



Research article

Stability and Hopf bifurcation analysis of a delayed phytoplankton-zooplankton model with Allee effect and linear harvesting

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Abstract: In this article, a delayed phytoplankton-zooplankton system with Allee effect and linear harvesting is proposed, where phytoplankton species protects themselves from zooplankton by producing toxin and taking shelter. First, the existence and stability of the possible equilibria of system are explored. Next, the existence of Hopf bifurcation is investigated when the system has no time delay. What's more, the stability of limit cycle is demonstrated by calculating the first Lyapunov number. Then, the condition that Hopf bifurcation occurs is obtained by taking the time delay describing the maturation period of zooplankton species as a bifurcation parameter. Furthermore, based on the normal form theory and the central manifold theorem, we derive the direction of Hopf bifurcation and the stability of bifurcating periodic solutions. In addition, by regarding the harvesting effort as control variable and employing the Pontryagin's Maximum Principle, the optimal harvesting strategy of the system is obtained. Finally, in order to verify the validity of the theoretical results, some numerical simulations are carried out.

Keywords: phytoplankton-zooplankton model; Allee effect; time delay; Hopf bifurcation

1. Introduction

Blooms are considered to be a rapid and significant increase in marine phytoplankton populations. It is generally believed that zooplankton is the primary predator of blooms. The zooplankton dissolves phytoplankton blooms in the bud by feeding a large amount of phytoplankton, thereby achieving the purpose of preventing and controlling algal blooms. It is clearly stated in an international research plan that the dynamic changes of zooplankton in marine ecosystems control the change in the total amount of primary productivity. This indicates that zooplankton plays a very important role in the production of phytoplankton and the control of existing quantities. In addition, this is also one of the significant reasons that the interaction between phytoplankton and zooplankton has been widely concerned by many scholars [1–6] in recent years.

Due to the importance of the interaction between phytoplankton and zooplankton in marine biological systems, many scholars have proposed a large number of models to describe this interaction. Moreover, they have continued to study this topic for the past few decades and have achieved very significant results [7–13]. Chakraborty et al. [7] proposed a mathematical model incorporating nutrient concentration, toxin producing phytoplankton (TPP), non-toxic phytoplankton (NTP), and toxin concentration and explained that TPP species controls the outbreaks of other NTP species when nutrient-deficient conditions are beneficial for the TPP species to release toxin. Saha and Bandyopadhyay [8] analyzed a toxin producing phytoplankton-zooplankton model in which the release of toxins from phytoplankton species follows a discrete time variation and discussed the basic dynamics of the system. Banerjee and Venturino [9] proposed a phytoplankton-toxic phytoplankton-zooplankton model and found that toxic phytoplankton population does not cause zooplankton population to become extinct. Javidi and Ahmad [10] investigated the dynamics of a time fractional order toxic-phytoplankton-phytoplankton-zooplankton system (TPPZS) and did some numerical simulations to validate the theoretical results. Han and Dai [11] studied the spatiotemporal pattern caused by cross-diffusion of a toxic-phytoplankton-zooplankton model with nonmonotonic functional response and discussed the effect of toxin-producing rate of toxic-phytoplankton (TPP) species and natural mortality rate of zooplankton species on pattern selection. In addition, Han and Dai proposed a spatiotemporal pattern formation and selection driven by nonlinear cross-diffusion of a toxic-phytoplankton-zooplankton model with Allee effect in [12]. Zheng and Sugie [13] studied a three-dimensional system including phytoplankton, zooplankton and fish and gave a sufficient condition to ensure that the equilibrium of this three-dimensional system is globally asymptotically stable. Furthermore, they also proved that the equilibrium is asymptotically stable under relatively weak conditions.

Because the seabed has a variety of sediments that can be used as a refuge for the prey, some phytoplankton populations reduce the risk of being caught by zooplankton through refuge. Therefore, some scholars [14–18] have considered the prey refuge in the predator-prey model in recent years. Kar [14] proposed a prey-predator model incorporating a prey refuge and studied the influence of prey refuge on the prey-predator model. Chen and Chen [15] investigated a predator-prey model with Holling type II functional response incorporating a constant prey refuge and gave the basic dynamical behaviors of the model. Tripathi et al. [16] studied a prey-predator model with reserved area and found that the predator species would exist if the value of prey reserve does not exceed a threshold value, beyond which the predator species would become extinct. Ghosh et al. [17] analyzed the effect of additional food for predator on the dynamical behaviors of a prey-predator model with prey refuge, they suggested that the possibility of extinction of predator species in high prey refuge ecosystems can be eliminated by providing additional food to predator species. Samanta et al. [18] proposed a fractional-order prey-predator model with prey refuge and derived some sufficient conditions to guarantee the global asymptotic stability of predator-extinction equilibrium and co-existing equilibrium.

Li et al. [19] considered the factors that phytoplankton protects themselves from being eaten by zooplankton by releasing toxin and taking refuge, and proposed a toxic phytoplankton-zooplankton

model with refuge

$$\begin{cases} \dot{P}(t) = rP\left(1 - \frac{P}{K}\right) - \frac{\beta_1(P-m)}{a_1 + (P-m)}Z, \\ \dot{Z}(t) = \frac{\beta_2(P-m)}{a_1 + (P-m)}Z - dZ - \frac{\theta P}{a_2 + P}Z, \end{cases} \quad (1.1)$$

where $P(t)$ and $Z(t)$ denote population size of phytoplankton and zooplankton at time t , respectively; r is the intrinsic growth rate; K is the environmental carrying capacity in the absence of zooplankton; β_1, β_2 ($0 < \beta_2 < \beta_1$) and d describe the predation rate, the ratio of biomass conservation and the natural death rate of zooplankton species, respectively; a_1 and a_2 are the half saturation constants; θ represents the rates of toxin production per phytoplankton species; m is the constant refuge capacity of phytoplankton population, then $P - m$ denotes the number of unprotected phytoplankton is captured by zooplankton species. Their results ultimately showed that phytoplankton refuge and toxin have important effects on the occurrence and termination of algal blooms in freshwater lakes.

Actually, the choice of prey species growth function and predator species function response is considered to be the most important element in the prey-predator model. Generally speaking, many modelers choose Logistic growth form as prey species growth function without considering the predator species. However, we all know that the resources in an ecosystem are limited, such as space, food, basic nutrition and so on. Thus the average growth rate becomes a decreasing function of population size as the population size increases gradually. When the number of population reaches the environmental capacity K , the average growth rate decreases to zero; in addition, any population number above the value K will have a negative growth rate. However, there are a lot of evidences that the low population density is the opposite [20–25]. This phenomenon is the so-called Allee effect, which is the positive density dependence of population growth at low density [22, 26]. The main causes of the Allee effect are the lack of a spouse, the reduced vigilance against predators, the adjustment of the environment, the reduction of defenses against predators, and many other reasons [20, 22]. In general, the Allee effect is expressed by an equation of the following form [27]

$$\frac{dX(t)}{dt} = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)}{K_0} - 1\right),$$

where $X(t)$ represents population size at time t ; r and K are the intrinsic growth rate and the environmental carrying capacity, respectively; K_0 is the critical level of phytoplankton population. When the population density is below the critical level, the population growth rate will decline and population will tend to become extinct.

In recent years, more and more people consider time delay into population biological model [28–35] to study its influence on dynamical behavior of system. What's more, it is well known that most countries of the world achieve the economic benefits of natural resource management by restoring and maintaining the ecosystem's health, the productivity and biodiversity and the overall quality of life in a manner that integrates social and economic objectives. Of course, it also satisfies the need for humans to benefit from natural resources. The prey-predator systems with harvesting have been widely studied by a large number of scholars [6, 34–40]. Meng et al. [40] studied a predator-prey system with harvesting prey and disease in prey species and found that the optimal harvesting effort is closely related to the incubation period of the infectious disease, and the maximum value of the optimal harvesting decreases with the increase of the time delay. Furthermore, Meng et al. [34] investigated

a nutrient-plankton model with Holling type IV, delay, and harvesting, their results indicated that the nutrient increases first and then stabilizes as the harvesting time increases, the number of phytoplankton and zooplankton decreases and even stabilizes as the harvesting time increases. Zhang and Zhao [36] considered a diffusive predator-prey system with delays and interval biological parameters and believed that the overfishing can lead to species extinction.

Motivated by the work mentioned-above, we propose a delayed phytoplankton-zooplankton model with Allee effect and linear harvesting. The results of this paper can be seen as a complement to system (1.1). As far as we know, although a large number of scholars have studied the interaction between phytoplankton and zooplankton, system (1.1) including the Allee effect, time delay and linear harvesting has not been studied yet.

The content of this paper is organized as follows. A delayed phytoplankton-zooplankton model with Allee effect and linear harvesting is described in section 2. In this section, we give the boundedness of the model. Section 3 demonstrates the existence and stability of the equilibria of system (2.1). A detailed discussion of the Hopf bifurcation in section 4. We not only give the existence and property of Hopf bifurcation when the system has no time delay, but also give the existence and property of Hopf bifurcation when the system has time delay. In section 5, the optimal policy is derived by using Pontryagin's Maximum Principle. In section 6, some numerical simulations are given for illustrating the theoretical results. The problem ends with a brief concluding remark.

2. Model and its basic properties

In this paper, based on the work of the reference [19], we propose a delayed phytoplankton-zooplankton system with Allee effect and linear harvesting described by

$$\begin{cases} \dot{P}(t) = rP(1 - \frac{P}{K})(\frac{P}{K_0} - 1) - \frac{\beta_1(P-m)Z}{a_1 + (P-m)} - q_1EP, \\ \dot{Z}(t) = \frac{\beta_2(P(t-\tau) - m)Z}{a_1 + (P(t-\tau) - m)} - dZ - \frac{\theta PZ}{a_2 + P} - q_2EZ, \end{cases} \quad (2.1)$$

where K_0 ($0 < K_0 \ll K$) is the critical level of the growth of phytoplankton; q_1 and q_2 are catchability coefficients of the two species; E represents the harvesting effort. The delay τ in system (2.1) can be regarded as the maturation period of zooplankton species. For biological significance, system (2.1) must satisfy the following initial conditions

$$P(\theta) = \vartheta_1(\theta) \geq m, \quad Z(\theta) = \vartheta_2(\theta) \geq 0, \quad \vartheta_1(0) > m, \quad \vartheta_2(0) > 0, \quad \theta \in [-\tau, 0],$$

where $(\vartheta_1(\theta), \vartheta_2(\theta)) \in C([-\tau, 0], R_+^2)$ and $C([-\tau, 0], R_+^2)$ represents the Banach space formed by all continuous functions from $[-\tau, 0]$ to R_+^2 , here $R_+^2 = \{(x_1, x_2) : x_1 > m, x_2 > 0\}$.

When $\tau = 0$, system (2.1) becomes

$$\begin{cases} \dot{P}(t) = rP(1 - \frac{P}{K})(\frac{P}{K_0} - 1) - \frac{\beta_1(P-m)Z}{a_1 + (P-m)} - q_1EP, \\ \dot{Z}(t) = \frac{\beta_2(P-m)Z}{a_1 + (P-m)} - dZ - \frac{\theta PZ}{a_2 + P} - q_2EZ, \end{cases} \quad (2.2)$$

and system (2.2) must satisfy the following initial conditions

$$P(0) > m, \quad Z(0) > 0.$$

As we all know, the boundedness of the model can ensure that the model has good dynamical behaviors. So we have the following result when $\tau = 0$.

Lemma 2.1. *All solutions of system (2.2) with initial conditions $P(0) > m, Z(0) > 0$ that start in $\Theta = \{(P, Z) \mid (m, +\infty) \times (0, +\infty)\} \subset R_+^2$ are uniformly bounded for all $t > 0$.*

Proof. Let $(P(t), Z(t))$ be any solution of system (2.2) with positive initial conditions $P(0) > m, Z(0) > 0$. Notice that there is $\Omega_1 = \{(P, Z) \mid K \geq P(t) > m, Z(t) > 0 \text{ for all } t > 0\}$ for system (2.2). So we have $P(t) \leq K$ for all $t > 0$.

We define $W(P, Z) = cP(t) + Z(t)$, here $c = \frac{\beta_2}{\beta_1}$, thus

$$\begin{aligned} \dot{W}(t) &= c\dot{P}(t) + \dot{Z}(t) \\ &= crP\left(\frac{K + K_0}{KK_0}P - 1 - \frac{P^2}{KK_0}\right) - cq_1EP - dZ - \frac{\theta P}{a_2 + P}Z - q_2EZ \\ &\leq -\left[dZ + \frac{cr(K + K_0)}{KK_0}P\right] + \frac{2cr(K + K_0)}{KK_0}P \\ &\leq -\alpha W + \frac{2cr(K + K_0)}{K_0}, \end{aligned}$$

where $\alpha = \min\left\{r, \frac{r(K+K_0)}{KK_0}\right\}$.

That is

$$\dot{W}(t) + \alpha W \leq \frac{2cr(K + K_0)}{K_0}.$$

So,

$$0 < W(P(t), Z(t)) \leq \frac{2cr(K + K_0)}{\alpha K_0} + e^{-\alpha t}W(P(0), Z(0)).$$

When $t \rightarrow \infty$, we have

$$0 < W \leq \frac{2cr(K + K_0)}{\alpha K_0}.$$

Therefore, all solutions of system (2.2) enter into the invariant set $\Omega = \{(P, Z) \in \Theta \mid 0 < W \leq \frac{2cr(K+K_0)}{\alpha K_0}\}$. The Lemma 2.1 is proved. □

Next, we use the method in [41] to give the positiveness of solution of system (2.1) in the case of $\tau > 0$.

Lemma 2.2. *All solutions of system (2.1) with initial conditions that start in $\Theta = \{(P, Z) \mid (m, +\infty) \times (0, +\infty)\} \subset R_+^2$ are positive invariant.*

Proof. We consider (P, Z) a noncontinuable solution of system (2.1), defined on $[-\tau, \Gamma)$, where $\Gamma \in (0, \infty]$. We have to prove that for all $t \in [0, \Gamma)$, $P(t) > m$, and $Z(t) > 0$. Suppose that is not true. Then, there exists $0 < T < \Gamma$ such that for all $t \in [0, T)$, $P(t) > m$, and $Z(t) > 0$ and either $P(T) = m$ or $Z(T) = 0$. For all $t \in [0, T)$, under initial conditions we have

$$P(t) = P(0)\exp\left\{\int_0^t \left[r\left(1 - \frac{P(s)}{K}\right)\left(\frac{P(s)}{K_0} - 1\right) - \frac{\beta_1(P(s) - m)Z(s)}{P(s)(a_1 + P(s) - m)} - q_1E\right] ds\right\},$$

$$Z(t) = Z(0)\exp\left\{\int_0^t \left[\frac{\beta_2(P(s) - m)}{a_1 + (P(s) - m)} - d - \frac{\theta P(s)}{a_2 + P(s)} - q_2 E\right] ds\right\}.$$

As (P, Z) is continuous on $[-\tau, T]$, there exists a $M \geq 0$ such that for all $t \in [-\tau, T]$,

$$\begin{aligned} P(t) &= P(0)\exp\left\{\int_0^t \left[r\left(1 - \frac{P(s)}{K}\right)\left(\frac{P(s)}{K_0} - 1\right) - \frac{\beta_1(P(s) - m)Z(s)}{P(s)(a_1 + P(s) - m)} - q_1 E\right] ds\right\} \\ &\geq P(0)\exp(-TM), \\ Z(t) &= Z(0)\exp\left\{\int_0^t \left[\frac{\beta_2(P(s) - m)}{a_1 + (P(s) - m)} - d - \frac{\theta P(s)}{a_2 + P(s)} - q_2 E\right] ds\right\} \\ &\geq Z(0)\exp(-TM). \end{aligned}$$

Taking the limit, as $t \rightarrow T$, we get

$$P(T) \geq P(0)\exp(-TM) > m, \quad Z(T) \geq Z(0)\exp(-TM) > 0,$$

which contradicts the fact that either $P(T) = m$, or $Z(T) = 0$. Thus, for all $t \in [0, \Gamma]$, $P(t) > m$ and $Z(t) > 0$. The Lemma 2.2 is proved. \square

3. The existence and stability of equilibria

3.1. The existence of equilibria

In order to get the conditions for the existence of the equilibria of system (2.2), we analyze the following Eq (3.1) which is given by

$$\begin{cases} rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{K_0} - 1\right) - \frac{\beta_1(P - m)Z}{a_1 + (P - m)} - q_1 EP = 0, \\ \frac{\beta_2(P - m)Z}{a_1 + (P - m)} - dZ - \frac{\theta PZ}{a_2 + P} - q_2 EZ = 0. \end{cases} \quad (3.1)$$

Obviously, the equilibria of system (2.2) are the intersections of the two equations of (3.1). In the absence of zooplankton, that is $Z = 0$, the first equation of (3.1) becomes

$$\frac{r}{K_0 K} P^2 - \frac{r(K_0 + K)}{K_0 K} P + r + q_1 E = 0.$$

Let $\Delta_1 = \frac{r^2(K_0 + K)^2 - 4K_0 K r(r + q_1 E)}{K_0^2 K^2}$, then there is the following conclusion.

Theorem 3.1. *The boundary equilibria of system (2.2) are as follows.*

(i) If $\Delta_1 = 0$, i.e., $E = \frac{r(K_0 - K)^2}{4K_0 K q_1}$, then system (2.2) has a unique boundary equilibrium given by $\tilde{E}_1(\tilde{P}_1, 0)$, here $\tilde{P}_1 = \frac{K_0 + K}{2}$.

(ii) If $\Delta_1 > 0$, i.e., $E < \frac{r(K_0 - K)^2}{4K_0 K q_1}$, then system (2.2) has two distinct boundary equilibria $\tilde{E}_{2,3}(\tilde{P}_{2,3}, 0)$, here $\tilde{P}_{2,3} = \frac{K_0 + K}{2} \pm \frac{1}{2} \sqrt{(K_0 + K)^2 - \frac{4K_0 K(r + q_1 E)}{r}}$.

For positive equilibria, from the second equation of (3.1), we have

$$A_1P^2 + A_2P + A_3 = 0, \quad (3.2)$$

where $A_1 = \beta_2 - \theta - d - q_2E$, $A_2 = \beta_2(a_2 - m) + \theta(m - a_1) + (d + q_2E)(m - a_1 - a_2)$,
 $A_3 = a_2(m - a_1)(d + q_2E) - \beta_2ma_2$.

Let $\Delta_2 = A_2^2 - 4A_1A_3$ and $h(P) = A_1P^2 + A_2P + A_3$. From the first equation of (3.1), we have

$$Z = \frac{B_1P^4 + B_2P^3 + B_3P^2 + B_4P}{K_0K\beta_1(P - m)},$$

where $B_1 = -r$, $B_2 = r(m - a_1 + K_0 + K)$, $B_3 = -[r(K_0 + K)(m - a_1) + K_0K(r + q_1E)]$,
 $B_4 = K_0K(r + q_1E)(m - a_1)$.

Since the signs of A_1 , A_2 and A_3 are uncertain, there are many cases where $h(P) = 0$ has positive root(s). For the sake of discussion, let $A_2 > 0$, i.e., $\beta_2 < \frac{\theta(m-a_1)+(d+q_2E)(m-a_1-a_2)}{m-a_2}$. Furthermore, as long as

$m - a_1 - a_2 > 0$. Thus, when $A_2 > 0$, $h(P) = 0$ will have possible positive root(s).

Through simple analyses, we obtain the following results about the existence of the positive equilibria.

Theorem 3.2. *Under the assumption that $A_2 > 0$ holds, the possible positive equilibria of system (2.2) are as follows.*

(i) *System (2.2) has one positive equilibrium when one of the following nine conditions is satisfied*

(a) $A_3 > 0, A_1 > 0, \Delta_2 = 0, h(m) > 0, h(K) > 0$;

(b) $A_3 > 0, A_1 > 0, \Delta_2 > 0, h(m) < 0, h(K) > 0$;

(c) $A_3 > 0, A_1 > 0, \Delta_2 > 0, h(m) > 0, h(K) < 0$;

(d) $A_3 > 0, A_1 < 0, \Delta_2 > 0, h(m) > 0, h(K) < 0$;

(e) $A_3 < 0, A_1 = 0$;

(f) $A_3 < 0, A_1 > 0, \Delta_2 > 0, h(m) < 0, h(K) > 0$;

(g) $A_3 < 0, A_1 < 0, \Delta_2 = 0, h(m) < 0, h(K) < 0$;

(h) $A_3 < 0, A_1 < 0, \Delta_2 > 0, h(m) > 0, h(K) < 0$;

(j) $A_3 < 0, A_1 < 0, \Delta_2 > 0, h(m) < 0, h(K) > 0$.

(ii) *System (2.2) has two distinct positive equilibria when one of the following two conditions is satisfied*

(k) $A_3 > 0, A_1 > 0, \Delta_2 > 0, h(m) > 0, h(K) > 0$;

(l) $A_3 < 0, A_1 < 0, \Delta_2 > 0, h(m) < 0, h(K) < 0$.

In the case of (e) in Theorem 3.2, that is, when $\beta_2 = \theta + d + q_2E$ and $\frac{(m-a_1)(d+q_2E)}{m} < \beta_2 < \frac{\theta(m-a_1)+(d+q_2E)(m-a_1-a_2)}{m-a_2}$ simultaneously hold, system (2.2) has a positive equilibrium $E^*(P^*, Z^*)$, where

$$P^* = -\frac{A_3}{A_2}, \quad Z^* = \frac{B_1P^{*4} + B_2P^{*3} + B_3P^{*2} + B_4P^*}{K_0K\beta_1(P^* - m)}.$$

Remark 1. *Due to the density of phytoplankton in system (2.2) is greater than the refuge constant m at any time, so the existence of other possible positive equilibria of system (2.2) is complex, we will not list them here. Therefore, we will focus our discussion about system (2.2) at positive equilibrium $E^*(P^*, Z^*)$.*

3.2. The local stability of equilibria

3.2.1. The local stability of boundary equilibria

When $\tau = 0$, we only discuss the local stability of the boundary equilibrium $\tilde{E}_1(\tilde{P}_1, 0)$ and can use the similar methods to obtain the local stability of the other boundary equilibria.

Theorem 3.3. *The boundary equilibrium $\tilde{E}_1(\tilde{P}_1, 0)$ of system (2.2) is locally asymptotically stable if and only if (H_1) holds, where*

$$(H_1) : \gamma_1 = K_0 + K - 2m > 0, r(K_0 + K)^2 > 4K_0K(r + q_1E + d + q_2E).$$

Proof. The Jacobian matrix of system (2.2) at the boundary equilibrium $\tilde{E}_1(\tilde{P}_1, 0)$ is given by

$$J_{\tilde{E}_1} = \begin{pmatrix} \frac{-3r\tilde{P}_1^2 + 2r(K_0 + K)\tilde{P}_1}{K_0K} - (r + q_1E) & -\frac{\beta_1(\tilde{P}_1 - m)}{a_1 + \tilde{P}_1 - m} \\ 0 & \frac{\beta_2(\tilde{P}_1 - m)}{a_1 + \tilde{P}_1 - m} - \frac{\theta\tilde{P}_1}{a_2 + \tilde{P}_1} - (d + q_2E) \end{pmatrix}.$$

Thus, we can get

$$\det(J_{\tilde{E}_1}) = \frac{C_1 + C_2 + C_3 + C_4 + C_5 + C_6}{4K_0K(2a_1 + \gamma_1)(2a_2 + K_0 + K)},$$

$$\text{tr}(J_{\tilde{E}_1}) = \frac{D_1 + D_2 + D_3 + D_4}{4K_0K(2a_1 + \gamma_1)(2a_2 + K_0 + K)},$$

where

$$\begin{aligned} \gamma_1 &= K_0 + K - 2m, C_1 = -r(K_0 + K)^2(d + q_2E)(2a_1 + \gamma_1)(2a_2 + K_0 + K), \\ C_2 &= \beta_2r\gamma_1(K_0 + K)^2(2a_2 + K_0 + K), C_3 = -r\theta(K_0 + K)^3(2a_1 + \gamma_1), \\ C_4 &= 4K_0K(r + q_1E)(d + q_2E)(2a_1 + \gamma_1)(2a_2 + K_0 + K), \\ C_5 &= -4K_0K\beta_2\gamma_1(r + q_1E)(2a_2 + K_0 + K), C_6 = 4K_0K\theta(K_0 + K)(r + q_1E)(2a_1 + \gamma_1), \\ D_1 &= r(K_0 + K)^2(2a_1 + \gamma_1)(2a_2 + K_0 + K), D_2 = 4K_0K\beta_2\gamma_1(2a_2 + K_0 + K), \\ D_3 &= -4K_0K\theta(K_0 + K)(2a_1 + \gamma_1), \\ D_4 &= -4K_0K(2a_1 + \gamma_1)(2a_2 + K_0 + K)(r + q_1E + d + q_2E). \end{aligned}$$

From the above analysis, it is easy to know that if and only if (H_1) holds, we have $\text{tr}(J_{\tilde{E}_1}) < 0$ and $\det(J_{\tilde{E}_1}) > 0$. Hence, the boundary equilibrium $\tilde{E}_1(\tilde{P}_1, 0)$ of system (2.2) is locally asymptotically stable if and only if (H_1) holds. This completes the proof. \square

3.2.2. The local stability of the positive equilibrium

Here, we discuss the local stability of system (2.2) at the positive equilibrium $E^*(P^*, Z^*)$ when $\tau = 0$.

Theorem 3.4. *If (H_2) and (H_3) simultaneously hold, then the positive equilibrium $E^*(P^*, Z^*)$ of system (2.2) is locally asymptotically stable, where*

$$(H_2) : \frac{N_1 + N_2 + N_3 + N_4 + N_5 + N_6}{L_3} < q_1 < \frac{M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 M_8}{L_1 + L_2},$$

$$(H_3) : \gamma_2 = 3P^* - 2(K_0 + K) > 0.$$

Proof. The Jacobian matrix of system (2.2) at the positive equilibrium $E^*(P^*, Z^*)$ is given by

$$J_{E^*} = \begin{pmatrix} m_{110} & m_{101} \\ m_{210} & m_{201} \end{pmatrix},$$

where

$$m_{110} = \frac{-3rP^{*2} + 2r(K_0 + K)P^*}{K_0K} - \frac{\beta_1 a_1 Z^*}{(a_1 + P^* - m)^2} - (r + q_1 E), \quad m_{101} = -\frac{\beta_1(P^* - m)}{a_1 + P^* - m},$$

$$m_{210} = \frac{\beta_2 a_1 Z^*}{(a_1 + P^* - m)^2} - \frac{\theta a_2 Z^*}{(a_2 + P^*)^2}, \quad m_{201} = \frac{\beta_2(P^* - m)}{a_1 + P^* - m} - \frac{\theta P^*}{a_2 + P^*} - (d + q_2 E).$$

Then characteristic equation of system (2.2) around $E^*(P^*, Z^*)$ is

$$\lambda^2 - \text{tr}(J_{E^*})\lambda + \det(J_{E^*}) = 0, \quad (3.3)$$

where

$$\det(J_{E^*}) = m_{110}m_{201} - m_{101}m_{210} = \frac{M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 M_8 - q_1(L_1 + L_2)}{K_0K(a_2 + P^*)^2(a_1 + P^* - m)^2},$$

$$\text{tr}(J_{E^*}) = m_{110} + m_{201} = \frac{N_1 + N_2 + N_3 + N_4 + N_5 + N_6 - q_1 L_3}{K_0K(P^* - m)(a_2 + P^*)(a_1 + P^* - m)^2},$$

here

$$\begin{aligned} \gamma_2 &= 3P^* - 2(K_0 + K), \\ L_1 &= K_0KE(P^* - m)(a_2 + P^*)(a_1 + P^* - m)[\beta_2(P^* - m)(a_2 + P^*) - \theta P^*(a_1 + P^* - m)], \\ L_2 &= K_0KE[M_7P^*(P^* + a_1 - m) - (P^* - m)(d + q_2E)(a_2 + P^*)^2(a_1 + P^* - m)^2], \\ L_3 &= K_0KE(a_2 + P^*)[(P^* - m)(a_1 + P^* - m)^2 + a_1P^*(m - a_1 - P^*)], \\ M_1 &= \gamma_2rP^*(P^* - m)(d + q_2E)(a_2 + P^*)^2(a_1 + P^* - m)^2, \\ M_2 &= \gamma_2r\theta P^{*2}(P^* - m)(a_2 + P^*)(a_1 + P^* - m)^2, \\ M_3 &= -\gamma_2r\beta_2P^*(P^* - m)^2(a_2 + P^*)^2(a_1 + P^* - m), \\ M_4 &= -K_0Kr\beta_2(P^* - m)^2(a_1 + P^* - m)(a_2 + P^*)^2, \\ M_5 &= K_0Kr(P^* - m)(d + q_2E)(a_2 + P^*)^2(a_1 + P^* - m)^2, \\ M_6 &= K_0Kr\theta P^*(P^* - m)(a_2 + P^*)(a_1 + P^* - m)^2, \\ M_7 &= a_1\theta P^*(a_2 + P^*) + a_1(d + q_2E)(a_2 + P^*)^2 - \theta a_2(P^* - m)(a_1 + P^* - m), \\ M_8 &= -rP^{*4} + [r(m - a_1 + K_0 + K)]P^{*3} - [r(K_0 + K)(m - a_1) + K_0Kr]P^{*2} + K_0KrP^*(m - a_1), \\ N_1 &= -\gamma_2rP^*(P^* - m)(a_2 + P^*)(a_1 + P^* - m)^2, \\ N_2 &= -K_0K(P^* - m)(a_2 + P^*)(a_1 + P^* - m)^2(r + d + q_2E), \\ N_3 &= K_0K\beta_2(P^* - m)^2(a_2 + P^*)(a_1 + P^* - m), \\ N_4 &= -K_0K\theta P^*(P^* - m)(a_1 + P^* - m)^2, \\ N_5 &= a_1r(a_2 + P^*)\{P^{*4} - (m - a_1 + K_0 + K)P^{*3} + [(K_0 + K)(m - a_1) + K_0K]P^{*2}\}, \\ N_6 &= -a_1K_0Kr(a_2 + P^*)(m - a_1)P^*. \end{aligned}$$

Through analysis and calculation, we can obtain $\text{tr}(J_{E^*}) < 0$ and $\det(J_{E^*}) > 0$ if and only if (H_2) and (H_3) hold. Thus, we easily get that the positive equilibrium $E^*(P^*, Z^*)$ of system (2.2) is locally asymptotically stable when (H_2) and (H_3) hold. This completes the proof. \square

4. Hopf bifurcation of the positive equilibrium

4.1. Hopf bifurcation of system without time delay

In this section, we investigate the existence of Hopf bifurcation around $E^*(P^*, Z^*)$ of system (2.2). Taking prey refuge m as the bifurcation parameter, the critical value of Hopf bifurcation is a positive root of $\text{tr}(J_{E^*}) = 0$, thus $m = m_H$ which satisfies $\det(J_{E^*})_{m=m_H} > 0$. As we know, when the value of m exceeds its critical value $m = m_H$, the stability of the positive equilibrium $E^*(P^*, Z^*)$ will be changed (Figure 1).

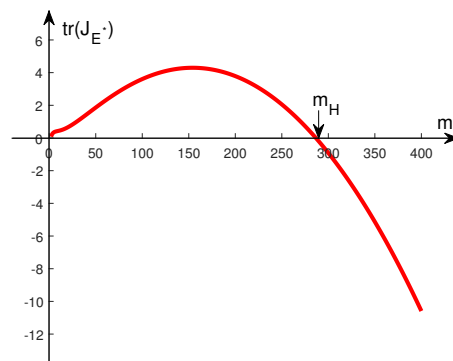


Figure 1. Parameter diagram of the existence of Hopf bifurcation.

In addition, we verify the transversality condition under which Hopf bifurcation occurs. By some simple calculations, we have that $\frac{d}{dm}\text{tr}(J_{E^*})_{m=m_H} \neq 0$, which implies that the stability of the positive equilibrium $E^*(P^*, Z^*)$ changes when the parametric restriction $\text{tr}(J_{E^*}) = 0$ and the transversality condition are satisfied simultaneously.

Therefore, we draw the following conclusion about the occurrence of Hopf bifurcation at the positive equilibrium $E^*(P^*, Z^*)$.

Theorem 4.1. *Under the condition that the positive equilibrium $E^*(P^*, Z^*)$ exists, the stability of system (2.2) at the positive equilibrium $E^*(P^*, Z^*)$ is changed through critical value $m = m_H$.*

In order to facilitate the discussion of the direction of Hopf bifurcation, we use the related theory of [42] to calculate the first Lyapunov number l_1 at the positive equilibrium $E^*(P^*, Z^*)$ of system (2.2).

We transform the positive equilibrium $E^*(P^*, Z^*)$ of system (2.2) to the origin by translation $\bar{P} = P - P^*$ and $\bar{Z} = Z - Z^*$. Then, system (2.2) in a neighborhood of the origin can be obtained

$$\begin{cases} \dot{\bar{P}} = m_{110}\bar{P} + m_{101}\bar{Z} + m_{120}\bar{P}^2 + m_{111}\bar{P}\bar{Z} + m_{130}\bar{P}^3 + m_{121}\bar{P}^2\bar{Z} + R(\bar{P}, \bar{Z}), \\ \dot{\bar{Z}} = m_{210}\bar{P} + m_{201}\bar{Z} + m_{220}\bar{P}^2 + m_{211}\bar{P}\bar{Z} + m_{230}\bar{P}^3 + m_{221}\bar{P}^2\bar{Z} + S(\bar{P}, \bar{Z}), \end{cases} \quad (4.1)$$

where m_{110} , m_{101} , m_{210} and m_{201} are the elements of the Jacobian matrix at the equilibrium $E^*(P^*, Z^*)$. So system (2.2) experiences the Hopf bifurcation at the equilibrium $E^*(P^*, Z^*)$, we have $m_{110} + m_{201} = 0$ and $\Delta = \det(J_{E^*}) = m_{110}m_{201} - m_{101}m_{210} > 0$. The coefficients m_{kij} ($k = 1, 2$) are given by

$$m_{120} = \frac{(K_0 + K)r - 3rP^*}{K_0K} + \frac{\beta_1 a_1 Z^*}{(a_1 + P^* - m)^3}, \quad m_{111} = -\frac{\beta_1 a_1}{(a_1 + P^* - m)^2},$$

$$\begin{aligned}
 m_{130} &= -\left[\frac{r}{K_0 K} + \frac{\beta_1 a_1 Z^*}{(a_1 + P^* - m)^4}\right], \quad m_{121} = \frac{\beta_1 a_1}{(a_1 + P^* - m)^3}, \\
 m_{220} &= -\frac{\beta_2 a_1 Z^*}{(a_1 + P^* - m)^3} + \frac{\theta a_2 Z^*}{(a_2 + P^*)^3}, \quad m_{211} = \frac{\beta_2 a_1}{(a_1 + P^* - m)^2} - \frac{\theta a_2}{(a_2 + P^*)^2}, \\
 m_{230} &= \frac{\beta_2 a_1 Z^*}{(a_1 + P^* - m)^4} - \frac{\theta a_2 Z^*}{(a_2 + P^*)^4}, \quad m_{221} = -\frac{2\beta_2 a_1}{3(a_1 + P^* - m)^3} + \frac{2\theta a_2}{3(a_2 + P^*)^3},
 \end{aligned}$$

and $R(\bar{P}, \bar{Z})$ and $S(\bar{P}, \bar{Z})$ are power series in (\bar{P}, \bar{Z}) with terms \bar{P}^i, \bar{Z}^j satisfying $i + j \geq 4$.

The calculation formula of the first Lyapunov number l_1 [42] determining the stability of the limit cycle in the planar system is described by

$$\begin{aligned}
 l_1 &= \frac{-3\Pi}{2m_{101}\Delta^{\frac{3}{2}}}\{[m_{110}m_{210}(m_{111}^2 + m_{111}m_{202} + m_{102}m_{211}) + m_{110}m_{101}(m_{211}^2 + m_{120}m_{211} \\
 &\quad + m_{111}m_{202}) + m_{210}^2(m_{111}m_{102} + 2m_{102}m_{202}) - 2m_{110}m_{210}(m_{202}^2 - m_{120}m_{102}) \\
 &\quad - 2m_{110}m_{101}(m_{120}^2 - m_{220}m_{202}) - m_{101}^2(2m_{120}m_{220} + m_{211}m_{220}) + (m_{101}m_{210} - 2m_{110}^2) \\
 &\quad (m_{211}m_{202} - m_{111}m_{120})] - (m_{110}^2 + m_{101}m_{210})[3(m_{210}m_{203} - m_{101}m_{130}) \\
 &\quad + 2m_{110}(m_{121} + m_{212}) + (m_{210}m_{112} - m_{101}m_{221})]\} \\
 &= \frac{-3\Pi}{2m_{101}\Delta^{\frac{3}{2}}}\{[m_{110}m_{210}m_{111}^2 + m_{110}m_{101}(m_{211}^2 + m_{120}m_{211}) - 2m_{110}m_{101}m_{120}^2 - m_{101}^2 \\
 &\quad (2m_{120}m_{220} + m_{211}m_{220}) - m_{111}m_{120}(m_{101}m_{210} - 2m_{110}^2)] - (m_{110}^2 + m_{101}m_{210}) \\
 &\quad [-3m_{101}m_{130} + 2m_{110}m_{121} - m_{101}m_{221}]\}.
 \end{aligned}$$

Based on the above analyses, we conclude the following conclusion about the direction of the limit cycle.

Theorem 4.2. *If $l_1 < 0$, then Hopf bifurcation is supercritical. If $l_1 > 0$, then Hopf bifurcation is subcritical.*

4.2. Hopf bifurcation of system with time delay

In this section, we will discuss the effect of maturation period of zooplankton τ on the dynamical behavior of system (2.1).

4.2.1. Local stability and the existence of Hopf bifurcation

The local stability of the positive equilibrium $E^*(P^*, Z^*)$ and the existence of Hopf bifurcation are studied by considering the maturity delay τ as bifurcation parameter. First, system (2.1) is linearized at positive equilibrium $E^*(P^*, Z^*)$, we obtain

$$\begin{cases} P' = a_{11}P(t) + a_{12}Z(t), \\ Z' = a_{21}P(t) + a_{22}Z(t) + b_{21}P(t - \tau), \end{cases} \tag{4.2}$$

where

$$a_{11} = \frac{-3rP^{*2} + 2r(K_0 + K)P^*}{K_0 K} - \frac{\beta_1 a_1 Z^*}{(a_1 + P^* - m)^2} - (r + q_1 E), \quad a_{12} = -\frac{\beta_1 (P^* - m)}{a_1 + P^* - m},$$

$$a_{21} = -\frac{\theta a_2 Z^*}{(a_2 + P^*)^2}, a_{22} = \frac{\beta_2(P^* - m)}{a_1 + P^* - m} - \frac{\theta P^*}{a_2 + P^*} - (d + q_2 E), b_{21} = \frac{\beta_2 a_1 Z^*}{(a_1 + P^* - m)^2}.$$

It is easy to see that corresponding characteristic equation of system (4.2) at positive equilibrium $E^*(P^*, Z^*)$ is

$$f(\lambda, \tau) = \lambda^2 + A\lambda + B + Ce^{-\lambda\tau} = 0, \quad (4.3)$$

where $A = -a_{11} - a_{22}$, $B = a_{11}a_{22} - a_{12}a_{21}$, $C = -a_{12}b_{21}$.

When $\tau = 0$, Eq (4.3) becomes

$$f(\lambda, 0) = \lambda^2 + A\lambda + B + C = 0, \quad (4.4)$$

then Eq (4.4) is equivalent to Eq (3.3). Hence, the positive equilibrium $E^*(P^*, Z^*)$ is locally asymptotically stable when (H_2) and (H_3) hold.

When $\tau \neq 0$, in order to get the existence of Hopf bifurcation, let $\lambda = i\omega_0$ ($\omega_0 > 0$) be a root of Eq (4.3) and substitute it into Eq (4.3), then we have

$$-\omega_0^2 + B + C \cos \omega_0\tau + i(A\omega_0 - C \sin \omega_0\tau) = 0.$$

By separating the real and imaginary parts, we obtain

$$\begin{cases} -\omega_0^2 + B + C \cos \omega_0\tau = 0, \\ A\omega_0 - C \sin \omega_0\tau = 0. \end{cases} \quad (4.5)$$

From which, we have

$$\omega_0^4 - (2B - A^2)\omega_0^2 + B^2 - C^2 = 0. \quad (4.6)$$

Let $u = \omega_0^2$, then Eq (4.6) deduces to

$$u^2 - (2B - A^2)u + B^2 - C^2 = 0. \quad (4.7)$$

Record Eq (4.7) as $f(u) = u^2 - (2B - A^2)u + B^2 - C^2$. From Eq (4.6), it follows that if $(H4) : 2B - A^2 > 0$, $B^2 - C^2 > 0$, and $A^4 - 2A^2B + 4C^2 = 0$

holds, then Eq (4.6) has a unique positive root ω_0^2 . Substituting ω_0^2 into Eq (4.5), we obtain

$$\tau_{2n} = \frac{1}{\omega_0} \arccos\left[\frac{\omega_0^2 - B}{C}\right] + \frac{2n\pi}{\omega_0}, n = 0, 1, 2 \dots$$

If

$(H5) : 2B - A^2 > 0$, $B^2 - C^2 > 0$, and $(2B - A^2)^2 > 4(B^2 - C^2)$

holds, then Eq (4.6) has two different positive roots ω_+^2 and ω_-^2 . We substitute ω_{\pm}^2 into Eq (4.5), then

$$\tau_{2k} = \frac{1}{\omega_{\pm}} \arccos\left[\frac{\omega_{\pm}^2 - B}{C}\right] + \frac{2k\pi}{\omega_{\pm}}, k = 0, 1, 2 \dots$$

Let λ be the root of Eq (4.3) satisfying $\text{Re}\lambda(\tau_{2n}) = 0$ (rep. $\text{Re}\lambda(\tau_{2k}^{\pm}) = 0$) and $\text{Im}\lambda(\tau_{2n}) = \omega_0$ (rep. $\text{Im}\lambda(\tau_{2k}^{\pm}) = \omega_{\pm}$). Then, when $\tau_0 = \min\{\tau_{2n}, \tau_{2k}\}$, $n, k = 0, 1, 2 \dots$, we have the following conclusion.

Theorem 4.3. *If $f'(u) > 0$, since $\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\tau=\tau_0}$ and $f'(u)$ have the same sign, then $\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\tau=\tau_0} > 0$ holds.*

Proof. From Eq (4.3), we can obtain

$$\frac{d\lambda}{d\tau} = \frac{C\lambda e^{-\lambda\tau}}{2\lambda + A - C\tau e^{-\lambda\tau}}.$$

Notice that $Ce^{-\lambda\tau} = -(\lambda^2 + A\lambda + B)$. Thus, we obtain that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A}{-\lambda(\lambda^2 + A\lambda + B)} - \frac{\tau}{\lambda}. \tag{4.8}$$

Substituting $\lambda = i\omega_0$ into (4.8), we have

$$\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0} = \frac{2\omega_0^4 - (2B - A^2)\omega_0^2}{(A\omega_0^2)^2 + (\omega_0^3 - B\omega_0)^2} = \frac{\omega_0^2 f'(\omega_0^2)}{(A\omega_0^2)^2 + (\omega_0^3 - B\omega_0)^2}.$$

Therefore,

$$\text{sign}\left\{\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\tau=\tau_0}\right\} = \text{sign}\left\{\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0}\right\} = \text{sign}\{f'(\omega_0^2)\}.$$

If $f'(\omega_0^2) \neq 0$, we have $\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\tau=\tau_0} \neq 0$, then $\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\tau=\tau_0} > 0$. If $\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\tau=\tau_0} < 0$, then Eq (4.3) has the roots of the real part when $\tau < \tau_0$. This contradicts that the positive equilibrium $E^*(P^*, Z^*)$ is locally asymptotically stable when $\tau \in (0, \tau_0)$. □

According to the Hopf bifurcation theorem given in [43], we can get the following result on stability and bifurcation of system (2.1).

Theorem 4.4. *For system (2.1), suppose (H4) and (H5) are satisfied. Then the positive equilibrium $E^*(P^*, Z^*)$ is locally asymptotically stable when $\tau \in (0, \tau_0)$, and system (2.1) undergoes a Hopf bifurcation at $E^*(P^*, Z^*)$ when τ passes through its critical value $\tau = \tau_0$.*

4.2.2. Direction and stability of Hopf bifurcation

In the previous section, we have already obtained sufficient conditions to guarantee that system (2.1) has a periodic solution at positive equilibrium $E^*(P^*, Z^*)$ when the critical value $\tau = \tau_0$. In this section, we will use the normal form theory and the center manifold theorem presented in Hassard et al. [43] to obtain the properties of the bifurcating periodic solution.

Let $\tau = \tau_0 + \mu$, $\mu \in R$. Then $\mu = 0$ is a Hopf bifurcation value of system (2.1). Without loss of generality, we can choose the phase space as $C = C([-\tau, 0], \Theta)$. Let $u(t) = (u_1(t), u_2(t))^T \in \Theta, u_t(\theta) = u(t + \theta) \in C$, system (2.1) can be rewritten as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{4.9}$$

where $L_\mu : C \rightarrow \Theta$ and $f : R \times C \rightarrow \Theta$ are defined by

$$L_\mu(\varphi) = G_1\varphi(0) + G_2\varphi(-\tau),$$

$$G(1) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, G(2) = \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix}, f(\mu, \varphi) = \begin{pmatrix} f_1(\mu, \varphi) \\ f_2(\mu, \varphi) \end{pmatrix},$$

where

$$f_1(\mu, \varphi) = \rho_1 \varphi_1(0) \varphi_2(0) + \rho_2 \varphi_1^2(0),$$

$$f_2(\mu, \varphi) = \sigma_1 \varphi_1(0) \varphi_2(0) + \sigma_2 \varphi_1^2(0) + \sigma_3 \varphi_1^2(-1) + \sigma_4 \varphi_1(-1) \varphi_2(0),$$

here

$$\rho_1 = -\frac{\beta_1 a_1}{(a_1 + P^* - m)^2}, \quad \rho_2 = \frac{2r(K_0 + K) - 6rP^*}{K_0 K} + \frac{2\beta_1 a_1 Z^*}{(a_1 + P^* - m)^3},$$

$$\sigma_1 = -\frac{\theta a_2}{(a_2 + P^*)^2}, \quad \sigma_2 = \frac{2\theta a_2 Z^*}{(a_2 + P^*)^3}, \quad \sigma_3 = \frac{-2\beta_2 a_1 Z^*}{(a_1 + P^* - m)^3}, \quad \sigma_4 = \frac{\beta_2 a_1}{(a_1 + P^* - m)^2}.$$

Based to the Reisz representation theorem, there is a bounded variation function $\eta(\bullet, \mu) : [-\tau, 0] \rightarrow \Theta^{2 \times 2}$ such that

$$L_\mu(\varphi) = \int_{-\tau}^0 d\eta(\theta, \mu) \varphi(\theta), \quad \varphi \in C.$$

Then, we have a choice that

$$d\eta(\theta, \mu) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \delta(\theta) d\theta + \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \delta(\theta + \tau) d\theta,$$

where δ is the Dirac function. For $\varphi \in C^1([-\tau, 0], \Theta^2)$, we define the operator $A(\mu)$ as

$$A(\mu)\varphi(\theta) = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-\tau, 0), \\ \int_{-\tau}^0 d\eta(\xi, \mu) \varphi(\xi), & \theta = 0. \end{cases} \quad (4.10)$$

Further, we can define the operator $R(\mu)$ as

$$R(\mu)\varphi(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ f(\varphi, \mu), & \theta = 0. \end{cases}$$

Then, we can rewrite Eq (4.9) as the following operator equation

$$\dot{u}(t) = A_\mu u_t + R_\mu u_t. \quad (4.11)$$

For $\psi \in C^1([-\tau, 0], \Theta^2)$, the adjoint operator A_0^* of A_0 can be defined as

$$A_0^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, \tau], \\ \int_{-\tau}^0 d\eta^T(t, 0) \psi(-t), & s = 0, \end{cases}$$

where η^T is the transpose of η , and the domains of A_0 and A_0^* are $C^1([-\tau, 0], \Theta^2)$ and $C^1([0, \tau], \Theta^2)$, respectively.

For $\varphi \in C^1([-\tau, 0], \Theta^2)$ and $\psi \in C^1([0, \tau], \Theta^2)$, a bilinear form is given by

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{-\tau}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \varphi(\xi) d\xi, \quad (4.12)$$

where $\eta(\theta) = \eta(\theta, 0)$. Then A_0 and A_0^* are a pair of adjoint operators. By the results obtained in the last section, we know that $\pm i\omega_0$ are the eigenvalues of A_0 . Thus, they are also eigenvalues of A_0^* . What's more, we assume that $q(\theta)$ is the the eigenvalues of A_0 corresponding to $i\omega_0$, then

$$A_0 q(\theta) = i\omega_0 q(\theta). \quad (4.13)$$

By Eq (4.10), Eq (4.13) becomes

$$\begin{cases} \frac{dq(\theta)}{d\theta} = i\omega_0 q(\theta), & \theta \in [-\tau, 0], \\ L_0 q(0) = i\omega_0 q(0), & \theta = 0. \end{cases} \quad (4.14)$$

From Eq (4.14) we can get

$$q(\theta) = V e^{i\omega_0 \theta}, \quad \theta \in [-\tau, 0], \quad (4.15)$$

where $V = (v_1, v_2)^T \in \Theta^2$ is a constant vector. By virtue of Eq (4.15), we may obtain

$$G_1 V + G_2 e^{-i\omega_0 \tau} V - i\omega_0 I V = 0,$$

which yields

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\omega_0 - a_{11}}{a_{12}} \end{pmatrix}.$$

On the other hand, if $-i\omega_0$ is the eigenvalue of A_0^* , then we have

$$A_0^* q^*(\xi) = i\omega_0 q^*(\xi).$$

For the non-zero vector $q^*(\xi)$, $\xi \in [0, \tau_0]$, we can obtain

$$G_1^T V^* + G_2^T e^{i\omega_0 \tau} V^* + i\omega_0 I V^* = 0.$$

Let $q^*(\xi) = D V^* e^{i\omega_0 \xi}$, where $\xi \in [0, \tau_0]$, and $V^* = (v_1^*, v_2^*)^T$ be a constant vector. Similarly, we get

$$V^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} = \begin{pmatrix} -\frac{i\omega_0 + a_{22}}{a_{12}} \\ 1 \end{pmatrix}.$$

In fact, by Eq (4.12), we have

$$\begin{aligned} \langle q^*, q \rangle &= \bar{q}^{*T}(0)q(0) - \int_{-\tau_0}^0 \int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi - \theta) d\eta(\theta) q(\xi) d\xi, \\ &= \bar{D}[\bar{V}^{*T} V - \int_{-\tau_0}^0 \int_{\xi=0}^{\theta} \bar{V}^{*T} e^{-i\omega_0(\xi - \theta)} d\eta(\theta) V e^{i\omega_0 \xi} d\xi] \\ &= \bar{D}[\bar{V}^{*T} V + \tau_0 e^{-i\omega_0 \tau_0} \bar{V}^{*T} G_2 V] \\ &= \bar{D}[v_2 + \bar{v}_1^* + \tau_0 b_{21} e^{-i\omega_0 \tau_0}]. \end{aligned} \quad (4.16)$$

Actually, we can choose

$$\bar{D} = [v_2 + \bar{v}_1^* + \tau_0 b_{21} e^{-i\omega_0 \tau_0}]^{-1}.$$

From Eq (4.16) we obtain that $\langle q^*, q \rangle = 1$. More importantly

$$-i\omega_0 \langle q^*, \bar{q} \rangle = \langle q^*, A_0 \bar{q} \rangle = \langle A_0^* q^*, \bar{q} \rangle = \langle -i\omega_0 q^*, \bar{q} \rangle = i\omega_0 \langle q^*, \bar{q} \rangle,$$

thus, $\langle q^*, \bar{q} \rangle = 0$ is easily verified.

Based on the above analysis, we can come to the following simple conclusion.

Theorem 4.5. *Let $q(\theta) = V e^{i\omega_0 \theta}$, where $\theta \in [-\tau, 0]$, is the eigenvector of A_0 corresponding to $i\omega_0$, and let $q^*(s) = D V^* e^{i\omega_0 s}$, where $s \in [0, \tau]$ is the eigenvector of A_0^* corresponding to $-i\omega_0$. Then*

$$\langle q^*, q \rangle = 1, \quad \langle q^*, \bar{q} \rangle = 0, \quad V = (1, v_2)^T, \quad V^* = (v_1^*, 1)^T,$$

where

$$v_2 = \frac{i\omega_0 - a_{11}}{a_{12}}, \quad v_1^* = -\frac{i\omega_0 + a_{22}}{a_{12}}, \quad \bar{D} = [v_2 + \bar{v}_1^* + \tau_0 b_{21} e^{-i\omega_0 \tau_0}]^{-1}.$$

Next, we use the same notations as the previous part to study the stability of bifurcating periodic solution. We shall calculate the coordinates to express the center manifold C_0 at $\mu = 0$. Let u_t be the solution of $\dot{u}(t) = A_\mu u_t + R_\mu u_t$ at $\mu = 0$ and define

$$z(t) = \langle q^*, u_t \rangle, \tag{4.17}$$

$$W(t, \theta) = u_t(\theta) - 2\text{Re}[z(t)q(\theta)]. \tag{4.18}$$

On the center manifold C_0 , we have

$$W(t, \theta) = W[z(t), \bar{z}(t), \theta],$$

and

$$W[z, \bar{z}, \theta] = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + W_{30}(\theta) \frac{z^3}{6} + \dots,$$

where z and \bar{z} are the local coordinates for the center manifold C_0 in the direction of q^* and \bar{q}^* . Notice that W is real if u_t is real. So here we only consider the real solutions.

For the solution $u_t \in C_0$ of Eq (4.11), we have

$$\dot{z}(t) = \langle q^*, \dot{u}_t \rangle = \langle q^*, A_0 u_t + R_0 u_t \rangle = \langle A_0^* q^*, u_t \rangle + \bar{q}^*(0) f_0(0, u_t) = i\omega_0 z + \bar{q}^{*T}(0) f(z, \bar{z}). \tag{4.19}$$

Then Eq (4.19) can be rewritten as

$$\dot{z}(t) = i\omega_0 z + g(z, \bar{z}), \tag{4.20}$$

where

$$g(z, \bar{z}) = g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta) z\bar{z} + g_{02}(\theta) \frac{\bar{z}^2}{2} + g_{30}(\theta) \frac{z^3}{6} + \dots \tag{4.21}$$

According to Eqs (4.11), (4.17) and (4.18), we may obtain

$$\begin{aligned} \dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} A_0(W) - 2\text{Re}[\bar{q}^* f_0 q(\theta)], & \theta \in [-\tau, 0), \\ A_0(W) - 2\text{Re}[\bar{q}^* f_0 q(0)] + f_0, & \theta = 0. \end{cases} \end{aligned} \tag{4.22}$$

We can rewrite Eq (4.22) as

$$\dot{W} = A_0(W) + H(z, \bar{z}, \theta), \quad (4.23)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + H_{30}(\theta) \frac{z^3}{6} + \dots \quad (4.24)$$

Substituting the corresponding series into Eq (4.22) and comparing the coefficients, we have

$$\begin{aligned} (A_0 - 2i\omega_0 I)W_{20}(\theta) &= -H_{20}(\theta), \\ A_0 W_{11}(\theta) &= -H_{11}(\theta), \\ (A_0 + 2i\omega_0 I)W_{02}(\theta) &= -H_{02}(\theta). \end{aligned} \quad (4.25)$$

According to Eqs (4.19) and (4.20), we know

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0)f(0)(W(z, \bar{z}, 0) + 2Re(z(t)q(\theta))) \\ &= \bar{D}(\bar{v}_1^*, 1) \left(\begin{array}{c} \rho_1 \varphi_1(0)\varphi_2(0) + \rho_2 \varphi_1^2(0) \\ \sigma_1 \varphi_1(0)\varphi_2(0) + \sigma_2 \varphi_1^2(0) + \sigma_3 \varphi_1^2(-1) + \sigma_4 \varphi_1(-1)\varphi_2(0) \end{array} \right) \\ &= \bar{D}((\bar{v}_1^* \rho_1 + \sigma_1)(W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + v_1 z + \bar{v}_1 \bar{z}) \\ &\quad \times (W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + z + \bar{z}) + (\bar{v}_1^* \rho_2 + \sigma_2) \\ &\quad \times (W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + v_1 z + \bar{v}_1 \bar{z})^2 \\ &\quad + \sigma_3 (W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + v_1 z e^{-i\omega_0 \tau_0} + \bar{v}_1 \bar{z} e^{i\omega_0 \tau_0})^2 \\ &\quad + \sigma_4 (W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + v_1 z e^{-i\omega_0 \tau_0} + \bar{v}_1 \bar{z} e^{i\omega_0 \tau_0}) \\ &\quad \times (W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + z + \bar{z}). \end{aligned}$$

Comparing the coefficients with Eq (4.21), it follows that

$$\begin{aligned} g_{20} &= 2\bar{D}[\bar{v}_1^* K_{11} + K_{21}], & g_{11} &= \bar{D}[\bar{v}_1^* K_{12} + K_{22}], \\ g_{02} &= 2\bar{D}[\bar{v}_1^* K_{13} + K_{23}], & g_{21} &= 2\bar{D}[\bar{v}_1^* K_{14} + K_{24}], \end{aligned}$$

where

$$\begin{aligned} K_{11} &= \rho_1 v_1 + \rho_2 v_1^2, & K_{12} &= \rho_1(v_1 + \bar{v}_1) + 2\rho_2 v_1 \bar{v}_1, \\ K_{21} &= \sigma_1 v_1 + \sigma_2 v_1^2 + \sigma_3 v_1^2 e^{-2i\omega_0 \tau_0} + \sigma_4 v_1 e^{-i\omega_0 \tau_0}, \\ K_{22} &= \sigma_4(v_1 e^{-i\omega_0 \tau_0} + \bar{v}_1 e^{i\omega_0 \tau_0}) + \sigma_1(v_1 + \bar{v}_1) + 2\sigma_2 v_1 \bar{v}_1 + 2\sigma_3 v_1 \bar{v}_1, \\ K_{13} &= \rho_1 \bar{v}_1 + \rho_2 \bar{v}_1^2, & K_{23} &= \sigma_1 \bar{v}_1 + \sigma_2 \bar{v}_1^2 + \sigma_3 \bar{v}_1^2 e^{2i\omega_0 \tau_0} + \sigma_4 v_1 e^{i\omega_0 \tau_0}, \\ K_{14} &= \rho_1 \left(\frac{1}{2} W_{20}^{(1)}(0) + W_{11}^{(1)}(0) + \frac{1}{2} W_{20}^{(2)}(0) \bar{v}_1 + W_{11}^{(2)}(0) v_1 \right) + 2\rho_2 \left(\frac{1}{2} W_{20}^{(1)}(0) \bar{v}_1 + W_{11}^{(1)}(0) v_1 \right), \\ K_{24} &= \sigma_1 \left(\frac{1}{2} W_{20}^{(1)}(0) + W_{11}^{(1)}(0) + \frac{1}{2} W_{20}^{(2)}(0) \bar{v}_1 + W_{11}^{(2)}(0) v_1 \right) \end{aligned}$$

$$\begin{aligned}
 &+ 2\sigma_2\left(\frac{1}{2}W_{20}^{(1)}(0)\bar{v}_1 + W_{11}^{(1)}(0)v_1\right) + 2\sigma_3\left(\frac{1}{2}W_{20}^{(1)}(-1)\bar{v}_1e^{i\omega_0\tau_0} + W_{11}^{(1)}(-1)v_1e^{-i\omega_0\tau_0}\right) \\
 &+ \sigma_4\left(\frac{1}{2}W_{20}^{(1)}(-1) + W_{11}^{(1)}(-1) + \frac{1}{2}W_{20}^{(2)}(0)\bar{v}_1e^{i\omega_0\tau_0} + W_{11}^{(2)}(0)v_1e^{-i\omega_0\tau_0}\right).
 \end{aligned}$$

Since $W_{20}(\theta)$ and $W_{11}(\theta)$ appear in g_{21} , we need to compute them further. From Eqs (4.22) and (4.23), we have that for $\theta \in [-\tau, 0)$,

$$H(z, \bar{z}, \theta) = -2Re[\bar{q}^*(0)f_0(z, \bar{z})q(\theta)] = -2Re[g(z, \bar{z})q(\theta)] = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta).$$

Comparing the coefficients with Eq (4.24) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{20}\bar{q}(\theta), \quad H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).$$

From Eqs (4.10) and (4.25), we have

$$\dot{W}_{20}(\theta) = 2i\omega_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \tag{4.26}$$

and

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta). \tag{4.27}$$

Solving for $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\omega_0}e^{i\omega_0\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\omega_0}e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta}, \tag{4.28}$$

and

$$W_{11}(\theta) = \frac{-ig_{11}q(0)}{\omega_0}e^{i\omega_0\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\omega_0}e^{-i\omega_0\theta} + E_2, \tag{4.29}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})$ and $E_2 = (E_2^{(1)}, E_2^{(2)})$ are the following two-dimensional constant vectors, and can be determined by setting $\theta = 0$ in H .

In fact, the following formula is true at $\theta = 0$,

$$H(z, \bar{z}, 0) = -2Re[\bar{q}^*(0)f_0(z, \bar{z})q(0)] + f_0,$$

we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix}, \tag{4.30}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix} K_{12} \\ K_{22} \end{pmatrix}. \tag{4.31}$$

Substituting Eqs (4.28) and (4.29) into Eqs (4.26) and (4.27), respectively, we get

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}W_{20}(0) + \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix}W_{20}(-\tau) = 2i\omega_0W_{20}(0) - H_{20}(0), \tag{4.32}$$

and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} W_{11}(0) + \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} W_{11}(-\tau) = -H_{11}(0). \quad (4.33)$$

Substituting Eq (4.28) into Eq (4.32), we have

$$\begin{aligned} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left[\frac{ig_{20}}{\omega_0} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} + \frac{i\bar{g}_{02}}{3\omega_0} \begin{pmatrix} 1 \\ \bar{v}_2 \end{pmatrix} + E_1 \right] + \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \left[\frac{ig_{20}}{\omega_0} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} e^{-i\omega_0\tau} \right. \\ & \left. + \frac{i\bar{g}_{02}}{3\omega_0} \begin{pmatrix} 1 \\ \bar{v}_2 \end{pmatrix} e^{i\omega_0\tau} + E_1 e^{-2i\omega_0\tau} \right] = -2i\omega_0 \left[\frac{ig_{20}}{\omega_0} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} + \frac{i\bar{g}_{02}}{3\omega_0} \begin{pmatrix} 1 \\ \bar{v}_2 \end{pmatrix} + E_1 \right] - H_{20}. \end{aligned} \quad (4.34)$$

Since $i\omega_0$ is the eigenvalue of A_0 and $q(0)$ is the corresponding eigenvector, we get

$$\left(i\omega_0 I - \int_{-\tau_0}^0 e^{i\omega_0\theta} d\eta(\theta) \right) q(0) = 0, \quad \left(-i\omega_0 I - \int_{-\tau_0}^0 e^{-i\omega_0\theta} d\eta(\theta) \right) \bar{q}(0) = 0,$$

that is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} e^{-i\omega_0\tau} = i\omega_0 \begin{pmatrix} 1 \\ v_2 \end{pmatrix}, \quad (4.35)$$

and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{v}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \bar{v}_2 \end{pmatrix} e^{i\omega_0\tau} = i\omega_0 \begin{pmatrix} 1 \\ \bar{v}_2 \end{pmatrix}. \quad (4.36)$$

By Eqs (4.30) and (4.34)–(4.36), we obtain

$$\begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} \\ -a_{21} - b_{21}e^{-2i\omega_0\tau} & 2i\omega_0 - a_{22} \end{pmatrix} E_1 = \begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix},$$

which leads to

$$E_1^{(1)} = \frac{M_1^{(1)}}{M_1}, \quad E_1^{(2)} = \frac{M_1^{(2)}}{M_1},$$

where

$$\begin{aligned} M_1 &= (2i\omega_0 - a_{11})(2i\omega_0 - a_{22}) - a_{12}(a_{21} + b_{21}e^{-2i\omega_0\tau}), \\ M_1^{(1)} &= K_{11}(2i\omega_0 - a_{22}) + K_{21}a_{12}, \quad M_1^{(2)} = K_{21}(2i\omega_0 - a_{11}) + K_{11}(a_{21} + b_{21}e^{-2i\omega_0\tau}). \end{aligned}$$

Similarly, substituting Eqs (4.29) and (4.31) into Eq (4.33), we have

$$\begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} - b_{21} & -a_{22} \end{pmatrix} E_1 = \begin{pmatrix} K_{12} \\ K_{22} \end{pmatrix},$$

which leads to

$$E_2^{(1)} = \frac{M_2^{(1)}}{M_2}, \quad E_2^{(2)} = \frac{M_2^{(2)}}{M_2},$$

where

$$M_2 = a_{11}a_{22} - a_{12}(a_{21} + b_{21}), \quad M_2^{(1)} = -K_{12}a_{22} + K_{22}a_{12}, \quad M_2^{(2)} = -K_{22}a_{11} + K_{12}(a_{21} + b_{21}).$$

Through the above calculation and analysis, it is not difficult to see that each g_{ij} depends on the parameters and time delay of system (2.1). Thus, we can obtain the following expressions

$$\begin{cases} C_1(0) = \frac{i}{2\omega_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \\ \beta = 2\operatorname{Re}\{C_1(0)\}, \\ T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0}, \end{cases} \quad (4.37)$$

which determine the direction of Hopf bifurcation and stability of bifurcating periodic solutions of system (2.1) on the center manifold at the critical values τ_0 . From the conclusion of Hassard et al. [43], we summarize the following main findings.

Theorem 4.6. *The values of the parameters μ_2 , β and T_2 of (4.37) will determine the properties of Hopf bifurcation of system (2.1).*

(i) *The sign of μ_2 determines the direction of Hopf bifurcation: The Hopf bifurcation is supercritical if $\mu_2 > 0$ and the Hopf bifurcation is subcritical if $\mu_2 < 0$.*

(ii) *The sign of β determines the stability of bifurcating periodic solutions: The periodic solutions are stable if $\beta < 0$ and unstable if $\beta > 0$.*

(iii) *The sign of T_2 determines the period of bifurcating periodic solutions: The period increases if $T_2 > 0$ and the period decreases if $T_2 < 0$.*

5. Optimal harvesting policy

When $\tau = 0$, let \hat{c} be the constant harvesting cost per unit effort and p_1 , p_2 are the constant price per unit biomass of phytoplankton and zooplankton, respectively. Then the net economic revenue to the society is given by

$$\pi(P, Z, E, t) = (p_1q_1P + p_2q_2Z - \hat{c})E.$$

In order to study the optimal harvesting yield of system (2.2), we will consequently maximize the full return from resource management. Hence we take the harvesting effort E as a control variable and consider a objective function defined by the present value J of a continuous time-stream of revenues

$$J = \int_0^{\infty} L(X, E, t)dt, \quad (5.1)$$

where

$$L(X, E, t) = e^{-\delta t}(p_1q_1P + p_2q_2Z - \hat{c})E,$$

where δ is the instantaneous annual discount rate. We consider that the present value of capital flow over time depends on discount rate δ . Here $X = [P, Z]^T$ is the vector of state variables, it can be written as $\dot{X} = f(X, E)$, $X(0) = X_0$ by according to the state equations of (2.2).

Our optimal control problem is to maximize Eq (5.1) subject to the state equations of (2.2) and to the control constraint

$$0 \leq E(t) \leq E_{\max},$$

where E_{\max} is the maximum harvesting effort.

Based on the Pontryagin's Maximum Principle [44], we can obtain the optimal solution of this problem. We know that the convexity of objective function with respect to $E(t)$, the linearity of the control differential equations and the compactness of the range values of the state variables to ensure the existence of optimal control.

The present value Hamiltonian function of the optimal problem is formulated by

$$H(X, E, t) = L(X, E, t) + \lambda^T(t)f(X, E),$$

where $\lambda(t) = [\lambda_1(t), \lambda_2(t)]^T$ is the vector of constant or adjoint variables.

Next, we substitute $\lambda^T(t)$ and $f(X, E)$ into above Hamiltonian function. Then, the Hamiltonian function becomes

$$\begin{aligned} H(P, Z, E, t) = & e^{-\delta t}(p_1q_1P + p_2q_2Z - \hat{c})E \\ & + \lambda_1(t)\left[rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{K_0} - 1\right) - \frac{\beta_1(P-m)}{a_1 + (P-m)}Z - q_1EP\right] \\ & + \lambda_2(t)\left[\frac{\beta_2(P-m)}{a_1 + (P-m)}Z - dZ - \frac{\theta P}{a_2 + P}Z - q_2EZ\right]. \end{aligned}$$

To make H reach the maximum on the control set $0 \leq E(t) \leq E_{\max}$, the condition that the Hamiltonian function H must satisfy is presented by

$$\frac{\partial H}{\partial E} = e^{-\delta t}(p_1q_1P + p_2q_2Z - \hat{c}) - \lambda_1(t)q_1P - \lambda_2(t)q_2Z = 0. \quad (5.2)$$

Pontryagin's Maximum Principle [44] states that the optimal state trajectory, optimal control, and corresponding adjoint variable vector must satisfy the following adjoint equations

$$-\dot{\lambda}(t) = H_X.$$

Obviously, the adjoint equations are

$$\begin{aligned} -\frac{d\lambda_1}{dt} &= \frac{\partial H}{\partial P} \\ &= e^{-\delta t}p_1q_1E + \lambda_1(t)\left[\frac{-3rP^2 + 2(K_0 + K)rP}{K_0K} - \frac{\beta_1a_1Z}{(a_1 + P - m)^2} - r - q_1E\right] \\ &\quad + \lambda_2(t)\left[\frac{\beta_2a_1Z}{(a_1 + P - m)^2} - \frac{\theta a_2Z}{(a_2 + P)^2}\right], \\ -\frac{d\lambda_2}{dt} &= \frac{\partial H}{\partial Z} \\ &= e^{-\delta t}p_2q_2E - \lambda_1(t)\frac{\beta_1(P-m)}{a_1 + P - m} + \lambda_2(t)\left[\frac{\beta_2(P-m)}{a_1 + P - m} - \frac{\theta P}{a_2 + P} - d - q_2E\right]. \end{aligned}$$

For the positive optimal solutions, which satisfy $\dot{P} = \dot{Z} = 0$ (in other words, P, Z are not dependent on t) and from two equations of system (2.2), we have

$$r\left(1 - \frac{P}{K}\right)\left(\frac{P}{K_0} - 1\right) - \frac{\beta_1(P-m)Z}{P(a_1+P-m)}Z - q_1E = 0, \quad (5.3)$$

$$\frac{\beta_2(P-m)}{a_1+P-m} - d - \frac{\theta P}{a_2+P} - q_2E = 0. \quad (5.4)$$

From the above analysis, it is obvious that E is also independent of t . Furthermore, we get

$$\begin{aligned} -\frac{d\lambda_1}{dt} = & \lambda_1(t) \left[\frac{-2rP^2 + (K_0 + K)rP}{K_0K} + \frac{\beta_1Z(P-m)^2 - \beta_1ma_1Z}{P(a_1+P-m)^2} \right] \\ & + \lambda_2(t) \left[\frac{\beta_2a_1Z}{(a_1+P-m)^2} - \frac{\theta a_2Z}{(a_2+P)^2} \right] + e^{-\delta t} p_1 q_1 E, \end{aligned}$$

and

$$-\frac{d\lambda_2}{dt} = e^{-\delta t} p_2 q_2 E - \lambda_1(t) \frac{\beta_1(P-m)}{a_1+P-m}. \quad (5.5)$$

Differentiating Eq (5.2) and replacing value of λ_1, λ_2 , we get

$$\begin{aligned} & \left[\frac{-2rq_1P^3 + (K_0 + K)rq_1P^2}{K_0K} + \frac{\beta_1PZ(q_1 - q_2)[(P-m)^2 - a_1m] - \beta_1q_2a_1P^2Z}{P(a_1+P-m)^2} \right] \\ & \times \lambda_1 e^{\delta t} + \lambda_2 e^{\delta t} \left[\frac{q_1\beta_2a_1PZ}{(a_1+P-m)^2} - \frac{q_1\theta a_2PZ}{(a_2+P)^2} \right] = \delta F - p_1q_1^2EP - p_2q_2^2EZ. \end{aligned} \quad (5.6)$$

By Eqs (5.2) and (5.6), we can get

$$\lambda_1 e^{\delta t} = \frac{P_1 + P_2}{Q_1 + Q_2 + Q_3}, \quad (5.7)$$

$$\lambda_2 e^{\delta t} = \frac{P_3 + P_4 + P_5}{Q_4 + Q_5 + Q_6}, \quad (5.8)$$

where

$$\begin{aligned} P_1 &= K_0Kq_2PZ(a_2+P)^2(a_1+P-m)^2[\delta F - E(p_1q_1^2P + p_2q_2^2Z)], \\ P_2 &= -K_0KFq_1\beta_2a_1P^2Z(a_2+P)^2 + K_0KFq_1\theta a_2P^2Z(a_1+P-m)^2, \\ P_3 &= K_0Kq_1P^2(a_2+P)^2(a_1+P-m)^2[\delta F - E(p_1q_1^2P + p_2q_2^2Z)], \\ P_4 &= -rq_1FP^3(a_2+P)^2(a_1+P-m)^2(K_0+K-2P), \\ P_5 &= -\beta_1FK_0KPZ(q_1-q_2)(a_2+P)^2[(P-m)^2 - a_1m] + FK_0K\beta_1q_2P^2Z(a_2+P)^2, \\ Q_1 &= rq_1q_2P^3Z(K_0+K-2P)(a_2+P)^2(a_1+P-m)^2, \\ Q_2 &= K_0Kq_2\beta_1PZ^2(q_1-q_2)(a_2+P)^2[(P-m)^2 - a_1m] - K_0K\beta_1q_2^2a_1P^2Z^2(a_2+P)^2, \\ Q_3 &= -K_0Kq_1^2\beta_2a_1P^3Z(a_2+P)^2 + K_0Kq_1^2\theta a_2P^2Z(a_1+P-m)^2, \\ Q_4 &= K_0Kq_1^2P^3Z[\beta_2a_1(a_2+P)^2 - \theta a_2(a_1+P-m)^2], \\ Q_5 &= -rq_1q_2P^3Z(a_2+P)^2(a_1+P-m)^2(K_0+K-2P), \end{aligned}$$

$$Q_6 = -\beta_1 K_0 K q_1 P Z^2 (q_1 - q_2) (a_2 + P)^2 [(P - m)^2 - a_1 m] + K_0 K \beta_1 q_2^2 a_1 P^2 Z^2 (a_2 + P)^2.$$

Now removing E from Eqs (5.3) and (5.4), we obtain

$$r \left(1 - \frac{P}{K}\right) \left(\frac{P}{K_0} - 1\right) - \frac{\beta_1 (P - m)}{P(a_1 + P - m)} Z = \frac{q_1}{q_2} \left[\frac{\beta_2 (P - m)}{a_1 + P - m} - d - \frac{\theta P}{a_2 + P} \right], \quad (5.9)$$

which is the optimal trajectory of the steady state given by the optimal solutions $P = P_\delta$, $Z = Z_\delta$. Then, we substitute λ_1 and λ_2 into Eq (5.5) and obtain optimal equilibrium level of effort given by

$$E_\delta = \frac{\delta \lambda_2 (a_1 + P - m) + \lambda_1 \beta_1 (P - m)}{p_2 q_2 (a_1 + P - m)} e^{\delta t}. \quad (5.10)$$

By solving Eqs (5.9) and (5.10) when assigning a certain value to δ , we can obtain the optimal equilibrium level (P_δ, Z_δ) .

The optimal harvesting effort at any time is determined by

$$E(t) = \begin{cases} E_{\min}, & \frac{\partial H}{\partial E} < 0, \\ E_\delta, & \frac{\partial H}{\partial E} = 0, \\ E_{\max}, & \frac{\partial H}{\partial E} > 0, \end{cases}$$

where E_{\min} is the minimum harvesting effort.

6. Numerical simulations

In this section, we will do some numerical simulations to verify the theoretical results. The values of all parameters in system (2.2) are sourced from Table 1. And the initial values of the system (2.2) are assumed to be $P(0) = 400$, $Z(0) = 800$.

Table 1. Parameter estimation of system (2.2).

Parameter	Description	Value
r	The intrinsic growth rate of phytoplankton	0.8
K	Environmental carrying capacity of phytoplankton	500
β_1	Predation rate of zooplankton	1
β_2	Growth efficiency of zooplankton	0.89
d	Mortality rate of zooplankton	0.2
a_1	Half saturation constant	0.5
a_2	Half saturation constant	9.2
θ	Toxin production rate	0.39
m	Refuge capacity	defaulted
K_0	Critical level of the growth of phytoplankton	25
q_1	Catchability coefficient of phytoplankton	0.03
q_2	Catchability coefficient of zooplankton	0.03
E	Combined harvesting effort	10

First, according to the case (e) in Theorem 3.2, there is a positive equilibrium $E^* = (343.2, 1004.4)$ of system (2.2). In addition, if assumptions (H2) and (H3) hold, the positive equilibrium E^* is locally asymptotically stable (Figure 2). We can clearly see that the solution curve starting from different initial values eventually tends to the black point, that is, the positive equilibrium $E^* = (343.2, 1004.4)$ (Figure 2b).

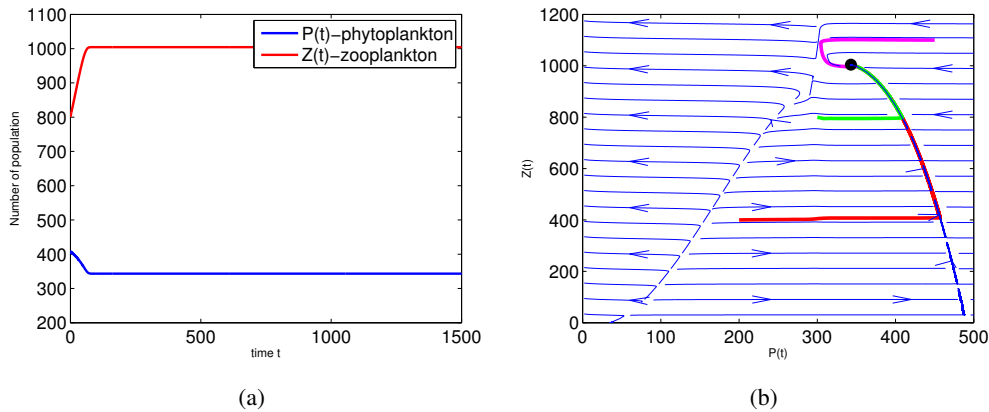


Figure 2. When $m = 300$, local asymptotic stability of the positive equilibrium $E^* = (343.2, 1004.4)$ of system (2.2). (a) Stable behavior $P(t)$ and $Z(t)$ with time, (b) phase portrait.

Second, for the parameter values given above, we obtain $m_H = 286.6$ and the first Lyapunov number $l_1 = -0.0187 < 0$ by simple calculation. This indicates that system (2.2) has a stable limit cycle around the positive equilibrium E^* . This result is shown in Figure 3.

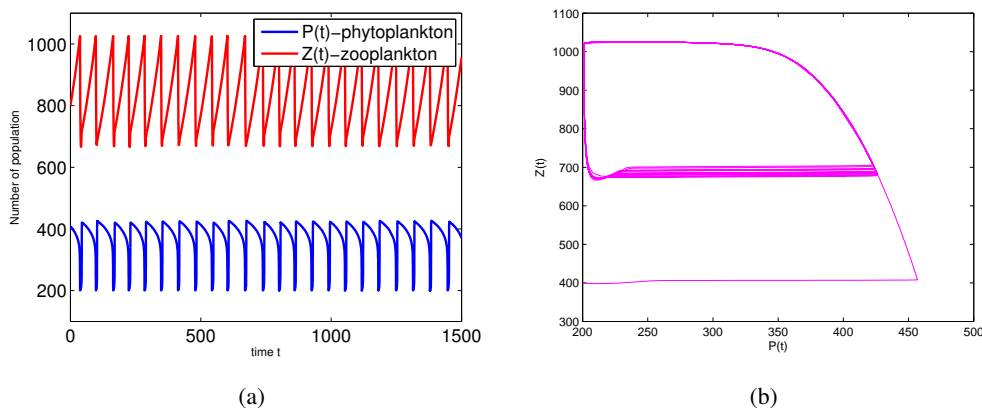


Figure 3. When $m = 200$, system (2.2) undergoes a supercritical Hopf bifurcation around positive equilibrium E^* . (a) Dynamical behavior of $P(t)$ and $Z(t)$, (b) phase portrait.

Next, for the given parameters, we get $\omega_0 = 0.2354$ and $\tau_0 = 0.011$ when $\tau \neq 0$. According to Theorem 4.4, we obtain that the positive equilibrium $E^* = (343.2, 1004.4)$ is locally asymptotically stable when $\tau = 0.01 < \tau_0 = 0.011$ (Figure 4). Then, we choose the value of τ as $\tau = 0.1 > \tau_0 = 0.011$

and obtain that $C_1(0) = -0.0019 - 0.0152i$, $\mu_2 = 6.4946 \times 10^{-21} > 0$, $\beta = -0.0038 < 0$, $T_2 = 0.0065 > 0$ by using formula (4.37). On the basis of Theorem 4.6, we know that the system (2.1) experiences a supercritical Hopf bifurcation when τ passes its critical value τ_0 . Other than this, system (2.1) has stable bifurcating periodic solutions and the period of the bifurcating periodic solutions is increasing. We clearly see from the Figure 5 that the positive equilibrium $E^* = (343.2, 1004.4)$ is destabilized through a Hopf bifurcation.

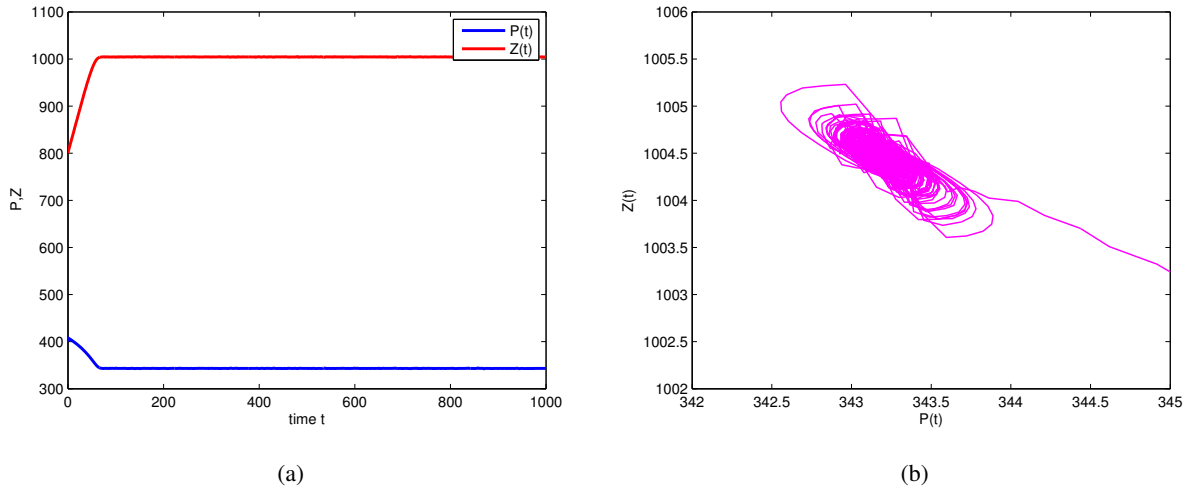


Figure 4. When $\tau = 0.01 < \tau_0 = 0.011$, the positive equilibrium $E^* = (343.2, 1004.4)$ is locally asymptotically stable. (a) Stable behavior of $P(t)$ and $Z(t)$, (b) phase portrait.

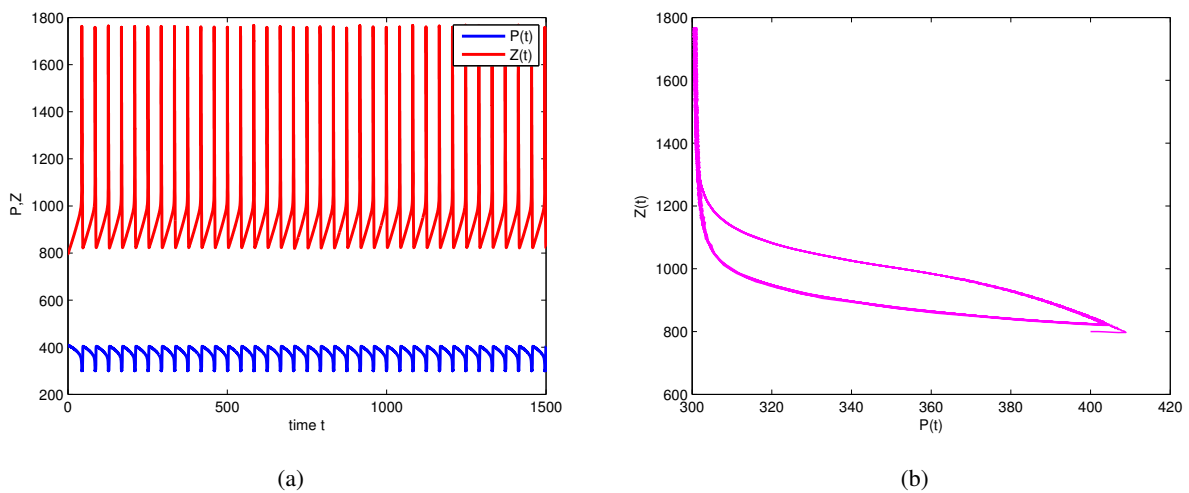


Figure 5. When $\tau = 0.1 > \tau_0 = 0.011$, the positive equilibrium $E^* = (343.2, 1004.4)$ is destabilized through a Hopf bifurcation. (a) Dynamical behavior of $P(t)$ and $Z(t)$, (b) phase portrait.

Finally, we consider the following parameter values: $p_1 = 1$, $p_2 = 5$, $\hat{c} = 0.1$, $\delta = 0.2$ and the other parameters remain unchanged. Figure 6 shows the solution curve of the state variables with time. Figure 7a,b show the variation curve of the adjoint variables λ_1 and λ_2 , respectively. It is easy to see from the Figure 7 that the adjoint variables λ_1 and λ_2 tend ultimately to 0 with the increase of time. In addition, the effect of the constant refuge capacity of phytoplankton population m , the critical value of the growth of phytoplankton K_0 , the half saturation constant a_1 and the half saturation constant a_2 on the optimal harvesting effort in Figure 8. It is not difficult to see that the optimal harvesting effort decreases as m increases (Figure 8a), but the optimal harvesting effort increases as K_0 , a_1 and a_2 increase (Figure 8b–d).

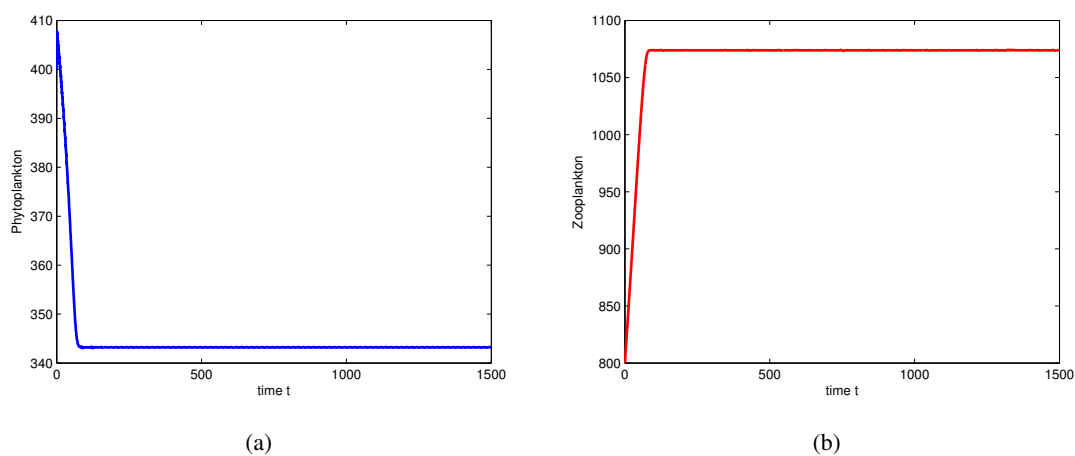


Figure 6. The solution curve of state variables of the control system (2.2): (a) phytoplankton, (b) zooplankton.

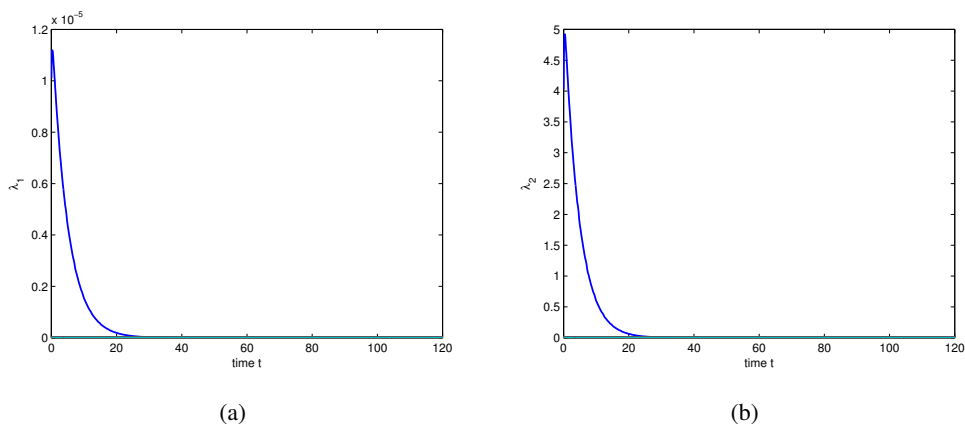


Figure 7. The curve of the adjoint variables of system (2.2): (a) λ_1 , (b) λ_2 .

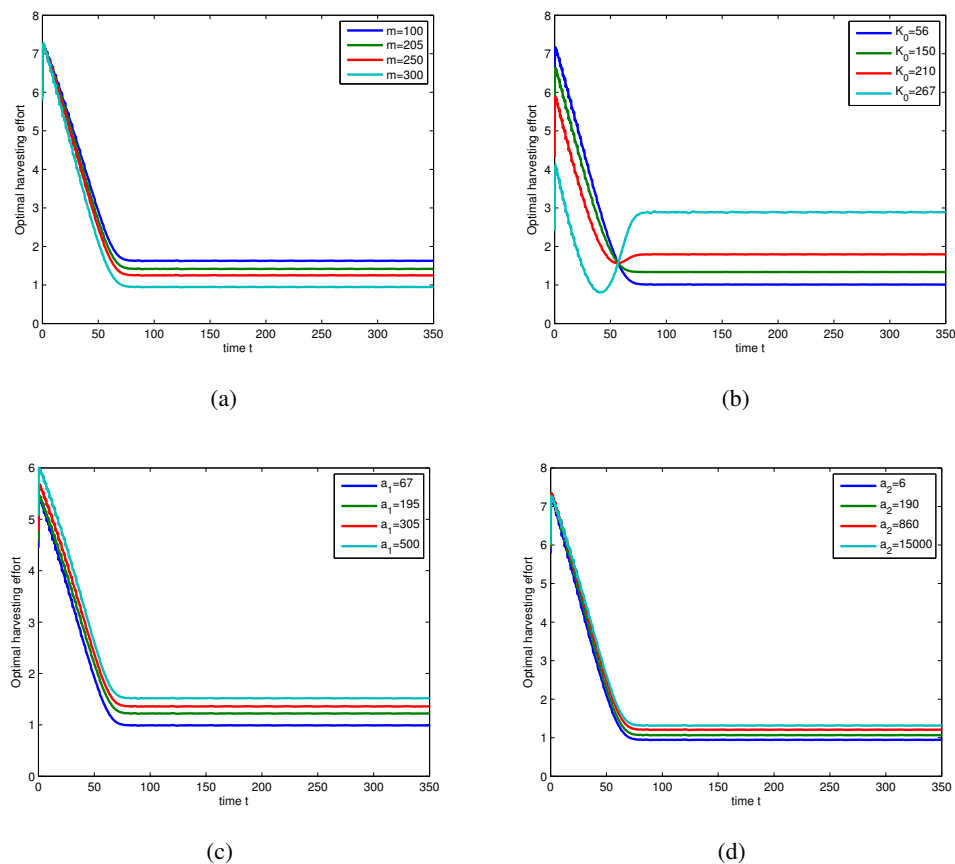


Figure 8. The curve of the optimal harvesting of system (2.2) with respect to different parameters: (a) constant refuge capacity of phytoplankton population m , (b) critical value of the growth of phytoplankton K_0 , (c) half saturation constant a_1 , (d) half saturation constant a_2 .

7. Conclusions

In this paper, we studied the dynamics of the phytoplankton-zooplankton system in which the growth of phytoplankton is affected by Allee effect and the growth of zooplankton is affected by maturation delay. For the positive equilibrium E^* , due to the expression of the trace and determinant of its Jacobian matrix are very complicated, so the stability of the positive equilibrium E^* is verified by the combination of theoretical derivation and numerical simulation. When the maturation delay of zooplankton is not considered, the strict mathematical proof of the existence of Hopf bifurcation is given by using the relevant bifurcation theory. In addition, we derive the expression of the first Lyapunov number l_1 that determines the direction of the Hopf bifurcation. Furthermore, when considering the maturation period of phytoplankton, we obtain some properties of the Hopf bifurcation through the normal form theory and the center manifold theorem. Because plankton has certain economic significance, we also consider linear harvesting for both phytoplankton and zooplankton and obtain the optimal harvesting policy by Pontryagin's Maximum Principle in this paper.

Comparing with the study on the phytoplankton-zooplankton system in [19], our model is more

realistically by considering Allee effect and maturation delay into growth of phytoplankton and zooplankton, respectively. What's more, both populations are linearly harvested. For phytoplankton, we can predict the stability of system (2.2) at the positive equilibrium E^* by selecting the parameter values to determine the critical value m_H of the refuge capacity m . When we choose $m > m_H$, system (2.2) is stable at the positive equilibrium E^* , but when $m < m_H$, system (2.2) experiences a supercritical Hopf bifurcation and loses stability. This indicates that increasing the refuge capacity m is important for balancing ecosystem. For zooplankton, we regard the maturation delay of zooplankton as a parameter to predict the stability of system (2.1). The positive equilibrium E^* of the system (2.1) is stable, if the maturation delay of zooplankton $\tau < \tau_0$, which implies that the densities of phytoplankton and zooplankton will tend to be stable constants, indicating the ecosystem is balanced. But system (2.1) undergoes a Hopf bifurcation at the positive equilibrium E^* if the the maturation delay of the zooplankton $\tau > \tau_0$, which shows that the densities of phytoplankton and zooplankton species will oscillate periodically.

Through the above analysis, we know that if the refuge capacity $m < m_H$ of phytoplankton or the maturation delay $\tau > \tau_0$ of zooplankton, the densities of phytoplankton and zooplankton will change periodically, which indicates the system becomes unstable. If the refuge capacity $m > m_H$ of phytoplankton or the maturation delay $\tau < \tau_0$ of zooplankton, the densities of phytoplankton and zooplankton will tend to be stable, which suggests a state of ecosystem balance. Thus, we can maintain the ecological balance by adjusting the values of the refuge capacity m of phytoplankton and the maturation delay τ of zooplankton determined by system parameters.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. J. Norberg, D. Deangelis, Temperature effects on stocks and stability of a phytoplankton-zooplankton model and the dependence on light and nutrients, *Ecol. Model.*, **95** (1997), 75–86.
2. B. Mukhopadhyay, R. Bhattacharyya, Modelling phytoplankton allelopathy in a nutrient-plankton model with spatial heterogeneity, *Ecol. Model.*, **198** (2006), 163–173.
3. Y. F. Lv, Y. Z. Pei, S. J. Gao, C. G. Li, Harvesting of a phytoplankton-zooplankton model, *Nonlinear Anal.: Real World Appl.*, **11** (2010), 3608–3619.
4. M. Bengfort, U. Feudel, F. M. Hilker, H. Malchow, Plankton blooms and patchiness generated by heterogeneous physical environments, *Ecol. Complex.*, **20** (2014), 185–194.
5. S. Rana, S. Samanta, S. Bhattacharya, K. Alkhaled, A. Goswami, J. Chattopadhyay, The effect of nanoparticles on plankton dynamics: A mathematical model, *Biosystems*, **127** (2015), 28–41.
6. X. Y. Meng, Y. Q. Wu, Bifurcation and control in a singular phytoplankton-zooplankton-fish model with nonlinear fish harvesting and taxation, *Int. J. Bifurcat. Chaos*, **28** (2018), 1850042.

7. S. Chakraborty, S. Roy, J. Chattopadhyay, Nutrient-limited toxin production and the dynamics of two phytoplankton in culture media: A mathematical model, *Ecol. Model.*, **213** (2008), 191–201.
8. T. Saha, M. Bandyopadhyay, Dynamical analysis of toxin producing phytoplankton-zooplankton interactions, *Nonlinear Anal.: Real World Appl.*, **10** (2009), 314–332.
9. M. Banerjee, E. Venturino, A phytoplankton-toxic phytoplankton-zooplankton model, *Ecol. Complex.*, **8** (2011), 239–248.
10. M. Javidi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, *Ecol. Model.*, **318** (2015), 8–18.
11. R. J. Han, B. X. Dai, Cross-diffusion induced Turing instability and amplitude equation for a toxic-phytoplankton-zooplankton model with nonmonotonic functional response, *Int. J. Bifurcat. Chaos*, **27** (2017), 1750088.
12. R. J. Han, B. X. Dai, Spatiotemporal pattern formation and selection induced by nonlinear cross-diffusion in a toxic-phytoplankton-zooplankton model with Allee effect, *Nonlinear Anal.: Real World Appl.*, **45** (2019), 822–853.
13. W. Zheng, J. Sugie, Global asymptotic stability and equiasymptotic stability for a time-varying phytoplankton-zooplankton-fish system, *Nonlinear Anal.: Real World Appl.*, **46** (2019), 116–136.
14. T. K. Kar, Stability analysis of a prey-predator model incorporating a prey refuge, *Commun. Nonlinear Sci. Numer. Si.*, **10** (2005), 681–691.
15. L. J. Chen, F. D. Chen, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge, *Nonlinear Anal.: Real World Appl.*, **11** (2010), 246–252.
16. J. P. Tripathi, S. Abbas, M. Thakur, Dynamical analysis of a prey-predator model with Beddington-DeAngelis type function response incorporating a prey refuge, *Nonlinear Dynam.*, **80** (2015), 177–196.
17. J. Ghosh, B. Sahoo, S. Poria, Prey-predator dynamics with prey refuge providing additional food to predator, *Chaos Soliton Fract.*, **96** (2017), 110–119.
18. G. P. Samanta, A. Maiti, M. Das, Stability analysis of a prey-predator fractional order model incorporating prey refuge, *Ecol. Genet. Genomi.*, **7-8** (2018), 33–46.
19. J. Li, Y. Z. Song, H. Wan, H. P. Zhu, Dynamical analysis of a toxin-producing phytoplankton-zooplankton model with refuge, *Math. Biosci. Eng.*, **14** (2017), 529–557.
20. L. Berec, E. Angulo, F. Courchamp, Multiple Allee effects and population management, *Trends Ecol. Evol.*, **22** (2007), 185–191.
21. C. Franck, C. B. Tim, G. Bryan, Inverse density dependence and the Allee effect, *Trends Ecol. Evol.*, **14** (1999), 405–410.
22. F. Courchamp, L. Berec, J. Gascoigne, *Allee Effects in Ecology and Conservation*, Oxford University Press, Oxford, 2008.
23. C. B. Tim, D. Gaynor, G. M. McIlrath, A. Maccoll, R. Kansky, P. Chadwick, et al., Predation, group size and mortality in a cooperative mongoose, *Suricata suricatta*, *J. Anim. Ecol.*, **68** (1999), 672–683.
24. M. S. Mooring, T. A. Fitzpatrick, T. T. Nishihira, D. D. Reisig, Vigilance, predation risk, and the Allee effect in desert bighorn sheep, *J. Wildlife Manage.*, **68** (2004), 519–532.
25. D. J. Rinella, M. S. Wipfli, C. A. Stricker, R. A. Heintz, M. J. Rinella, Pacific salmon (*Oncorhynchus* spp.) runs and consumer fitness: Growth and energy storage in stream-dwelling salmonids increase with salmon spawner density, *Can. J. Fish. Aquat. Sci.*, **69** (2011), 73–84.
26. P. A. Stephens, W. J. Sutherland, R. P. Freckleton, What is the Allee effect?, *Oikos*, **87** (1999), 185–190.

27. A. Maiti, P. Sen, D. Manna, G. P. Samanta, A predator-prey system with herd behaviour and strong Allee effect, *Nonlinear Dyn. Syst. Theory*, **16** (2016), 86–101.
28. H. F. Huo, W. T. Li, J. J. Nieto, Periodic solutions of delayed predator-prey model with the Beddington-DeAngelis functional response, *Chaos Soliton Fract.*, **33** (2007), 505–512.
29. W. Y. Wang, L. J. Pei, Stability and Hopf bifurcation of a delayed ratio-dependent predator-prey system, *Acta Mech. Sinica*, **27** (2011), 285–296.
30. G. D. Zhang, Y. Shen, B. S. Chen, Positive periodic solutions in a non-selective harvesting predator-prey model with multiple delays, *J. Math. Anal. Appl.*, **395** (2012), 298–306.
31. H. Xiang, Y. Y. Wang, H. F. Huo, Analysis of the binge drinking models with demographics and nonlinear infectivity on networks, *J. Appl. Anal. Comput.*, **8** (2018), 1535–1554.
32. R. Chinnathambi, F. A. Rihan, Stability of fractional-order prey-predator system with time-delay and Monod-Haldane functional response, *Nonlinear Dynam.*, **92** (2018), 1–12.
33. X. Y. Meng, J. G. Wang, Analysis of a delayed diffusive model with Beddington-DeAngelis functional response, *Int. J. Biomath.*, **12** (2019), 1950047.
34. X. Y. Meng, J. G. Wang, H. F. Huo, Dynamical behaviour of a nutrient-plankton model with Holling type IV, delay, and harvesting, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 9232590.
35. X. Y. Meng, Y. Q. Wu, Bifurcation analysis in a singular Beddington-DeAngelis predator-prey model with two delays and nonlinear predator harvesting, *Math. Biosci. Eng.*, **16** (2019), 2668–2696.
36. X. B. Zhang, H. Y. Zhao, Bifurcation and optimal harvesting of a diffusive predator-prey system with delays and interval biological parameters, *J. Theor. Biol.*, **363** (2014), 390–403.
37. M. Sen, P. Srinivasu, M. Banerjee, Global dynamics of an additional food provided predator-prey system with constant harvest in predators, *Appl. Math. Comput.*, **250** (2015), 19–211.
38. H. Y. Zhao, X. X. Huang, X. B. Zhang, Hopf bifurcation and harvesting control of a bioeconomic plankton model with delay and diffusion terms, *Physica A*, **421** (2015), 300–315.
39. D. P. Hu, H. J. Cao, Stability and bifurcation analysis in a predator-prey system with Michaelis-Menten type predator harvesting, *Nonlinear Anal.: Real World Appl.*, **33** (2017), 58–82.
40. X. Y. Meng, N. N. Qin, H. F. Huo, Dynamics analysis of a predator-prey system with harvesting prey and disease in prey species, *J. Biol. Dynam.*, **12** (2018), 342–374.
41. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.
42. L. Perko, *Differential Equations and Dynamical Systems*, Springer Science and Business Media, New York, 2013.
43. B. D. Hassard, N. D. Kazarinoff, Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981.
44. L. S. Pontryagin, *Mathematical Theory of Optimal Processes*, Routledge, London, 2018.



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