infty(\Omega)} ds \\
\leq & \chi \int_0^t (t-s)^{-l-\frac{1}{2}}(2R+1)\|\hat{u}_1(s) - \hat{u}_2(s)\|_X \|\nabla v_2(s)\|_{L^p(\Omega)} ds \\
& + \chi \int_0^t (R+1)(R+2)\left(\beta\|w_1(s) - w_2(s)\|_{L^p(\Omega)} + \gamma\|\hat{u}_1(s) - \hat{u}_2(s)\|_{L^p(\Omega)}\right) ds \\
& + \int_0^t (2R+1)\|\hat{u}_1(s) - \hat{u}_2(s)\|_X ds \\
\leq & \chi \int_0^t (t-s)^{-l-\frac{1}{2}}(2R+1)\|\hat{u}_1(s) - \hat{u}_2(s)\|_X \|\nabla v_2(s)\|_{L^p(\Omega)} ds \\
& + 2\beta\delta\chi \int_0^t (R+1)(R+2)t\|\hat{u}_1(s) - \hat{u}_2(s)\|_{L^p(\Omega)} + \gamma\|\hat{u}_1(s) - \hat{u}_2(s)\|_{L^p(\Omega)} ds \\
& + \int_0^t (2R+1)\|\hat{u}_1(s) - \hat{u}_2(s)\|_X ds \\
\leq & \left(C\chi T^{\frac{1}{2}-l}(2R+1)(\beta|\Omega|^{\frac{1}{p}} + \gamma(R+1)) + 2\beta\delta\chi T(R^2+3R+\gamma+2) + T(2R+1)\right)\|\hat{u}_1(s) - \hat{u}_2(s)\|_X.
\end{aligned}$$

Fixing $T \in (0, 1)$ small enough such that

$$\left(C\chi T^{\frac{1}{2}-l}(2R+1)(\beta|\Omega|^{\frac{1}{p}} + \gamma(R+1)) + 2\beta\delta\chi T(R^2+3R+\gamma+2) + T(2R+1)\right) \leq \frac{1}{2}.$$

It follows from the Banach fixed point theorem that there exists a unique fixed point of Ψ.

(ii) **Regularity.** Since the above of T depends on $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{L^\infty(\Omega)}$ only, it is clear that (u, v, w) can be extended up to some maximal $T_{\max} \in (0, \infty]$. Let $Q_T = \Omega \times (0, T]$ for all $T \in (0, T_{\max})$. From $u \in C^0(\bar{Q}_T)$, we know that $w \in C^{0,1}(\bar{Q}_T)$ by the expression (2.2) and $v \in C^0([0, T], W_p^2(\Omega))$ by Agmon-Douglas-Nirenberg Theorem [22,23]. From parabolic L^p-estimate and the embedding relation $W_p^1(\Omega) \hookrightarrow C^\nu(\bar{\Omega})$, $p > n$, we can get $u \in W_p^{2,1}(Q_T)$. By applying the following embedding relation

$$W_p^{2,1}(Q_T) \hookrightarrow C^{\nu,\nu/2}(\bar{Q}_T),\ p > \frac{n+2}{2}, \tag{2.8}$$

we can derive $u(x, t) \in C^{\nu,\nu/2}(\bar{Q}_T)$ with $0 < \nu \leq 2 - \frac{n+2}{p}$. The conclusion $w \in C^{\nu,1+\nu/2}(\bar{Q}_T)$ can be obtained by substituting $u \in C^{\nu,\nu/2}(\bar{Q}_T)$ into the formulation (2.2). The regularity $u \in C^{2+\nu,1+\nu/2}(\bar{Q}_T)$ can be deduced by using further bootstrap argument and the parabolic Schauder estimate. Similarly, we can get $v \in C^0((0, T), C^{2+\nu}(\bar{\Omega}))$ by using Agmon-Douglas-Nirenberg Theorem [22, 23]. From the regularity of u we have $w \in C^{2+\nu,1+\nu/2}(\bar{Q}_T)$.

Moreover, the maximal principle entails that $0 < u(x, t) \leq 1$, $0 \leq v(x, t) \leq \frac{\beta+\gamma}{\alpha}$. It follows from the positivity of u that $\tilde{F}(u) = F(u)$ and because of the uniqueness of solution we infer the existence of the solution to the original problem.

(iii) **Uniqueness.** Suppose (u_1, v_1, w_1) and (u_2, v_2, w_2) are two deferent solutions of model (1.3) in $\Omega \times [0, T]$. Let $U = u_1 - u_2$, $V = v_1 - v_2$, $W = w_1 - w_2$ for $t \in (0, T)$. Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega U^2 dx + \int_\Omega |\nabla U|^2 dx \\
\leq &\chi \int_\Omega |u_1(1-u_1) - u_2(1-u_2)||\nabla v_1||\nabla U| + u_2(1-u_2)|\nabla V||\nabla U| dx \\
&+ \int_\Omega |u_1(1-u_1) - u_2(1-u_2)||U| dx \\
\leq &\chi \int_\Omega |U||\nabla v_1||\nabla U| + \frac{1}{4}|\nabla V||\nabla U| dx + \int_\Omega |U|^2 dx \\
\leq &\int_\Omega |\nabla U|^2 dx + \frac{\chi^2}{32}\int_\Omega |\nabla V|^2 dx + \frac{\chi^2 K^2 + 2}{2}\int_\Omega |U|^2 dx,
\end{aligned} \tag{2.9}$$

where we have used that $|\nabla v_1| \leq K$ results from $\nabla v_1 \in C^0([0, T], C^0(\bar{\Omega}))$.

Similarly, by Young inequality and $w_0 \leq w_1 \leq 1$, we can estimate

$$\int_\Omega |\nabla V|^2 dx + \frac{\alpha}{2}\int_\Omega |V|^2 dx \leq \frac{\beta^2}{\alpha}\int_\Omega |W|^2 dx + \frac{\gamma^2}{\alpha}\int_\Omega |U|^2 dx, \tag{2.10}$$

and

$$\frac{d}{dt}\int_\Omega W^2 dx \leq \delta \int_\Omega |U|^2 + |W|^2 dx. \tag{2.11}$$

Finally, adding to the inequalities (2.9)–(2.11) yields

$$\frac{d}{dt}\left(\int_\Omega U^2 dx + \int_\Omega W^2 dx\right) \leq C\left(\int_\Omega U^2 dx + \int_\Omega W^2 dx\right) \text{ for all } t \in (0, T).$$

The results $U \equiv 0$, $W \equiv 0$ in $\Omega \times (0, T)$ are obtained by Gronwall's lemma. From the inequality (2.10), we have $V \equiv 0$. Hence $(u_1, v_1, w_1) = (u_2, v_2, w_2)$ in $\Omega \times (0, T)$.

(iv) **Uniform estimates.** We use the Agmon-Douglas-Nirenberg Theorem [22, 23] for the second equation of the model (1.3) to get

$$\|v\|_{C^0([t,t+1],W_p^2(\Omega))} \le C\left(\|u\|_{L^p(\Omega\times[t,t+1])} + \|w\|_{L^p(\Omega\times[t,t+1])}\right) \le C_2 \tag{2.12}$$

for all $t \ge 1$ and C_2 is independent of t. From the embedded relationship $W_p^1(\Omega) \hookrightarrow C^0(\bar{\Omega})$, $p > n$, the parabolic L^p-estimate and the estimation (2.12), we have

$$\|u\|_{W_p^{2,1}(\Omega\times[t,t+1])} \le C_3$$

for all $t \ge 1$. The estimate $\|u\|_{C^{\nu,\frac{\nu}{2}}(\bar{\Omega}\times[t,t+1])} \le C_4$ for all $t \ge 1$ obtained by the embedded relationship (2.8). We can immediately compute $\|w\|_{C^{\nu,1+\frac{\nu}{2}}(\bar{\Omega}\times[t,t+1])} \le C_5$ for all $t \ge 1$ according to the regularity of u and the specific expression of w. Further, bootstrapping argument leads to $\|v\|_{C^0([t,t+1],C^{2+\nu}(\bar{\Omega}))} \le C_6$ and $\|u\|_{C^{2+\nu,1+\frac{\nu}{2}}(\bar{\Omega}\times[t,t+1])} \le C_7$ for all $t \ge 1$. Thus the uniform estimation (1.7) is proved. □

Remark 2.1. Assume the initial data $0 < u_0(x) \le 1$ and $w_0(x) = 0$. Then the BCS model (1.3) has a unique classical solution.

3. Exponential stability of positive equilibrium point

In this section we investigate the global asymptotic stability of the unique positive equilibrium point $(1, \frac{\beta+\gamma}{\alpha}, 1)$ to model (1.3). To this end, we first introduce following auxiliary problem

$$\begin{cases} u_{\epsilon t} = \Delta u_\epsilon - \nabla\cdot(u_\epsilon(1-u_\epsilon)\nabla v_\epsilon) + u_\epsilon(1-u_\epsilon), & x\in\Omega,\ t>0,\\ -\Delta v_\epsilon + \alpha v_\epsilon = \beta w_\epsilon + \gamma u_\epsilon, & x\in\Omega,\ t>0,\\ w_{\epsilon t} = \delta\dfrac{u_\epsilon^2+\epsilon}{1+u_\epsilon}(1-w_\epsilon), & x\in\Omega,\ t>0,\\ \partial_\eta u_\epsilon = \partial_\eta v_\epsilon = 0, & x\in\partial\Omega,\ t>0,\\ u_\epsilon(x,0) = u_0(x),\ w_\epsilon(x,0) = w_0(x), & x\in\Omega. \end{cases} \tag{3.1}$$

By a similar proof of Theorem 1.1, we get that the problem (3.1) has a unique global classical solution $(u_\epsilon, v_\epsilon, w_\epsilon)$, and there exist a $\nu \in (0,1)$ and $M_1 > 0$ which is independent of ϵ such that

$$\|u_\epsilon\|_{C^{2+\nu,1+\nu/2}(\bar{\Omega}\times[1,\infty))} + \|v_\epsilon\|_{C^{2+\nu,1+\nu/2}(\bar{\Omega}\times[1,\infty))} + \|w_\epsilon\|_{C^{\nu,1+\nu/2}(\bar{\Omega}\times[1,\infty))} \le M_1. \tag{3.2}$$

Then, motivated by some ideas from [25,26], we construct a Lyapunov function to study the uniform convergence of homogeneous steady state for the problem (3.1).

Let us give following lemma which is used in the proof of Lemma 3.2.

Lemma 3.1. *Suppose that a nonnegative function f on $(1,\infty)$ is uniformly continuous and $\int_1^\infty f(t)dt < \infty$. Then $f(t) \to 0$ as $t \to \infty$.*

Lemma 3.2. *Assume that the condition (1.8) is satisfied. Then*

$$\|u_\epsilon(\cdot,t) - 1\|_{L^2(\Omega)} + \|v_\epsilon(\cdot,t) - v^*\|_{L^2(\Omega)} + \|w_\epsilon(\cdot,t) - 1\|_{L^2(\Omega)} \to 0,\ \ t\to\infty, \tag{3.3}$$

where $v^* = \frac{\beta+\gamma}{\alpha}$.

Proof We construct a positive function

$$E(t) := \int_\Omega (u_\varepsilon - 1 - \ln u_\epsilon) + \frac{1}{2\delta\epsilon}\int_\Omega (w_\epsilon - 1)^2, \ \ t > 0.$$

From the problem (3.1) and Young's inequality, we can compute

$$\frac{d}{dt}E(t) \le \frac{\chi^2}{4}\int_\Omega |\nabla v_\epsilon|^2 dx - \int_\Omega (u_\epsilon - 1)^2 dx - \int_\Omega (w_\epsilon - 1)^2 dx, \ \ t > 0. \tag{3.4}$$

We multiply the second equations in system (3.1) by $v_\epsilon - v^*$, integrate by parts over Ω and use Young's inequality to obtain

$$\int_\Omega |\nabla v_\epsilon|^2 dx \le \frac{\gamma^2}{2\alpha}\int_\Omega (u_\epsilon - 1)^2 dx + \frac{\beta^2}{2\alpha}\int_\Omega (w_\epsilon - 1)^2 dx, \ \ t > 0, \tag{3.5}$$

and

$$\int_\Omega (v_\epsilon - v^*)^2 dx \le \frac{2\gamma^2}{\alpha^2}\int_\Omega (u_\epsilon - 1)^2 dx + \frac{2\beta^2}{\alpha^2}\int_\Omega (w_\epsilon - 1)^2 dx, \ \ t > 0. \tag{3.6}$$

Substituting inequality (3.5) into inequality (3.4) to get

$$\frac{d}{dt}E(t) \le -C_8\left(\int_\Omega (u_\epsilon - 1)^2 dx + \int_\Omega (w_\epsilon - 1)^2 dx\right), \ \ t > 0,$$

where $C_8 = \min\left\{1 - \frac{\chi^2\beta^2}{8\alpha}, 1 - \frac{\chi^2\gamma^2}{8\alpha}\right\} > 0$.

Let $f(t) := \int_\Omega (u_\epsilon - 1)^2 + (w_\epsilon - 1)^2 dx$. Then

$$\int_1^\infty f(t)dt \le \frac{E(1)}{C_8} < \infty, \ \ t > 1.$$

It follows from the uniform estimation (3.2) and the Arzela-Ascoli theorem that $f(t)$ is uniformly continuous in $(1, \infty)$. Applying Lemma 3.1, we have

$$\int_\Omega (u_\epsilon(\cdot, t) - 1)^2 + (w_\epsilon(\cdot, t) - 1)^2 dx \to 0, \ \ t \to \infty. \tag{3.7}$$

Combining inequality (3.6) and the limit (3.7) to obtain

$$\int_\Omega (v_\epsilon(\cdot, t) - v^*)^2 dx \to 0, \ \ t \to \infty.$$

□

Proof of Theorem 1.2 As we all known, each bounded sequence in $C^{2+\nu, 1+\frac{\nu}{2}}(\bar{\Omega} \times [1, \infty))$ is precompact in $C^{2,1}(\bar{\Omega} \times [1, \infty))$. Hence there exists some subsequence $\{u_{\epsilon_n}\}_{n=1}^\infty$ satisfying $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \|u_{\epsilon_n} - u_*\|_{C^{2,1}(\bar{\Omega}\times[1,\infty))} = 0.$$

Similarly, we can get

$$\lim_{n\to\infty} \|v_{\epsilon_n} - v_*\|_{C^2(\bar{\Omega})} = 0,$$

and

$$\lim_{n\to\infty} \|w_{\epsilon_n} - w_*\|_{C^{0,1}(\bar{\Omega}\times[1,\infty))} = 0.$$

Combining above limiting relations yields that (u_*, v_*, w_*) satisfies model (1.3). The conclusion $(u_*, v_*, w_*) = (u, v, w)$ is directly attributed to the uniqueness of the classical solution of the model (1.3). Furthermore, according to the conclusion, the strong convergence (3.3) and Diagonal line method, we can deduce

$$\|u(\cdot,t) - 1\|_{L^2(\Omega)} + \|v(\cdot,t) - v^*\|_{L^2(\Omega)} + \|w(\cdot,t) - 1\|_{L^2(\Omega)} \to 0, \quad t \to \infty. \tag{3.8}$$

By applying Gagliardo-Nirenberg inequality

$$\|z\|_{L^\infty} \le C\|z\|_{L^2(\Omega)}^{2/(n+2)}\|z\|_{W^{1,\infty}(\Omega)}^{n/(n+2)}, \quad z \in W^{1,\infty}(\Omega), \tag{3.9}$$

comparison principle of ODE and the convergence (3.8), the uniform convergence (1.9) is obtained immediately.

Since $\lim_{t\to\infty} \|u(\cdot,t) - 1\|_{L^\infty(\Omega)} = 0$, so there exists a $t_1 > 0$ such that

$$u(x,t) \ge \frac{1}{2} \text{ for all } x \in \Omega, \quad t > t_1. \tag{3.10}$$

Using the explicit representation formula of w

$$w(x,t) = (w_0(x) - 1)e^{-\delta \int_0^t F(u)uds} + 1, \quad x \in \Omega, \ t > 0$$

and the inequality (3.10), we have

$$\|w(\cdot,t) - 1\|_{L^\infty(\Omega)} \le e^{-\frac{\delta}{6}(t-t_1)}, \quad t > t_1. \tag{3.11}$$

Multiply the first two equations in model (1.3) by $u - 1$ and $v - v^*$, respectively, integrate over Ω and apply Cauchy's inequality, Young's inequality and the inequality (3.10), to find

$$\frac{d}{dt}\int_\Omega (u-1)^2 dx \le \frac{\chi^2}{32}\int_\Omega |\nabla v|^2 dx - \int_\Omega (u-1)^2 dx, \quad t > t_1. \tag{3.12}$$

$$\int_\Omega |\nabla v|^2 dx + \frac{\alpha}{2}\int_\Omega (v - v^*)^2 dx \le \frac{\beta^2}{\alpha}\int_\Omega (w-1)^2 dx + \frac{\gamma^2}{\alpha}\int_\Omega (u-1)^2 dx, \quad t > 0. \tag{3.13}$$

Combining the estimations (3.11)–(3.13) leads us to the estimate

$$\frac{d}{dt}\int_\Omega (u-1)^2 dx \le \left(\frac{\chi^2\gamma^2}{32\alpha} - 1\right)\int_\Omega (u-1)^2 dx + \frac{\chi^2\beta^2}{32\alpha}e^{-\frac{\delta}{3}(t-t_1)}, \quad t > t_1.$$

Let $y(t) = \int_\Omega (u-1)^2 dx$. Then

$$y'(t) \le \left(\frac{\chi^2\gamma^2}{32\alpha} - 1\right)y(t) + \frac{\chi^2\beta^2}{32\alpha}e^{-\frac{\delta}{3}(t-t_1)}, \quad t > t_1.$$

From comparison principle of ODE, we get

$$y(t) \le \left(y(t_1) - \frac{3\chi^2\beta^2}{32\alpha(3-\delta)-\chi^2\gamma^2}\right)e^{-\left(1-\frac{\chi^2\gamma^2}{32\alpha}\right)(t-t_1)} + \frac{3\chi^2\beta^2}{32\alpha(3-\delta)-\chi^2\gamma^2}e^{-\frac{\delta}{3}(t-t_1)}, \quad t > t_1.$$

This yields

$$\int_\Omega (u-1)^2 dx \le C_9 e^{-\lambda_2(t-t_1)}, \quad t > t_1, \tag{3.14}$$

where $\lambda_2 = \min\{1 - \frac{\chi^2\gamma^2}{32\alpha}, \frac{\delta}{3}\}$ and $C_9 = \max\left\{|\Omega| - \frac{3\chi^2\beta^2}{32\alpha(3-\delta)-\chi^2\gamma^2}, \frac{3\chi^2\beta^2}{32\alpha(3-\delta)-\chi^2\gamma^2}\right\}$.

From the inequalities (3.11), (3.13) and (3.14), we derive

$$\int_\Omega \left(v - \frac{\beta+\gamma}{\alpha}\right)^2 dx \le C_{10} e^{-\lambda_2(t-t_1)}, \quad t > t_1, \tag{3.15}$$

where $C_{10} = \max\left\{\frac{2\gamma^2}{\alpha^2}C_9, \frac{2\beta^2}{\alpha^2}\right\}$. By employing the uniform estimation (1.7), the inequalities (3.9), (3.14) and (3.15), the exponential decay estimation (1.10) can be obtained.

The proof is complete. □

4. Concluding remarks

In this paper, we mainly study the uniform boundedness of classical solutions and exponential asymptotic stability of the unique positive equilibrium point to the chemotactic cellular model (1.3) for Baló's concentric sclerosis (BCS). For model (1.1), by numerical simulation, Calveza and Khonsarib in [7] shown that demyelination patterns of concentric rings will occur with increasing of chemotactic sensitivity. By the Theorem 1.1 we know that systems (1.1) and (1.2) are uniformly bounded and dissipative. By the Theorem 1.2 we also find that the constant equilibrium point of model (1.1) is exponentially asymptotically stable if

$$\tilde{\chi} < \frac{2}{\bar{w}\tilde{\beta}}\sqrt{\frac{2D\mu\tilde{\alpha}\tilde{\epsilon}}{\bar{u}}},$$

and the constant equilibrium point of the model (1.2) is exponentially asymptotically stable if

$$\tilde{\chi} < 2\sqrt{\frac{2D\mu\tilde{\alpha}\tilde{\epsilon}}{\bar{u}}}\min\left\{\frac{1}{\bar{w}\tilde{\beta}}, \frac{1}{\bar{u}\tilde{\gamma}}\right\}.$$

According to a pathological viewpoint of BCS, the above stability results mean that if chemoattractive effect is weak, then the destroyed oligodendrocytes form a homogeneous plaque.

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Conflict of interest

We have no conflict of interest in this paper.

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