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Research article

Discrete-time population dynamics on the state space of measures

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Abstract: If the individual state space of a structured population is given by a metric space S, measures μ on the σ-algebra of Borel subsets T of S offer a modeling tool with a natural interpretation: $\mu(T)$ is the number of individuals with structural characteristics in the set T. A discrete-time population model is given by a *population turnover map* F on the cone of finite nonnegative Borel measures that maps the structural population distribution of a given year to the one of the next year. Under suitable assumptions, F has a first order approximation at the zero measure (the extinction fixed point), which is a positive linear operator on the ordered vector space of real measures and can be interpreted as a *basic population turnover operator*. For a semelparous population, it can be identified with the *next generation operator*. A spectral radius can be defined by the usual Gelfand formula.We investigate in how far it serves as a threshold parameter between population extinction and population persistence. The variation norm on the space of measures is too strong to give the basic turnover operator enough compactness that its spectral radius is an eigenvalue associated with a positive eigenmeasure. A suitable alternative is the flat norm (also known as (dual) bounded Lipschitz norm), which, as a trade-off, makes the basic turnover operator only continuous on the cone of nonnegative measures but not on the whole space of real measures.

Keywords: extinction; basic reproduction number; spectral radius; flat norm; measure kernels; Feller property; eigenvectors; fixed point

1. Introduction

In the most elementary population models, all members of a population are equal, except that they may live at different times. In reality, they are different in many aspects like spatial location, chronological age, development stage, size etc., and the distributions of their individual characteristics have a considerable influence on the population dynamics.

Let the state space of individual characteristics [1] be described by a metric space (S, d), i.e., a nonempty set S with a metric d. A point in S gives an individual's characteristics, and d describes how close the characteristics of two different individuals are to each other. From a population point of view, S represents the population structure. To be specific, we consider the spatial structure of a population, but it could be any other structure given by one or several individual characteristics.

A natural way to describe the structural distribution is to consider a collection $\mathcal{B}$ of subsets of S and *set-functions* $\mu : \mathcal{B} \to \mathbb{R}_+$ with the interpretation that $\mu(T)$ is the number of individuals located within the set $T \in \mathcal{B}$. The total population size is given by $\mu(S)$. Here are a few properties of $\mathcal{B}$ and μ that are compatible with this interpretation:

Definition 1.1.

- $\emptyset \in \mathcal{B}$, $\quad \mu(\emptyset) = 0$.
- If $T_1, T_2 \in \mathcal{B}$, then $T_1 \cup T_2 \in \mathcal{B}$ and $T_1 \cap T_2 \in \mathcal{B}$ and $\mu(T_1 \cup T_2) = \mu(T_1) + \mu(T_2) - \mu(T_1 \cap T_2)$.
- If $T \in \mathcal{B}$, then $S \setminus T \in \mathcal{B}$ and $\mu(S \setminus T) = \mu(S) - \mu(T)$.
- If (T_n) is a sequence of sets in $\mathcal{B}$ and $T_n \subseteq T_{n+1}$ for all $n \in \mathbb{N}$, then $T := \bigcup_{n\in\mathbb{N}} T_n \in \mathcal{B}$ and $\mu(T) = \lim_{n\to\infty} \mu(T_n)$.

If $\mathcal{B}$ and $\mu : \mathcal{B} \to \mathbb{R}_+$ have these properties, $\mathcal{B}$ is called a σ-algebra and μ is called a *nonnegative measure*; $\mu : \mathcal{B} \to \mathbb{R}$ with these properties is called a *real measure*.

See any of the many books covering measure theory like [2–6] [7, App.B] and Section 6. The population state space is the set (actually cone) of nonnegative measures, denoted by $\mathcal{M}_+(S)$.

In spite of this natural setting, most of the time the special case is considered that $\mu(T) = \int_T f(x)\nu(dx)$, $T \in \mathcal{B}$, with some master measure ν and a density f. The dynamics of the population are then discussed in terms of densities [8, 9], maybe because densities are perceived as mathematically easier than measures.

Still, there is a substantial body of work in population or other dynamics in terms of measures. The reasons, among others, are the natural conceptuality of the setting [1, 10–13] or the circumstance that, even if the initial conditions are given by densities, the solution can become measure-valued in finite time [14, 15] [16, Sec.9], or in the limit as time tends to infinity [17–20], or via some other limiting procedure [21]. Another reason is the inclusion of random structural transitions ([22] and the references therein) like random mutations [17, 23, 24] or, in the case of this paper, yearly random spatial movements described by a measure kernel $P : \mathcal{B} \times S \to \mathbb{R}_+$ (Sections 9 and 10). Here, $P(T, s)$ is the probability that an individual that is at $s \in S$ at the beginning at the year is still alive at the end of the year and located within the set $T \in \mathcal{B}$.

Similarly as in [25,26], we want to consider $\mathcal{M}_+(S)$ as the closed cone of the ordered normed vector space $\mathcal{M}(S)$ of real measures (Section 6) in order to take advantage of the rich mathematical theory that is available in this framework [27–31, 33, 34] (see Section 5), in particular the theory of homogeneous order-preserving operators and their spectral radius [35–41, 43–46] (see Sections 4.2 and 7).

Since S is a metric space, $\mathcal{M}(S)$ is an ordered normed vector space not only under the variation norm, but also under the weaker flat (aka dual bounded Lipschitz) norm (Section 6.1). The variation norm makes $\mathcal{M}(S)$ a Banach lattice which seems as good a framework as one can wish, except that compact sets are hard to come by. The flat norm provides applicable conditions for the compactness of subsets of $\mathcal{M}_+(S)$ but has other challenges: $\mathcal{M}(S)$ is only complete if S is a uniformly discrete metric space and $\mathcal{M}_+(S)$ is only complete if S is complete (Theorem 6.12, Theorem 6.14, [25, 26, 48]).

Most of the population dynamics models with measure-valued solutions are formulated in continuous time. There, the flat norm offers the additional advantage that certain solutions depend continuously on time, while they do not do so with respect to the variation norm, and is also useful in obtaining approximation results ([26, Sec.7] and the references therein).

Discrete-time models bypass the often quite difficult existence and uniqueness problems that continuous-time models with measures face [11, 14, 23, 49–51] and, almost from the outset, are able to concentrate on threshold conditions for population extinction and survival, and more readily deal with the mathematical techniques that help with this fundamental issue (Sections 8 and 12, [16, 36, 47]). Biologically, discrete-time models are appropriate idealization for populations with very short reproductive seasons. Their essential ingredient is a population turnover map $F : \mathcal{M}_+(S) \to \mathcal{M}_+(S)$ that maps the structural population distribution at the census period of a given year to that at the census period of the next year (Section 2). If the sequence (μ_n) in $\mathcal{M}_+(S)$ is constituted by the structural distributions μ_n in the nth year, the population development is described by the difference equation

$$\mu_n = F(\mu_{n-1}), \qquad n \in \mathbb{N}, \tag{1.1}$$

with a given initial distribution μ_0.

F has a linear first order approximation A at the origin, the order derivative (Section 8.1), which can be continuously extended to all of $\mathcal{M}(S)$ with respect to the variation norm. The map A is continuous on $\mathcal{M}_+(S)$ with respect to the flat norm if and only if it is induced by a Feller kernel (Section 10); in general, it is not continuous with respect to the flat norm on all of $\mathcal{M}(S)$ (Example 10.14). The spectral radius of this *basic population turnover operator* A, denoted by r_0 and called *basic population turnover number*, is the threshold parameter that decides about population extinction: If $r_0 < 1$, the zero measure (the extinction state) is locally asymptotically stable; the extinction state is unstable and a nonzero equilibrium state exists if $r_0 > 1$ (Sections 8 and 12). To establish the latter, one shows that r_0 is an eigenvalue of the basic turnover operator (Sections 7 and 10).

2. Modeling population dynamics with measures

The dynamics of a structured population are governed by the processes of birth, death, and structural development, with the last being spatial movement in our case.

We consider a population that reproduces during a very short annual reproduction season. For ease of exposition, we assume that the population is semelparous, i.e., individuals only reproduce once during their life-time, one year after birth, and die shortly thereafter. We count the years in such a way that the census period is just before reproductive season.

Births and deaths can be affected by competition for resources. We restrict our exposition to the reduction of per capita birth rates by inner-species competition. If μ is the spatial distribution of the adult population, the competition level at the location s generated by μ, $(Q\mu)(s)$, is

$$(Q\mu)(s) = \int_S q(s,t)\mu(dt), \quad s \in S. \tag{2.1}$$

Here $q : S^2 \to \mathbb{R}_+$ is bounded and continuous and $q(s,t)$ describes the competitive influence an individual at location t exerts on an individual at s. We call $Q\mu$ the *competitive force* generated by the population distribution μ.

Let $g : S \times \mathbb{R}_+ \to \mathbb{R}_+$ be the per capita birth function, i.e., $g(s,q)$ is the per capita number of offspring produced at $s \in S$ by an adult located at s when the competition level for reproductive resources at s is

$q \in \mathbb{R}_+$. The number of neonates in a set $T \in \mathcal{B}$ is then given by

$$\nu(T) = \int_T g(s, (Q\mu)(s))\mu(ds) \tag{2.2}$$

provided that μ is the spatial distribution of adults.

Equation (2.2) defines a measure ν on $\mathcal{B}$ that represents the spatial distribution of neonates resulting from the distribution of adults, μ. The movement of individuals during a year is described by a measure kernel

$$P : \mathcal{B} \times S \to [0, 1] \tag{2.3}$$

where $P(T, s)$ is the probability that a neonate born at $s \in S$ at the beginning of the year is still alive at the end of the year and located at some point in the set T.

Definition 2.1. $P : \mathcal{B} \times S \to \mathbb{R}_+$ is called *a measure kernel*, if

- for any $s \in S$, $P(\cdot, s)$ is a measure on $\mathcal{B}$ and
- for any $T \in \mathcal{B}$, $P(T, \cdot)$ is a Borel measurable function on S.

If the measure ν represents the spatial distribution of neonates shortly after the reproductive season at the beginning of the year,

$$(\tilde{A}\nu)(T) = \int_S P(T, s)\nu(ds), \qquad T \in \mathcal{B}, \tag{2.4}$$

provides the resulting number of surviving adults at the end of the year that are located within the set T.

If there were no deaths over the year, $P(\cdot, s)$ would be a *probability measure*, $P(S, s) = 1$, for all $s \in S$. In such a case, the measure kernel P is sometimes called a *Markov kernel* or a *stochastic kernel* [2, Def.19.11]. But we will not make this unrealistic assumption but rather assume that $P(S, s) \in [0, 1]$ for all $s \in S$. Such measure kernels are called *transition kernels* in [52].

If the measure μ represents the spatial distribution of adults at the beginning of a given year, immediately before the reproductive season, the number of adults in a set $T \in \mathcal{B}$ at the end of the year (and the beginning of the next year), is given by

$$\begin{aligned} \int_S P(T, s)\nu(ds) &\overset{(2.2)}{=} \int_S P(T, s)\, g\big(s, (Q\mu)(s)\big)\, \mu(ds) \\ &=: F(\mu)(T). \end{aligned} \tag{2.5}$$

Here $Q\mu$ is the competitive force due to the population distribution μ given by (2.1). (2.5) defines a map F from $\mathcal{M}_+(S)$ to itself, called the *yearly population turnover map*. In (2.5), for $\mu \in \mathcal{M}_+(S)$, the measure kernel κ^μ,

$$\kappa^\mu(T, s) = P(T, s)\, g\big(s, (Q\mu)(s)\big), \qquad T \in \mathcal{B}, s \in S, \tag{2.6}$$

does not necessarily satisfy $\kappa^\mu(T, s) \in [0, 1]$ and is not a transition kernel in the sense of [52].

Though individual development (here spatial movement) is modeled stochastically by the measure kernel P, population development is ultimately modeled deterministically by the map F.

3. A general measure-theoretic framework

In the previous section, we have made some simplifying biological assumptions like semelparity of the population and competition that only affects reproduction. Instead, we may like to consider iteroparous populations and competitive effects on both reproduction and survival as we will do in a future publication. To have a general framework, let S be a non-empty set and $\mathcal{B}$ a σ-algebra of subsets of S.

Definition 3.1. A map $\kappa : \mathcal{B} \times S \to \mathbb{R}_+$ is called a *measure kernel* if
$\kappa(\cdot, s) \in \mathcal{M}_+(S)$ for all $s \in S$ and $\kappa(T, \cdot) \in M_+^b(S)$ for all $T \in \mathcal{B}$.

Here $M^b(B)$ is the Banach space of bounded measurable functions with the supremum norm and $M_+^b(S)$ the cone on nonnegative function in $M^b(S)$.

We consider yearly turnover maps F of the following form,

$$F(\mu)(T) = \int_S \kappa^\mu(T, s)\, \mu(ds), \qquad \mu \in \mathcal{M}_+(S), \qquad T \in \mathcal{B}, \tag{3.1}$$

where $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ is a family of measure kernels $\kappa^\mu : \mathcal{B} \times S \to \mathbb{R}_+$. Consider a typical individual that, in a given year, is located at the point $s \in S$, and let $\mu \in \mathcal{M}_+(S)$ be the spatial distribution of the population in that year. Then, under the environment affected by the population distribution μ, $\kappa^\mu(T, s)$ is the probability that this individual is still alive and located within the set $T \in \mathcal{B}$ in the next year plus the amount of its offspring that is produced during this given year and is still alive and also located in T in the next year.

In the special case of a semelparous population, where only the offspring is accounted for, κ^μ takes the form (2.6).

If μ is the zero measure, we use the notation κ^o. κ^o is the *basic population turnover kernel* describing the yearly turnover in a competition-free environment. In the case of (2.6), $\kappa^o(T, s) = P(T, s)g(s, 0)$ for $T \in \mathcal{B}$ and $s \in S$.

Assumption 3.2. For each $\mu \in \mathcal{M}_+(S)$, κ^μ is a measure kernel and $\{\kappa^\mu(S, t); \mu \in \mathcal{M}_+(S), t \in S\}$ is a bounded subset of $\mathbb{R}$.

Standard measure-theoretic arguments imply the following result.

Proposition 3.3. *Let the Assumption 3.2 be satisfied. Then F maps $\mathcal{M}_+(S)$ into itself.*

3.1. Convolutions and spectral radius of measure kernels

The *convolution* of two measure kernels $\kappa_j : \mathcal{B} \times S \to \mathbb{R}_+$, $j = 1, 2$, is defined by

$$(\kappa_1 \star \kappa_2)(T, s) = \int_S \kappa_1(T, t)\kappa_2(dt, s), \qquad T \in \mathcal{B}, s \in S. \tag{3.2}$$

$\kappa_1 \star \kappa_2$ is again a measure kernel.

Definition 3.4. Let $\kappa : \mathcal{B} \times S \to \mathbb{R}_+$ be a measure kernel. We inductively define the *multiple convolution kernels* $\kappa^{n\star}$ by $\kappa^{1\star} = \kappa$ and $\kappa^{(n+1)\star} = \kappa^{n\star} \star \kappa$.

The *spectral radius* of the kernel κ is defined by

$$\mathbf{r}(\kappa) = \inf_{n \in \mathbb{N}} \Big(\sup_{s \in S} \kappa^{n\star}(S, s) \Big)^{1/n}. \tag{3.3}$$

We will see later that, in (3.3), $\inf_{n\in\mathbb{N}}$ can be replaced by $\lim_{n\to\infty}$. See (9.7).

Definition 3.5. The kernel family $\{\kappa^\mu; \mu \in M_+(S)\}$ is called *upper semicontinuous at the zero measure* if for any $\epsilon \in (0,1)$ there is some $\delta > 0$ such that

$$\kappa^\mu(T,s) \le (1+\epsilon)\kappa^o(T,s), \qquad T \in \mathcal{B}, s \in S,$$

for all $\mu \in \mathcal{M}_+(S)$ with $\mu(S) \le \delta$.

The kernel family $\{\kappa^\mu; \mu \in M_+(S)\}$ is called *lower semicontinuous at the zero measure* if for any $\epsilon \in (0,1)$ there is some $\delta > 0$ such that

$$\kappa^\mu(T,s) \ge (1-\epsilon)\kappa^o(T,s), \qquad T \in \mathcal{B}, s \in S,$$

for all $\mu \in \mathcal{M}_+(S)$ with $\mu(S) \le \delta$.

The kernel family $\{\kappa^\mu; \mu \in M_+(S)\}$ is called *continuous at the zero measure* if for any $\epsilon \in (0,1)$ there is some $\delta > 0$ such that

$$(1-\epsilon)\kappa^o(T,s) \le \kappa^\mu(T,s) \le (1+\epsilon)\kappa^o(T,s), \qquad T \in \mathcal{B}, s \in S,$$

for all $\mu \in \mathcal{M}_+(S)$ with $\mu(S) \le \delta$.

In a preview of results, we will showcase the spectral radius of the basic turnover kernel κ^o as a crucial threshold parameter between local stability and instability of the extinction state represented by the zero measure; $\mathbf{r}(\kappa^o)$ is called the *basic population turnover number*. For a semelparous population, the *basic turnover number* coincides with the *basic reproduction number*.

The proofs can be found in Section 11.

3.2. Local (global) stability of the zero measure in the subthreshold case

Theorem 3.6. *Make Assumption 3.2. Let the kernel family $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)$ be upper semicontinuous at the zero measure.*

(a) If $r = \mathbf{r}(\kappa^o) < 1$, the extinction state is locally asymptotically stable in the following sense:

For each $\alpha \in (r,1)$, there exist some $\delta_\alpha > 0$ and $M_\alpha \ge 1$ such that, for each solution $(\mu_n)_{n\in\mathbb{Z}_+}$ of the difference equation $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$,

$$\mu_n(S) \le M_\alpha \alpha^n \mu_0(S), \qquad n \in \mathbb{N},$$

if $\mu_0 \in \mathcal{M}_+(S)$ with $\mu_0(S) \le \delta_\alpha$.

(b) If $r = \mathbf{r}(\kappa^o) < 1$ and $\kappa^\mu(T,s) \le \kappa^o(T,s)$ for all $T \in \mathcal{B}$, $s \in S$, the extinction state is globally stable in the following sense:

For each $\alpha \in (r,1)$ there exists some $M_\alpha \ge 1$ such that for each solution $(\mu_n)_{n\in\mathbb{Z}_+}$ of the difference equation $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$, $\mu_n(S) \le M_\alpha \alpha^n \mu_0(S)$ for all $n \in \mathbb{N}$.

Recall that $\mu_n(S)$ is the total population size in the nth year.

Corollary 3.7. *Let Assumption 3.2 be satisfied. Let the family of measure kernels $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ be upper semicontinuous at the zero measure.*

Let S be the union of pairwise disjoint nonempty sets $T_1, \dots, T_m$ in $\mathcal{B}$ and

$$\beta_{jk} = \sup_{t \in T_j} \kappa^o(T_k, t), \qquad j, k = 1, \dots, m,$$

and assume that the matrix of size m with coefficients β_{jk} is irreducible and has a spectral radius $r < 1$.

Then $\mathbf{r}(\kappa^o) \le r < 1$.

(a) Further, the extinction state is locally asymptotically stable in the following sense:

For each $\alpha \in (r, 1)$, there exist some $\delta_\alpha > 0$ and $M_\alpha \ge 1$ such that, for each solution $(\mu_n)_{n \in \mathbb{Z}_+}$ of the difference equation $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$,

$$\mu_n(S) \le M_\alpha \alpha^n \mu_0(S), \qquad n \in \mathbb{N},$$

if $\mu_0 \in \mathcal{M}_+(S)$ with $\mu_0(S) \le \delta_\alpha$.

(b) If $\kappa^\mu(T, s) \le \kappa^o(T, s)$ for all $T \in \mathcal{B}$, $s \in S$ the extinction state is globally stable in the following sense:

For each $\alpha \in (r, 1)$ there exists some $M_\alpha \ge 1$ such that for each solution $(\mu_n)_{n \in \mathbb{Z}_+}$ of the difference equation $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$, $\mu_n(S) \le M_\alpha \alpha^n \mu_0(S)$ for all $n \in \mathbb{N}$.

3.3. Instability of the zero measure in the superthreshold case

The next instability result only weakly highlights the threshold role of the spectral radius of κ^o, but may be more practical and is a counterpart to Corollary 3.7.

Theorem 3.8. *Let Assumption 3.2 be satisfied. Let the family of measure kernels $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ be lower semicontinuous at the zero measure.*

Let $T_1, \dots, T_m$ be pairwise disjoint nonempty sets in $\mathcal{B}$ and

$$\alpha_{jk} = \inf_{t \in T_j} \kappa^o(T_k, t), \qquad j, k = 1, \dots, m,$$

and assume that the matrix of size m with coefficients α_{jk} has a spectral radius $r > 1$.

Then $\mathbf{r}(\kappa^o) \ge r > 1$. Further, there is some $\delta_0 > 0$ such for that any solution $(\mu_n)_{\in \mathbb{Z}_+}$ of $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$, with $\mu_0(T_k) > 0$, $k = 1, \dots, m$, there is some $n \in \mathbb{Z}_+$ with $\mu_n(S) \ge \delta_0$.

Notice that there is some $\mu_0 \in \mathcal{M}_+(S)$ with $\mu_0(T_k) > 0$ for $k = 1, \dots, m$, namely $\mu = \sum_{k=1}^m \delta_{s_k}$ with suitable points s_k from the nonempty sets T_k.

Remark 3.9. Assume that S is the union of pairwise disjoint nonempty sets $T_1, \dots, T_m$, $m \in \mathbb{N}$, such that, for $j, k = 1, \dots, m$, $\kappa^o(T_k, \cdot)$ is constant on T_j,

$$\kappa^o(T_k, s) = \alpha_{jk}, \qquad s \in T_j.$$

If the matrix of size m with coefficients α_{jk} is irreducible, then its spectral radius, r_0, equals $\mathbf{r}(\kappa^o)$ and is a threshold parameter:

If $r_0 < 1$, the zero measure is locally stable in the sense of Theorem 3.6; it is unstable in the sense of Theorem 3.8 if $r_0 > 1$.

Further $r_0 = \mathbf{r}(\kappa^o)$ is associated with an eigenfunction f in the interior of $M_+^b(S)$, $r_0 f(s) = \int_S f(t) \kappa^o(dt, s)$ for all $s \in S$.

In order to prove an instability result that shows that the stability result in Theorem 3.6 is almost sharp, we assume that S is a metric space and $\mathcal{B}$ the σ-algebra of Borel sets.

We consider the following concepts [25, 48].

Definition 3.10. Consider a subset $\mathcal{N}$ of $\mathcal{M}_+(S)$.

- $\mathcal{N}$ is called *tight* if for any $\epsilon > 0$ there exists a compact subset K of S such that $\mu(S \setminus K) < \epsilon$ for all $\mu \in \mathcal{N}$.
- A single measure $\mu \in \mathcal{M}_+(S)$ is called *tight*, and we write $\mu \in \mathcal{M}_+^t(S)$, if $\{\mu\}$ is tight.
- $\mathcal{N}$ is called *pre-tight* if for any $\epsilon > 0$ there exists a closed totally bounded subset T of S such that $\mu(S \setminus T) < \epsilon$ for all $\mu \in \mathcal{N}$.
- A single measure $\mu \in \mathcal{M}_+(S)$ is called *separable*, and we write $\mu \in \mathcal{M}_+^s(S)$, if there exists a countable subset T of S such that $\mu(S \setminus \bar{T}) = 0$.
- A single measure $\mu \in \mathcal{M}(S)$ is called *separable*, and we write $\mu \in \mathcal{M}^s(S)$, if its absolute value $|\mu|$ is separable.

By definition, a subset T of S is *totally bounded* if for any $\epsilon > 0$ there exists a finite subset K of T such that $T \subseteq \bigcup_{s\in K} U_\epsilon(s)$. Here $U_\epsilon(s) = \{t \in S\,; d(t, s) < \epsilon\}$ is the open neighborhood with center s and radius ϵ. $T \subseteq S$ is compact if and only if T is totally bounded and complete [2, Sec.3.7].

If S is a compact metric space, $\mathcal{M}_+(S)$ is trivially tight. If S is a separable metric space, $\mathcal{M}_+(S) = \mathcal{M}_+^s(S)$.

Definition 3.11. A measure kernel κ is called a *tight kernel* if $\{\kappa(\cdot, s); s \in S\}$ is a tight set of measures.

A measure kernel κ is called a *kernel of separable measures* if all measures $\kappa(\cdot, s)$, $s \in S$, are separable.

Definition 3.12. $\kappa : \mathcal{B} \times S \to \mathbb{R}_+$ is called a *Feller kernel* if

$\kappa(\cdot, s) \in \mathcal{M}_+(S)$ for all $s \in S$ and if κ has the *Feller property*

$\int_S f(y)\kappa(dy, \cdot) \in C^b(S)$ for any $f \in C^b(S)$.

Cf. [2, Sec.19.3]. See Example 10.12. By Proposition 6.3, if κ is a Feller kernel, $\kappa(U, \cdot)$ is a Borel measurable function on S for all open subsets U of S and thus for all Borel sets U in S.

Theorem 3.13. *Let Assumption 3.2 be satisfied. Let the kernel family* $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ *be lower semicontinuous at the zero measure.*

Assume that $\kappa^o = \kappa_1 + \kappa_2$ *with two Feller kernels* κ_j *of separable measures and assume that* κ_1 *is a tight kernel and* $r := \mathbf{r}(\kappa^o) > 1 \geq \mathbf{r}(\kappa_2)$.

Then there exists some eigenmeasure $\nu \in \mathcal{M}_+^s(S)$, $\nu(S) = 1$, *such that*

$$r\nu(T) = \int_S \kappa^o(T, s)\nu(ds), \qquad T \in \mathcal{B}.$$

Further, there is some $\delta_0 > 0$ *such for that any solution* $(\mu_n)_{\in\mathbb{Z}_+}$ *of* $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$, *with* ν*-positive* $\mu_0 \in \mathcal{M}_+(S)$ *there is some* $n \in \mathbb{Z}_+$ *with* $\mu_n(S) \geq \delta_0$.

$\mu_0 \in \mathcal{M}_+(S)$ is called ν-positive if there exists some $\delta > 0$ such that $\mu(T) \geq \delta\nu(T)$ for all $T \in \mathcal{B}$.

In an iteroparous population, the kernel κ_1 may be associated with reproduction and first year development and the kernel κ_2 with adult survival and adult development. If $\mathbf{r}(\kappa_2) < 1$, $\kappa_2^\infty = \sum_{n=1}^\infty \kappa_2^{n\star}$ is a measure kernel, and the measure kernel

$$\kappa_1 + \kappa_1 \star \kappa_2^\infty = \kappa^1 + \sum_{n=1}^\infty \kappa^1 \star \kappa_2^{n\star}$$

can be interpreted as *next generation kernel* and its spectral radius as *basic [53] (or inherent net [54, 55]) reproduction number*. We again like to think of $\kappa^o = \kappa_1 + \kappa_2$ as *basic population turnover kernel* and its spectral radius as *basic turnover number*; this spectral radius has also been called *inherent population growth rate* [55].

Remark 3.14. The following trichotomy holds (Theorem 7.16):

- $\mathbf{r}(\kappa_1 + \kappa_1 \star \kappa_2^\infty) > 1$ and $\mathbf{r}(\kappa_1 + \kappa_2) > 1$
  or
- $\mathbf{r}(\kappa_1 + \kappa_1 \star \kappa_2^\infty) = 1$ and $\mathbf{r}(\kappa_1 + \kappa_2) = 1$
  or
- $\mathbf{r}(\kappa_1 + \kappa_1 \star \kappa_2^\infty) < 1$ and $\mathbf{r}(\kappa_1 + \kappa_2) < 1$.

3.4. Existence of a nonzero equilibrium measure in the superthreshold case

Assumption 3.15. For all $\mu \in \mathcal{M}_+^s(S)$, the kernel κ^μ in (3.1) is a Feller kernel of separable measures.

We will derive from the preceding assumption that F maps $\mathcal{M}_+^s(S)$ into itself and from the next one that F is compact on $\mathcal{M}_+^s(S)$ with respect to the flat norm.

Assumption 3.16. If $\mathcal{N}$ is a bounded subset of $\mathcal{M}_+^s(S)$, then the set of measures $\{\kappa^\mu(\cdot, s); s \in S, \mu \in \mathcal{N}\}$ is tight and the set $\{\kappa^\mu(S, s); s \in S, \mu \in \mathcal{N}\}$ is bounded in $\mathbb{R}$.

The next assumption looks rather technical, but is often satisfied; the technicality is the prize we pay for the generality of the framework. We will derive from it that F is continuous on $\mathcal{M}_+^s(S)$ with respect to the flat norm.

The Banach space of bounded continuous real-valued functions on S with the supremum norm is denoted by $C^b(S)$ and the cone of nonnegative functions in it by $C_+^b(S)$. $\mathcal{L}$ denotes the following convex set of Lipschitz continuous functions,

$$\mathcal{L} = \{h \in [0,1]^S;\ \forall t, \tilde{t} \in S\ :\ |h(t) - h(\tilde{t})| \le d(t, \tilde{t})\}. \tag{3.4}$$

Recall that M^S denotes the set of functions from S to a set M.

Assumption 3.17. If $\mu \in \mathcal{M}_+^s(S)$ and (μ_n) is a sequence in $\mathcal{M}_+^s(S)$ such that $\int_S f d\mu_n \to \int_S f d\mu$ for all $f \in C_+^b(S)$, then, for all $h \in \mathcal{L}$,

$$\int_S h(t)\, \kappa^{\mu_n}(dt, s) \to \int_S h(t)\, \kappa^\mu(dt, s) \tag{3.5}$$

uniformly for s in every closed totally bounded subset of S.

Assumption 3.18. $\limsup_{\mu(S)\to\infty} \sup_{s\in S} \kappa^\mu(S,s) < 1$.

We will derive from this assumption that F and some perturbations of F map a bounded convex subset of $\mathcal{M}_+^s(S)$ into itself. Applying the Schauder-Tychonov fixed point theorem [58, Thm.10.1] to perturbations of F, we will derive the following fixed-point result.

Theorem 3.19. *Let the Assumptions 3.2 and 3.15 – 3.18 be satisfied and* $\mathbf{r}(\kappa^o) > 1$. *Let the kernel family* $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ *be lower semicontinuous at the zero measure.*
Then there exists a nonzero fixed point $\mu \in \mathcal{M}_+^s(S)$ *of* F, $\mu(T) = \int_S \kappa^\mu(T,s)\mu(ds)$, $T \in \mathcal{B}$.

4. Discrete-time population models in normed vector spaces

To prove the results in Section 3, we want to use the rich theory of homogeneous operators and their spectral radius [35–41, 43–46].

Given a yearly population turnover map $F : X_F \to X_F$ like in (2.5) or (3.1) on a nonempty set $X_F \ni 0$ of a real vector space X, the dynamics of the population can be studied by a difference equation

$$x_n = F(x_{n-1}), \qquad n \in \mathbb{N}, \quad x_0 \in X_F, \tag{4.1}$$

with the population structure being encoded in the set X_F [47]. The vector x_n describes the structural distribution of the population in year n while $F : X_F \to X_F$ formulates the rule how the structural distribution in a given year follows from the structural distribution of the previous year. If X is a normed vector space, the norm $\|x_n\|$ is some measure of the population size in year n.

A fundamental question (Sections 8 and 12) is as to whether or not the population dies out, $\|x_n\| \to 0$ as $n \to \infty$.

4.1. Homogeneous operators

In order to motivate the framework in which we will address this question, we assume that X_F is *a (positively) homogeneous subset* of X:

If $x \in X_F$ and $\alpha \in \mathbb{R}_+$, then $\alpha x \in X_F$.

Let us assume for the moment that F is Gateaux-differentiable at 0, i.e., that all directional derivatives

$$B(x) = \partial F(0,x) = \lim_{\mathbb{R}_+ \ni b \to 0} \frac{1}{b} F(bx), \qquad x \in X_F, \tag{4.2}$$

exist. Then $B : X_F \to X_F$ is (positively) *homogeneous* (of degree one) [36, Thm.3.1]:

If $x \in X_F$ and $\alpha \in \mathbb{R}_+$, then $B(\alpha x) = \alpha B(x)$.

Since we rarely consider homogeneity in a different sense, an operator B with this property is simply called *homogeneous*. B is a *first order approximation of* F in a weak sense, and we will need B to be a first order approximation in a stronger sense. In Section 3, $X_F = \mathcal{M}_+(S)$ and B is given by

$$(B\mu)(T) = \int_S \kappa^o(T,s)\mu(ds), \qquad \mu \in \mathcal{M}_+(S), T \in \mathcal{B},$$

with Definition 3.5 describing in what sense B is a first order approximation of F. We call B the *basic population turnover operator* because it approximates the turnover operator at low population densities.

Lemma 4.1. *Assume that there are $\delta > 0$ and $c > 0$ such $F : X_F \to X_F$ satisfies $\|F(x)\| \le c\|x\|$ for all $x \in X_F$ with $\|x\| \le \delta$. Then the Gateaux derivative B of F at 0 is bounded: $\|Bx\| \le c\|x\|$ for all $x \in X_F$.*

4.2. The spectral radius of homogeneous operators

Let $B : X_B \to X_B$ be a *homogeneous* operator: X_B is a subset of a real normed vector space X and

$$x \in X_B, \alpha \in \mathbb{R}_+ \quad \Longrightarrow \quad \alpha x \in X_B, \quad B(\alpha x) = \alpha B(x). \tag{4.3}$$

The *operator norm* of the homogenous operator B is defined as

$$\|B\| := \sup\{\|B(x)\|; x \in X_B, \|x\| \le 1\}, \tag{4.4}$$

and B is called *bounded* if this supremum exists. By the homogeneity,

$$\|B(x)\| \le \|B\| \, \|x\|, \qquad x \in X_B, \tag{4.5}$$

when B is bounded. The powers (iterates) $B^n : X_B \to X_B$ of a bounded homogeneous B are also bounded homogeneous operators and

$$\|B^{n+m}\| \le \|B\|^n \|B^m\|, \qquad n, m \in \mathbb{N}. \tag{4.6}$$

The *spectral radius* of a bounded homogeneous $B : X_B \to X_B$ is defined by the *Gelfand* formula [56]

$$\mathbf{r}(B) = \inf_{n \in \mathbb{N}} \|B^n\|^{1/n} = \lim_{n \to \infty} \|B^n\|^{1/n}. \tag{4.7}$$

The last equality is shown in the same well-known way as for a bounded linear everywhere-defined map (Theorem 3 in [57, Sec.VIII.2]). The name "spectral radius" is motivated by the following fact. Let B be a bounded linear map on X. If $X_{\mathbb{C}}$ is the complexification of X and $B_{\mathbb{C}}$ the extension of B to $X_{\mathbb{C}}$, then

$$\mathbf{r}(B) = \sup\{|\lambda|; \lambda \in \sigma(B_{\mathbb{C}})\}, \tag{4.8}$$

where $\sigma(B_{\mathbb{C}}) \subseteq \mathbb{C}$ denotes the spectrum of B [40, 56] [58, Sec.9.8].

Equation (4.8) is partially preserved for bounded homogeneous B. Recall that, for linear bounded everywhere-defined B, eigenvalues are special elements of the spectrum of B.

For homogeneous $B : X_B \to X_B$, we call $\lambda \in (0, \infty)$ an *eigenvalue* of B and $v \in X_B$ an *associated eigenvector* of B if $B(v) = \lambda v \ne 0$.

Proposition 4.2. *Let $B : X_B \to X_B$ be homogeneous and bounded, $\lambda \in (0, \infty)$, $0 \ne v \in X_B$ and $B(v) = \lambda v$. Then $\lambda \le \mathbf{r}(B)$.*

Proof. Since B is homogeneous, we can assume that $\|v\| = 1$. By induction, $\lambda^n v = B^n(v)$ for all $n \in \mathbb{N}$. Since B^n is homogeneous and bounded, $\lambda^n \le \|B^n\| \, \|v\| \le \|B^n\|$ and $\lambda \le \|B^n\|^{1/n}$ for all $n \in \mathbb{N}$. The assertion now follows from (4.7). □

Mallet-Paret and Nussbaum [38, 39] suggest an alternative definition of a spectral radius for homogeneous (not necessarily bounded) maps $B : X_B \to X_B$. First, define asymptotic least upper bounds for the geometric growth factors of B-orbits,

$$\gamma(x, B) := \gamma_B(x) := \gamma_x(B) := \limsup_{n \to \infty} \|B^n(x)\|^{1/n}, \qquad x \in X_B, \tag{4.9}$$

and then the *orbital spectral radius*

$$\mathbf{r}_o(B) = \sup_{x \in X_B} \gamma_B(x). \tag{4.10}$$

Here $\gamma_B(x) := \infty$ if the sequence $(\|B^n(x)\|^{1/n})$ is unbounded and $\mathbf{r}_o(B) = \infty$ if $\gamma_B(x) = \infty$ for some $x \in X_B$ or the set $\{\gamma_B(x); x \in X_B\}$ is unbounded. Since $\|B^n(x)\| \le \|B^n\| \, \|x\|$ by (4.6),

$$\gamma(x, B) \le \mathbf{r}(B), \quad x \in X_B, \qquad \mathbf{r}_o(B) \le \mathbf{r}(B). \tag{4.11}$$

If X_B is a closed cone in X, the number $\mathbf{r}(B)$ has been called *partial spectral radius* by Bonsall [35], X_B spectral radius by Schaefer [31, 33], and *cone spectral radius* by Nussbaum [41]. Mallet-Paret and Nussbaum [38, 39] call $\mathbf{r}(B)$ the *Bonsall cone spectral radius* and $\mathbf{r}_o(B)$ the cone spectral radius. For $x \in X_B$, the number $\gamma_B(x)$ has been called *local spectral radius* of B at x by Förster and Nagy [59].

Since $\mathbf{r}(B)$ makes sense as long as the domain of definition X_B is homogeneous (including the familiar special case $X_B = X$) and the domain of definition is part of the concept of an operator, we simply call $\mathbf{r}(B)$ the spectral radius of B and $\mathbf{r}_o(B)$ the orbital spectral radius. If a name were to be attached to $\mathbf{r}(B)$, *Gelfand spectral radius* might be as appropriate as Bonsall cone spectral radius.

The spectral radius and orbital spectral radius coincide for many applications [43]. For the purposes of this paper, the following condition for $\mathbf{r}_o(B) = \mathbf{r}(B)$ is the most relevant.

Theorem 4.3. *Let X_B be a closed homogeneous set in the normed vector space X and $B : X_B \to X_B$ be continuous, compact and homogeneous. Then $\mathbf{r}_o(B) = \mathbf{r}(B)$.*

This result is proved in [38, Thm.2.3] under the overall assumptions that X is a Banach space and X_B a closed cone in X (see Section 5), but the proof also works for the weaker assumptions in Theorem 4.3.

Various other conditions for equality are given in [38, 43]. But they require more mathematical structure than homogeneity, namely also order and monotonicity (Section 5). Gripenberg [60], gives an example for $\mathbf{r}_o(B) < \mathbf{r}(B)$.

There are at least three motivations to consider the more general situation of a bounded homogenous order-preserving operator rather than of a bounded linear positive operator. The first is of mathematical nature, namely that the Gateaux derivative (4.2) is homogeneous but not linear and that homogenous operators are not Frechet differentiable at 0 unless they are linear [36, Sec.3].

The second, biological, motivation are two-sex population models [61, Ch.4] which often use homogeneous mating functions resulting in homogeneous first order approximations of the population turnover operator [36, 44–46, 62, 63].

The third motivation comes from the population turnover map F in (2.5) that has the first order approximation

$$(B\mu)(T) = \int_S P(T, s) g(s, 0) \mu(ds), \qquad T \in \mathcal{B}, s \in S, \mu \in \mathcal{M}_+(S). \tag{4.12}$$

B can readily be extended to a linear map A on $\mathcal{M}(S)$. If $\mathcal{M}(S)$ is endowed with the variation norm (Section 6.1), A is a bounded linear operator but B lacks needed compactness properties. If S is a metric space, $\mathcal{M}(S)$ can alternatively be endowed with the flat norm which makes B continuous on $\mathcal{M}_+(S)$ (Theorem 10.4 (e)) with useful compactness properties (Proposition 10.10), but the extension A of B to $\mathcal{M}(S)$ is not bounded in general (Example 10.14).

5. Wedges, cones, and associated orders

Let X be a vector space over $\mathbb{R}$. A nonempty subset W of X is called *homogeneous*, if $\alpha x \in W$ for all $\alpha \in \mathbb{R}_+$ and $x \in W$.

Any homogeneous subset contains the vector 0. If W is a homogeneous subset of X, we use the notation

$$\dot{W} = W \setminus \{0\}. \tag{5.1}$$

X and $\{0\}$ are obvious homogeneous subsets of X.

A subset W of X is called *convex*, if $\alpha x + (1-\alpha)y \in W$ for all $x, y \in W$ and $\alpha \in (0,1)$.

Geometrically, convexity of W means that a line segment is contained in W whenever its endpoints are elements of W.

W is called *additive* if $x + y \in W$ for all $x, y \in W$. A homogeneous subset of X is convex if and only if it is additive.

5.1. Wedges and cones

$W \subseteq X$ is called a *wedge* if it is convex and homogeneous. A wedge W is called a *cone* if $W \cap (-W) = \{0\}$, i.e., $x = 0$ is the only vector $x \in X$ such that x and $-x$ are elements in W.

X is a wedge and $\{0\}$ is a cone. If $x \in X$, $\{\alpha x; \alpha \in \mathbb{R}_+\}$ is a cone.

If X is a normed vector space over $\mathbb{R}$ and a cone (wedge) W is a closed subset of X, W is called a *closed cone (wedge)*.

5.2. Partial orders

If W is a cone, the definition

$$x \le y \iff y - x \in W \tag{5.2}$$

provides a partial order $\le$ on X. We also write $y \ge x$ for $x \le y$. This order is compatible with addition and with the multiplication by positive real numbers. We recover the cone from the order by

$$W = \{x \in X; x \ge 0\}. \tag{5.3}$$

A real vector space X has many cones. When we talk about an *ordered vector space* we typically have a specific cone in mind which induces the order. This cone is denoted by X_+, and we will talk about it as the *order cone* in order to single it out from the other cones contained in X.

X is called an *ordered Banach space* if X is an ordered normed vector space which is complete. If X is a vector space of real-valued functions, X_+ typically is the cone of nonnegative functions.

5.3. Lattices

Let X be an ordered vector space with order cone X_+ and $S \subseteq X$.

S is called an inf-*semilattice* [64] (or *minihedral* [27]) if $x \wedge y = \inf\{x, y\}$ exist and are elements of S for all $x, y \in S$.

S is called a sup-*semilattice* if $x \vee y = \sup\{x, y\}$ exist and are elements of S for all $x, y \in S$. S is called a *lattice* if $x \wedge y$ and $x \vee y$ exist and are elements of S for all $x, y \in S$.

X is a *lattice* if $x \vee y$ exist for all $x, y \in X$. Since $x \wedge y = -((-x) \vee (-y))$, also $x \wedge y$ exist for all x, y in a lattice X.

Equivalently, X is a lattice if for any $x \in X$ the *absolute value*

$$|x| = x \vee (-x) = \sup\{x, -x\} \tag{5.4}$$

exists. The following connections hold if X is a lattice:

$$x \vee y = (1/2)(x + y + |x - y|), \qquad x \wedge y = (1/2)(x + y - |x - y|). \tag{5.5}$$

An ordered normed vector space X that is a lattice is called a *normed lattice* if

$$x, y \in X, |x| \le |y| \implies \|x\| \le \|y\|. \tag{5.6}$$

In a normed lattice X,

$$\big\| |x| - |y| \big\| \le \|x - y\|, \qquad x, y \in X. \tag{5.7}$$

A normed lattice that is complete is called a *Banach lattice* [2, 34].

5.4. Normal cones

The following result for normed vector spaces is well-known [27, Sec.1.2].

Theorem 5.1. *Let X be a normed vector space with wedge W. Then the following three properties are equivalent:*

(i) There exists some $\delta > 0$ such that $\|x + z\| \ge \delta$ whenever $x \in W$, $z \in W$ and $\|x\| = 1 = \|z\|$.
(ii) The norm is semi-monotonic: There exists some $M \ge 0$ such that $\|x\| \le M\|x + z\|$ for all $x, z \in W$.
(iii) There exists some $\tilde{M} \ge 0$ such that $\|x\| \le \tilde{M}\|y\|$ whenever $x \in X$, $y \in W$, and $y + x$ and $y - x$ are elements in W.

A wedge W in a normed vector space X is called

- *normal* if it satisfies one (and then all) of (i), (ii), (iii) in Theorem 5.1.

Remark 5.2. By (i), any normal wedge is a cone. We have formulated these equivalences for wedges to make clear that any of the properties (i), (ii), (iii) makes a wedge a cone. In a normed lattice, the order cone X_+ is normal (see (5.6).

5.5. Generating, total, solid, serially complete and non-flat wedges

If X is a real vector space and W a wedge in X, then

$$W - W = \{w - v; v, w \in W\} \tag{5.8}$$

is a linear subspace of X.

- W is called *generating* if $X = W - W$.

Now let W be a wedge in a normed vector space X with norm $\|\cdot\|$.

- W is called *total* if $W - W$ is dense in X.
- W is called *solid* if W contains an interior point.
- W is called *serially complete* if every series $\sum_{n=1}^{\infty} x_n$ with $x_n \in W$ and $\sum_{n=1}^{\infty} \|x_n\|$ converging in $\mathbb{R}$ converges in W.
- W is called *non-flat* [28, Sec.1.8] if there exists some $c \geq 1$ such that for any $x \in W - W$ there exist $v, w \in W$ such that $x = v - w$ and $\|v\| + \|w\| \leq c\|x\|$.
- W is called *flat* if there is a bounded sequence (x_n) in $W - W$ such that, for all sequences (v_n) and (w_n) in W with $x_n = v_n - w_n$ for all $n \in \mathbb{N}$, at least one of the sequences (v_n) or (w_n) is unbounded.

Bonsall [35] says that $W - W$ has the strict bounded decomposition property if W is non-flat.

One easily checks the following relation between the last two concepts.

Proposition 5.3. *Let W be a wedge in a normed vector space X. Then W is non-flat if and only if it is not flat.*

5.6. Operators between ordered normed vector spaces

Let X and Y be vector spaces and $B : X_B \to Y$ with $X_B \subseteq X$.

Definition 5.4. B is called *additive* if $B(x + z) = B(x) + B(z)$ for all $x, z \in X_B$ with $x + z \in X_B$.

Definition 5.5. Let Y be an ordered vector space with order cone Y_+.

- B is called *superadditive* if $B(x + z) \geq B(x) + B(z)$ for all $x, z \in X_B$ with $x + z \in X_B$.
- B is called *subadditive* if $B(x + z) \leq B(x) + B(z)$ for all $x, z \in X_B$ with $x + z \in X_B$.
  Let X_B be a convex subset of X.
- B is called a *convex operator* if $B(\alpha x + (1 - \alpha)z) \leq \alpha B(x) + (1 - \alpha)B(z)$ for all $x, z \in X_B$ and $\alpha \in (0, 1)$.
- B is called a *concave operator* if $B(\alpha x + (1 - \alpha)z) \geq \alpha B(x) + (1 - \alpha)B(z)$ for all $x, z \in X_B$ and $\alpha \in (0, 1)$.

Remark 5.6. If X_B is convex and B is homogeneous, superadditivity (subadditivity) of B is equivalent to concavity (convexity) of B.

Definition 5.7. Let X and Y be ordered normed vector spaces with order cones X_+ and Y_+. We use the same symbols $\| \cdot \|$ and $\leq$ for the norms on X and Y and for the orders induced by X_+ on X and by Y_+ on Y, respectively.

- B is called a positive operator if $B(X_B \cap X_+) \subseteq Y_+$.
- B is called *order-preserving* if $B(x) \leq B(z)$ for all $x, z \in X_B$ with $x \leq z$.
- B is called *order-preserving on* $Z \subseteq X_B$ if $B(x) \leq B(z)$ for all $x, z \in Z$ with $x \leq z$.

Example 5.8. Let X and Y be as above and $B : X_B \to Y$ with $X_+ \subseteq X_B \subseteq X$. If B is positive and superadditive, B is order-preserving.

Proof. Let $x, z \in X_B$ and $x \leq z$. By (5.2), $z - x \in X_+ \subseteq X_B$. Since B is positive, $B(z - x) \in Y_+$ and, since B is superadditive, $B(x) \leq B(x) + B(z - x) \leq B(x + (z - x)) = B(z)$. □

6. The ordered vector space of real measures

Let S be a nonempty set and $\mathcal{B}$ a σ-algebra on S.

Let $\mathcal{M}(S)$ denote the set of real measures on $\mathcal{B}$ (Definition 1.1).

$\mathcal{M}(S)$ becomes a real vector space by the definitions $(\mu + \nu)(T) = \mu(T) + \nu(T)$ and $(\alpha\mu)(T) = \alpha\mu(T)$ where $T \in \mathcal{B}$ and $\alpha \in \mathbb{R}$ and $\mu, \nu \in \mathcal{M}(S)$.

$\mathcal{M}(S)$ has the cone of all nonnegative measures, $\mathcal{M}_+(S)$. $\mathcal{M}(S)$ is a vector lattice which is even order-complete: Each subset $\mathcal{N}$ of $\mathcal{M}(S)$ which has an lower (upper) bound has an infimum (supremum). For this, it is enough to provide an infimum (largest lower bound) to any subset $\mathcal{N}$ of $\mathcal{M}_+(S)$,

$$\mu(T) = \inf\Big\{\sum_{j=1}^{m} \mu_j(T_j)\Big\}, \qquad T \in \mathcal{B}, \tag{6.1}$$

where the infimum is taken over all $m \in \mathbb{N}$, all subsets $\{\mu_1, \ldots, \mu_m\}$ of $\mathcal{N}$ and all subsets $\{T_1, \ldots, T_m\}$ of $\mathcal{B}$ such that T is their disjoint union [5, III.7.5]. The same construction works for any subset $\mathcal{N}$ with lower bound. If $\mathcal{N}$ has an upper bound, the infimum in (6.1) is replaced by the supremum.

The absolute value $|\mu|$ of a measure (in this context also called the variation of the measure) is given by $|\mu| = \sup\{\mu, -\mu\}$,

$$\begin{aligned} |\mu|(T) &= \sup\{\mu(U) - \mu(T \setminus U); \mathcal{B} \ni U \subseteq T\} \\ &= \sup\{|\mu(U)| + |\mu(T \setminus U)|; \mathcal{B} \ni U \subseteq T\} = \sup\Big\{\sum_{j=1}^{n} |\mu(T_j)|\Big\}, \end{aligned} \tag{6.2}$$

where the supremum is taken over all $n \in \mathbb{N}$ and subsets $\{T_1, \ldots, T_n\}$ of $\mathcal{B}$ such that T is its disjoint union [2, Cor.10.54 & Thm.10.56].

The following is a consequence of the Vitali-Hahn-Saks Theorem [5, III.7] [57, II.2], but can also be proved by more elementary means.

Corollary 6.1. *Let (μ_n) be an increasing sequence in $\mathcal{M}_+(S)$ such that $(\mu_n(S))$ is a bounded sequence in $\mathbb{R}$. Then $\mu(T) = \lim_{n\to\infty} \mu_n(T)$ exists for every $T \in \mathcal{B}$ and provides a measure $\mu \in \mathcal{M}_+(S)$.*

6.1. Measures under the variation norm and the flat norm

The *variation norm* (also called *total variation*) on $\mathcal{M}(S)$ is defined by

$$\|\mu\|_\sharp = |\mu|(S), \qquad \mu \in \mathcal{M}(S), \tag{6.3}$$

where $|\mu|$ is the absolute value of μ defined by (6.2).

If $\mu \in \mathcal{M}_+(S)$, $\|\mu\|_\sharp = \mu(S)$. So the variation norm is additive and order-preserving on $\mathcal{M}_+(S)$, and $\mathcal{M}_+(S)$ is a normal cone. The variation norm makes $\mathcal{M}(S)$ a Banach lattice (Section 5.3); in particular, $\mathcal{M}_+(S)$ is a non-flat generating cone: Every real-valued measure μ can be written as the difference of its positive and negative variation, $\mu = \mu_+ - \mu_-$, and $\|\mu_\pm\|_\sharp \le \|\mu\|_\sharp$. By Corollary 6.1, the cone $\mathcal{M}_+(S)$ is serially complete (Section 5.5). The variation norm is equivalent to the supremum norm $\|\mu\|_\infty = \sup_{T\in\mathcal{B}} |\mu(T)|$, and the two norms are equal on $\mathcal{M}_+(S)$.

Let (S, d) be a metric space. $\mathcal{B}$ now denotes the *Borel* σ-algebra of S which is the smallest σ-algebra that contains all open and closed sets. The sets in the Borel σ-algebra are called *Borel sets*. In a metric space, the Borel σ-algebra is also the smallest σ-algebra for which all (bounded) continuous functions are measurable [4, Thm.7.1.1]. This second σ-algebra is often [4] but not always [2] called the *Baire-σ*-algebra.

The following is a summary of results needed later. For more details, we refer to [26]. Many of the results can already been found in [4, 48]. See also [22, 25].

The subsequent lemma guarantees that the two σ-algebras mentioned above coincide in a metric space for all common definitions of Baire sets. Recall $\mathcal{L}$ in (3.4).

Lemma 6.2. *$\mathcal{B}$ is the smallest σ-algebra that contains all sets $f^{-1}(\{0\})$, $f \in \mathcal{L}$.*

Proposition 6.3. *Let T be a closed subset of S. Then there exists a decreasing sequence of Lipschitz continuous functions (f_n) with values between 0 and 1 such that $f_n \to \chi_T$ pointwise.*

Let T be a open subset of S. Then there exists an increasing sequence of Lipschitz continuous functions (f_n) with values between 0 and 1 such that $f_n \to \chi_T$ pointwise.

We introduce the following functional on $\mathcal{M}(S)$,

$$\|\mu\|_\flat = \sup_{f\in\mathcal{L}} \Big| \int_S f d\mu \Big|, \tag{6.4}$$

with $\mathcal{L}$ in (3.4). $\|\cdot\|_\flat$ is a norm on $\mathcal{M}(S)$ [26], which we call the *flat norm*, and

$$\|\mu\|_\flat \le \|\mu\|_\sharp, \qquad \mu \in \mathcal{M}(S). \tag{6.5}$$

In the literature, definitions different from (6.4) are used [4, 50, 51, 66] that lead to equivalent norms. For instance, $[0, 1]^S$ is replaced by $[-1, 1]^S$. Also different names are used for the flat norm or its equivalent definitions. For details see [26].

All the definitions have in common that

$$\|\mu\|_\flat = \mu(S) = \|\mu\|_\sharp, \qquad \mu \in \mathcal{M}_+(S). \tag{6.6}$$

This implies that the flat norm is additive and order-preserving on $\mathcal{M}_+(S)$.

In the following, all topological notions concerning $\mathcal{M}(S)$ and $\mathcal{M}_+(S)$ are meant with respect to the flat norm unless it is explicitly said otherwise.

Theorem 6.4. *$\mathcal{M}_+(S)$ is a generating, normal, closed, and serially complete cone.*

Serial completeness (Section 5.5) follows from Corollary 6.1.

Lemma 6.5. *Let (S, d) be a metric space. For $x \in S$, let δ_x denote the Dirac measure at x. Then $1 = \|\delta_x\|_\flat$ and, for $y, x \in S$,*

$$\|\delta_x - \delta_y\|_\flat = \min\{1, d(x, y)\}.$$

6.2. Convergence in $\mathcal{M}_+(S)$

Recall Definition 3.10.

Theorem 6.6. *Let (μ_n) in $\mathcal{M}_+(S)$ and $\mu \in \mathcal{M}_+^s(S)$. Equivalent are*

(i) $\|\mu_n - \mu\|_\flat \to 0$,

(ii) $\int_S f\,d(\mu_n - \mu) \to 0$ *for all continuous functions* $f \in C^b(S)$,
(iii) $\int_S f\,d(\mu_n - \mu) \to 0$ *for all Lipschitz continuous functions* $f : S \to [0, 1]$.

Cf. [4, Thm.11.3.3].

Corollary 6.7. $\mathcal{M}_+^s(S)$ *is a closed cone of* $\mathcal{M}(S)$.

The next result follows from [25, Thm.3.9] and [26, Thm.4.19] and its proof.

Theorem 6.8. *The set of measures* $\sum_{j=1}^n q_j\delta_{s_j}$ *with* $n \in \mathbb{N}$, $q_j \in \mathbb{Q}_+$ *and* $s_j \in S$ *is dense in* $\mathcal{M}_+^s(S)$.
The set of measures $\sum_{j=1}^n q_j\delta_{s_j}$ *with* $n \in \mathbb{N}$, $q_j \in \mathbb{Q}$ *and* $s_j \in S$ *is dense in* $\mathcal{M}^s(S)$. *If* (S, d) *is separable, then* $\mathcal{M}(S)$ *is separable.*

For a function $f : S \to \mathbb{R}$ that is bounded below define [2, Thm.3.13]

$$[f]_k(s) = \inf\{f(t) + kd(s, t);\ t \in S\}, \qquad s \in S, k \in \mathbb{N}. \tag{6.7}$$

Then $\inf f(S) \le [f]_k(s) \le [f]_{k+1}(s) \le f(s)$ for all $k \in \mathbb{N}$, and $[f]_k$ is Lipschitz continuous with k being a Lipschitz constant. If f is lower semicontinuous on S, $[f]_k \to f$ pointwise on S; if f is uniformly continuous, $[f]_k \to f$ uniformly on S.

Lemma 6.9. *Let* $\mathcal{F}$ *be an equicontinuous family of functions* $f : S \to \mathbb{R}_+$ *[6, Def.8.3]. For each* $f \in \mathcal{F}$, *let* $([f]_k)$ *be the approximation (6.7). Then, for each* $s \in S$ *for which* $\{f(s); f \in \mathcal{F}\}$ *is bounded,* $\sup_{f\in\mathcal{F}} \big|f(s) - [f]_k(s)\big| \to 0$ *as* $k \to \infty$.

Proof. Let $s \in S$ and $\{f(s); f \in \mathcal{F}\}$ be bounded. Suppose that $\sup_{f\in\mathcal{F}} \big|f(s) - [f]_k(s)\big| \to 0$ as $k \to \infty$ does not hold. Since $f(s) \ge [f]_k(s)$, there exists some $\epsilon > 0$ and a sequence (k_n) in $\mathbb{N}$ such that $k_n \to \infty$ as $n \to \infty$ and $\sup_{f\in\mathcal{F}} (f(s) - [f]_{k_n}(s)) > \epsilon$ for all $n \in \mathbb{N}$. Then there exist $f_n \in \mathcal{F}$ such that

$$f_n(s) > [f_n]_{k_n}(s) + \epsilon, \qquad n \in \mathbb{N}.$$

By (6.7), there exists a sequence $(t_n) \in S$ such that

$$f_n(s) > f_n(t_n) + k_n d(s, t_n) + \epsilon, \qquad n \in \mathbb{N}.$$

Since $k_n \to \infty$ and $\{f_n(s); n \in \mathbb{N}\}$ is bounded, $d(s, t_n) \to 0$ as $n \to \infty$. Since $\{f_n; n \in \mathbb{N}\} \subseteq \mathcal{F}$ is equicontinuous, $f_n(s) - f_n(t_n) \to 0$ as $n \to \infty$ which implies $0 \ge \epsilon$, a contradiction. □

Proposition 6.10. *Let* $\mathcal{F}$ *be an equicontinuous family of functions* $f : S \to \mathbb{R}_+$ *such that* $\{f(s); f \in \mathcal{F}, s \in S\}$ *is bounded. Let* $\mu \in \mathcal{M}(S)$ *and* (μ_n) *be a sequence in* $\mathcal{M}_+(S)$ *such that* $\|\mu_n - \mu\|_\flat \to 0$ *as* $n \to \infty$. *Then* $\int_S f d\mu_n \to \int_S f d\mu$ *as* $n \to \infty$ *uniformly for* $f \in \mathcal{F}$.

Proof. We can assume that $0 \le f(s) \le 1$ for all $f \in \mathcal{F}$ and $s \in S$. For $f \in \mathcal{F}$, let $([f]_k)$ be the approximation in (6.7). Then

$$0 \le [f]_k(s) \le f(s) \le 1, \qquad k \in \mathbb{N},\ f \in \mathcal{F},\ s \in S. \tag{6.8}$$

By Lemma 6.9,

$$h_k(s) := \sup_{f\in\mathcal{F}} \big|f(s) - [f]_k(s)\big| \to 0, \qquad k \to \infty, s \in S.$$

Since $\{f - [f]_k; f \in \mathcal{F}\}$ is equicontinuous, h_k is continuous and thus integrable. By (6.8), $h_k(s) \le \sup_{f\in\mathcal{F}} f(s) \le 1$. By the dominated convergence theorem,

$$\sup_{f\in\mathcal{F}}\Big(\int_S f d\mu - \int_S [f]_k d\mu\Big) = \sup_{f\in\mathcal{F}} \int_S \big|f - [f]_k\big| d\mu \le \int_S h_k d\mu \to 0, \qquad k \to \infty.$$

Let $\epsilon > 0$. Then there exists $k \in \mathbb{N}$ such that

$$0 \le \int_S f d\mu - \int_S [f]_k d\mu \le \frac{\epsilon}{4}, \qquad f \in \mathcal{F}.$$

For all $n \in \mathbb{N}$,,

$$\int_S f d\mu \le \int_S [f]_k d\mu + \frac{\epsilon}{4} = \int_S [f]_k d(\mu - \mu_n) + \int_S [f]_k d\mu_n + \frac{\epsilon}{4}.$$

Since k is a Lipschitz constant for $[f]_k$ and $[f]_k \le f$,

$$\int_S f d\mu \le k\|\mu - \mu_n\|_\flat + \int_S f d\mu_n + \frac{\epsilon}{4}.$$

Since $\|\mu-\mu_n\|_\flat \to 0$ as $n \to \infty$, there exists some $N = N_k \in \mathbb{N}$ such that $\|\mu-\mu_n\| \le \epsilon/(4k)$ for all $n \ge N_k$ and so

$$\int_S f d\mu \le \int_S f d\mu_n + \frac{\epsilon}{2}, \qquad n \ge N_k, f \in \mathcal{F}. \tag{6.9}$$

Define $g_f(s) = 1 - f(s)$ for all $s \in S$ and all $f \in \mathcal{F}$. Then $\{g_f; f \in \mathcal{F}\}$ is an equicontinuous family in $C_{b+}(S)$ and, by (6.8), $g_f(s) \in [0, 1]$ for all $f \in \mathcal{F}$, $s \in S$. By (6.9), applied to $\{g_f; f \in \mathcal{F}\}$ instead of $\mathcal{F}$, there exists some $N \in \mathbb{N}$ such that

$$\mu(S) - \int_S f d\mu = \int_S g_f d\mu \le \int_S g_f d\mu_n + \frac{\epsilon}{2} \le \mu_n(S) - \int_S f d\mu_n + \frac{\epsilon}{2}, \qquad n \ge N,$$

for all $f \in \mathcal{F}$. We rearrange

$$\int_S f d\mu \ge \int_S f d\mu_n - \frac{\epsilon}{2} + \mu(S) - \mu_n(S), \qquad n \ge N, f \in \mathcal{F}.$$

Since $\mu_n(S) \to \mu(S)$ as $n \to \infty$, there is some $\tilde{N} \in \mathbb{N}$ such that

$$\int_S f d\mu \ge \int_S f d\mu_n - \epsilon, \qquad n \ge \tilde{N}, f \in \mathcal{F}.$$

In combination with (6.9), $\int_S f d\mu_n \to \int_S f d\mu$ uniformly for $f \in \mathcal{F}$. □

6.3. Compactness and completeness in $\mathcal{M}_+(S)$

Theorem 6.11. *Let (μ_n) be a tight sequence in $\mathcal{M}_+(S)$ such that $(\mu_n(S))$ is bounded. Then (μ_n) has a converging subsequence (with the limit measure being tight as well).*

Theorem 6.12. *If S is not uniformly discrete (for any $\epsilon > 0$ there exist $s, t \in S$ with $0 < d(s, t) < \epsilon$), then the ordered normed vector space $\mathcal{M}(S)$ is not complete.*

Proposition 6.13. *Let $\mathcal{N} \subseteq \mathcal{M}_+^s(S)$ be a totally bounded set of separable measures. Then $\mathcal{N}$ is pre-tight and, if S is complete, tight.*

Theorem 6.14 ([25, Thm.3.8]). *$\mathcal{M}_+^s(S)$ is complete if and only if S is complete.*

7. Homogeneous operators on ordered normed vector spaces

In the following, let X be an ordered normed vector space with closed order cone X_+.

7.1. Upper estimates for growth factors

In this subsection, let $B : X_B \to X_B$ be a homogeneous operator on a homogeneous set X_B contained in X_+.

Lemma 7.1 (Bonsall [35])**.** *Let (a_n) be an unbounded sequence in $\mathbb{R}_+$. Then there exists a sequence (n_j) in $\mathbb{N}$ such that $n_j \to \infty$ and*

$$a_{n_j} \to \infty, \quad j \to \infty, \qquad a_k \le a_{n_j}, \qquad k = 1, \dots, n_j, \quad j \in \mathbb{N}.$$

Proof. For each $j \in \mathbb{N}$, choose $n_j \le j$ such that $a_{n_j} = \max\{a_1, \dots, a_j\}$. Since (a_n) is unbounded, $n_j \to \infty$ and $a_{n_j} \to \infty$ as $j \to \infty$. By construction, $a_k \le a_{n_j}$ for $k = 1, \dots, j \ge n_j$. □

Lemma 7.2. *Let X be an ordered normed vector space with closed order cone X_+. Let $B : X_B \to X_B$ be a homogeneous operator on a homogeneous set X_B contained in X_+ and let some power of B be compact.*

Further let $x \in X_B$ and (y_n) be a bounded sequence in X_+. Let $m \in \mathbb{N}$, and $B^n(x) \le y_n$ for all $n \in \mathbb{N}$, $n \ge m$.

Then the sequence $(B^n(x))$ is bounded.

Proof. Assume that $x \in X_B$ and $(B^n(x))$ is unbounded. Set $a_n = \|B^n(x)\|$. By Lemma 7.1, there exists a sequence (n_j) in $\mathbb{N}$ such that

$$a_{n_j} \to \infty, \quad j \to \infty, \qquad a_k \le a_{n_j}, \qquad k = 1, \dots, n_j, \quad j \in \mathbb{N}.$$

Set $w_j = \frac{B^{n_j}(x)}{a_{n_j}}$. Then $w_j \in X_B$ and $\|w_j\| = 1$. Choose $\ell \in \mathbb{N}$ such that B^ℓ is compact. Since B is homogeneous, $w_j = B^\ell(v_j)$ with $v_j = \frac{B^{n_j-\ell}(x)}{a_{n_j}} \in X_B$. By the properties of the (a_n), $\|v_j\| \le 1$. After choosing a subsequence, $w_j \to w$ with $w \in X_+$, $\|w\| = 1$. By assumption, after choosing subsequences again, $w_j \le \frac{y_{n_j}}{a_{n_j}}$ for $j \in \mathbb{N}$. Since (y_{n_j}) is bounded and $a_{n_j} \to \infty$, $\frac{y_{n_j}}{a_{n_j}} \to 0$. Since X_+ is closed, $w = \lim_{j\to\infty} w_j \le 0$. Since $w \in X_+$ and X_+ is a cone, $w = 0$ contradicting $\|w\| = 1$. □

Recall the large-time bound of the geometric growth factor for initial value $u \in X_B$,

$$\gamma_B(u) := \limsup_{n\to\infty} \|B^n(u)\|^{1/n}. \tag{7.1}$$

Lemma 7.3. *Let X be an ordered normed vector space with closed order cone X_+. Let $B : X_B \to X_B$ be a homogeneous operator on a homogeneous set X_B contained in X_+ and let some power of B be compact. Further let (x_n) be a bounded sequence in X_+, x a point in X_B, $m, k \in \mathbb{N}$, $\lambda \ge 0$, and $B^{n+m}(x) \le \lambda^{n+k} x_n$ for all $n \in \mathbb{N}$, $n \ge m$. Then $\gamma_B(x) \le \lambda$.*

Proof. If $\lambda = 0$, then $B^{n+m}(x) = 0$ for all $n \in \mathbb{N}$ and $\gamma^B(x) = 0$.

So we can assume that $\lambda > 0$. Set $C(x) = \lambda^{-1}B(x)$. Then C is homogeneous, and some power of C is compact. Further,

$$C^{n+m}(x) = \lambda^{-(n+m)}B^{n+m}(x) \leq \lambda^{k-m}x_n, \qquad n \in \mathbb{N},$$

with the right hand side of this inequality forming a bounded sequence. By Lemma 7.2, applied to C, $\gamma^C(x) \leq 1$. By definition of the growth bounds, (4.9), and the homogeneity of B, $\gamma^B(x) \leq \lambda$. □

7.2. *A uniform boundedness principle*

Theorem 7.4. *Let X be an ordered normed vector space with closed order cone X_+ and Z be an ordered normed vector space with a normal order cone Z_+. Let $X_\mathcal{H} \subseteq X_+$ be a serially complete cone and $\mathcal{H}$ be a family of homogeneous order-preserving operators $H : X_\mathcal{H} \to Z_+$.*

If $\{\|H(x)\|; H \in \mathcal{H}\}$ is bounded for each $x \in X_\mathcal{H}$, then $\{\|H\|; H \in \mathcal{H}\}$ is bounded.

Proof. Assume that $\{\|H(x)\|; H \in \mathcal{H}\}$ is bounded for each $x \in X_\mathcal{H}$, but $\{\|H\|; H \in \mathcal{H}\}$ is not bounded. Then, for every $n \in \mathbb{N}$, there exists some $H_n \in \mathcal{H}$ such that $\|H_n\| > n^4$ and, by (4.4), some $x_n \in X_\mathcal{H}$ with $\|x_n\| \leq 1$ such that $\|H_n(x_n)\| > n^4$. Since $X_\mathcal{H}$ is serially complete, $x = \sum_{n=1}^\infty n^{-2}x_n$ converges and is an element in $X_\mathcal{H} \subseteq X_+$ by the definitions in Section 5.5. Since X_+ is closed, $x \geq n^{-2}x_n$ for all $n \in \mathbb{N}$. Since each H_n is order-preserving and homogeneous,

$$H_n(x) \geq n^{-2}H_n(x_n), \qquad n \in \mathbb{N}.$$

Since Z_+ is normal, there exists some $c > 0$ such that

$$c\|H_n(x)\| \geq \|n^{-2}H_n(x_n)\| \geq n^2,$$

a contradiction. □

7.3. *A series characterization of the spectral radius*

We give a characterization of Datko/Pazy type. The proof uses an adaption of the one of [65, Prop.9.4] in combination with Theorem 7.4. See [65] for bibliographic notes.

Theorem 7.5. *Let X be an ordered normed vector space with normal order cone X_+. Let B be a homogeneous order-preserving operator $B : X_B \to X_B$ on a serially complete cone $X_B \subseteq X_+$. Then the following are equivalent for $\lambda > 0$ and $p > 0$:*

(i) $\mathbf{r}(B) < \lambda$. (ii) $\displaystyle\sum_{n=1}^\infty \lambda^{-pn}\|B^n\|^p < \infty$.

(iii) $\displaystyle\sum_{n=1}^\infty \lambda^{-pn}\|B^n(x)\|^p < \infty, \qquad x \in X_B.$

Proof. Since $\mathbf{r}(B) = \lambda$ if and only if $\mathbf{r}((1/\lambda)B) = 1$ by (4.7), it is sufficient to prove the statement for $\lambda = 1$.

(i) ⇒ (ii). Let $\mathbf{r}(B) < 1$. Choose some $r \in (\mathbf{r}(B), 1)$. Then there exists some $c > 1$ such that $\|B^n\| \leq cr^n$ for all $n \in \mathbb{N}$ and (ii) follows.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i). (iii) implies that $\|B^n(x)\| \to 0$ for each $x \in X_B$. By the Uniform Boundedness Theorem 7.4, there is some $M > 0$ such that $\|B^n\| \le M$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ and $x \in X_B$,

$$n\|B^n(x)\|^p = \sum_{k=1}^{n} \|B^n(x)\|^p = \sum_{k=1}^{n} \|B^{n-k}(B^k(x))\|^p.$$

By (4.5),

$$n\|B^n(x)\|^p \le \sum_{k=1}^{n} \|B^{n-k}\|^p \; \|B^k(x)\|^p \le M^p \sum_{k=1}^{\infty} \|B^k(x)\|^p < \infty.$$

So, for each $x \in X_B$, $\{\|n^{1/p}B^n(x)\|, n \in \mathbb{N}\}$ is bounded. Again by Theorem 7.4, $\{\|n^{1/p}B^n\|, n \in \mathbb{N}\}$ is bounded. So there exists some $n \in \mathbb{N}$ such that $\|B^n\| < 1$. By (4.7), $\mathbf{r}(B) = \inf_{n\in\mathbb{N}} \|B^n\|^{1/n} < 1$. □

Corollary 7.6. *Let X be an ordered normed vector space with normal order cone X_+. Let B be a homogeneous order-preserving operator $B : X_B \to X_B$ on a serially complete cone $X_B \subseteq X_+$. Then, for all $p > 0$,*

$$\begin{aligned}\mathbf{r}(B) &= \inf\Big\{\lambda > 0; \sum_{n=1}^{\infty} \lambda^{-pn}\|B^n(x)\|^p < \infty, \quad x \in X_B\Big\}\\ &= \inf\Big\{\lambda > 0; \sum_{n=1}^{\infty} \lambda^{-pn}\|B^n\|^p < \infty\Big\}.\end{aligned}$$

Corollary 7.7. *Let X be an ordered normed vector space with normal closed order cone X_+. Let B be a homogeneous order-preserving operator $B : X_B \to X_B$ on a serially complete cone $X_B \subseteq X_+$. Let $r = \mathbf{r}(B) > 0$. Then there exists some $x \in X_B$ such that $\sum_{n=1}^{\infty} r^{-n}\|B^n(x)\| = \infty$.*

7.4. Eigenvectors of compact homogeneous operators

In the following, let X be an ordered normed vector space with closed order cone X_+ and $B : X_B \to X_B$ be a homogeneous, compact and order-preserving operator on a closed cone X_B contained in X_+.

Theorem 7.8. *Assume that $B : X_B \to X_B$ is compact and continuous and that $r = \mathbf{r}(B) > 0$. Then there exists $x \in X_B$, $\|x\| = 1$, such that $B(x) = rx$.*

Notice that this theorem does not assume that X_+ is complete and can be applied to $\mathcal{M}^s(S)$ with the flat norm even if the metric space S and so $\mathcal{M}^s_+(S)$ is not complete (Theorem 6.14). If X_+ is complete, the compactness assumption for B can considerably be relaxed [39].

Theorem 7.8 is proved in [42] under the additional assumption that X_+ is normal. The normality assumption can be avoided by using Lemma 7.3.

To set the stage for the proof of Theorem 7.8, we recall (7.1).

Proposition 7.9. *Let $B : X_B \to X_B$ be compact and continuous and let $u \in \dot{X}_B$. Then there exist $x \in X_B$, $\|x\| = 1$, and $\lambda \ge \gamma_u(B)$ such that $B(x) + u = \lambda x$.*

The idea of perturbing B and using one of the classical fixed point theorems seems as old as the theory of positive operators [27, Sec.2.2] [30, Sec.3] [67, Thm.3.6].

Proof. For $u \in \dot{X}_B$, define

$$B_u(x) = B(x) + u, \qquad x \in X_B.$$

Then $B_u(x) \ge u$ and $\{\|B_u(x)\|; x \in X_B\}$ is bounded away from 0 because X_+ is closed. We define

$$K(x) = \frac{B_u(x)}{\|B_u(x)\|}, \qquad x \in X_B.$$

Since B is continuous on X_B, K is continuous on X_B. K maps the set $C = X_B \cap \bar{U}_1$, $\bar{U}_1 = \{x \in X; \|x\| \le 1\}$, into itself. C is a closed convex subset of X and $K(C)$ has compact closure. By Tychonov's fixed point theorem [58, Thm.10.1], K has a fixed point $v \in X_B$, $\|v\| = 1$,

$$B(v) + u = \lambda v, \qquad \lambda = \|B(v) + u\| > 0. \tag{7.2}$$

Since $B(v) \in X_+$, $u \le \lambda v$. Since $u \in X_+$, $B(v) \le \lambda v$. Since B is order preserving and homogeneous, $B^n(v) \le \lambda^n v$ for all $n \in \mathbb{N}$. Hence

$$B^n(u) \le \lambda^n \lambda v, \qquad n \in \mathbb{N}. \tag{7.3}$$

Since B is compact, $\gamma_B(u) \le \lambda$ by Lemma 7.3. □

Recall that the orbital spectral radius is defined by $\mathbf{r}_o(B) = \sup_{x \in X_B} \gamma_B(x)$.

Proof of Theorem 7.8. Since B is compact, $\mathbf{r}_o(B) = \mathbf{r}(B) > 0$ by Theorem 4.3.

Since $\gamma_B(\alpha u) = \gamma_B(u)$ for all $u \in X_B$, $\alpha > 0$, we can choose a sequence (u_n) in $\dot{X}_B$ such that $\|u_n\| \to 0$ and $\gamma_B(u_n) \to \mathbf{r}_o(B) > 0$. By Proposition 7.9, for each $n \in \mathbb{N}$, there exist $x_n \in X_B$ such that $\|x_n\| = 1$ and $\lambda_n \ge \gamma_B(u_n)$ such that

$$\lambda_n x_n = B(x_n) + u_n. \tag{7.4}$$

Since B is compact, after choosing a subsequence, $B(x_n)$ converges to some $y \in X_B$. The corresponding subsequence of (λ_n) is bounded and $\lambda_n \to \lambda \ge \mathbf{r}_o(B) > 0$ after choosing another subsequence. Since $u_n \to 0$, by (7.4) $x_n \to x \in X_B$ with $\|x\| = 1$ and, since B is continuous at x, $\lambda x = B(x)$. By Proposition 4.2, $\lambda = \mathbf{r}_o(B) = \mathbf{r}(B)$. □

7.5. *Eigenvectors for additive homogeneous maps with some compactness*

Corollary 7.10. *Assume that $B : X_B \to X_B$ is additive and continuous and some power of B is compact and $r = \mathbf{r}(B) > 0$.*

Then there exists some $x \in X_B$, $\|x\| = 1$, such that $B(x) = rx$.

Proof. Let $m \in \mathbb{N}$ such that B^m is compact. B^m is also continuous, homogeneous and order-preserving and $\mathbf{r}(B^m) = (\mathbf{r}(B))^m = r^m > 0$. By Theorem 7.8, there exists some $y \in \dot{X}_B$ such that $B^m y = r^m y$. Set $x = \sum_{j=0}^{m-1} r^{-j} B^j y$ [28, Thm.9.3] [41, Thm.2.2]. Since B is additive,

$$B(x) = \sum_{j=0}^{m-1} r^{-j} B^{j+1}(y) = \sum_{j=1}^{m} r^{1-j} B^j(y) + ry = rx.$$ □

For a moment, we consider the more general situation that X is a normed vector space and W is a wedge in X. Then $\tilde{X} = W - W$ is a linear subspace of X. The following construction is well-known from the literature [32, 40] and will be put to new use.

Define a seminorm on $\tilde{X}$ by

$$\|x\|^{\sim} = \inf\{\|v\| + \|w\|; x = v - w, v, w \in W\}. \tag{7.5}$$

We have

$$\|x\| \le \|x\|^{\sim}, \quad x \in \tilde{X}, \qquad \|x\| = \|x\|^{\sim}, \quad x \in W. \tag{7.6}$$

The inequality shows that $\|\cdot\|^{\sim}$ is a norm and follows from $\|v - w\| \le \|v\| + \|w\|$. For $x \in W$, $x = x - 0$ and so $\|x\| = \|x\|^{\sim}$. The inequality also shows that W is closed under $\|\cdot\|^{\sim}$ if it is closed under the original norm. The equality of the norms on W shows that W is normal under $\|\cdot\|^{\sim}$ if it is normal under $\|\cdot\|$.

Remark 7.11. W is a generating non-flat cone of $\tilde{X}$ under $\|\cdot\|^{\sim}$.

Proposition 7.12. *If W is a serially complete wedge in X under $\|\cdot\|$, then $\tilde{X}$ is a Banach space under $\|\cdot\|^{\sim}$.*

Proof. We use the characterization of a Banach space in terms of series: A normed vector space is a Banach space if and only if any absolutely convergent series converges in the space. [6, Prop.9.3].

Let (x_k) be a sequence in $\tilde{X}$ such that $\sum_{n=1}^{\infty} \|x_n\|^{\sim}$ converges in $\mathbb{R}$. For any $n \in \mathbb{N}$, by (7.5), there exist $v_n, w_n \in W$ such that $x_n = v_n - w_n$ and $\|v_n\| + \|w_n\| < \|x_n\|^{\sim} + 2^{-n}$. Since W is serially complete, $v = \sum_{n=1}^{\infty} v_n$ and $w = \sum_{n=1}^{\infty} w_n$ converge in W with respect to $\|\cdot\|$. Since $\|\cdot\|$ and $\|\cdot\|^{\sim}$ coincide on W, these series also converge with respect to $\|\cdot\|^{\sim}$. So $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty}(v_n - w_n)$ converges with respect to $\|\cdot\|^{\sim}$. □

Corollary 7.13 ([28, Thm.1.5]). *Let X be an ordered normed Banach space and the order cone X_+ be closed and generating. Then X_+ is non-flat. In particular, there exists some $c > 0$ such that for all $x \in X$ there exist $v, w \in X_+$ such that $x = v - w$ and $\|v\| + \|w\| \le c\|x\|$.*

Proof. Since $W = X_+$ is generating, $X = \tilde{X}$ as sets. By (7.6), the identity is continuous from X with $\|\cdot\|^{\sim}$ to X with $\|\cdot\|$. Since X is a Banach space under both norms (Proposition 7.12), the identity is continuous from X with $\|\cdot\|$ to X with $\|\cdot\|^{\sim}$ by the open mapping theorem. By Remark 7.11, X_+ is a non-flat cone with respect to $\|\cdot\|$. □

Example 7.14. Let S be a non-empty set and $\mathcal{B}$ a σ- algebra of subsets of S. Let $\|\cdot\|$ be a norm on $\mathcal{M}(S)$ such that $\|\mu\| = \mu(S)$ for all $\mu \in \mathcal{M}_+(S)$. Then $\|\cdot\|^{\sim}$ is the variation norm on $\mathcal{M}(S)$.

Let $B : W \to W$ be a homogeneous additive operator on W. We define an extension $A : \tilde{X} \to \tilde{X}$ by

$$Ax = Bv - Bw, \qquad x = v - w \in \tilde{X}, \qquad v, w \in W.$$

Since B is additive, the definition does not depend on the choice of v and w. Indeed, if $v - w = \tilde{v} - \tilde{w}$ for $v, \tilde{v}, w, \tilde{w} \in W$, then $v + \tilde{w} = \tilde{v} + w \in W$ and $B(v) + B(\tilde{w}) = B(\tilde{v}) + B(w)$ and $B(v) - B(w) = B(\tilde{v}) - B(\tilde{w})$. It is obvious from the definition that $A : \tilde{X} \to \tilde{X}$.

One readily checks that A is additive and (positively) homogeneous on $\tilde{X}$. Let $\alpha < 0$. Then

$$\begin{aligned} A(\alpha x) =& A(\alpha v - \alpha w) = A((-\alpha)w - (-\alpha)v) \\ =& B((-\alpha w) - B((-\alpha)v) = -\alpha B(w) + \alpha B(v) = \alpha Ax. \end{aligned}$$

This shows that A is linear. The linear extension A is uniquely determined by B.

Proposition 7.15. *Let $B : W \to W$ be a bounded homogeneous additive operator on W. Then the linear extension $A : \tilde{X} \to \tilde{X}$ is bounded with respect to $\|\cdot\|^\sim$ and $\|A\|^\sim = \|B\|$ for the respective operator norms. This equality translates to the respective spectral radii.*

Proof. We can assume that B is not the zero operator on W. Otherwise, A is the zero operator on $\tilde{X}$ and the assertion is valid.

Let $x \in \tilde{X}$ and $x = v - w$ with $v, w \in W$. Then $Ax = B(v) - B(w)$ with

$$\|B(v)\| + \|B(w)\| \leq \|B\|(\|v\| + \|w\|).$$

Since $B(v), B(w) \in W$, by (7.5),

$$\|A(x)\|^\sim \leq \|B\|(\|v\| + \|w\|).$$

Now $\|B\| > 0$ and so

$$\|v\| + \|w\| \geq \|Ax\|^\sim / \|B\|.$$

Since $x = v - w$, again by (7.5), $\|x\|^\sim \geq \|Ax\|^\sim/\|B\|$. This implies that A is bounded with respect to $\|\cdot\|^\sim$ and $\|A\|^\sim \leq \|B\|$. The oppositive inequality is obvious because A extends B, and we obtain equality. □

The next result is an extension of [68, Thm.3.10] which, in turn, is a generalization of [54, Thm.3] to infinite dimensions.

Theorem 7.16. *Let X be an ordered normed vector space and let its order cone X_+ be closed, normal and serially complete. Let Q and B be bounded homogeneous additive operators on X_+, $\mathbf{r}(Q) < 1$. Then $\mathbf{r}(Q + B) - 1$ and $\mathbf{r}(B \sum_{n=0}^\infty Q^n) - 1$ have the same sign.*

Proof of Theorem 7.16. By Proposition 7.12, Remark 7.11 and the preceding remarks, $\tilde{X}$ under $\|\cdot\|^\sim$ is an ordered Banach space and $W = X_+$ is a normal generating closed cone of $\tilde{X}$ with $\|\cdot\|^\sim$. Let $\tilde{Q}$ be the bounded linear extension of Q to $\tilde{X}$ and A the bounded linear extension of B to $\tilde{X}$ with $\|\cdot\|^\sim$. Then $\tilde{Q} + A$ is the extension of $Q + B$ to $\tilde{X}$ and $A(\mathbb{I} - \tilde{Q})^{-1}$ the extension of $B \sum_{n=0}^\infty Q^n$ to $\tilde{X}$. By Proposition 7.15, the respective spectral radii of the extensions are equal to those of the original operators. By [68, Thm.3.10], $\mathbf{r}(\tilde{Q} + A) - 1 = \mathbf{r}(A(\mathbb{I} - \tilde{Q})^{-1}) - 1$ have the same sign, which implies the assertion. □

Theorem 7.17. *Let X be an ordered normed vector space and let its order cone X_+ be closed, normal and serially complete. Let Q and B be continuous homogeneous additive operators on X_+, $\mathbf{r}(Q) < \mathbf{r}(B + Q) =: \lambda$. Assume that B^2 and at least one of BQ or QB are compact on X_+.*

Then there exists some $w \in \dot{X}_+$ such that $(B + Q)(w) = \lambda w$.

Proof. We first assume that $\mathbf{r}(Q + B) =: \lambda = 1$. So $\mathbf{r}(Q) < 1$.

By Theorem 7.16, $\mathbf{r}(BR) = 1$ with

$$R = \sum_{n=0}^{\infty} Q^n = \mathbb{I} + QR = \mathbb{I} + RQ. \tag{7.7}$$

Since B is additive,

$$BR = B + BQR = B + BRQ.$$

Since all ingredients are additive and $QR = RQ$ by (7.7) ,

$$(BR)^2 = B^2R + BQR(BR) = B^2R + BR(QB)R.$$

Since by assumption, B^2 and at least one of BQ or QB are compact on X_+, $(BR)^2$ is compact (and continuous) on X_+. By Corollary 7.10, there exists some $v \in \dot{X}_+$ such that $v = BRv$. Set $w = Rv$. By (7.7), $w = v + Qw$ and $w - Qw = v = Bw$, i.e., $w = (B + Q)(w)$.

We return to the general case $\lambda = \mathbf{r}(Q + B) > \mathbf{r}(Q)$. Set $B_\lambda = \frac{1}{\lambda}B$ and $Q_\lambda = \frac{1}{\lambda}Q$. Then

$$\mathbf{r}(Q_\lambda + B_\lambda) = \frac{1}{\lambda}\mathbf{r}(Q + B) = 1 > \frac{1}{\lambda}\mathbf{r}(Q) = \mathbf{r}(Q_\lambda).$$

By our previous consideration, there exists some $w \in \dot{X}_+$ such that $w = (Q_\lambda + B_\lambda)w = \frac{1}{\lambda}(Q + B)w$. □

8. Nonlinear dynamics on cones

Let X_+ be the closed order cone of an ordered normed vector space X and $F : X_+ \to X_+$, $F(0) = 0$.

8.1. Order derivatives

$B : X_+ \to X_+$ is called an *order derivative* of $F : X_+ \to X_+$ at 0 if

$$\begin{gathered}\text{for any } \epsilon \in (0,1) \text{ there is some } \delta > 0 \text{ such that} \\ (1-\epsilon)B(x) \le F(x) \le (1+\epsilon)B(x) \text{ for all } x \in X_+ \text{ with } \|x\| \le \delta.\end{gathered} \tag{8.1}$$

The following is shown in [36], Prop.3.3, Lemma 3.6 and 3.7.

Lemma 8.1. *The order derivative B satisfies $B(0) = 0$. B is uniquely determined if it is homogeneous. If X_+ is normal and B is homogeneous, B is the Gateaux-derivative of F at 0.*

A homogeneous $B : X_+ \to X_+$ is called a *lower order derivative* of F at 0 if one part of (8.1) holds:

$$\begin{gathered}\text{For any } \epsilon \in (0,1) \text{ there is some } \delta > 0 \text{ such that} \\ F(x) \ge (1-\epsilon)B(x) \text{ for all } x \in X_+ \text{ with } \|x\| \le \delta.\end{gathered} \tag{8.2}$$

B is called an *upper order derivative* if the other part of (8.1) holds:

$$\begin{gathered}\text{For any } \epsilon \in (0,1) \text{ there is some } \delta > 0 \text{ such that} \\ F(x) \le (1+\epsilon)B(x) \text{ for all } x \in X_+ \text{ with } \|x\| \le \delta.\end{gathered} \tag{8.3}$$

8.2. An instability result on a cone

Theorem 8.2. *Let $F, B : X_+ \to X_+$ and let B be homogeneous and order-preserving and a lower order-derivative of F at 0 as in (8.2).*

Let $r > 1$ and let $\theta : X_+ \to \mathbb{R}_+$ be a homogeneous bounded order-preserving functional with $\theta(B(x)) \geq r\theta(x)$ for all $x \in X_+$.

Then there exists some $\delta_0 > 0$ such that

$$\sup_{n \in \mathbb{Z}_+} \|F^n(x)\| \geq \delta_0 \quad \text{for all } x \in X_+ \text{ with } \theta(x) > 0.$$

In particular, 0 is an unstable equilibrium of F.

Proof. Choose some $\epsilon \in (0, 1)$ such that $1 < (1 - \epsilon)r =: s$. Then choose some $\delta > 0$ such that $F(x) \geq (1 - \epsilon)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.

Suppose that the statement is false. By contraposition, there exists some $x \in X_+$ such that $\theta(x) > 0$ and $\sup_{n \in \mathbb{Z}_+} \|F^n(x)\| < \delta$.

Since θ is bounded, this implies that $(\theta(F^n(x)))$ is a bounded sequence.

We will show by induction that $\theta(F^n(x)) \geq s^n\theta(x)$ for all $n \in \mathbb{Z}_+$, which is a contradiction because $s > 1$.

This statement holds for $n = 0$. Let $n \in \mathbb{Z}_+$ and $\theta(F^{m+n}(x)) \geq s^n\theta(F^m(x))$. Then

$$F^{n+1}(x) = F(F^n(x)) \geq (1 - \epsilon)B(F^n(x)).$$

Since B and θ are order-preserving and homogeneous,

$$\begin{aligned} \theta(F^{n+1}(x)) \geq &(1 - \epsilon)\theta(B(F^n(x))) \geq (1 - \epsilon)r\theta(F^n(x)) \\ \geq &ss^n\theta(x) = s^{n+1}\theta(x). \end{aligned}$$

□

If $x, v \in X_+$, x is called *v-positive*

$$\text{if there exists some } \delta > 0 \text{ such that } x \geq \delta v. \tag{8.4}$$

Theorem 8.3. *Let B be a lower order derivative of F at 0, B order-preserving, and let $r > 1$ and $v \in \dot{X}_+$ such that $B(v) \geq rv$.*

Then there exists some $\delta_0 > 0$ such that for any v-positive $x \in X_+$ with $\|x\| \leq \delta_0$ there is some $n \in \mathbb{N}$ such that $\|F^n(x)\| > \delta_0$.

In particular, 0 is unstable.

Proof. Let $r > 1$ and $v \in \dot{X}_+$ such that $B(v) \geq rv$. We define [44, (2.14)]

$$\theta(x) = \sup\{\beta \in \mathbb{R}_+; \beta v \leq x\} =: [x]_v.$$

By [44, Rem.3.4], $\theta(B(x)) \geq r\theta(x)$ for all $x \in X_+$. Apply Theorem 8.2. □

8.3. Existence of a nonzero fixed point

For discrete time-dynamics, 'persistence at equilibrium'
comes in the form of a fixed point of the population turnover map.

Theorem 8.4. *Let B be a lower order derivative of F at 0, B order-preserving, and let F be continuous and compact on $\dot{X}_+$. Assume that there is some continuous homogeneous subadditive functional $\theta : X_+ \to \mathbb{R}_+$ and the following hold:*

(i) There is some $\zeta \in (0,1)$ such that $\theta(x) \geq \zeta\|x\|$ for all $x \in X_+$.
(ii) $\limsup_{\|x\|\to\infty} \theta(F(x))/\theta(x) < 1$.
(iii) There is some $v \in \dot{X}_+$ and $r > 1$ such that $B(v) \geq rv$.

Then there exists some $x \in \dot{X}_+$ such that $F(x) = x$.

We emphasize that X_+ does not need to be complete.

Proof. For $\lambda \in (0,1)$, we define $F_\lambda : X_+ \to X_+$ by

$$F_\lambda(x) = F(x + \lambda v) + \lambda v, \qquad x \in X_+, \tag{8.5}$$

where v is chosen according to assumption (iii).

We claim: There exists some $R > 0$ such that $\theta(F_\lambda(x)) \leq R$ for all $\lambda \in (0,1]$ and all $x \in X_+$ with $\theta(x) \leq R$.

Suppose not. Then, for any $n \in \mathbb{N}$, there exist $x_n \in X_+$ and $\lambda_n \in (0,1]$ such that $\theta(x_n) \leq n$ and $\theta(F_{\lambda_n}(x_n)) \geq n$. Since θ is subadditive and homogeneous, for all $n \in \mathbb{N}$,

$$n \leq \theta\big(F(x_n + \lambda_n v) + \lambda_n v\big) \leq \theta(F(x_n + \lambda_n v)) + \theta(v).$$

Since F is compact on $\dot{X}_+$ and θ bounded on X_+, (x_n) is unbounded. After choosing subsequences, $\|x_n\| \to \infty$ as $n \to \infty$. By assumption (ii), there exists some $\alpha \in (0,1)$ and $\beta > 0$ such that

$$\theta(F(x)) \leq \alpha\theta(x), \qquad x \in X_+, \|x\| \geq \beta.$$

So, for large enough n, $n \leq \alpha\theta(x_n + \lambda_n v) + \theta(v) \leq \alpha n + 2\theta(v)$, a contradiction because $\alpha \in (0,1)$.

Thus, we have shown that there is an $R > 0$ such that, for any $\lambda \in (0,1]$, F_λ maps the convex closed bounded set $\{x \in X_+; \theta(x) \leq R\}$ into itself. Since F_λ is compact and continuous on X_+, by Tychonov's fixed point theorem [58, Thm.10.1], there exists $x_\lambda \in X_+$ with $\theta(x_\lambda) \leq R$ and $F_\lambda(x_\lambda) = x_\lambda$.

Now, choose a sequence (λ_n) in $(0,1]$ such that $\lambda_n \to 0$ for $n \to \infty$. By assumption (i) and (8.5), for each $n \in \mathbb{N}$, there exists some x_n such that

$$F(x_n + \lambda_n v) + \lambda_n v = x_n, \qquad 0 < \|x_n\| \leq R/\zeta. \tag{8.6}$$

We choose $\epsilon \in (0,1)$ such that $(1-\epsilon)r > 1$ with the number r from assumption (iii). Since B is a lower order derivative of F, by (8.2) we choose $\delta > 0$ such that $F(x) \geq (1-\epsilon)B(x)$ for all $x \in \dot{X}_+$ with $\|x\| \leq 2\delta$. We claim that

$$\liminf_{n\to\infty} \|x_n\| \geq \delta. \tag{8.7}$$

Suppose not. Then, for some large $n \in \mathbb{N}$, $\|x_n\| \le \delta$ and $\lambda_n \|v\| \le \delta$ and so $\|x_n + \lambda_n v\| \le 2\delta$ and

$$x_n \ge (1-\epsilon)B(x_n + \lambda_n v) + \lambda_n v.$$

Since $v \in X_+$ and B is order-preserving, $x_n \ge (1-\epsilon)B(x_n)$. Since B is homogeneous and order-preserving, $x_n \ge (1-\epsilon)^k B^k(x_n)$, $k \in \mathbb{N}$. We also have that $x_n \ge \lambda_n v$ and $B(v) \ge rv$. So

$$x_n \ge (1-\epsilon)^k B^k(\lambda_n v) \ge [(1-\epsilon)r]^k \lambda_n v, \qquad k \in \mathbb{N}.$$

Since X_+ is a closed cone and $(1-\epsilon)r > 1$, this implies that $v = 0$, a contradiction. So (8.7) holds. Since F is compact on $\dot{X}_+$ and the sequence $(x_n + \lambda_n v)$ in $\dot{X}_+$ is bounded and $\lambda_n \to 0$, $F(x_n + \lambda_n v) \to y \in X_+$ after choosing a subsequence. By (8.6) and (8.7), $x_n \to y$ as $n \to \infty$ with $y \in X_+$ and $\delta \le \|y\|$. Since F is continuous on $\dot{X}_+$, $F(y) = y$. □

By Theorem 7.8, there is the following consequence.

Corollary 8.5. *Let $F : X_+ \to X_+$ be continuous and compact on $\dot{X}_+$. Assume that $\limsup_{\|x\|\to\infty} \|F(x)\|/\|x\| < 1$. Let B be a lower order derivative of F and let $B : X_+ \to X_+$ be compact, continuous and order-preserving, and $\mathbf{r}(B) > 1$. Then there exists some $x \in \dot{X}_+$ such that $F(x) = x$.*

Again, we emphasize that X_+ does not need to be complete.

9. Linear maps induced by measure kernels

Let $(S, \mathcal{B})$ and $(\tilde{S}, \tilde{\mathcal{B}})$ be measurable spaces with $\mathcal{B}$ and $\tilde{\mathcal{B}}$ σ-algebras of subsets of S and $\tilde{S}$, respectively.

Definition 9.1. A function $\kappa : \mathcal{B} \times \tilde{S} \to \mathbb{R}_+$ is called a *measure kernel* if
$\kappa(\cdot, s) \in \mathcal{M}_+(S)$ for all $s \in \tilde{S}$ and $\kappa(T, \cdot) \in M^b(\tilde{S})$ for all $T \in \mathcal{B}$.

Here $M(\tilde{S})$ denotes the vector space of measurable functions from $\tilde{S}$ to $\mathbb{R}$ and $M^b(\tilde{S})$ the Banach space of bounded measurable functions from $\tilde{S}$ to $\mathbb{R}$ with the supremum norm. Cf. [2, Sec.19.2] [3, Sec.10.3] [69]. Measure kernels are simply called measurable maps in [70, Sec.5.3]. See also [22].

The measure kernel κ induces a linear bounded map A_* from $M^b(S)$ to $M^b(\tilde{S})$ with the supremum norm and a linear bounded map A from $\mathcal{M}(\tilde{S})$ to $\mathcal{M}(S)$ with the variation norm by

$$(A_* f)(s) = \int_S f(t)\kappa(dt, s), \qquad s \in \tilde{S}, f \in M^b(S), \tag{9.1}$$

$$(A\mu)(T) = \int_{\tilde{S}} \kappa(T, s)\mu(ds), \qquad T \in \mathcal{B}, \mu \in \mathcal{M}(\tilde{S}). \tag{9.2}$$

The following duality relation holds,

$$\int_{\tilde{S}} (A_* f) d\mu = \int_S f\, d(A\mu), \qquad f \in M^b(S), \mu \in \mathcal{M}(\tilde{S}). \tag{9.3}$$

This formula is first proved for simple measurable functions, then for nonnegative bounded measurable functions, which are uniform limits of increasing sequences of simple functions, and finally for bounded measurable functions, which are differences of nonnegative measurable bounded functions.

There is equality of the operator norms,

$$\|A_*\| = \|A\| = \sup_{s \in \tilde{S}} \kappa(S, s). \tag{9.4}$$

9.1. Convolution of measure kernels

Let $S = \tilde{S}$ and $\mathcal{B} = \tilde{\mathcal{B}}$. The *convolution* of two measure kernels κ_j, $j = 1, 2$, is defined by (3.2). $\kappa_1 \star \kappa_2$ is again a measure kernel,

$$\int_S f(z)(\kappa_1 \star \kappa_2)(dz, s) = \int_S \Big(\int_S f(z)\kappa_1(dz, t)\Big)\kappa_2(dt, s), \quad f \in M^b(S), s \in S. \tag{9.5}$$

This follows from (9.3) with $\mu = \kappa_2(\cdot, s)$, $s \in S$. (9.5) implies that this convolution is associative, i.e., for three measure kernels κ_j, $j = 1, 2, 3$,

$$\kappa_1 \star (\kappa_2 \star \kappa_3) = (\kappa_1 \star \kappa_2) \star \kappa_3. \tag{9.6}$$

Lemma 9.2. *For $j = 1, 2$, let κ_j be measure kernels and A_{j*} and A_j be the linear bounded operators on $M^b(S)$ and $\mathcal{M}(S)$, respectively. Then $A_{2*}A_{1*}$ and A_1A_2 are induced by $\kappa_1 \star \kappa_2$ via (9.1) and (9.2).*

9.2. Spectral radius of a measure kernel

Multiple convolutions $\kappa^{n\star}$ of a measure kernel $\kappa : \mathcal{B} \times S \to \mathbb{R}_+$ are defined in Definition 3.4.

By Lemma 9.2, $\kappa^{n\star}$ induces A_*^n on $M^b(S)$ and A^n on $\mathcal{M}(S)$. Guided by (9.4) and (4.7), the spectral radius of the kernel κ, $\mathbf{r}(\kappa)$ is defined by (3.3), and we have

$$\mathbf{r}(\kappa) = \mathbf{r}(A_*) = \mathbf{r}(A) = \lim_{n\to\infty} \Big(\sup_{s\in S} \kappa^{n^\star}(S, s)\Big)^{1/n}. \tag{9.7}$$

Since $\mathcal{M}(S)$ with the variation norm is an ordered Banach space with the closed generating order cone $\mathcal{M}_+(S)$, by Proposition 7.15 and Example 7.14,

$$\mathbf{r}(\kappa) = \mathbf{r}(A) = \mathbf{r}(A_+), \tag{9.8}$$

where $A_+ = B$ is the restriction of the bounded linear map A in (9.2) to $\mathcal{M}_+(S)$. By [43, Thm.1.7],

$$\mathbf{r}(\kappa) = \mathbf{r}(A_+) \ge \sup_{\mu \in \dot{\mathcal{M}}_+(S)} \inf \Big\{ \int_S \frac{\kappa(T, s)}{\mu(T)} \mu(ds); T \in \mathcal{B}, \mu(T) > 0 \Big\}. \tag{9.9}$$

Further, $M^b(S)$ with the supremum norm is an ordered Banach space with the closed generating order cone $M_+^b(S)$, which is non-flat, and

$$\mathbf{r}(\kappa) = \mathbf{r}(A_*) = \mathbf{r}(A_{*+}), \tag{9.10}$$

where A_{*+} is the restriction of the bounded linear map A_* in (9.1) to $M_+^b(S)$. By [43, Thm.1.7],

$$\mathbf{r}(\kappa) = \mathbf{r}(A_{*+}) \ge \sup_{f \in \dot{M}_+^b(S)} \inf \Big\{ \int_S \frac{f(s)}{f(t)} \kappa(ds, t); t \in S, f(t) > 0 \Big\}. \tag{9.11}$$

$M_+^b(S)$ is a solid cone and $f \in \breve{M}_+^b(S)$, the open interior of $M_+^b(S)$, if and only if $f \in M(S)$ and $0 < \inf f(S) \le \sup f(S) < \infty$. By [43, Thm.1.15],

$$\mathbf{r}(\kappa) = \mathbf{r}(A_{*+}) \le \inf_{f \in \breve{M}_+^b(S)} \sup \Big\{ \int_S \frac{f(s)}{f(t)} \kappa(ds, t); t \in S \Big\}. \tag{9.12}$$

Since $\|\mu\| = \mu(S)$ for $\mu \in \mathcal{M}_+(S)$ in the variation norm, the series characterization of the spectral radius in Corollary 7.6, with $p = 1$, translates to

$$\begin{aligned}
\mathbf{r}(\kappa) &= \inf\Big\{\lambda > 0;\ \forall \mu \in \mathcal{M}_+(S) : \sum_{n=1}^{\infty} \lambda^{-n} \int_S \kappa^{n\star}(S,t)\mu(dt) < \infty\Big\} \\
&= \inf\Big\{\lambda > 0;\ \sum_{n=1}^{\infty} \lambda^{-n} \sup_{s\in S} \kappa^{n\star}(S,s) < \infty\Big\} \\
&= \inf\Big\{\lambda > 0;\ \sup_{s\in S} \sum_{n=1}^{\infty} \lambda^{-n}\kappa^{n\star}(S,s) < \infty\Big\}.
\end{aligned} \tag{9.13}$$

The third equality follows from the first two. The formulas are reminiscent of what has been called the Perron root of κ by Shurenkov ("On the relationship between spectral radii and Perron roots", preprint) [69],

$$\mathbf{r}^{\wp}(\kappa) = \inf\Big\{\lambda > 0;\ \forall s \in S : \sum_{n=1}^{\infty} \lambda^{-n}\kappa^{n\star}(\cdot, s) \text{ is } \sigma\text{-finite}\Big\}. \tag{9.14}$$

Obviously, $\mathbf{r}^{\wp}(\kappa) \le \mathbf{r}(\kappa)$ and equality does not always hold (even for Feller kernels, Examples 10.12 and 10.13). Conditions for equality appear to involve irreducibility conditions (Shurenkov) or communication conditions [69] that are beyond the scope of this paper. Actually, in view of the fact that the equalities in (9.13) hold without any such conditions, the question whether or under which conditions $\mathbf{r}(\kappa)$ equals

$$\inf\Big\{\lambda > 0;\ \forall s \in S : \sum_{n=1}^{\infty} \lambda^{-n}\kappa^{n\star}(S, s) < \infty\Big\}$$

may be more compelling. Corollary 7.7 implies the following for measure kernels.

Remark 9.3. Let $r = \mathbf{r}(\kappa) > 0$. Then there exists some $\mu \in \mathcal{M}_+(S)$ such that $\sum_{n=1}^{\infty} r^{-n} \int_S \kappa^{n\star}(S,t)\mu(dt) = \infty$.

10. Feller kernels

Let S and $\tilde S$ be metrizable topological spaces and $\mathcal{B}$ and $\tilde{\mathcal{B}}$ the respective Borel σ-algebras.

Definition 10.1. A function $\kappa : \mathcal{B} \times \tilde S \to \mathbb{R}_+$ is called a *Feller kernel* if

$\kappa(\cdot, s) \in \mathcal{M}_+(S)$ for all $s \in \tilde S$ and if κ has the *Feller property*
$\int_S f(y)\kappa(dy, \cdot) \in C^b(\tilde S)$ for any $f \in C^b(S)$.

A Feller kernel κ is called a *Feller kernel of separable measures* if

$\kappa(\cdot, s) \in \mathcal{M}_+^s(S)$ for all $s \in \tilde S$.

Cf. [2, Sec.19.3]. See Example 10.12. By Proposition 6.3, if κ is a Feller kernel, $\kappa(U, \cdot)$ is a Borel measurable function on $\tilde S$ for all open subsets U of S and thus for all Borel sets U in $\tilde S$. This implies that every Feller kernel is a measure kernel and induces the maps $A : \mathcal{M}(\tilde S) \to \mathcal{M}(S)$ and $A_* : M^b(S) \to M^b(\tilde S)$. Since κ is a Feller kernel, A_* maps $C^b(S)$ to $C^b(\tilde S)$.

Theorem 10.2. *Let $\kappa : \mathcal{B} \times \tilde{S} \to \mathbb{R}_+$ be a Feller kernel. Let $\mu \in \mathcal{M}_+(\tilde{S})$ and $A(\mu) \in \mathcal{M}_+^s(S)$. Then A is continuous at μ with respect to the flat norm.*

Proof. Let (μ_n) be a sequence in $\mathcal{M}_+(\tilde{S})$ with $\|\mu_n - \mu\|_\flat \to 0$. Let $f \in \mathrm{LC}_b(S)$. By our Feller type property, the definition $g(s) = \int_S f(t)\kappa(dt, s) = (A_* f)(s)$, $s \in \tilde{S}$, provides a function $g \in C^b(\tilde{S})$. By (9.3) and Theorem 6.6,

$$\int_S f(t)(A\mu_n)(dt) = \int_S A_* f \, d\mu_n \to \int_S A_* f \, d\mu = \int_S f(t)(A\mu)(dt).$$

By Proposition 6.6, $\|A\mu_n - A\mu\|_\flat \to 0$. □

Proposition 10.3. *Let $A : \mathcal{M}_+^s(\tilde{S}) \to \mathcal{M}_+^s(S)$ be additive, homogeneous and continuous with respect to the flat norm. Then A is induced by a Feller kernel of separable measures via (9.2).*

Proof. Set $\kappa(T, s) = (A\delta_s)(T)$ for $s \in \tilde{S}$ and $T \in \mathcal{B}$. Since $\delta_s \in \mathcal{M}_+^s(S)$, $\kappa(\cdot, s) \in \mathcal{M}_+^s(S)$ for any $s \in \tilde{S}$. Let (s_n) be a sequence in $\tilde{S}$ and $s_n \to s \in \tilde{S}$. By Lemma 6.5, $\|\delta_{s_n} - \delta_s\|_\flat \to 0$ and $\|A\delta_{s_n} - A\delta_s\|_\flat \to 0$ because A is continuous. By Theorem 6.6, κ has the Feller property. Let B be the additive homogenous operator induced by κ. Then $B\delta_s = A\delta_s$ for all $s \in \tilde{S}$. Let $\mathcal{N}$ be the set of measures $\sum_{j=1}^k q_j \delta_{s_j}$ with $s_j \in \tilde{S}$ and $q_j \in \mathbb{Q}_+$. By Theorem 6.8, $\mathcal{N}$ is dense in $\mathcal{M}_+^s(\tilde{S})$ and $B\nu = A\nu$ for all $\nu \in \mathcal{N}$. Let $\mu \in \mathcal{M}_+^s(\tilde{S})$. Choose a sequence (μ_n) in $\mathcal{N}$ such that $\|\mu_n - \mu\|_\flat \to 0$ as $n \to \infty$. Let $f \in C^b(S)$. Since κ has the Feller property, $g \in C^b(\tilde{S})$ where $g(s) = \int_S f(t)\kappa(dt, s)$ for $s \in \tilde{S}$. By Theorem 6.6,

$$\int_S f d(B\mu) = \int_S g d\mu = \lim_{n\to\infty} \int_S g d\mu_n = \lim_{n\to\infty} \int_S f d(B\mu_n) = \lim_{n\to\infty} \int_S f d(A\mu_n).$$

Since A is continuous by assumption, $\int_S f d(B\mu) = \int_S f d(A\mu)$. By Theorem 6.3, $(B\mu)(T) = (A\mu)(T)$ for all open subsets T of S and so for all $T \in \mathcal{B}$. □

Theorem 10.4. *Let $\kappa : \mathcal{B} \times \tilde{S} \to \mathbb{R}_+$ be a Feller kernel of separable measures. Then the following hold:*

(a) The map $\tilde{S} \ni s \mapsto \kappa(\cdot, s)$ from $\tilde{S}$ to $\mathcal{M}_+^s(S)$ is continuous with respect to the flat norm.
(b) For any compact subset K of $\tilde{S}$, the set of measures $\{\kappa(\cdot, s); s \in K\}$ in $\mathcal{M}_+^s(S)$ is pre-tight and, if S is complete, tight.
(c) If $\mathcal{N}$ is a tight bounded subset of $\mathcal{M}_+(\tilde{S})$, then $A(\mathcal{N})$ is a pre-tight set of measures in $\mathcal{M}_+(S)$ and, if S is complete, tight.
(d) For any separable set $\tilde{T} \in \tilde{\mathcal{B}}$, there exists a separable closed subset T of S such that $\kappa(S \setminus T, s) = 0$ for all $s \in \tilde{T}$.
(e) A maps $\mathcal{M}_+^s(\tilde{S})$ continuously into $\mathcal{M}_+^s(S)$.
(f) A maps $\mathcal{M}^s(\tilde{S})$ into $\mathcal{M}^s(S)$.

Proof. (a) Since κ has the Feller property, the map $\tilde{S} \ni x \mapsto \kappa(\cdot, x) \in \mathcal{M}_+^s(S)$ is continuous with respect to the flat norm by Theorem 6.6.

(b) If K is a compact subset of $\tilde{S}$, the set $\Gamma_K = \{\kappa(\cdot, s), s \in K\}$ is the continuous image of a compact set by (a) and thus a compact set in $\mathcal{M}_+^s(S)$ with the flat norm. By Proposition 6.13, it is a pre-tight set and, if S is complete, a tight set.

(c) Let $\mathcal{N}$ a tight bounded subset of $\mathcal{M}_+(\tilde{S})$ and $\epsilon > 0$. Since κ has the Feller property, there exists some $c_1 > 0$ such that $\kappa(S, x) \le c_1$ for all $x \in \tilde{S}$. Since $\mathcal{N}$ is a bounded subset of $\mathcal{M}_+(\tilde{S})$, there exists some $c_2 > 0$ such that $\mu(S) \le c_2$ for all $\mu \in \mathcal{N}$.

Since $\mathcal{N}$ is tight, there exists a compact set K such that $\mu(S \setminus K) < \frac{\epsilon}{2c_1}$ for all $\mu \in \mathcal{N}$. Further, there exists a closed totally bounded set T in S such that $\kappa(S \setminus T, x) < \frac{\epsilon}{2c_2}$ for all $x \in K$. Then, for all $\mu \in \mathcal{N}$,

$$\begin{aligned}(A\mu)(S \setminus T) &= \int_K \kappa(S \setminus T, x)\mu(dx) + \int_{\tilde{S}\setminus K} \kappa(S \setminus T, x)\mu(dx)\\ &\leq \sup_{x\in K} \kappa(S \setminus T, x)c_2 + c_1\mu(\tilde{S} \setminus K) \leq \epsilon.\end{aligned}$$

So the set of measures $A(\mathcal{N})$ is pre-tight and, if S is complete, tight.

(d) Let $\tilde{T} \in \tilde{\mathcal{B}}$ be separable and D be a dense countable subset of $\tilde{T}$. For any $t \in D$, there exists a separable subset $T_t \in \mathcal{B}$ such that $\kappa(S \setminus T_t, t) = 0$. Let T be the closure of $\bigcup_{t\in D} T_t$. Since D is countable, T is a closed separable subset of S. Further, for all $t \in D$, $\kappa(S \setminus T, t) \leq \kappa(S \setminus T_t, t) = 0$.

Let $s \in \tilde{T}$. Then there exists a sequence (t_j) in D such that $t_j \to s$. By (a), $\|\kappa(\cdot, t_j) - \kappa(\cdot, t)\|_\flat \to 0$ as $j \to \infty$. Since $S \setminus T$ is open,

$$\kappa(S \setminus T, s) \leq \liminf_{j\to\infty} \kappa(S \setminus T, t_j) = 0$$

by Theorem [26, Thm.4.10].

(e) Let $\mu \in \mathcal{M}^s_+(\tilde{S})$. By definition, there exists a separable set $\tilde{T} \in \tilde{\mathcal{B}}$ such that $\mu(S \setminus \tilde{T}) = 0$. Choose T according to (d). Then

$$(A\mu)(S \setminus T) = \int_T \kappa(S \setminus T, s)\mu(ds) = 0.$$

This shows that $A\mu$ is separable. By Theorem 10.2, A is continuous at μ.

(f) Let $\mu \in \mathcal{M}^s(\tilde{S})$. By Definition 3.10, $|\mu| \in \mathcal{M}^s_+(\tilde{S})$ and $\mu_\pm = \frac{1}{2}(|\mu| \pm \mu) \in \mathcal{M}^s_+(\tilde{S})$. Since A is linear, by (e), $A\mu = A\mu_+ - A\mu_- \in \mathcal{M}^s(S)$. □

Theorem 10.5. *Let d be a metric that induces the topology of S. Let $\kappa : \mathcal{B} \times \tilde{S} \to \mathbb{R}_+$ be a measure kernel of separable measures. Assume that $\int_S f(y)\kappa(dy, \cdot) \in C^b(\tilde{S})$ for any $f : S \to [0, 1]$ with $|f(y) - f(z)| \leq d(y, z)$ for all $y, z \in S$. Then κ has the Feller property.*

Further, the map $s \mapsto \kappa(\cdot, s)$ from $(\tilde{S}, d)$ to $\mathcal{M}^s_+(S)$ endowed with the associated flat norm is continuous.

Proof. Let (s_n) be a sequence in $\tilde{S}$ and $s \in \tilde{S}$ and $s_n \to s$. By assumption, $\int_S f(y)\kappa(dt, s_n) \to \int_S f(y)\kappa(dt, s)$ for all $f : S \to [0, 1]$ with $|f(y) - f(z)| \leq d(y, z)$ for all $y, z \in S$. Since $\kappa(\cdot, s)$ is separable, by Proposition 6.6, $\int_S g(y)\kappa(dy, s_n) \to \int_S g(y)\kappa(dy, s)$ for all $g \in C^b(S)$.

The last statement follows from Theorem 10.4 (a). □

Remark 10.6. Let κ be a Feller kernel of separable measures and A^s denote the restriction of A from $\mathcal{M}^s(\tilde{S})$ to $\mathcal{M}^s(S)$ and A^s_+ the restriction of A from $\mathcal{M}^s_+(\tilde{S})$ to $\mathcal{M}^s_+(S)$. Since the Dirac measures are separable, we still have for the operator norms that $\|A^s\| = \|A^s_+\| = \sup_{s\in\tilde{S}} \kappa(S, s)$, (see (9.4)). If $S = \tilde{S}$ and $\mathcal{B} = \tilde{\mathcal{B}}$, by (3.3),

$$\mathbf{r}(A^s) = \mathbf{r}(A) = \mathbf{r}(A^s_+) = \mathbf{r}(\kappa).$$

Remark 10.7. The map A induced by a measure kernel via (9.2) is continuous from $\mathcal{M}_+(\tilde{S})$ to $\mathcal{M}_+(S)$ with respect to the variation norms even without the Feller type property. But it seems difficult to come up with conditions for A to be compact with respect to the variation norm.

10.1. Tight measure kernels

Definition 10.8. A measure kernel $\kappa : \mathcal{B} \times \tilde{S} \to \mathbb{R}_+$ is called a *tight measure kernel* if the set of measures $\{\kappa(\cdot, x); x \in \tilde{S}\}$ is tight.

A measure kernel κ is called a *pre-tight measure kernel* if set of measures $\{\kappa(\cdot, x); x \in \tilde{S}\}$ is pre-tight.

Proposition 10.9. *Let $\kappa : \mathcal{B} \times \tilde{S} \to \mathbb{R}_+$ be a Feller kernel of separable measures. Then κ is a pre-tight Feller kernel (tight Feller kernel if S is complete) if for any $\epsilon > 0$ there is a compact subset K of $\tilde{S}$ such that $\kappa(S, t) < \epsilon$ for all $t \in \tilde{S} \setminus K$.*

Proof. Let $\epsilon > 0$. By assumption, there is a compact subset K of $\tilde{S}$ such that $\kappa(S, t) < \epsilon$ for all $t \in \tilde{S} \setminus K$. By Theorem 10.4 (b), the set of measures $\{\kappa(\cdot, s); s \in K\}$ in $\mathcal{M}_+^s(S)$ is pre-tight and, if S is complete, tight. So there exists some totally bounded closed (compact if S is complete) subset T of S such the $\kappa(S \setminus T, t) < \epsilon$ for all $t \in K$. But also $\kappa(S \setminus T, t) \le \kappa(S, t) < \epsilon$ for all $t \in \tilde{S} \setminus K$. □

Proposition 10.10. *Let $\kappa : \mathcal{B} \times \tilde{S} \to \mathbb{R}_+$ be a tight measure kernel. Then A is continuous and compact from $\mathcal{M}_+(\tilde{S})$ to $\mathcal{M}_+(S)$ with respect to the flat norm and maps $\mathcal{M}_+(\tilde{S})$ into $\mathcal{M}_+^t(S)$.*

Proof. To show that A is compact, let (μ_n) be a bounded sequence in $\mathcal{M}_+(\tilde{S})$ with the flat norm.

Since $\{\kappa(\cdot, t); t \in \tilde{S}\}$ is tight, for any $\epsilon > 0$ there exists some compact set K in S such that $\kappa(S \setminus K; t) < \epsilon$ for all $t \in \tilde{S}$. By (9.2),

$$(A\mu_n)(S \setminus K) = \int_S \kappa(S \setminus K, t)\mu_n(dt) \le \epsilon\mu_n(S), \qquad n \in \mathbb{N}.$$

Since $(\mu_n(\tilde{S}))$ is a bounded sequence, the sequence $(A\mu_n)$ is tight.

Finally, $(A\mu_n)(S) \le \sup_{t\in\tilde{S}} \kappa(S, t)\mu_n(S)$ and the set $\{(A\mu_n)(S); n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. By Proposition 6.11, $(A\mu_n)$ has a convergent subsequence.

By the same arguments as above, A maps $\mathcal{M}_+(S)$ into $\mathcal{M}_+^t(S)$ and is continuous with respect to the flat norm by Theorem 10.2. □

Proposition 10.11. *Let $P : \mathcal{B}\times\tilde{S} \to \mathbb{R}_+$ be a tight Feller kernel and $g \in C_{b+}(\tilde{S} \times S)$. Then $\tilde{\kappa} : \mathcal{B}\times\tilde{S} \to \mathbb{R}_+$,*

$$\tilde{\kappa}(T, s) = \int_T g(s, t)P(dt, s), \qquad s \in \tilde{S}, T \in \mathcal{B}, \tag{10.1}$$

is a tight Feller kernel. In particular, $\tilde{\kappa}(S, \cdot) \in C^b(\tilde{S})$.

Proof. $\tilde{\kappa}$ inherits tightness from P via the boundedness of g.

Let $f \in C^b(S)$. Set $h(s) = \int_S f(t)\tilde{\kappa}(dt, s)$, $s \in \tilde{S}$. Then

$$h(s) = \int_S f(t)g(s, t)P(dt, s), \qquad s \in \tilde{S}.$$

Our task is to show that $h \in C^b(\tilde{S})$. Since g and f are bounded and $P(S, \cdot)$ is bounded, h is bounded.

To demonstrate the continuity of h on $\tilde{S}$, let $s \in \tilde{S}$ and (s_n) a sequence in $\tilde{S}$ with $s_n \to s$. To show that $h(s_n) \to h(s)$, let $\epsilon > 0$. Since P is tight, there exists a compact subset K of S such that

$P(S \setminus K, s) \leq \epsilon$ for all $s \in \tilde{S}$. By the triangle inequality,

$$\begin{aligned}|h(s_n) - h(s)| \leq & \int_S |f(t)|\, |g(s_n, t) - g(s,t)| P(dt, s_n) \\ & + \Big| \int_S f(t) g(s,t) P(dt, s_n) - \int_S f(t) g(s,t) P(dt, s) \Big|.\end{aligned}$$

Since $f(t)g(s,t)$ is a continuous bounded function of $t \in S$ and P is a Feller kernel, the second expression on the right hand side of this inequality converges to 0 as $n \to \infty$ and

$$\begin{aligned}\limsup_{n\to\infty} |h(s_n) - h(s)| \leq & \limsup_{n\to\infty} \int_S |f(t)|\, |g(s_n,t) - g(s,t)|\, P(dt, s_n) \\ \leq & 2 \sup|f| \ \sup|g|\ \epsilon \ + \ \sup|f| \ \limsup_{n\to\infty} \ \sup_{t\in K} |g(s_n,t) - g(s,t)|\, P(S, s_n).\end{aligned}$$

Since g is uniformly continuous on the compact set $\big(\{s_n; n \in \mathbb{N}\} \cup \{s\}\big) \times K$,

$$\limsup_{n\to\infty} \ \sup_{t\in K} |g(s_n,t) - g(s,t)| = 0 \qquad \text{and so}$$

$$\limsup_{n\to\infty} |h(s_n) - h(s)| \ \leq \ 2 \sup|f| \ \sup|g|\ \epsilon.$$

Since this holds for any $\epsilon > 0$, $|h(s_n) - h(s)| \to 0$ as $n \to \infty$. □

Example 10.12 (Survival and deterministic movement (development))**.** Let

$$\kappa(T, s) = \chi_T\big(\xi(s)\big)\, g(s), \qquad s \in S, T \in \mathcal{B},$$

where $\xi : S \to S$ is continuous, $g \in C^b_+(S)$. Notice that

$$f \in C^b(S) \implies \int_S f(t)\kappa(dt, \cdot) = f(\xi(\cdot))g(\cdot) \in C^b(S).$$

For each $s \in S$, $\kappa(\cdot, s) = g(s)\delta_{\xi(s)}$ is a tight measure and κ is a Feller kernel of separable measures. By Theorem 10.4, the map A induced by κ on $\mathcal{M}(S)$ maps $\mathcal{M}^s(S)$ into itself and maps $\mathcal{M}^s_+(S)$ continuously into itself with respect to the flat norm. One readily derives from (6.4) that A maps $\mathcal{M}(S)$ continuously into itself if ξ and g are both Lipschitz continuous.

κ is tight if and only if for any $\epsilon > 0$ there is a compact subset T of S such that $g(s) < \epsilon$ if $s \in S$ and $\xi(s) \notin T$.

A^n is associated with the kernel $\chi_T(\xi^n(s)) \ \prod_{j=0}^{n-1} g(\xi^j(s))$,

$$\mathbf{r}(A) = \lim_{n\to\infty} \Big(\sup_{s\in S} \prod_{i=0}^{n-1} g(\xi^j(s)) \Big)^{1/n}. \tag{10.2}$$

There may or may not be an eigenvector of A associated with $\mathbf{r}(A)$.

Suppose that $s^* \in S$ and $\xi(s^*) = s^*$ and $g(s^*) \geq g(s)$ for all $s \in S$. Then $g(s^*) = \mathbf{r}(A)$ and $A\delta_{s^*} = g(s^*)\delta_{s^*}$.

Now assume that for any compact subset K of S there is some $n \in \mathbb{N}$ such that $\xi^n(s) \in S \setminus K$ for all $s \in S$. We claim that there is no $r > 0$ and $\mu \in \mathcal{M}_+^t(S)$, $\mu \neq 0$, such that $A\mu = r\mu$. Suppose that such $r > 0$ and μ exist. Then $\mu = r^{-n} A^n \mu$ for all $n \in \mathbb{N}$ and

$$\mu(T) = r^{-n} \int_S \chi_T(\xi^n(s)) \prod_{j=0}^{n-1} g(\xi^j(s))\mu(ds), \qquad n \in \mathbb{N}, T \in \mathcal{B}. \tag{10.3}$$

By our assumption, $\mu(T) = 0$ for any compact subset T of S. Since $\mu \in \mathcal{M}_+^t(S)$, $\mu = 0$.

Let us assume that for any compact subset T of S and any $s \in S$, $\xi^n(s) \notin T$ for all but finitely many $n \in \mathbb{N}$ and $g(s) = 1$ for all $s \in S$. Then $\chi_T(\xi^n(s)) \to 0$ as $n \to \infty$ and the right hand side of (10.3) converges to 0 by the dominated convergence theorem and $\mu(T) = 0$. So μ is the zero measure if μ is tight.

Let us finally assume in addition that S is σ-compact, i.e., K is a countable union of compact sets. Then any $\mu \in \mathcal{M}_+(S)$ is tight. So, the spectral radius is not associated with an eigenmeasure; however it is associated with all constant positive functions as eigenfunctions.

Further, for any compact set T of S and any $s \in S$, $\kappa^{n\star}(T, s) = \chi_T(\xi^n(s)) = 0$ for all but finitely many $n \in \mathbb{N}$. This implies that $\sum_{n=1}^{\infty} \lambda^{-n} \kappa^{n\star}(T, s) \in \mathbb{R}_+$ for all $\lambda \in (0, \infty)$ and the so-called Perron root of κ in (9.14) equals $\mathbf{r}^{\wp}(\kappa) = 0$ while $\mathbf{r}(\kappa) = 1$.

Example 10.13 (Bonsall's kernel)**.** A Feller kernel on the σ-compact metric space $S = (0, 1]$ that has got some attention without the Feller kernel perspective is $\kappa(\cdot, s) = \delta_{s/2}$, $s \in (0, 1]$ [35, 44, 71]. Obviously, $\mathbf{r}(\kappa) = 1$, and according to our considerations in Example 10.12, $\mathbf{r}^{\wp}(\kappa) = 0$. There are no eigenmeasures of κ associated with positive eigenvalues, but plenty of eigenfunctions, $f(s) = s^\alpha$, $\alpha \geq 0$, associated with the eigenvalues $(1/2)^\alpha$. For $\alpha > 0$, these eigenfunctions are also in $C_0(0, 1]$, the space of continuous functions f on $(0, 1]$ with $f(s) \to 0$ as $s \to 0$, which is a Banach space under the supremum norm and has $\mathcal{M}((0, 1])$ as dual space. A_* maps $C_0(0, 1]$ into itself, and the spectral radius of its restriction A_0 to $C_0(0, 1]$ still equals 1. Let $C_{con}(0, 1]$ be the closed cone of nonnegative convex function in $C_0(0, 1]$. $C_{con}(0, 1]$ is a total but not generating cone of $C_0(0, 1]$ that is invariant under A_0 and the restriction of A_0 to $C_{con}(0, 1]$, A_{con}, is compact and $\mathbf{r}(A_{con}) = 1/2$ [35, 71].

Example 10.14 (Multiplication operator)**.** Consider the special case $\xi(s) = s$ for all $s \in S$ in Example 10.12 with $g \in C_{b+}(S)$. The induced map A on $\mathcal{M}(S)$ takes the form of a multiplication operator $(A\mu)(T) = \int_T g(s)\mu(ds)$ for $T \in \mathcal{B}$. A maps $\mathcal{M}^s(S)$ into itself. A is a bounded linear map on $\mathcal{M}(S)$ with respect to the variation norm. A maps $\mathcal{M}_+^s(S)$ into itself continuously with respect to the flat norm.

A is continuous on $\mathcal{M}^s(S)$ with respect to the flat norm if and only if g is Lipschitz continuous.

Proof. One readily derives from (6.4) that A maps $\mathcal{M}(S)$ continuously into itself if g is Lipschitz continuous. Now let A be continuous and thus a bounded linear operator on $\mathcal{M}^s(S)$ with the respect to the flat norm. Let $s, t \in S$. Since $\|\delta_s\|_\flat = 1$,

$$|g(s) - g(t)| = \|(g(s) - g(t))\delta_s\|_\flat \leq \|g(s)\delta_s - g(t)\delta_t\|_\flat + \|g(t)(\delta_t - \delta_s)\|_\flat.$$

Since $A\delta_s = g(s)\delta_s$, $|g(s) - g(t)| = \|A\delta_s - A\delta_t\|_\flat + g(t)\|\delta_t - \delta_s\|_\flat$. Since A is a bounded linear operator on $\mathcal{M}^s(S)$ with respect to the flat norm and $\delta_s \in \mathcal{M}^s(S)$, by Lemma 6.5,

$$|g(s) - g(t)| \leq (\|A\|_\flat + \sup_S |g|)\|\delta_t - \delta_s\|_\flat = (\|A\|_\flat + \sup g(S)) \min\{1, d(t, s)\}.$$

So g is Lipschitz continuous with a Lipschitz constant $\|A\|_\flat + \sup g(S)$. $\square$

10.2. Eigenmeasures of tight Feller kernels

We return to general Feller kernels κ and the maps A induced by them via (9.2). A is a bounded linear map on $\mathcal{M}(S)$ with the variation-norm. Let A_+ denote the restriction of A to $\mathcal{M}_+(S)$. Then $\|A\| = \|A_+\| = \|A_+\|_\flat$ and $\mathbf{r}(A) = \mathbf{r}(A_+)$. Since the variation norm and the flat norm coincide on $\mathcal{M}_+(S)$, this means that the spectral radius of A with respect to the variation norm coincides with the spectral radius of A_+ with respect to the flat norm and with the spectral radius of κ. Recall (3.3), (9.8), (9.11) and (9.12).

Theorem 10.15. *Assume that κ is a tight Feller kernel and $r = \mathbf{r}(A) = \mathbf{r}(\kappa) > 0$. Then there exists some tight $\mu \in \mathcal{M}_+(S)$, $\mu \neq 0$, such that $\int_S \kappa(\cdot, s)\mu(ds) = r\mu$.*

This result follows from Theorem 7.8 and Proposition 10.10. It does not follow from the Krein-Rutman theorem [72] because A is only compact and continuous on $\mathcal{M}_+(S)$ and not necessarily on $\mathcal{M}(S)$. It would follow from [35] if S were complete, but not if S (and then $\mathcal{M}_+(S)$) is not complete (or cannot be made complete by transition to a topologically equivalent metric, Theorem 6.14). See [26] for details and further remarks.

Theorem 10.16. *Let S be a metric space. Let $\kappa_j : \mathcal{B}\times S \to \mathbb{R}_+$, $j = 1, 2$, be Feller kernels of separable measures and $\kappa = \kappa_1 + \kappa_2$. Assume that κ_1 is a tight measure kernel and $r := \mathbf{r}(\kappa) > \mathbf{r}(\kappa_2)$. Then there exists some $\nu \in \mathcal{M}_+^s(S)$ with $\nu(S) = 1$ and $\int_S \kappa(T, s)\nu(ds) = r\nu(T)$, $T \in \mathcal{B}$. Further,*

$$\mathbf{r}(\kappa) = \sup_{\mu\in\dot{\mathcal{M}}_+(S)} \inf\Big\{\frac{1}{\mu(T)}\int_S \kappa(T, s)\mu(ds); T \in \mathcal{B}, \mu(T) > 0\Big\}.$$

This result would follow from [39] if S were complete or completely remetrizable because completeness of S is equivalent to the completeness of $\mathcal{M}_+^s(S)$ with respect to the flat norm (Theorem 6.14).

Proof. We apply Theorem 7.17 with $X = \mathcal{M}^s(S)$ and $X_+ = \mathcal{M}_+^s(S)$, endowed with the flat norm. X_+ is a closed, normal and serially complete cone in X with respect to flat norm (Corollary 6.7). Let $B_j : \mathcal{M}_+^s(S) \to \mathcal{M}(S)$ be given by

$$(B_j\mu)(T) = \int_S \kappa_j(T, s)\mu(ds), \qquad \mu \in \mathcal{M}_+^s(S), \quad T \in \mathcal{B}.$$

By Theorem 10.4, for $j = 1, 2$, B_j maps $\mathcal{M}_+^s(S)$ into itself and is a continuous homogeneous additive map on $\mathcal{M}_+^s(S)$. By Proposition 10.10, B_1 is compact on X_+. By Theorem 7.17, there exists $\mu \in \dot{\mathcal{M}}_+^s(S)$ such that $(B_1 + B_2)(\mu) = r\mu$.

The formula for $\mathbf{r}(\kappa)$ follows from the inequality (9.9) which is turned into an equality by the existence of the eigenmeasure. $\square$

11. Nonlinear dynamics on measures: proofs for the general framework

Proof of Theorem 3.6. Apply [36, Thm.4.2] or [63, Thm.4.1] where B is the homogeneous additive operator on $\mathcal{M}_+(S)$ given by

$$(B\mu)(T) = \int_S \kappa^o(T, s)\mu(ds), \qquad T \in \mathcal{B}, \mu \in M_+(S),$$

that is continuous with respect to the variation norm. Then $\mathbf{r}(B) = \mathbf{r}(\kappa^o) < 1$ and also all other assumptions are satisfied. □

Proof of Corollary 3.7. By Theorem 3.6 it is sufficient to show that $\mathbf{r}(\kappa^o) \le r$.

Let S be the union of pairwise disjoint nonempty sets $T_1, \ldots, T_m$ in $\mathcal{B}$ and

$$\beta_{jk} = \sup_{t \in T_j} \kappa^o(T_k, t)$$

and assume that the matrix of size m with coefficients β_{jk} is irreducible and has a spectral radius $r < 1$.

By the Perron-Frobenius theorem, there exists a vector $w \in (0, \infty)^m$ such that

$$\sum_{k=1}^m \beta_{jk} w_k = r w_j, \qquad j = 1, \ldots, m. \tag{11.1}$$

Define $f : S \to \mathbb{R}_+$ by

$$f = \sum_{k=1}^m w_k \chi_{T_k}.$$

Then f is in the interior of $M_+^b(S)$. For $j \in \{1, \ldots, m\}$ and $t \in T_j$,

$$\int_S f(s) \kappa^o(ds, t) \le \sum_{k=1}^m \beta_{jk} w_k = r w_j = r f(t).$$

By (9.12), $\mathbf{r}(\kappa^o) \le r$. □

Proof of Theorem 3.8. Let $T_1, \ldots, T_m$, $m \in \mathbb{N}$, be pairwise disjoint nonempty sets in $\mathcal{B}$ and

$$\alpha_{jk} = \inf_{t \in T_j} \kappa^o(T_k, t)$$

and assume that the matrix of size m with coefficients α_{jk} has a spectral radius $r > 1$. Then there exists an eigenvector $v \in \mathbb{R}_+^m$ such that

$$\sum_{k=1}^m \alpha_{jk} v_k = r v_j, \qquad j = 1, \ldots, m.$$

Set $f = \sum_{k=1}^m v_k \chi_{T_k}$. Since v is not the zero vector, $f \in \dot{M}_+^b(S)$. Similarly as in the proof before, one shows that $\int_S f(s) \kappa^o(ds, t) \ge r f(t)$. By (9.11), $\mathbf{r}(\kappa^o) \ge r$. We define the linear nonnegative functional $\theta : \mathcal{M}(S) \to \mathbb{R}$ by

$$\theta(\mu) = \int_S f d\mu = \sum_{k=1}^m v_k \mu(T_k).$$

Let A and A_* be the maps on $\mathcal{M}(S)$ and $M^b(S)$ that are induced by κ^o. Then $A_* f \ge r f$ and, by (9.3),

$$\theta(A\mu) = \int_S A_* f \, d\mu \ge \int_S r f d\mu = r\theta(\mu).$$

The statement now follows from Theorem 8.2. □

Proof of Theorem 3.13. Combine Theorem 8.3 and Theorem 10.16. □

Proof of Remark 3.14. Combine Theorem 7.16 with Section 9. □

Proposition 11.1. *Let the Assumption 3.2 be satisfied. Then F maps $\mathcal{M}_+^s(S)$ into itself.*

Proof. Theorem 10.4 (b). □

Lemma 11.2. *Let $(\tilde{f}_n)$ be a bounded sequence in $C^b(S)$ and (μ_n) be a bounded pre-tight sequence in $\mathcal{M}_+(S)$. Then*

$$\int_S \tilde{f}_n d\mu_n \stackrel{n\to\infty}{\longrightarrow} 0 \qquad \textit{if} \quad \tilde{f}_n \stackrel{n\to\infty}{\longrightarrow} 0$$

uniformly on every totally bounded subset of S.

Proof. Let $\epsilon > 0$. Since $\{\mu_n; n \in \mathbb{N}\}$ is pre-tight, there exists a closed totally bounded subset T of S such that $\mu_n(S \setminus T) < \epsilon$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$,

$$\begin{aligned}\Big|\int_S \tilde{f}_n d\mu_n\Big| &\le \int_T |\tilde{f}_n| d\mu_n + \int_{S\setminus T} |\tilde{f}_n| d\mu_n \\ &\le \sup_T |\tilde{f}_n| \sup_{k\in\mathbb{N}} \mu_k(S) + \sup_{k\in\mathbb{N}} \sup_S |\tilde{f}_k|\, \mu_n(S \setminus T).\end{aligned}$$

Since $\tilde{f}_n \to 0$ uniformly on T, the last but one expression converges to 0 as $n \to \infty$ and

$$\limsup_{n\to\infty} \Big|\int_S \tilde{f}_n d\mu_n\Big| \le \sup_{k\in\mathbb{N}} \sup_S |\tilde{f}_k|\, \epsilon.$$

Since this holds for arbitrary $\epsilon > 0$, the limit superior is zero and we have proved the assertion. □

Proposition 11.3. *Let the family of Feller kernels $\{\kappa^\mu); \mu \in \mathcal{M}_+^s(S)\}$ satisfy the Assumptions 3.2 and 3.17. Then $F : \mathcal{M}_+^s(S) \to \mathcal{M}_+^s(S)$ is continuous with respect to the flat norm.*

Proof. Let $\mu \in \mathcal{M}_+^s(S)$ and (μ_n) be a sequence in $\mathcal{M}^s(S)$ such that $\|\mu_n - \mu\|_\flat \to 0$. By Theorem 6.6,

$$\int_S \tilde{f} d\mu_n \to \int_S \tilde{f} d\mu, \qquad \tilde{f} \in C_+^b(S). \tag{11.2}$$

Then $\{\mu_n; n \in \mathbb{N}\}$ is a compact subset of $\mathcal{M}_+(S)$ with respect to the flat norm and pre-tight by Proposition 6.13 and a bounded subset of $\mathcal{M}_+(S)$.

Let $f \in \mathcal{F}$. By (3.1),

$$\Big|\int_S f\, dF(\mu_n) - \int_S f\, dF(\mu)\Big| = \Big|\int_S f_n d\mu_n - \int_S \tilde{f} d\mu\Big| \tag{11.3}$$

with

$$f_n(s) = \int_S f(t)\kappa^{\mu_n}(dt, s), \qquad \tilde{f}(s) = \int_S f(t)\kappa^{\mu}(dt, s).$$

By Theorem 6.6, it is sufficient that the expression on the right hand side of the equation converges to 0 as $n \to \infty$.

By the triangle inequality and (11.3),

$$\Big|\int_S f\,dF(\mu_n) - \int_S f\,dF(\mu)\Big| \le \Big|\int_S (f_n - \tilde{f})d\mu_n\Big| + \Big|\int_S \tilde{f}d\mu_n - \int_S \tilde{f}d\mu\Big|.$$

Since κ^μ is a Feller kernel, $\tilde{f} \in C_+^b(S)$ and the second term on the right hand side of the last inequality converges to 0 as $n \to \infty$ by (11.2). As for the first term, by Assumption 3.17, for any closed totally bounded subset T of S,

$$f_n(s) - \tilde{f}(s) \to 0, \qquad n \to \infty, \text{ uniformly for } s \in T. \tag{11.4}$$

Further, by Assumption 3.2, $(f_n - \tilde{f})$ is a bounded sequence in $C^b(S)$. Now the first term of the last inequality converges to 0 by Lemma 11.2. □

Proposition 11.4. *Under the Assumptions 3.2 and 3.16, the yearly population turnover map F : $\mathcal{M}_+^s(S) \to \mathcal{M}_+^s(S)$ is compact; for any bounded subset $\mathcal{N}$ of $\mathcal{M}_+^s(S)$, $F(\mathcal{N})$ is a tight bounded subset of $\mathcal{M}_+^s(S)$.*

Proof. By Proposition 11.1, F maps $\mathcal{M}_+^s(S)$ into itself. Let $\mathcal{N}$ be a bounded subset of $\mathcal{M}_+^s(S)$. For any set $T \in \mathcal{B}$ and $\mu \in \mathcal{N}$,

$$F(\mu)(S \setminus T) = \int_S \kappa^\mu(S \setminus T, s)\,\mu(ds) \le \sup_{s\in S}\ \kappa^\mu(S \setminus T, s)\,\mu(S). \tag{11.5}$$

For $T = \emptyset$, we obtain that $\{F(\mu)(S); \mu \in \mathcal{N}\}$ is bounded in $\mathbb{R}$ by Assumption 3.16.

Let $\epsilon > 0$. By Assumption 3.16, there exists some compact set T in S such that

$$\kappa^\mu(S \setminus T, s) \le \epsilon\Big(1 + \sup_{\mu\in\mathcal{N}} \mu(S)\Big)^{-1}.$$

By (11.5), $F(\mu)(S \setminus T) \le \epsilon$ for all $\mu \in \mathcal{N}$. By Definition 3.10, $F(\mathcal{N})$ is a tight subset of $\mathcal{M}_+^s(S)$.

By Theorem 6.11, $F(\mathcal{N})$ has compact closure in $\mathcal{M}_+^s(S)$. □

Proposition 11.5. *Let the Assumptions 3.2 and 3.18 be satisfied. Then*

$$\limsup_{\mu(S)\to\infty} \frac{F(\mu)(S)}{\mu(S)} < 1.$$

Proof. For all $\mu \in \mathcal{M}_+^s(S)$,

$$\mathcal{F}(\mu)(S) = \int_S \kappa^\mu(S, s)\,\mu(ds) \le \sup_{s\in S} \kappa^\mu(S, s)\,\mu(S).$$

This implies the assertion. □

Proof of Theorem 3.19. We apply Corollary 8.5 with

$$(B\mu)(T) = \int_S \kappa^o(T, s)\mu(ds), \qquad T \in \mathcal{B}, \mu \in \mathcal{M}_+(S).$$

Then $B : \mathcal{M}_+^s(S) \to \mathcal{M}_+(S)$ is homogeneous, additive, continuous and compact by Proposition 10.10, and $\mathbf{r}(B) = \mathbf{r}(\kappa^o) > 1$. Further, F is continuous and compact by Propositions 11.3 and 11.4. Since $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ is lower semicontinuous at measure zero, B is the lower order derivative of F. Further use Proposition 11.5. Then the assumptions of Corollary 8.5 are satisfies from which the existence of a non-zero fixed point follows. □

12. Nonlinear dynamics on measures: semelparous populations

The metric space S with the σ-algebra $\mathcal{B}$ of Borel sets represents the habitat of a spatially distributed population. We consider a difference equation on $\mathcal{M}_+(S)$, the cone of nonnegative finite measures on $\mathcal{B}$,

$$\mu_n(T) = F(\mu_{n-1})(T), \qquad n \in \mathbb{N}, T \in \mathcal{B}, \tag{12.1}$$

with $\mu_0 \in \mathcal{M}_+(S)$ and $F : \mathcal{M}_+(S) \to \mathcal{M}_+(S)$ given by

$$\left.\begin{aligned} F(\mu)(T) = & \int_S \kappa^\mu(T, s)\mu(ds), \\ \kappa^\mu(T, s) = & \; P(T, s) g\Big(s, \int_S q(s, t)\mu(dt)\Big) \end{aligned}\right\} \quad \left\{ \begin{array}{c} \mu \in \mathcal{M}_+(S), \\ T \in \mathcal{B}. \end{array}\right. \tag{12.2}$$

The difference equation describes the year-to-year development of a spatially distributed population with a short reproductive season. μ_n is the spatial distribution at the beginning of the n^{th} year. See Section 2 for the interpretation of the measure kernel P, and the functions $g : S \times \mathbb{R}_+ \to \mathbb{R}_+$ and $q : S^2 \to \mathbb{R}_+$.

Assumption 12.1. For the per capita reproduction function g,

(g1) $g : S \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded.
(g2) $g(s, 0) > 0$ for all $s \in S$ and $\dfrac{g(s, u)}{g(s, 0)} \to 1$ as $u \to 0$ uniformly for $s \in S$.

For the competitive influence function q,

(q1) $q : S^2 \to \mathbb{R}_+$ is continuous and bounded.

For the survival/migration kernel P,

(P1) $P : \mathcal{B} \times S \to \mathbb{R}_+$ is a measure kernel (Definition 9.1).
(P2) $0 \le P(S, s) \le 1$ for all $s \in S$.

Proposition 12.2. *Assume Assumptions 12.1. Then Assumption 3.2 is satisfied.*
Further, the kernel family $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ is continuous at the zero measure (Definition 3.5).
If $s \in S$ and $g(s, w) \le g(s, 0)$ for all $w \in \mathbb{R}_+$, $\kappa^\mu(T, s) \le \kappa^o(T, s)$ for all $T \in \mathcal{B}$.

Proof. Let $\mu \in \mathcal{M}_+(S)$. Set

$$v(s) = \int_S q(s, t)\mu(dt), \qquad w(s) = g(s, v(s)), \qquad s \in S. \tag{12.3}$$

By the Assumption 12.1 (q1) and the dominated convergence theorem, $v : S \to \mathbb{R}_+$ is continuous and bounded, and so is $w : S \to \mathbb{R}_+$ by Assumption 12.1 (g1). This implies that

$$\kappa^\mu(T, s) = P(T, s)w(s), \qquad T \in \mathcal{B}, \quad s \in S, \tag{12.4}$$

is a measure kernel and $\kappa^\mu(S, s) \le \sup g$ for all $\mu \in \mathcal{M}_+(S)$, $s \in S$.

Let $\epsilon \in (0, 1)$. By Assumption 12.1 (g2), there exists some $\tilde{\delta} > 0$ such that $(1 - \epsilon)g(s, 0) \le g(s, u) \le (1 + \epsilon)g(s, 0)$ for all $u \in [0, \tilde{\delta}]$ and $s \in S$. Since q is bounded by Assumption 12.1 (q1), there exist some $\delta > 0$ such that $q(s,t)\delta \leq \tilde{\delta}$ for all $s,t \in S$. Let $\mu \in \mathcal{M}_+(S)$ and $\mu(S) \leq \delta$. Then $\int_S q(s,t)\mu(dt) \leq \sup_{t\in S} q(s,t)\delta \leq \tilde{\delta}$ for all $s \in S$. This implies that

$$(1-\epsilon)\kappa^o(T,s) \leq \kappa^\mu(T,s) \leq (1+\epsilon)\kappa^o(T,s), \qquad T \in \mathcal{B}, s \in S,$$

and the kernel family $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ is continuous at the zero measure. □

12.1. Local (global) asymptotic stability of the extinction state in the subthreshold case

Theorem 12.3. *Assume Assumptions 12.1 and* $r = \mathbf{r}(\kappa^o) < 1$.

Then the zero measure (extinction fixed point) is locally asymptotically stable in the sense of Theorem 3.6 (a).

If, in addition, $g(s,u) \leq g(s,0)$ *for all* $u \geq 0$, $s \in S$, *then the origin is globally stable in the sense of Theorem 3.6 (b).*

Proof. Combine Theorem 3.6 and Proposition 12.2. □

Remark 12.4. Notice that we have only used the variation norm so far. Therefore, the previous results remain valid if S is a measurable space with σ-algebra $\mathcal{B}$ and the assumptions are slightly modified.

Instead of Assumption 12.1 (g1) we would assume

(g1)' $g : S \times \mathbb{R}_+ \to \mathbb{R}_+$ is bounded and measurable where $S \times \mathbb{R}_+$ is equipped with the appropriate product σ-algebra,

and instead of Assumption 12.1 (q1) we would assume

(q1)' $q : S^2 \to \mathbb{R}_+$ is bounded and measurable where S^2 is equipped with the appropriate product σ-algebra.

Notice that continuity of q on S^2 does not imply that q is measurable with respect to the product σ-algebra without additional assumptions (like S being separable [2, Sec.4.10]).

12.2. Superthreshold instability of the extinction state

Proposition 12.5. *Assume Assumptions 12.1.*

(a) *If P is a Feller kernel of separable measures, κ^o is a Feller kernel of separable measures.*
(b) *If P is a tight measure kernel, then the set of measures* $\{\kappa^\mu(\cdot,s); \mu \in \mathcal{M}_+(S), s \in S\}$ *is tight and the set* $\{\kappa^\mu(S,s); \mu \in \mathcal{M}_+(S), s \in S\}$ *is bounded in* $\mathbb{R}$.

Proof. (a) κ^o inherits these properties from P via (12.4); recall that $g(s,0)$ is a continuous function of $s \in S$.

(b) Let P be a tight measure kernel. Let $K \in \mathcal{B}$ and $\mu \in \mathcal{M}_+(S)$. Since g is bounded by Assumption 12.1 (g1),

$$\kappa^\mu(S \setminus K, s) \leq (\sup g)\, P(S \setminus K, s), \qquad \mu \in \mathcal{M}_+(S).$$

With $K = \emptyset$, we see that $\kappa^\mu(S,s) \leq \sup g$ for all $s \in S$, $\mu \in \mathcal{M}_+(S)$.

Let $\epsilon > 0$. Since P is a tight kernel, there exists a compact subset K of S such that $P(S \setminus K, s) < \delta$ with $\delta \sup g < \epsilon$ and $\kappa^\mu(S \setminus K, s) < \epsilon$ for all $\mu \in \mathcal{M}_+(S)$ and $s \in S$. This implies that the set of measures $\{\kappa^\mu(\cdot,s); \mu \in \mathcal{M}_+(S), s \in S\}$ is tight. □

Theorem 12.6. *Make Assumption 12.1.*

Assume that $P = P_1 + P_2$ with Feller kernels P_1 and P_2 of separable measures where P_1 is a tight Feller kernel. Let κ_1 and κ_2 and κ be related to P_1, P_2 and P by (12.4), respectively, and $r = \mathbf{r}(\kappa) > 1$ and $r > \mathbf{r}(\kappa_2)$.

Then there exists some $\nu \in \mathcal{M}_+^s(S)$ such that $\int_S \kappa(\cdot, s)\nu(ds) = r\nu$, and there is some $\delta_0 > 0$ such that for any ν-positive $\mu_0 \in \mathcal{M}_+(S)$ and any solutions (μ_n) in $\mathcal{M}_+(S)$ of $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$, there is some $n \in \mathbb{Z}_+$ with $\mu_n(S) > \delta_0$. In particular, the zero measure is unstable.

By (8.4), μ is ν-positive if there is some $\delta > 0$ such that $\mu(T) \geq \delta\nu(T)$ for all $T \in \mathcal{B}$.

Proof. The measure kernels κ_j related to P_j by (12.4) are also Feller kernels of separable measures because w is continuous, and κ_1 inherits tightness from P_1.

Combine Proposition 12.2 with Theorem 3.13. □

12.3. Existence of a nonzero equilibrium measure

Assumption 12.7. Assume:

(i) For any closed totally bounded subset T of S, the set of functions $\{q(s, \cdot); s \in T\}$ is equicontinuous on S: For any $t_0 \in S$ and $\epsilon > 0$, there exists $\delta > 0$ such that $|q(s,t) - q(s,t_0)| < \epsilon$ whenever $s, t \in S$ and $d(t, t_0) < \delta$.

(ii) For any closed totally bounded subset T of S, the set of functions $\{g(s, \cdot); s \in S\}$ is uniformly equicontinuous on bounded subsets of $\mathbb{R}_+$: For any $c \in (0, \infty)$ and $\epsilon > 0$, there exists $\delta > 0$ such that $|g(s, w) - g(s, v)| < \epsilon$ whenever $s \in S$ and $w, v \in [0, c]$ and $|w - v| < \delta$.

(iii) $P : \mathcal{B} \times S \to \mathbb{R}_+$ is a Feller kernel of separable measures.

Lemma 12.8. *Assumption 12.7 (i), (ii) are satisfied if S is completely metrizable, and Assumption 12.1 holds.*

Proof. Let T be a closed totally bounded subset of S, which is completely metrizable. Then T is compact.

(ii) Let $c > 0$. Then the set $T \times [0, c]$ is compact. Since g is continuous on $S \times \mathbb{R}_+$, g is uniformly continuous on $T \times [0, c]$. This implies (ii)

(i) Suppose that Assumption 12.7 (i) is false. Then there is some $\tilde{s} \in S$ such that $\{q(s, \cdot); s \in T\}$ is not equicontinuous at $\tilde{s}$.

Then there exists some $\epsilon > 0$ and a sequence (s_n) in T and a sequence $(\tilde{s}_n)$ in S such that $\tilde{s}_n \to \tilde{s}$ as $n \to \infty$ and

$$|q(s_n, \tilde{s}_n) - q(s_n, \tilde{s})| > \epsilon, \qquad n \in \mathbb{N}.$$

Since $T \times (\{\tilde{s}_n; n \in \mathbb{N}\} \cup \{\tilde{s}\})$ is a compact subset of S^2, q is uniformly continuous on this set, a contradiction. □

Lemma 12.9. *Let the Assumptions 12.1 and 12.7 (i) be satisfied. Further let $\mu \in \mathcal{M}_+^s(S)$ and (μ_n) be a sequence in $\mathcal{M}_+^s(S)$, $\|\mu_n - \mu\|_\flat \to 0$ as $n \to \infty$.*

Then, $(Q\mu_n)(s) \to (Q\mu)(s)$ as $n \to \infty$ uniformly for s in any closed totally bounded subset of S. Further $Q\mu_n$ and $Q\mu$ are bounded functions.

Proof. The convergence statement follows from Proposition 6.10. The boundedness statements are immediate. □

Lemma 12.10. *Let the Assumptions 12.1 and 12.7 (ii) be satisfied. Let T be a closed totally bounded subset of S and $f_n : T \to \mathbb{R}_+$, $n \in \mathbb{N}$, and $f : T \to \mathbb{R}_+$ be bounded functions such that $f_n \to f$ uniformly on T. Let $g : T \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. Then $g(s, f_n(s)) \to g(s, f(s))$ as $n \to \infty$ uniformly for $s \in T$.*

Proof. There exists some $c \in (0, \infty)$ such that $f_n(s), f(s) \le c$ for all $n \in \mathbb{N}$, $s \in T$. Since $\{g(s, \cdot); s \in T\}$ is uniformly equicontinuous on $[0, c]$, the assertion follows. □

Proposition 12.11. *Make the Assumptions 12.1 and Assumptions 12.7.*
Then Assumption 3.17 is satisfied for $\{\kappa^\mu; \mu \in \mathcal{M}_+(S)\}$ given by (12.2).

Proof. Let $f \in C_+^b(S)$. For $s \in S$,

$$\int_S f(t)\kappa^{\mu_n}(dt, s) - \int_S f(t)\kappa^\mu(dt, s)$$
$$= \int_S f(t)P(dt, s)\Big[g(s, (Q\mu_n)(s)) - g(s, (Q\mu)(s))\Big].$$

Since $\int_S f(t)P(dt, s) \le \sup f(S)$, it is sufficient to show that

$$g(s, (Q\mu_n)(s)) \to g(s, (Q\mu)(s)), \qquad n \to \infty,$$

uniformly for s in every closed totally bounded subset T of S. But this follows by combining Lemma 12.9 and Lemma 12.10. □

Assumption 12.12. We assume the following for the competitive influence function q and the per capita birth rate g:

(i) $\inf q(S^2) > 0$; (ii) $g(s, u) \to 0$ as $u \to \infty$ uniformly for $s \in S$.

Proposition 12.13. *Make the Assumptions 12.1 and Assumptions 12.12.*
Then Assumption 3.18 is satisfied.

Proof. For $\mu \in \dot{\mathcal{M}}_+(S)$, by Assumption 12.1 (g1) and (12.2),

$$\kappa^\mu(S, s) \le \sup_{s \in S} g\Big(s, \int_S q(s, t)\mu(dt)\Big).$$

Further, $\int_S q(s, t)\mu(dt) \ge \inf q(S^2)\, \mu(S)$. Since $\inf q(S^2) > 0$ by Assumption 12.12 (i), as $\mu(S) \to \infty$, $\int_S q(s, t)\mu(dt) \to \infty$ uniformly in $s \in S$. By Assumption 12.12 (ii), $\kappa^\mu(S, s) \to 0$ as $\mu(S) \to \infty$ uniformly for $s \in S$. □

Theorem 12.14. *Make the Assumptions 12.1, 12.7, and 12.12. Assume that the Feller kernel P is tight and that $\mathbf{r}(\kappa) > 1$ for the Feller kernel κ given by $\kappa(T, s) = P(T, s)g(s, 0)$. Then F has a nonzero fixed point in $\mathcal{M}_+^s(S)$.*

Proof. Combine Theorem 3.19 with Propositions 12.13, 12.11, 12.5, 12.2. □

13. Discussion

This paper establishes the spectral radius r_0 of a suitable Feller kernel as threshold parameter that decides about the extinction of the population the dynamics of which are modeled in the cone of nonnegative measures. This Feller kernel induces the order derivative of the yearly population turnover map F at the zero measure.

If $r_0 < 1$, the zero measure (the extinction state) is locally asymptotically stable and, under some extra conditions, even globally asymptotically stable (Theorem 12.3). If $r_0 > 1$, the zero measure is unstable (Theorem 12.6) and there exists a nonzero equilibrium measure (Theorem 12.14). While the first result is quite satisfactory, it is desirable to replace the instability of the extinction state by a persistence result analogously to population models in other state spaces [36] [73, Ch.7]: If $r_0 > 1$, then there exists some $\epsilon > 0$ such that $\liminf_{n\to\infty} \mu_n(S) \geq \epsilon$ for all solutions (μ_n) of $\mu_n = F(\mu_{n-1})$, $n \in \mathbb{N}$, with $\mu_0 \in \dot{\mathcal{M}}_+(S)$. Even more desirable is the existence of a compact persistence attractor [73, Sec.5.2]. The underlying technical problem consists in finding an eigenfunction f of the Feller kernel associated with r_0, $\quad r_0 f(s) = \int_S f(t)\kappa(dt, s), \ \ s \in S$. While we have found an eigenmeasure under acceptable assumptions (Section 10.2) and an eigenfunction in a special case (Remark 3.9), general assumptions for the existence of an eigenfunction we have found so far are quite restrictive and will be presented elsewhere.

The threshold role of r_0 makes estimates of this spectral radius desirable. Such estimates seem hard to come by, except in special cases [47, Sec.7] (Remark 3.9). Theoretically, formulas (9.11) and (9.9) provide lower estimates of the spectral radius and formula (9.12) provides upper estimates, but they may not be of very practical use. The inequality in (9.9) becomes an equality if there is an eigenmeasure associated with the spectral radius (Theorem 10.16). The inequality in (9.11) becomes an equality if there is a a bounded strictly positive eigenfunction associated with the spectral radius, and the one in (9.12) if the eigenfunction is also bounded away from zero. If the Feller kernel is the sum of two other Feller kernels, Remark 3.14 can be helpful provided that one of the kernels is of simple form.

We have strived for results that do not require that the individual state space S is separable or is completely metrizable (i.e., becomes complete after switching to a topologically equivalent metric). Depending on the application, individual state spaces could become quite intricate.

By results by Alexandrov and Mazurkiewicz, a subset of a complete metric space S is completely metrizable itself if and only if it is a countable intersection of open subsets (i.e., a so-called G_δ subset) of S [2, Sec.3.7] [4, Thm.2.5.4]. In particular, the set of rational numbers with the standard topology is not completely metrizable.

If S is the Banach space $S = C^b(K)$ of bounded continuous real-valued functions on a metric space K with the supremum norm, then S is separable if and only if K is compact [5, IV.13.16]. In particular, the Banach space ℓ^∞ of bounded real-valued sequences with the supremum norm is not separable.

The results in Section 3 and Section 12 appear to be new even if the individual state space S is separable or completely metrizable; some complicated looking assumptions like Assumption 12.7 (i) and (ii) are satisfied if S is completely metrizable.

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Conflict of interest

The author declares no conflict of interest.

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